Matched Filtering for Generalized Stationary Processes

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Abstract

The methods for solving optimal filtering problems in the case of the classical stationary processes are well-known since the late forties. Practice often gives rise to what is not a classical stationary process but a generalized one, and white noise is one simplest example. Hence it is of interest to describe the system action on the generalized stationary processes, and then to carry over filtering methods to them. For the arbitrary generalized stochastic processes this seems to be a challenging problem. In this correspondence we *identify* a rather general class of S_J*-generalized stationary* processes for which the desired extension can be done for matched filters. This class can be considered as a model of colored noise, and it is wide enough to include white noise, positive frequencies white noise, as well as certain generalized processes occurring in practice, namely when the smoothing effect gives rise the situation in which the distribution of probabilities may not exist at some time instances. One advantage of the suggested model is that it connects optimal filter design with inverting of integral operators; the methods for the latter can be found in the extensive literature.

I. MOTIVATION

1.1. Classical Stationary Processes. A complex-valued stochastic process $x(t)$ is called *stationary in the wide sense* (see, e.g., [D53]), if its mean is a constant,

$$E[x(t)] = const, \qquad\qquad -\infty < t < \infty$$

and the autocorrelation depends only on the difference $(t-s)$, i.e., $E[x(t)\overline{x(s)}] = k_x(t-s)$. We assume that $E[|x(t)|^2] < \infty$.

Let us consider a system with the memory depth of ω that maps the input stochastic process $x(t)$ into the output stochastic process $y(t)$ in accordance with the following rule:

$$y(t) = \int_{t-\omega}^{t} x(s)h(t-s)ds, \qquad h(t) \in L(0,w). \tag{1}$$

Wiener's seminal monograph [W49] described the construction of the optimal filter in the case $\omega = \infty$. His results were extended to the case $\omega < \infty$ in [ZR50]. Here w is called the memory depth.

1.2. Wiener filters and matched filters. Let the system receive a signal $a(t)$ corrupted by noise $x(t)$, which we assume, for a moment, to be a zero-mean stochastic process *stationary in the wide sense* (the more natural *generalized stationary* case is discussed below). In accordance with (1) the system output is

$$a_o(t) + y(t) = \int_0^T h(\tau)[a(t-\tau) + x(t-\tau)]d\tau, \tag{2}$$

as shown in the following picture

$$\xrightarrow{a(t)+x(t)} \boxed{h(t)} \xrightarrow{a_o(t)+y(t)}$$

Figure 1. Classical Optimal Filter.

The objective is to find the form of $h(t)$ that makes the system optimal in the sense of a certain criterion. Two particular criteria are recalled next.

Wiener filter. In the case of random signals the typical criterion is to minimize the mean-square value of the difference between the system output $a_o(t) + y(t)$ and the actual value $a(t)$ being observed. In this case $h(t)$ is called the *Wiener filter* [H00].

Matched filter. Here we consider a different approach mostly used when the signal is deterministic. In this case the standard criterion is to maximize, at the time moment t_0, the signal-to-noise ratio

$$S/N = \frac{a_o^2(t_0)}{\sigma^2}, \qquad \text{where} \qquad \sigma^2 = E[|y(t)|^2].$$

Such filters are referred to as *matched filters*, and the formulas for them were first derived in the classified reports [N43], [VVM44]. Matched filters are heavily used, e.g., in radar systems [LM04] and communications [H00]. Radar systems operate by periodically transmitting very short bursts of radio-frequency energy. The received signal is simply one or more replicas of the transmitted signal that are created by being reflected from any objects that the transmitted signal illuminates. Thus, the form of the received signals is known exactly. The things that are not known about received signals are the number of reflections, the time delay between the transmitted and received signals, the amplitude, and even whether there is a received signal or not. It can be shown that the probability of detecting a weak radar signal in the presence of noise is greatest when the signal-to-noise ratio is the greatest, see, e.g., [LU48], [K47].

