# CHARACTERIZATIONS OF QUASISEPARABLE MATRICES AND THEIR SUBCLASSES VIA RECURRENCE RELATIONS AND SIGNAL FLOW GRAPHS

T.BELLA*, Y.EIDELMAN†, I.GOHBERG† , AND V.OLSHEVSKY*

**Abstract.** The three-term recurrence relations satisfied by real-orthogonal polynomials (related to irreducible tridiagonal matrices) and the two- and three-term recurrence relations satisfied by the Szegö polynomials (related to unitary Hessenberg matrices) are all well-known. In this paper we consider more general two- and three-term recurrence relations, and prove that the related classes of matrices are all Hessenberg order one quasiseparable ($(H,1)$-quasiseparable) matrices.

Specifically, we give several characterizations of $(H,1)$-quasiseparable matrices and some subclasses including diagonal plus order-one upper semiseparable matrices. Characterizations are given in terms of the quasiseparable generators, in terms of the recurrence relations satisfied by their corresponding systems of polynomials, and in terms of the corresponding signal flow graphs.

**1. Introduction.** In this paper we present complete characterizations of several classes of matrices in terms of their polynomial systems. Throughout the paper, we will say that an $n \times n$ matrix $A = (a_{i,j})$ is *related to* or *corresponds to* a system of polynomials $\{r_k(x)\}$ provided

$$r_k(x) = \frac{1}{a_{2,1}a_{3,2}\cdots a_{n,n-1}} \det{(xI - A)_{(k\times k)}}, \quad (A \text{ is upper Hessenberg}). \tag{1.1}$$

We will also associate a set of recurrence relations with the system of polynomials satisfying them, and speak of the matrix related to or corresponding to a set of recurrence relations.

All matrices considered in this paper are assumed to be upper Hessenberg ($a_{i,j} = 0$ for $i > j+1$) and all elements of the subdiagonal nonzero ($a_{i+1,i} \neq 0$ for $i = 1, \ldots, n-1$). We suggest to call such matrices *strongly Hessenberg.*

We begin by considering two classical classes of matrices and the related systems of polynomials and the recurrence relations they satisfy.

**1.1. Real orthogonal polynomials & irreducible tridiagonal matrices.** It is well-known that systems of $n+1$ polynomials $\{r_k(x)\}_{k=0}^{n}$ orthogonal with respect to an inner product of the form

$$\langle p(x), q(x) \rangle = \int_a^b p(x)q(x)w^2(x)dx$$

satisfy three-term recurrence relations of the form

$$r_k(x) = (\alpha_k x - \delta_k)r_{k-1}(x) - \gamma_k \cdot r_{k-2}(x), \quad \alpha_k \neq 0, \gamma_k > 0. \tag{1.2}$$

Real orthogonal polynomials are also related to irreducible tridiagonal matrices. Specifically, the polynomials $\{r_k(x)\}$ satisfying (1.2) are related to the irreducible tridiagonal matrix

$$A = \begin{bmatrix} \frac{\delta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & 0 \\ \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} & \ddots & \ddots & \vdots \\ 0 & \frac{1}{\alpha_2} & \ddots & \frac{\gamma_{n-1}}{\alpha_{n-1}} & 0 \\ \vdots & & \ddots & \frac{\delta_{n-1}}{\alpha_{n-1}} & \frac{\gamma_n}{\alpha_n} \\ 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n} \end{bmatrix}. \tag{1.3}$$

via

$$r_k(x) = \alpha_1 \cdots \alpha_k \det{(xI - A)_{(k\times k)}}.$$

In particular, there is a bijection between irreducible tridiagonal matrices $A$ of the form (1.3) and recurrence relations (1.2), and hence they provide a complete characterization of each other. [1]

*Department of Mathematics, University of Connecticut, Storrs CT 06269-3009, USA. Email: bella@math.uconn.edu

†School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat-Aviv 69978, Israel.

[1]The standard result on such a bijection (see, for instance, [SB92]) is typically formulated for the **monic** polynomials $\{r_k(x)\}$ and for the symmetric irreducible tridiagonal matrices $\widetilde{A}$. It is easy to see, however, that for a symmetric irreducible

**1.2. Szegö polynomials & unitary Hessenberg matrices.** Next we consider the $n+1$ Szegö polynomials $\Phi^{\#} = \{\phi_k^{\#}(x)\}_{k=0}^{n}$, or polynomials orthonormal on the unit circle with respect to an inner product of the form

$$\langle p(x), q(x)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(e^{i\theta}) \cdot [q(e^{i\theta})]^* \cdot w^2(\theta) d\theta,$$

For any such inner product, it is known [GS58] that there exist a set of *reflection coefficients*[2] $\{\rho_k\}$ satisfying

$$\rho_0 = -1, \qquad |\rho_k| < 1, \quad k = 1, \ldots, n-1, \quad |\rho_n| \leqslant 1,$$

and *complementary parameters* $\{\mu_k\}$ defined by the reflection coefficients via

$$\mu_k = \begin{cases} \sqrt{1-|\rho_k|^2} & |\rho_k| < 1 \\ 1 & |\rho_k| = 1 \end{cases} \tag{1.4}$$

such that the corresponding Szegö polynomials satisfy the two-term recurrence relations

$$\begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_0}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_k}\begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix}\begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^{\#}(x) \end{bmatrix}, \quad (k = 1, 2, \ldots, n), \tag{1.5}$$

where the polynomial system $\{\phi_k(x)\}$ is a system of auxiliary polynomials. The Szegö polynomials are also known to be related to (almost) unitary Hessenberg matrices[3] of the form

$$H = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & \cdots & -\rho_0^*\mu_1\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & \cdots & -\rho_1^*\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ 0 & \mu_2 & -\rho_2^*\rho_3 & \cdots & -\rho_2^*\mu_3\cdots\mu_{n-1}\rho_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -\rho_{n-1}^*\rho_n \end{bmatrix} \tag{1.6}$$

via the equation

$$\phi_k^{\#}(x) = \frac{1}{\mu_1\cdots\mu_k}\det(xI - H)_{(k\times k)},$$

see, e.g, [G82, KP83, BC92, T92, ACR93, BH95, R95, O98, O01], and the references therein.