A similar situation arises in digital communication systems [H00]. In such a system the message to be transmitted is converted to a sequence of binary signals, say 0 and 1. Each of them is represented as a time function having a specified form. For example, a negative rectangular pulse can represent 0 while a positive rectangular pulse can represent 1. Again, in the presence of noise the probability to recognize the right pulse at the receiver is greatest when the signal-to-noise ratio is greatest, see, e.g., Chapter 4 of [H00].

1.3. Generalized stationary processes. Motivation. Since the early 1950's, when Schwartz published his theory of distributions, generalized functions have found numerous applications in various fields of science and engineering. An (ordinary) function is a function $f(t)$ of a real number t. A *generalized* function is a *functional* f assigning a *test function* φ the number $f(\varphi)$, see, e.g., [V02] for the detailed elaboration and exposition. For example, the Dirac delta function δ can be introduced as linear functional of φ by the relation

$$\delta(\varphi) = \varphi(0) = \int \delta(t)\varphi(t)dt, \tag{3}$$

and thus δ is a generalized function. One of the most useful aspects of this theory in applications is that discontinuous functions can be handled as easily as continuous or differentiable functions. In particular, the derivatives of (ordinary) continuous functions (while generally may not exist as ordinary functions) can be thought as generalized functions. This provided an extremely powerful tool in *formulating* and then solving many problems, e.g., of mathematical physics. Following this success story, Gelfand and Vilenkin introduced in [GV61] the concept of *generalized stochastic processes*. An (ordinary) stochastic process is a function $X(t)$ of t such that for each t, $X(t)$ is a random variable. A *generalized* stochastic process is a *functional* X assigning a *test function* φ a random variable $X(\varphi)$.

One of the advantages of this approach is that the derivatives of classical stochastic processes (while generally may not exist in the usual sense) can be thought as generalized functions. For example, *white noise* $X(t)$ (having equal intensity at all frequencies within a broad band) is not a stochastic process in the *classical sense*. In fact, white noise can be thought of as the *derivative of a Brownian motion*, which is a continuous stationary stochastic process $W(t)$. It can be shown that $W(t)$ is nowhere differentiable, a fact explaining the highly irregular motions that Robert Brown observed. This means that white noise $\frac{dW(t)}{dt}$ does not exist in the ordinary sense. In fact, white noise is a *generalized stochastic process*.

Generally, any receiving device has a certain "inertia" and hence instead of actually measuring the classical stochastic process $\xi(t)$ it measures its averaged value (for motivation cf. with (3))

$$\Phi(\varphi) = \int \varphi(t)\xi(t)dt, \tag{4}$$

where $\varphi(t)$ is a certain function characterizing the device. Small changes in φ yield small changes in $\Phi(\varphi)$ (small changes in the receiving devices yield closer measurements), hence Φ is a continuous linear functional, i.e., a generalized stochastic process [GV61]. Hence it is very natural and important to solve the optimal filtering problem in the case of generalized stochastic processes.

1.4. The main result. Solving filtering problems for arbitrary generalized stochastic processes seems to be a challenging problem, and there was no progress since their introduction in [GV61]. In this correspondence we identify a rather general new class of S_J*-generalized processes*, for which the matched filtering problem can be solved by methods based on integral equations [M77], [S97]. The process Φ is called S_J*-generalized* if for $\varphi(t), \psi(t)$ vanishing outside $J = [a, b]$ we have

$$E[\Phi(\varphi)\overline{\Phi(\psi)}] = (S_J\varphi, \psi)_{L^2}, \quad \text{where} \quad S_J\varphi = \frac{d}{dt}\int_a^b \varphi(u)s(t-u)du \in L^2(a,b). \tag{5}$$

We provide, for the first time, formulas that completely solve the problem of designing S_J-generalized matched filters. We show that if the equation $S_J f_0(u) = a(t_0 - t)$ is solvable then the S_J-matched filter is given by $h_{opt} = \frac{f_0}{(g_0, f_0)_{L^2}}$. Here t_0 is the time moment for which signal-to-noise-ratio is to be maximized. In particular, if S_J is invertible then the formula simplifies to $h_{opt} = \frac{S_J^{-1}a(t_0-t)}{(g_0, S_J^{-1}a(t_0-t))_{L^2}}$.

1.5. Examples. A new model of colored noise. PF-white noise. We have just showed that solving the S_J-generalized matched filtering problem is reduced to solving integral equations or inverting integral operators. There is an extensive literature on this subject [M77], [S97] whose methods can be now employed to solve a number of specific filtering problems. We show how the approach works in three cases.

- In examples 1, 2 we indicate that white noise corresponds to the trivial case in which S_J is just the identity operator. The kernel $s(t)$ of such $S_J = I$ in (5) is the delta function which is the Fourier transform of $f_W(z) = 1$ (i.e. equal intensity at all frequencies).
- We provide two more involved examples. In example 3 we consider a new model of *colored noise*, i.e., when the latter is approximated by $f_{cn}(z) = \sum_{m=1}^{N} \gamma_m \frac{1}{z^2+\alpha_m^2}, \alpha_m > 0, \gamma_m > 0.$ We use its Fourier transform $k(t) = \sum_{m=1}^{N} \frac{\pi\gamma_m}{\alpha_m} e^{-\alpha_m|t|}$ to define

$$S_J f = f(x)\mu + \int_0^w f(t)k(x-t)dt.$$

  For this case we obtain the algorithm to compute S_J^{-1}, and thus constructively solve the S_J-generalized matched filtering problem.
- In example 4 we consider $f_{PW}(z) = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0 \end{cases}$ representing the equal intensity at all *positive frequencies* and the zero intensity at the negative frequencies (hence the name *PF-white noise*). Then

$$f(z) = f_{PW}(z) + (D-1)f_W(z), \qquad D \geq 1 \tag{6}$$

represents a mix of white and PF-white noises. Let us now use the Fourier transform $s(t)$ of $f(z)$ in (6) as a kernel to define the operator S_J by (5). Again, we obtain the algorithm to compute S_J^{-1}, and thus constructively solve the S_J-generalized matched filtering problem.

II. Generalized Stationary Processes

Let $\mathcal{K}$ be the set of all infinitely differentiable finite functions. A stochastic functional Φ assigns to any $\varphi(t) \in \mathcal{K}$ a stochastic value $\Phi(\varphi)$. A stochastic functional Φ is called linear if $\Phi(\alpha\varphi + \beta\psi) = \alpha\Phi(\varphi) + \beta\Phi(\psi)$. Let us now assume that all the stochastic values $\Phi(\varphi)$ have expectations $m(\varphi)$ that depend continuously on φ as

$$m(\varphi) = E[\Phi(\varphi)] = \int_{-\infty}^{\infty} x dF(x), \quad \text{where} \quad F(x) = P[\Phi(\varphi) \leq x].$$

The functional $m(\varphi)$ is linear in the space $\mathcal{K}$. The bilinear functional

$$B(\varphi, \psi) = E[\Phi(\varphi)\overline{\Phi(\psi)}] \tag{7}$$

is a correlation functional of a stochastic process. It is supposed that $B(\varphi, \psi)$ is continuously dependent on each of the arguments.

The stochastic process Φ is called *generalized stationary in the wide sense* [GV61], [S97] if for any functions $\varphi(t)$ and $\psi(t)$ from $\mathcal{K}$ and for any number h the equalities

$$m[\varphi(t)] = m[\varphi(t+h)], \qquad B[\varphi(t), \psi(t)] = B[\varphi(t+h), \psi(t+h)] \tag{8}$$

hold true.

Let us denote by $\mathcal{K}_J$ the set of the functions from $\mathcal{K}$ such that $\varphi(t) = 0$ when $t \notin J = [a, b]$. The correlation functional $B_J(\varphi, \psi)$ is called a *segment* of the correlation functional $B(\varphi, \psi)$ if

$$B_J(\varphi, \psi) = B(\varphi, \psi), \qquad \varphi, \psi \in \mathcal{K}_J. \tag{9}$$

In what follows we consider the generalized stationary processes of the form

$$B_J(\varphi, \psi) = (S_J\varphi, \psi)_{L^2}, \tag{10}$$

where $(\cdot, \cdot)_{L^2}$ is the inner product in the space $L^2(a, b)$, and S_J is a bounded nonnegative operator acting in $L^2(a, b)$ and having the form

$$S_J\varphi = \frac{d}{dt}\int_a^b \varphi(u)s(t-u)du. \tag{11}$$

Definition 1: Generalized stationary processes satisfying (10) and (11) are called S_J*-generalized processes*.