Again, there is a bijection between strongly Hessenberg matrices $H$ of the form (1.6) and recurrence relations of the form (1.5), and hence they again provide a complete characterization of each other.

It was shown by Geronimus in [G48] that, under the additional restriction of $\rho_k \neq 0$ for each $k$, the corresponding Szegö polynomials satisfy the three-term recurrence relations

$$\begin{aligned} &\phi_0^{\#}(x) = \frac{1}{\mu_0}, \quad \phi_1^{\#}(x) = \frac{1}{\mu_1}(x\cdot\phi_0^{\#}(x) + \rho_1\rho_0^*\cdot\phi_0^{\#}(x)) \\ &\phi_k^{\#}(x) = \left[\frac{1}{\mu_k}\cdot x + \frac{\rho_k}{\rho_{k-1}}\frac{1}{\mu_k}\right]\phi_{k-1}^{\#}(x) - \frac{\rho_k}{\rho_{k-1}}\frac{\mu_{k-1}}{\mu_k}\cdot x\cdot\phi_{k-2}^{\#}(x), \quad (k = 2, \ldots, n) \end{aligned} \tag{1.7}$$

Obviously, the class of unitary Hessenberg matrices is partitioned into two classes: those with $\rho_k \neq 0$ for all $k$, and those with at least one $\rho_k = 0$. Motivated by this result of Geronimus, we make the following definition.

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tridiagonal $\widetilde{A}$ there is a unique invertible diagonal matrix $D$ such that $A = D^{-1}\widetilde{A}D$ has the same first subdiagonal shown in (1.3). Since the latter is completely determined by the leading coefficients of $\{r_k(x)\}$, the bijection (1.1) between (1.3) and (1.2) is just a variant of a classical result.

[2]Reflection coefficients are also known in various contexts as Schur parameters [S17], Verblunsky coefficients [S05], parcor coefficients.

[3]Throughout the paper, matrices referred to as unitary Hessenberg are almost unitary, differing from unitary in the last column. Specifically, $H = UD$ for a unitary matrix $U$ and diagonal matrix $D = \mathrm{diag}\{1, ..., 1, \rho_n\}$.

Definition 1.1. *A unitary Hessenberg matrix $H$ of the form (1.6) is a Geronimus-type unitary Hessenberg matrix provided the corresponding system of Szegö polynomials satisfy three-term recurrence relations, and non-Geronimus-type unitary Hessenberg matrix otherwise. Szegö polynomials corresponding to Geronimus-type unitary Hessenberg matrices are called Geronimus-type Szegö polynomials, and non-Geronimus-type Szegö polynomials otherwise.*

So the result of Geronimus implies that unitary Hessenberg matrices with $\rho_k \neq 0$ for each $k$ are Geronimus-type unitary Hessenberg matrices. In Section 5, we give a slightly larger class for which three-term recurrence relations exist for the corresponding Szegö polynomials, and give a complete characterization of Geronimus-type unitary Hessenberg matrices.

Next an interpretation of the Szegö polynomials is given in terms of signal flow graphs. We emphasize at this point that the reader is **not** assumed to have any knowledge of signal flow graphs, as all necessary definitions will be provided, and additionally the signal flow graphs could be skipped completely without any loss of continuity. The Szegö polynomials can also be realized, or represented via signal flow graphs, using the *Markel-Grey* filter structure, familiar to engineers. Such a signal flow graph, given in Figure 1.1, also characterizes the Szegö polynomials in the same way the corresponding unitary Hessenberg matrices do. Details about signal flow graphs and their uses in characterizations are given below in Section 4.4.

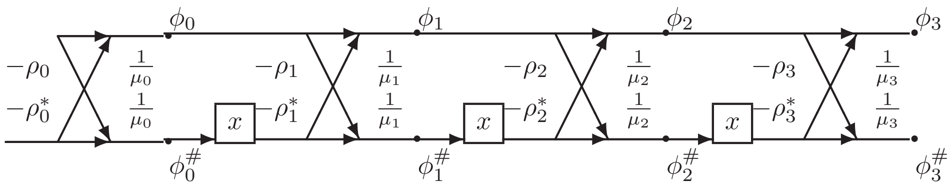



Fig. 1.1. *Markel-Grey filter structure: signal flow graph for the Szegö polynomials using two-term recurrence relations (1.5).*

**1.3. Quasiseparable matrices.** The following *rank definition* of $(H, 1)$-quasiseparable matrices will be followed by an equivalent definition in terms of *generators* of the matrix after demonstrating that the classes of tridiagonal and unitary Hessenberg matrices are special cases of $(H, 1)$-quasiseparable matrices.