Example 1: **White noise.** It is well-known [L68] that the correlation functional for the Wiener stochastic process Υ (Brownian motion) is given by

$$B(\varphi,\psi)=\int_0^\infty[\widehat{\varphi}(t)-\widehat{\varphi}(\infty)]\overline{[\widehat{\psi}(t)-\widehat{\psi}(\infty)]}dt,$$

where

$$\widehat{\varphi}(t)=\int_0^t\varphi(t)dt,\qquad \widehat{\psi}(t)=\int_0^t\psi(t)dt.$$

White noise W (which is the derivative of Υ) is not a continuous stochastic process. In fact, it is a generalized stationary process whose correlation functional is known [L68] to be

$$B'(\varphi,\psi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(t-s)\varphi(t)\overline{\psi(s)}ds$$

Thus, in this case we have $B'(\varphi,\psi)=(\varphi,\psi)_{L_2}$ and hence (10) implies that white noise Φ is a very special S_J-generalized stationary process with

$$S_J=I. \qquad (12)$$

III. System action on the generalized stationary processes

Let the system receive the generalized stationary signal Φ and the deterministic signal $a(t)$. We assume that Φ is zero-mean and the correlation functional $B(\varphi,\psi)$ is known. At the output we obtain the generalized process of the form

$$a_o(t)+\Psi, \qquad (13)$$

where (cf. with (1) and (2))

$$a_o(t)=\int_0^T h(\tau)a(t-\tau)d\tau. \qquad (14)$$

The problem is how to describe the system action for the generalized stationary processes shown in the Figure 2.

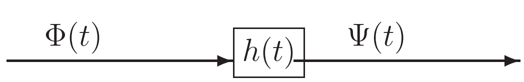

Figure 2. Generalized stationary processes

Here we answer the latter question and define it as follows

$$\Psi(\varphi)=\Phi[\int_0^T h(\tau)\varphi(t+\tau)d\tau], \qquad (15)$$

so that the overall system is described in Figure 3.

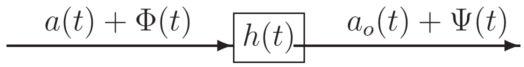

Figure 3. Generalized Matched Filter.

Proposition 1: Let $x(t)$ and $y(t)$ be the classical stationary processes. Then the formula (15) is equivalent to the relation

$$y(t) = \int_0^T x(t-\tau)h(\tau)d\tau. \tag{16}$$

Remark 1: The proposition 1 has the following meaning. The formula (16) describes the behavior of a a classical system, cf. with (1) and (2). It follows that the formula (15) suggested here indeed generalizes (16). In the case of the classical stationary processes our definition coincides with the standard one.

Proof of the proposition 1. The functionals

$$\Phi(\varphi) = \int_{-\infty}^{\infty} x(t)\varphi(t)dt, \qquad \Psi(\varphi) = \int_{-\infty}^{\infty} y(t)\varphi(t)dt \tag{17}$$

are associated with the processes $x(t)$ and $y(t)$. It follows from (16), (17) that

$$\Psi(\varphi) = \int_{-\infty}^{\infty}\int_0^T x(t-\tau)h(\tau)d\tau\varphi(t)dt = \int_{-\infty}^{\infty} x(t)\int_0^T h(\tau)\varphi(t+\tau)d\tau dt. \tag{18}$$

Hence (16) implies (15). In the same way we can prove the converse.

■

IV. Generalized Matched Filters

The objective is to choose the function $h(\tau)$ so that it characterizes the detected signal in an optimal way. If we consider the case of the classical stationary processes $x(t)$ then the criterium of the system quality is maximizing, at the time moment t_0, the signal-to-noise ratio defined by the formula

$$S/N = \frac{a_o^2(t_0)}{\sigma^2} \tag{19}$$

where

$$\sigma^2 = E[|y(t)|^2] = \int_0^T\int_0^T h(u)B(u-v)\overline{h(v)}dvdu \tag{20}$$

Here $B(u)$ is the correlation function of the process $x(t)$.

Problem 1: Let us now replace the formula (20) by

$$\sigma^2 = B(h,h), \qquad h \in L_2(0,T). \tag{21}$$

using correlation functional $B(\varphi, \psi)$ defined by (7), i.e., the one corresponding to a *generalized* stationary process. Then the formula (19) makes sense in the case of the generalized stationary process as well.

Find $h(t) \in L^2(0, T)$ such that the signal-to-noise ratio S/N at the time moment t_0 has the greatest value.

V. S_J-GENERALIZED MATCHED FILTERS

We solve the problem for the case of S_J-generalized processes where $J = [0, T]$. In this case the equality

$$\sigma^2 = (S_J h, h) \tag{22}$$

is valid. If $h(t)$ is a solution to the problem 1 then $ch(t)$ is a solution as well. Hence, without any loss of generality, we may assume that

$$\int_0^T h(t)a(t_0 - t)dt = 1. \tag{23}$$

Problem 1 is equivalent to the following problem.

Problem 2: Find the minimum of the form$(S_J h, h)$ under the constraint (23)

By solving the problem 2 we assume that there exists a function $f_0(t)$ which belongs to $L^2(0, T)$ and satisfies the relation

$$S_J f_0 = g_0, \qquad \text{where} \qquad g_0(t) = a(t_0 - t). \tag{24}$$

Remark 2: If the operator S_J is invertible then f_0 exists, and

$$f_0 = S_J^{-1} g_0. \tag{25}$$

Taking into account the equality (24) we can rewrite the condition (23) in the following form

$$(h, S_J f_0)_{L^2} = (\sqrt{S_J} h, \sqrt{S_J} f_0)_{L^2} = 1. \tag{26}$$

The latter and the Schwarz inequality imply $1 \leq (S_J h, h)_{L^2} (S_J f_0, f_0)_{L^2}$, i.e.,

$$(S_J h, h) \geq \frac{1}{(S_J f_0, f_0)_{L^2}} = \frac{1}{(g_0, f_0)_{L^2}} = \nu_{min}. \tag{27}$$

Equality in (27) takes place if and only if

$$\sqrt{S_J} h = \beta \sqrt{S_J} f_0. \tag{28}$$

Hence we obtain the minimal value

$$\nu_{min} = \frac{1}{(g_0, f_0)_{L^2}} \tag{29}$$

when

$$h = \beta f_0. \tag{30}$$

Now, in view of the condition (23) we have

$$\beta = \frac{1}{(g_0, f_0)_{L^2}}. \tag{31}$$

Thus, the solution to the problem 1 has the form

$$h_{opt} = \frac{f_0}{(g_0, f_0)_{L^2}}, \qquad \mu_{max} = S/N = (g_0, f_0)_{L^2}. \tag{32}$$

where g_0 and f_0 are defined by the formulas (24) and (25), respectively.

Theorem 1: If the operator S_J is invertible then there always exist a unique solution of the form (32) with $f_0 = S_J^{-1} g_0$, i.e.,

$$h_{opt} = \frac{S_J^{-1} g_0}{(g_0, S_J^{-1} g_0)_{L^2}}, \qquad \mu_{max} = S/N = (g_0, S_J^{-1} g_0)_{L^2}. \tag{33}$$

Proposition 2: If $(S_J h, h) > 0$ when $h \neq 0$ then the problem 1 has at most one solution.

Example 2: **White noise.** Recall that in white noise (12) case $S_J = I$ and hence the solution

$$h_{opt} = \frac{a(t_0 - t)}{\int_0^T |a(t_0 - t)|^2 dt}, \quad \text{and} \quad \mu_{max} = S/N = \int_0^T |a(t_0 - t)|^2 dt$$

is unique.