Definition 1.2 (Rank definition for $(H, 1)$-quasiseparable matrices). *A matrix $A = [a_{ij}]$ is called $(H, 1)$-quasiseparable (i.e., Hessenberg-1-quasiseparable) if* **(i)** *it is stringly upper Hessenberg ($a_{i+1,i} \neq 0$ for $i = 1, \ldots, n-1$ and $a_{i,j} = 0$ for $i > j + 1$), and* **(ii)** $\max(rank A_{12}) = 1$ *where the maximum is taken over all symmetric partitions of the form*

$$A = \left[\begin{array}{c|c} * & A_{12} \\ \hline * & * \end{array}\right]$$

It is easy to see that both tridiagonal matrices (1.3) and unitary Hessenberg matrices (1.6) are $(H, 1)$-quasiseparable by considering a typical submatrix of each:

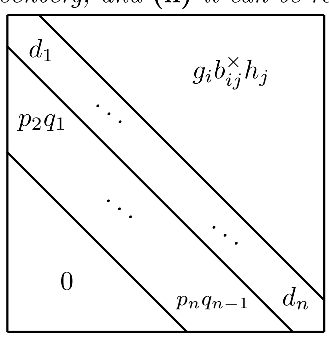
$$A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{\delta_k}{\alpha_k} & 0 & \cdots & 0 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} -\rho_4\mu_3\mu_2\mu_1\rho_0^* & -\rho_5\mu_4\mu_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_n\mu_{n-1}\cdots\mu_3\mu_2\mu_1\rho_0^* \\ -\rho_4\mu_3\mu_2\rho_1^* & -\rho_5\mu_4\mu_3\mu_2\rho_1^* & \cdots & -\rho_n\mu_{n-1}\cdots\mu_3\mu_2\rho_1^* \\ -\rho_4\mu_3\rho_2^* & -\rho_5\mu_4\mu_3\rho_2^* & \cdots & -\rho_n\mu_{n-1}\cdots\mu_3\rho_2^* \end{bmatrix},$$

both of which are rank one.

Another definition of quasiseparability that is well-known to be equivalent to Definition 1.2 is given next.

Definition 1.3 (Generator definition for $(H, 1)$-quasiseparable matrices). *A matrix $A$ is called $(H, 1)$-quasiseparable if* **(i)** *it is strongly upper Hessenberg, and* **(ii)** *it can be represented in the form*

$A =$



*where* $b_{ij}^{\times} = b_{i+1} \cdots b_{j-1}$ *for* $j > i+1$ *and* $b_{ij}^{\times} = 1$ *for* $j = i+1$. *The scalar elements* $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ *are called the generators of the matrix* $A$.

The reason for introducing this class of matrix is as follows. In the next section we will consider more general recurrence relations than those corresponding to tridiagonal matrices and unitary Hessenberg matrices. We will show that all recurrence relations we consider correspond to subclasses of $(H,1)$-quasiseparable matrices (or the exact class of $(H,1)$-quasiseparable matrices).

**1.4. Main results.** Following the characterization of two classical classes of matrices (tridiagonal in Section 1.1 and unitary Hessenberg in Section 1.2) in terms of recurrence relations for corresponding systems of polynomials, we determine the classes of polynomials and corresponding classes of matrices that result by considering recurrence relations more general than (1.2), (1.5) and (1.7). In particular we consider generalizations of the three-term recurrence relations of (1.2), their unrestricted variant

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k \cdot r_{k-2}(x), \quad \alpha_k \neq 0, \tag{1.8}$$

as well as (1.7) to the more general three-term recurrence relations

$$r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x) \tag{1.9}$$

and the two-term recurrence relations of (1.5) to

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k) r_{k-1}(x) \end{bmatrix}, \tag{1.10}$$

with invertible transfer matrices; i.e. $\alpha_k - \beta_k \gamma_k \neq 0$. Additionally, the two-term recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix} \tag{1.11}$$

will be considered. The motivation for considering this third type of recurrence relations will be given below in Section 6. Specifically, the questions answered in this paper are as follows:

- what are the structures of the strongly upper Hessenberg matrices whose corresponding polynomial systems $\{r_k\}$ satisfy recurrence relations of the forms (1.9), (1.10), and (1.11)?
- what are the recurrence relations satisfied by polynomials $\{r_k\}$ related to quasiseparable matrices?
- what are the recurrence relations satisfied by polynomials $\{r_k\}$ related to some subclasses of quasiseparable matrices, specifically well-free and semiseparable matrices (Definitions 4.1 & 3.1)?
- how are the classes of matrices mentioned above related to each other (i.e. are the classes disjoint, do they have an intersection, are they contained within one another, etc.)
- why are there no two-term recurrence relations for real orthogonal polynomials of the same form as those for the Szegö polynomials?
- what is the widest possible pattern of reflection coefficients $\rho_k$ for which there are three-term recurrence relations for the corresponding Szegö polynomials $\phi_k^{\#}(x)$?
- what are the realizations in terms of signal flow graphs such as in Figure 1.1 for the polynomial systems $\{r_k\}$ in the above questions?

The answers to the preceding questions are the main results of this paper, and many of them are summarized in Table 1.1. We additionally present proofs that the described classes of matrices coincide as in Figure 1.2. Also shown are the same results in terms of classes of corresponding polynomials, and the recurrence relations satisfied by them. We also propose new filter structures that generalize the Markel-Grey filter structures of Figure 1.1.

**2. Correspondences between strongly Hessenberg matrices and polynomial systems.** In this section we give details of the correspondences between Hessenberg matrices and systems of polynomials defined via (1.1), and how these correspondences can be used in characterizations of matrices in terms of recurrence relations and vice versa.