Problem 3: In this paper we solve the matched filtering problem for generalized stationary processes. We would like to conclude this section with the interesting open problem of extending the Kalman filtering method to generalized stationary processes.

VI. Some practical consequences. A connection to the techniques available in the integral equations literature

The main focus of the sections II - V had mostly a theoretical nature. In this section we indicate that theorem 1 offers a novel technique allowing one to work out practical problems. Specifically:

- Filtering problems for *classical* stationary processes typically lead to non-invertible operators S_J, and to find the solution (32) one needs to solve (24).

In the case of *generalized* stationary processes the operator S_J is often invertible and hence there are better formulas (33).

- Secondly, the operator S_J in (10) and (11) can be seen as a new way of modelling colored noise. This model is useful since the existing integral equations literature already describes many particular examples on inverting S_J, either explicitly or numerically. Hence (33) solves the corresponding filtering problem.

Before providing two such examples we shall rewrite the operator S_J of (11),

$$S_J f = \frac{d}{dx}\int_a^b f(t)s(x-t)dt, \qquad f(x) \in L^2(a,b)$$

in a more familiar form. Let $J = [a,b] := [0,w]$, and let $s(x)$ be a function differentiable when $x \neq 0$. Let

$$s'(x) = k(x) \in L^2(-w,w), \qquad s(+0) - s(-0) = \mu.$$

Then S_J has the more familiar form

$$S_J f = f(x)\mu + \int_0^w f(t)k(x-t)dt. \tag{34}$$

The integral equations literature (see, e.g., [GS72], [S96]) contains results on the inversion of the operator S_J of the form (34). The following theorem is well-known.

Theorem 2: **(Gohberg-Semencul)** Let the operator S_J has the form (34) with $k(x) \in L(-w,w)$. If there are two functions $\gamma_\pm(x) \in L(0,w)$ such that

$$S_J\gamma_+(x) = k(x), \qquad S_J\gamma_-(x) = k(x-w) \tag{35}$$

then $S_{[0,w]}$ is invertible in $L^p(0,w)$ ($p \geq 1$) and

$$S_J^{-1} f = f(x) + \int_0^w f(t)\gamma(x,t)dt, \tag{36}$$

where $\gamma(x,t)$ is given by (37).

$$\gamma(x,t) = \begin{cases} -\gamma_+(x-t) - \int_t^{w+t-x} [\gamma_-(w-s)\gamma_+(s+x-t) - \gamma_+(w-s)\gamma_-(s+x-t)]ds & x > t, \\ -\gamma_-(x-t) - \int_t^{w} [\gamma_-(w-s)\gamma_+(s+x-t) - \gamma_+(w-s)\gamma_-(s+x-t)]ds & x < t \end{cases} \tag{37}$$

The latter result leads to a number of special cases when the operator S_J can be explicitly inverted and hence the formulas (33) solve the matching filter problem in these cases. We provide two such examples next.

Example 3: **Colored noise approximated by rational functions. The exponential kernel.** Let us consider *colored noise* approximated by a combination of rational functions with the fixed poles $\{\pm i\alpha_m\}$,

$$f(t)=\sum_{m=1}^{N}\gamma_m\frac{1}{t^2+\alpha_m^2},\qquad \alpha_m>0,\qquad \gamma_m>0.$$

Let us use its Fourier transform

$$k(x)=\sum_{m=1}^{N}\beta_m e^{-\alpha_m|x|},\qquad \beta_m=\frac{\pi}{\alpha_m}\gamma_m \tag{38}$$

to define the operator S_J via (34).

Solution to the filtering problem. The situation is exactly the one captured in theorem 2 where the operator (34) has the special kernel (38). A procedure to solve (35), and hence to find the inverse of S_J is obtained next.

Theorem 3: **(Procedure)** Let S_J be given by (34) and its kernel $k(x)$ have the special form (38). Then S_J^{-1} is given by the formula (36), (37), (35), where

$$\gamma_+(x)=-\gamma(x,0),\qquad \gamma_-(x)=-\gamma(w-x,0). \tag{39}$$

Here

$$\gamma(x,0)=G(x)\begin{bmatrix}F_1\\ F_2\end{bmatrix}^{-1}B, \tag{40}$$

where a $1\times 2N$ row $G(x)$, $N\times 2N$ matrices F_1, F_2 and a $2N\times 1$ column B are defined by

$$G(x)=\begin{bmatrix} e^{\nu_1 x} & e^{\nu_2 x} & \cdots & e^{\nu_{2N}x}\end{bmatrix},\qquad F_1=\begin{bmatrix}\frac{1}{\alpha_i+\nu_k}\end{bmatrix}_{1\le i\le N,1\le k\le 2N},$$

$$F_2=\begin{bmatrix}\frac{-e^{\nu_k w}}{\alpha_i-\nu_k}\end{bmatrix}_{1\le i\le N,1\le k\le 2N},\qquad B=\begin{bmatrix}\underbrace{1\ \cdots\ 1}_{N} & \underbrace{0\ \cdots\ 0}_{N}\end{bmatrix}.$$

The numbers $\{\nu_k\}$ are the roots (we assume them to be pairwise different) of the polynomial

$$Q(z)=P(z)-2\sum_{m=1}^{N}\delta_m\sum_{s=1}^{m}z^{2(m-s)}\sum_{k=1}^{N}\alpha_k^{2s-1}\beta_k, \tag{41}$$

where the numbers $\{\delta_k\}$ are the coefficients of the polynomial $P(z)=\sum_{m=1}^{N}\delta_m z^{2m}$ defined by

$$P(z)=\prod_{m=1}^{N}(z^2-\alpha_m^2). \tag{42}$$

Hence, in this important case, the formulas (33) allow us to find the explicit solution h_{opt} and μ_{max} to the problem 1 via plugging in the formula (36), (37) together with (39) (40) that can be found using the methods of the integral equations theory.

Example 4: **Positive frequencies white noise (PF-white noise).** Recall (cf with the example 2) that the white noise case corresponds to $S_J = I$. The kernel $s(t)$ of such $S_J = I$ in (11) is the delta function which is the Fourier transform of $f_W(z) = 1$ (i.e. equal intensity at all frequencies). Similarly, $f_{PW}(z) = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0 \end{cases}$ represents the equal intensity at all *positive frequencies* and the zero intensity at the negative frequencies (hence the name *PF-white noise*). Then

$$f(z) = f_{PW}(z) + (D-1)f_W(z), \qquad D \geq 1 \tag{43}$$

represents a mix of white and PF-white noises. Let us now use the Fourier transform $s(t)$ of $f(z)$ in (43) as a kernel to define the operator S_J via (11) (cf. with (34)), i.e.,

$$S_J f = fD + \frac{j}{\pi} ⨍_0^T \frac{f(t)}{x-t}dt, \quad f \in L_2[0,T], \tag{44}$$

where $⨍_0^T$ is the Cauchy Principal Value integral, and $D \geq 1$.

Solution to the filtering problem. In the case of a pure PF-white noise (i.e., when $D = 1$) the operator S_J (44) is noninvertible and (32) should be used. In the case $D > 1$ the operator is invertible. Moreover, the following result is implicit in the literature. We provide a direct transparent proof in the appendix.

Theorem 4: **(Explicit formula)** If $D > 1$ then S_J is positive definite and invertible with

$$S_J^{-1} f = f(x)D_1 - \beta \frac{j}{\pi} ⨍_0^T (\frac{t}{T-y})^{j\alpha} (\frac{x}{T-x})^{-j\alpha} \frac{f(t)}{x-t}dt, \tag{45}$$

where

$$D_1 = \frac{D}{D^2-1}, \qquad \beta = \frac{1}{D^2-1}, \tag{46}$$

and the number α is obtained from

$$\cosh \alpha\pi = D|\sinh \alpha\pi|. \tag{47}$$

Hence, in this important case, plugging the formula (45) in (33) allow us to find the solution to the problem 1 explicitly.