**2.1. A bijection between strongly Hessenberg matrices and polynomial systems.** Let $\mathcal{H}$ be the set of strongly upper Hessenberg matrices. Then it is clear that (1.1) provides a mapping of $\mathcal{H}$ into $\mathcal{P}$,

Table 1.1
*Characterizations of subclasses of $(H,1)$-quasiseparable matrices.*

| | Characterizations | | |
|---|---|---|---|
| Class of matrices | $(H,1)$-q.s. generators of Definition 1.3 | Recurrence relations | Signal flow graphs |
| Irreducible tridiagonal matrices | (4.2) | three-term (1.2) | Figure 7.1 |
| Unitary Hessenberg matrices | (4.2) | two-term (1.5) | Figure 1.1 |
| Geronimus-type UH matrices | (4.2) | three-term (1.7) | |
| $(H,1)$-semiseparable matrices (Definition 3.1) | $b_k \neq 0$ (Lemma 3.2) | Szegö-type 2-term (1.10) (Theorem 3.4) | Figure 3.3 |
| $(H,1)$-well-free matrices (Definition 4.1) | $h_k \neq 0$ (Lemma 4.2) | general 3-term (1.9) (Theorem 4.4) | Figure 4.1 |
| $(H,1)$-quasiseparable matrices (Definitions 1.2 & 1.3) | no restrictions | [EGO05]-type 2-term (1.11) (Theorem 6.1) | Figure 6.1 |

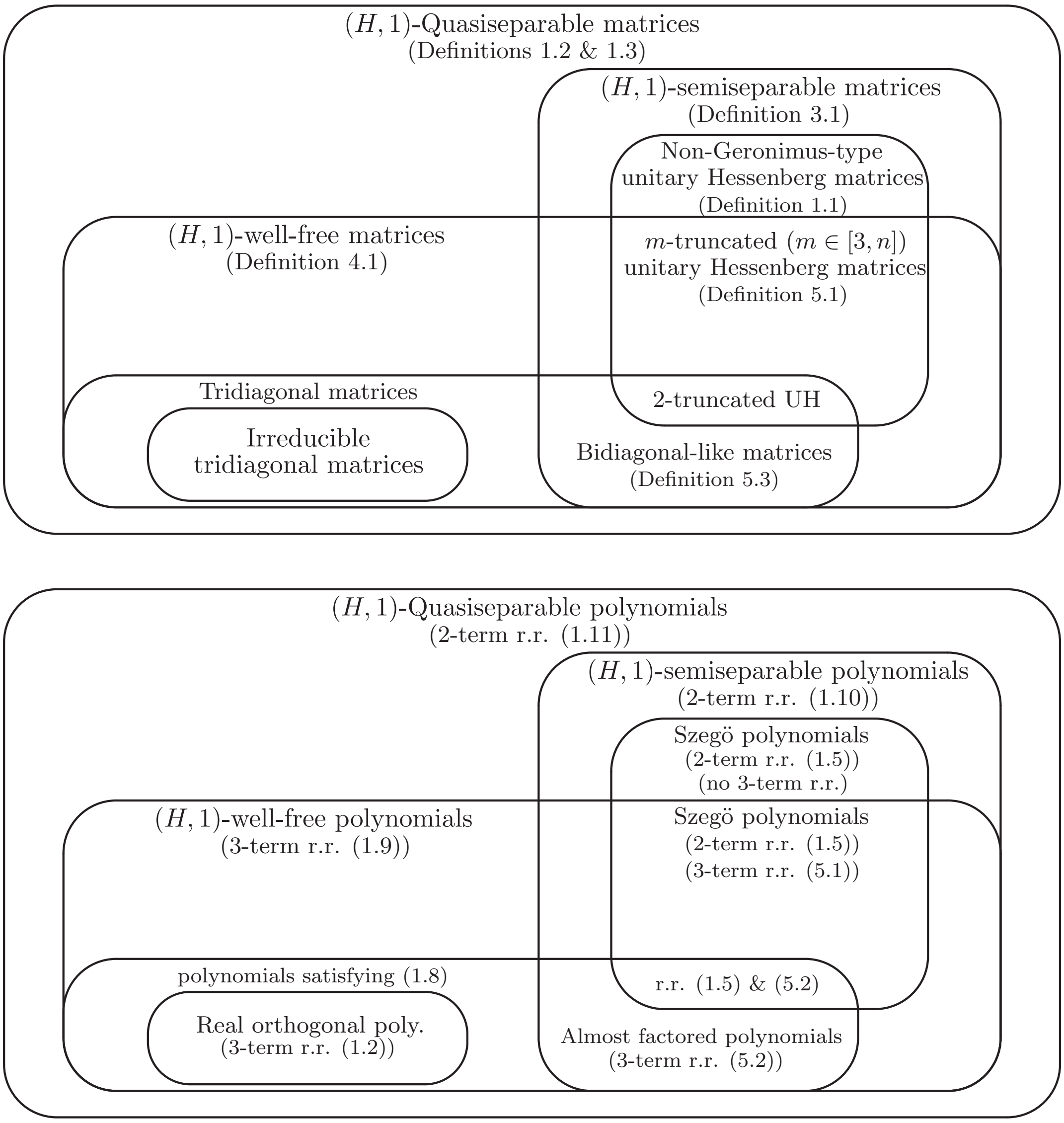



Fig. 1.2. *Relations between classes of matrices as given in Theorem 5.6 (above), and relations between corresponding classes of polynomials and the recurrence relations satisfied (below).*

the set of polynomial systems $\{r_k\}$ with $\deg r_k = k$; that is, to each such matrix $A$ corresponds a unique (up to scaling of the constant polynomial $r_0(x)$) system of polynomials $\{r_k\}$ with $\deg r_k = k$:

$$f : \mathcal{H} \to \mathcal{P}, \qquad \text{where } \; r_k(x) = \frac{1}{a_{2,1}a_{3,2}\cdots a_{n,n-1}} \det{(xI - A)_{(k\times k)}} \,. \tag{2.1}$$

The results of [MB79] allow one to observe that this mapping is, in fact, a bijection, allowing characterizations of one class in terms of the other.