VII. Appendix. Proof of theorem 3

By comparing (34) and (36) we obtain

$$\gamma(x,t)+k(x,t)+\int_0^w k(x,u)\gamma(u,t)du=0, \tag{48}$$

and when $t=0$

$$\gamma(x,0)+k(x,0)+\int_0^w k(x,u)\gamma(u,0)du=0. \tag{49}$$

Let us now apply to both sides of (49) the operator $P(D)$, where D is an operator of differentiation with respect to x, and $P(z)$ is a polynomial given in (42). As a result we obtain

$$Q(D)\gamma(x,0)=0, \tag{50}$$

where the polynomial $Q(z)$ is given by (41). The relation (50) implies

$$\gamma(x,0)=\sum_{m=1}^{2N} c_m e^{\nu_m x}. \tag{51}$$

Now, (51) and (49) yield (40) Finally, using (40) we solve (35) and obtain (39).

VIII. Appendix. Proof of theorem 4

It is well-known [M77] that the operator

$$Vf=\frac{j}{\pi}\int_0^T \frac{f(t)}{x-t}dt,$$

is a contraction: $\|V\|<1$. Hence the operator S_J in (44) is nonnegative if $D\geq 1$. If $D>1$ then S_J is invertible. A similar argument applied to the operator T_J defined by the right hand side[1] of (45) implies that if $D>1$ then T_J is bounded, positive definite and invertible.

Let us check if $S_J T_J = I$. To this end let us compute

$$S_J T_J f = DD_1 f+\frac{iD_1}{\pi}\int_0^T \frac{f(t)}{x-t}dt-\beta\frac{i}{\pi}D\int_0^T (\frac{t}{T-t})^{i\alpha}(\frac{x}{T-x})^{-i\alpha}\frac{f(t)}{x-t}dt+ \tag{52}$$

$$\frac{\beta}{\pi^2}\int_0^T (\frac{x}{T-x})^{-i\alpha}(\frac{t}{T-t})^{i\alpha}\frac{1}{x-t}\cdot\int_0^T\frac{f(u)}{t-u}du.$$

According to the Poincare-Bertran formula [M77]

$$\int_0^T (\frac{t}{T-t})^{i\alpha}\frac{1}{x-t}\int_0^T \frac{f(u)}{t-u}du=\int_0^T f(u)\int_0^T (\frac{t}{T-t})^{i\alpha}\frac{1}{(x-t)(t-u)}dtdu-\pi^2 f(x)(\frac{x}{T-x})^{i\alpha}. \tag{53}$$

[1]Let us denote by T_J a candidate for the inverse defined by the right hand side of (45).

Denoting $W(z) = e^{-i\alpha\int_0^T \frac{dt}{t-z}}$ we see that it admits a representation

$$W(z) = 1 + \frac{1}{2\pi}\int_0^T [W_+(\sigma) - W_-(\sigma)]\frac{1}{\sigma - z}d\sigma, \tag{54}$$

where

$$W_\pm(\sigma) = \lim_{\tau\to\pm 0} W(\sigma + i\tau) = (\frac{\sigma}{T-\sigma})^{i\alpha}e^{\pm\pi\alpha}, \quad \sigma = \overline{\sigma}, \alpha = \overline{\alpha}. \tag{55}$$

Now, the Sohotsky-Plemelj formula and (54) imply

$$W_+(s) - W_-(s) = 2 + \frac{1}{\pi i}\int_0^T \frac{W_+(\sigma) - W_-(\sigma)}{\sigma - s}d\sigma = 2 + \frac{1}{\pi i}\int_0^T (\frac{\sigma}{T-\sigma})^{i\alpha}(e^{\pi\alpha} - e^{-\pi\alpha})\frac{1}{\sigma - s}d\sigma.$$

From here we obtain

$$\int_0^T (\frac{t}{T-t})^{i\alpha}\frac{1}{(x-t)(t-u)} = \frac{\pi i}{u-x}\frac{\cosh\pi\alpha}{\sinh\pi\alpha}[(\frac{u}{T-u})^{i\alpha} - (\frac{x}{T-x})^{i\alpha}]. \tag{56}$$

Finally, comparing (52), (53) and (56), and using (46) and (47) we arrive at the desired identity

$$S_J T_J = I.$$

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