Indeed, given a polynomial system $R = \{r_0(x), r_1(x), \ldots, r_n(x)\} \in \mathcal{P}$, there exist unique $n$-term recurrence relations of the form

$$x \cdot r_{k-1}(x) = a_{k,k} \cdot r_k(x) + a_{k-1,k} \cdot r_{k-1}(x) + \cdots + a_{0,k} \cdot r_0(x), \quad a_{k,k} \neq 0, \quad (k = 1, \ldots, n). \tag{2.2}$$

This is because this formula represents $r_k \in \mathbb{P}_k$ ($\mathbb{P}_k$ being the space of all polynomials of degree at most $k$) in terms of $x \cdot r_{k-1}, r_{k-1}, r_{k-2}, \ldots, r_0$, which form a basis in $\mathbb{P}_k$, and hence the coefficients are unique. That is, the mapping

$$f_1 : \{\text{coefficients } a_{i,j} \text{ of } (2.2)\} \to \mathcal{P} \tag{2.3}$$

is a bijection. To show (2.1) is a bijection, it then suffices to show that it corresponds to a bijection from the set of strongly Hessenberg matrices to the set of all such $n$-term recurrence relations.

For a given set of $n$-term recurrence relations of the form (2.2), in [MB79] the authors introduced the so-called *confederate matrix*

$$A = \begin{bmatrix} a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,n} \\ a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{n-2,n} \\ 0 & \cdots & 0 & a_{n-1,n-1} & a_{n-1,n} \end{bmatrix}, \qquad a_{k,k} \neq 0, \quad (k = 1, \ldots, n-1) \tag{2.4}$$

and showed that this matrix is related to the system of polynomials satisfying those $n$-term recurrence relations via (1.1), and hence the mapping is a surjection. To see the mapping is an injection, given two different matrices of the form (2.4), they must differ in some position $(i, j)$ with minimal $j$ of all such positions. Then the first $(j-1)$ polynomials of the corresponding systems will coincide as the corresponding columns do, but then the $j$–th polynomials of the corresponding systems must involve a different coefficient of the $(i-1)$–th polynomial in $\{x \cdot r_{j-1}, r_{j-1}, r_{j-2}, \ldots, r_0\}$, and hence the corresponding systems must be different. This demonstrates the function

$$f_2 : \mathcal{H} \to \{\text{coefficients } a_{i,j} \text{ of } (2.2)\} \tag{2.5}$$

is a bijection, and combining this with (2.3) shows that (2.1) is a bijection.

This bijection combined with the following lemma giving $n$-term recurrence relations for systems of polynomials corresponding to $(H,1)$-quasiseparable matrices in terms of their generators together provide the tools for the desired characterizations. The following lemma is given in [BEGOT07], and is a consequence of Definition 1.3 and [MB79]:

Lemma 2.1. *Let $A$ be an $(H,1)$-quasiseparable matrix specified by its generators as in Definition 1.3. Then a system of polynomials $\{r_k(x)\}$ satisfies the recurrence relations*

$$r_k(x) = \frac{1}{p_{k+1}q_k} \left[ (x - d_k) r_{k-1}(x) - \sum_{j=0}^{k-2} g_{j+1} b^{\times}_{j+1,k} h_k r_j(x) \right], \tag{2.6}$$

*if and only if $\{r_k(x)\}$ is related to $A$ via (1.1).*

**2.2. Nonuniqueness of recurrence relation coefficients.** In contrast to the $n$-term recurrence relations (2.2), other recurrence relations such as the general three-term recurrence relations (1.9) corresponding to a given polynomial system are not unique. Indeed, for $R = \{r_k(x)\}$, the latter expresses the polynomial $r_k \in \mathbb{P}_k$ $(k = 1, \dots, n)$ in terms of $k+2$ different elements, and since $\dim \mathbb{P}_k = k+1$, we see there is a freedom in choosing the coefficients involved in the recurrence relations.

As a simple example of a system of polynomials satisfying more than one set of recurrence relations of the form (1.9), consider the monomials $R = \{1, x, x^2, \dots, x^n\}$, easily seen to satisfy the recurrence relations

$$r_0(x) = 1, \qquad r_k(x) = x \cdot r_{k-1}(x), \quad (k = 1, \dots, n)$$

as well as the recurrence relations

$$r_0(x) = 1, \qquad r_1(x) = x \cdot r_{k-1}(x), \qquad r_k(x) = (x+1) \cdot r_{k-1}(x) - x \cdot r_{k-2}(x), \quad (k = 2, \dots, n).$$

Hence a given system of polynomials may be expressed using the same (more general) recurrence relations but with different coefficients of those recurrence relations.

**2.3. Nonuniqueness of $(H,1)$-quasiseparable generators.** Similarly, given a $(H,1)$-quasiseparable matrix, there is a freedom in choosing the set of generators of Definition 1.3. As a simple example, if a $(H,1)$-quasiseparable matrix is given by the generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$, then the same $(H,1)$-quasiseparable matrix is generated by the set $\{p'_k, q'_k, d_k, g_k, b_k, h_k\}$ where $p'_k = 1$, $(k = 2, \dots, n)$ and $q'_k = p_{k+1}q_k$, $(k = 1, \dots, n-1)$.

**2.4. Characterizations of subclasses of matrices via recurrence relations.** In the sections below we construct mappings that map special classes of recurrence relations to special classes of $(H,1)$-quasiseparable generators, and vice versa. Specifically, given recurrence relation coefficients for a system of polynomials, the following formulas yield a set of generators for the corresponding $(H,1)$-quasiseparable matrix. These formulas, in both directions, are given in Tables 2.1 & 2.2

Table 2.1
*Conversion formulas: Recurrence relation coefficients $\Rightarrow$ $(H,1)$-quasiseparable generators.*

| Recurrence relation coefficients | $(H,1)$-quasiseparable generators | | | | |
|---|---|---|---|---|---|
| | $p_kq_k$ | $d_k$ | $g_k$ | $b_k$ | $h_k$ |
| Gen. 3-term (1.9) (Corollary 4.5) | $1/\alpha_k$ | $\frac{\delta_k}{\alpha_k} + \frac{\beta_k}{\alpha_{k-1}\alpha_k}$ | $\frac{d_k\beta_{k+1}+\gamma_{k+1}}{\alpha_{k+1}}$ | $\frac{\beta_{k+1}}{\alpha_{k+1}}$ | $1$ |
| Szegö-type (1.10) (Corollary 3.5) | $1/\delta_k$ | $-\frac{\theta_k+\gamma_k\beta_{k-1}}{\delta_k}$ | $\beta_{k-1}$ | $\alpha_{k-1} - \beta_{k-1}\gamma_{k-1}$ | $-\frac{\gamma_k(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1})}{\delta_k}$ |
| [EGO05]-type (1.11) (Corollary 6.2) | $1/\delta_k$ | $-\frac{\theta_k}{\delta_k}$ | $\beta_k$ | $\alpha_k$ | $-\frac{\gamma_k}{\delta_k}$ |

Table 2.2
*Conversion formulas: $(H,1)$-quasiseparable generators $\Rightarrow$ recurrence relations.*

| Matrix & recurrence relations | $\alpha_k$ | $\delta_k$ | $\beta_k$ | $\gamma_k$ |
|---|---|---|---|---|
| $(H,1)$-well-free, (1.9) | $\frac{1}{p_{k+1}q_k}$ | $\frac{d_k}{p_{k+1}q_k} - \frac{p_kq_{k-1}b_{k-1}h_k}{p_{k+1}q_kh_{k-1}}$ | $\frac{h_kb_{k-1}}{p_{k+1}q_kh_{k-1}}$ | $\frac{h_k}{p_{k+1}q_kh_{k-1}}(h_{k-1}g_{k-1} - d_{k-1}b_{k-1})$ |

| Matrix & recurrence relations | $\alpha_k$ | $\beta_k$ | $\gamma_k$ | $\delta_k$ | $\theta_k$ |
|---|---|---|---|---|---|
| $(H,1)$-semiseparable, (1.10) | $p_{k+1}q_kb_{k+1} - \frac{g_{k+1}h_k}{b_k}$ | $-g_{k+1}$ | $\frac{h_k}{b_k}$ | $\frac{1}{p_{k+1}q_k}$ | $\frac{g_kh_k}{p_{k+1}q_kb_k} - \frac{d_k}{p_{k+1}q_k}$ |
| $(H,1)$-quasiseparable, (1.11) | $\frac{p_kb_k}{p_{k+1}}$ | $-\frac{g_k}{p_{k+1}}$ | $\frac{p_kh_k}{p_{k+1}q_k}$ | $\frac{1}{p_{k+1}q_k}$ | $-\frac{d_k}{p_{k+1}q_k}$ |

Despite the nonuniqueness situations described in Sections 2.2 & 2.3, these conversion formulas do in fact provide the desired complete characterizations. This is because they implement the mapping (2.1), shown in Section 2.1 to be a bijection. The characterization of a given matrix is then found by choosing a representative in the equivalence class of generators for that given matrix and mapping it to a representative in the equivalence class of recurrence relations describing the corresponding polynomial system.

**3.** $(H,1)$**-semiseparable matrices & Szegö-type recurrence relations (1.10).** In this section we present the subclass of $(H,1)$-quasiseparable matrices that correspond to polynomials satisfying the Szegö-type recurrence relations (1.10):

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k) r_{k-1}(x) \end{bmatrix} \tag{3.1}$$

That is, we prove the characterizations listed in Table 1.1 for $(H,1)$-semiseparable matrices.

DEFINITION 3.1 ($(H,1)$-semiseparable matrices). *A matrix $A$ is called $(H,1)$-semiseparable if* **(i)** *it is strongly upper Hessenberg, and* **(ii)** *it is of the form*

$$A = B + \mathit{triu}(A_U)$$

*for a rank-one matrix $A_U$ and a lower bidiagonal matrix $B$, where $\mathit{triu}(A_U)$ denotes the strictly upper triangular portion of the matrix $A_U$.*

Paraphrased, a $(H,1)$-semiseparable matrix has arbitrary diagonal entries, arbitrary nonzero subdiagonal entries, and the strictly upper triangular part of a rank one matrix. It is clear from this definition that a $(H,1)$-semiseparable matrix is $(H,1)$-quasiseparable. Indeed, let $A$ be $(H,1)$-semiseparable and $n \times n$. Then it is clear that[4] $\operatorname{rank} A(1:k, k+1:n) = \operatorname{rank} A_U(1:k, k+1:n) \leqslant 1, \quad (k = 1, \dots, n-1)$ and $A$ is $(H,1)$-quasiseparable by Definition 1.2.

**3.1.** $(H,1)$**-semiseparable matrices. Generator characterization.** We next give a lemma that provides a characterization of $(H,1)$-semiseparable matrices in terms of a condition on the generators of a $(H,1)$-quasiseparable matrix.

LEMMA 3.2. *A $(H,1)$-quasiseparable matrix is $(H,1)$-semiseparable if and only if there exists a choice of generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ of the matrix such that $b_k \neq 0$ for $k = 2, \dots, n-1$.*

*Proof.* Let $A = (A_{i,j})$ be an irreducible $(H,1)$-quasiseparable matrix with generator representation $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ as in Definition 1.3, and suppose first that $A$ is $(H,1)$-semiseparable; in particular, $A = B + \operatorname{triu}(A_U)$ where $A_U$ is rank-one. If $b_k = 0$ for some $k$, then since $b_k$ is a factor of the elements $A_{i,j}$ for $k \in [i+1, j-1]$, we necessarily must have the upper right block $A(1:k-1, k+1:n) = 0$. Since $A_U$ is rank-one, $A(1:k-1, k+1:n) = A_U(1:k-1, k+1:n) = 0$ implies that at least one of the submatrices $A_U(k:n, k+1:n)$ and $A_U(1:k-1, 1:k)$ must be zero as well, as otherwise $\operatorname{rank}(A_U) > 1$. If we set $b'_k = 1$, and

$$g'_j = 0, \quad (j = 1, \dots, k-1) \quad \text{if} \quad A_U(1:k-1, 1:k) = 0$$

or

$$h'_j = 0, \quad (j = k+1, \dots, n) \quad \text{if} \quad A_U(k:n, k+1:n) = 0.$$

Repeating this process for each $b_k = 0$, the resulting set $\{p_k, q_k, d_k, g'_k, b'_k, h'_k\}$ can easily be confirmed to generate $A$ as well, and $b'_k \neq 0$ for $k = 2, \dots, n-1$.

Conversely, suppose the generators of $A$ are such that $b_k \neq 0$ for $k = 2, \dots, n-1$. Then the matrices

$$A_U = \begin{cases} g_i b^{\times}_{i,j} h_j & \text{if } 1 \leqslant i < j \leqslant n \\ g_i b_i^{-1} h_i & \text{if } 1 < i = j < n \\ g_i (b^{\times}_{j-1,i+1})^{-1} h_j & \text{if } 1 < j < i < n \\ 0 & \text{if } j = 1 \text{ or } i = n \end{cases} \qquad B = \begin{cases} d_i & \text{if } 1 \leqslant i = j \leqslant n \\ p_i q_j & \text{if } 1 \leqslant i+1 = j \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

are well defined, $\operatorname{rank}(A_U) = 1$, $B$ is lower bidiagonal, and $A = B + \operatorname{triu}(A_U)$. □

COROLLARY 3.3. *The class of $(H,1)$-semiseparable matrices contains as a subclass the class of unitary Hessenberg matrices.*

[4] The MATLAB notation $A(i:j, k:l)$ denotes the submatrix obtained from rows $i, i+1, \dots, j$ and columns $k, k+1, \dots, l$.

*Proof.* It suffices to produce a set of $(H,1)$-quasiseparable generators such that $b_j \neq 0,\ (j=2,\ldots,n-1)$. Such choices can be given by

| Matrix class | $p_j$ | $q_i$ | $d_l$ | $g_i$ | $b_k$ | $h_j$ |
|---|---|---|---|---|---|---|
| Unitary Hessenberg | 1 | $\mu_i$ | $-\rho_{l-1}^*\rho_l$ | $-\rho_{i-1}^*\mu_i$ | $\mu_k$ | $\rho_j$ |

for $i=1,\ldots,n-1$, $j=2,\ldots,n$, $k=2,\ldots,n-1$, and $l=1,\ldots,n$. Note that by (1.4), we necessarily must have $b_k=\mu_k\neq 0$ for each $k$. □

**3.2.** $(H,1)$**-semiseparable matrices. Recurrence relation characterization.** In this section we present a theorem giving the characterization of $(H,1)$-semiseparable matrices as those corresponding to systems of polynomials satisfying the Szegö-type two-term recurrence relations (1.10).

THEOREM 3.4. *Let $R=\{r_0(x),\ldots,r_{n-1}(x)\}$ be a system of polynomials satisfying* $\deg r_k = k$. *Then $R$ satisfies the recurrence relations (3.1):*

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x+\theta_k)r_{k-1}(x) \end{bmatrix}$$

*with $\alpha_k-\beta_k\gamma_k\neq 0$ if and only if there exists a $(H,1)$-semiseparable matrix $A$ corresponding to $R$ (i.e. related via (1.1)).*

*Proof.* Suppose $R$ satisfies the two-term recurrence relations (3.1) with $\alpha_k\neq\beta_k\gamma_k$. We will show that the $(H,1)$-quasiseparable matrix with generators

$$\begin{array}{c} d_1=-\frac{\theta_1+\gamma_1}{\delta_1}, \qquad d_k=-\frac{\theta_k+\gamma_k\beta_{k-1}}{\delta_k}, \quad (k=2,\ldots,n) \\ p_{k+1}q_k=\frac{1}{\delta_k}, \quad (k=1,\ldots,n-1), \qquad g_1=1, \quad g_k=\beta_{k-1}, \quad (k=2,\ldots,n-1) \\ b_k=\alpha_{k-1}-\beta_{k-1}\gamma_{k-1}\neq 0, \quad (k=2,\ldots,n-1), \qquad h_k=-\frac{\gamma_k}{\delta_k}\left(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1}\right), \quad (k=2,\ldots,n) \end{array}$$

is related to $R$ via (1.1). These generators in conjunction with the general $n$-term recurrence relations (2.6) give

$$\begin{aligned} r_k(x) &= (\delta_k x+\theta_k+\gamma_k\beta_{k-1})r_{k-1}(x)+\gamma_k(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1})\beta_{k-2}r_{k-2}(x) \\ &+\gamma_k(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1})(\alpha_{k-2}-\beta_{k-2}\gamma_{k-2})\beta_{k-3}r_{k-3}(x)+\cdots+ \\ &+\gamma_k(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1})(\alpha_{k-2}-\beta_{k-2}\gamma_{k-2})\cdots(\alpha_2-\beta_2\gamma_2)\beta_1 r_1(x)+ \\ &+\gamma_k(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1})(\alpha_{k-2}-\beta_{k-2}\gamma_{k-2})\cdots(\alpha_2-\beta_2\gamma_2)(\alpha_1-\beta_1\gamma_1)r_0(x) \end{aligned} \tag{3.2}$$

The proof is presented by showing that the polynomial system satisfying the two-term recurrence relations also satisfies these $n$-term recurrence relations. By applying the given two-term recursion, we have

$$\begin{bmatrix} G_1(x) \\ r_1(x) \end{bmatrix} = \begin{bmatrix} \beta_1 r_1(x)+\alpha_1-\beta_1\gamma_1 \\ \delta_1 x+\theta_1+\gamma_1 \end{bmatrix}$$

and

$$\begin{bmatrix} G_2(x) \\ r_2(x) \end{bmatrix} = \begin{bmatrix} (\beta_2\delta_2 x+\beta_2\theta_2+\alpha_2\beta_1)r_1(x)+\alpha_2(\alpha_1-\beta_1\gamma_1) \\ (\delta_2 x+\theta_2+\gamma_2\beta_1)r_1(x)+\gamma_2(\alpha_1-\beta_1\gamma_1) \end{bmatrix} \tag{3.3}$$

giving the result for $k=1,2$. From (3.3), we have

$$\delta_2 x r_1(x) = r_2(x)-(\theta_2+\gamma_2\beta_1)r_1(x)-\gamma_2(\alpha_1-\beta_1\gamma_1)$$

and inserting this into the expression for $r_3(x)$ of the form

$$r_3(x)=\gamma_3 G_2(x)+(\delta_3 x+\theta_3)r_2(x)$$

yields (3.2) for $k=3$. Continuing in this fashion, the result follows. Since $b_k=\alpha_{k-1}-\beta_{k-1}\gamma_{k-1}\neq 0$ for $k=2,\ldots,n-1$, the matrix $A$ is $(H,1)$-semiseparable by Lemma 3.2.

Conversely, suppose there exists an $n \times n$ $(H,1)$-semiseparable matrix $A$ in terms of its generators that corresponds to $R$. We may assume that the generators are such that $b_k \neq 0$ for $k = 2, \ldots, n-1$ by Lemma 3.2. We show that $R$ satisfies the recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{p_{k+1}q_k} \begin{bmatrix} v_k & -g_{k+1} \\ h_k/b_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ u_k(x) r_{k-1}(x) \end{bmatrix} \tag{3.4}$$

with

$$u_k(x) = (x - d_k) + \frac{g_k h_k}{b_k}, \qquad v_k = p_{k+1} b_{k+1} q_k - \frac{g_{k+1} h_k}{b_k}. \tag{3.5}$$

Suppose first that the generators are such that $h_k \neq 0$ for each $k$. The proof in this case will be given by showing that the system of polynomials generated these two-term recurrence relations coincides with those given by Theorem 4.6. From (3.4) and the relationship

$$\left( v_k + \frac{g_{k+1} h_k}{b_k} \right) \begin{bmatrix} v_k & -g_{k+1} \\ h_k/b_k & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & g_{k+1} \\ -h_k/b_k & v_k \end{bmatrix} \tag{3.6}$$

(noting that the required inverse exists because $b_k \neq 0$) we obtain

$$\left( v_k + \frac{g_{k+1} h_k}{b_k} \right) \begin{bmatrix} G_{k-1}(x) \\ u_k(x) r_{k-1}(x) +_{n-k} \end{bmatrix} = p_{k+1} q_k \begin{bmatrix} 1 & g_{k+1} \\ -h_k/b_k & v_k \end{bmatrix} \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} \tag{3.7}$$

Thus, we have the following expression for $u_k(x) r_{k-1}(x) +_{n-k}$,

$$\left( v_k + \frac{g_{k+1} h_k}{b_k} \right) (u_k(x) r_{k-1}(x) +_{n-k}) = p_{k+1} q_k \left( \frac{h_k}{b_k} G_k(x) + v_k r_k(x) \right) \tag{3.8}$$

Using (3.4) for $k+1$, we have that $p_{k+2} q_{k+1} r_{k+1}(x) = \frac{h_{k+1}}{b_{k+1}} G_k(x) + u_{k+1} r_k(x)$, which gives that $G_k(x)$ is given by

$$G_k(x) = \left( \frac{b_{k+1}}{h_{k+1}} \right) (p_{k+2} q_{k+1} r_{k+1}(x) - u_{k+1} r_k(x)). \tag{3.9}$$

Inserting (3.9) into (3.8) and shifting from $k+1$ to $k$, we arrive at (4.1) with (4.9) and (4.10) as desired. Using the assumptions and $h_k \neq 0$ for each $k$, Theorem 4.6 implies the result.

For the case of a polynomial system $R$ where $h_j = 0$ for some $j$, note that the coefficients of the polynomials generated by the two-term recurrence relations (3.4) depend continuously on the entries of the $2 \times 2$ transfer matrix. Let $\{\epsilon_k\}$ be a sequence tending to zero with $\epsilon_k \neq 0$ for each $k$, and consider a sequence of systems of polynomials $R_k$ with $h_j = \epsilon_k$ for each $j$ such that $h_j = 0$ in the original polynomial system $R$, and all other generators the same as in $R$. Then the result of the theorem holds for the system $R_k$ for every $k$ by above, and $R_k \to R$, so by continuity, the result must hold for $R$ as well. This completes the proof. ∎

**3.3.** $(H,1)$**-semiseparable matrices. Conversions between recurrence relation coefficients & $(H,1)$-quasiseparable generators.** The following corollary follows directly from the proof given for Theorem 3.4.

COROLLARY 3.5 (Recurrence relation coefficients ⇒ quasiseparable generators). *Let $R$ be a system of polynomials satisfying the Szegö-type two-term recurrence relations (3.1). Then the $(H,1)$-semiseparable matrix $A$ defined by*

$$\begin{bmatrix} -\frac{\theta_1+\gamma_1}{\delta_1} & -(\alpha_1-\beta_1\gamma_1)\frac{\gamma_2}{\delta_2} & -(\alpha_1-\beta_1\gamma_1)(\alpha_2-\beta_2\gamma_2)\frac{\gamma_3}{\delta_3} & & \cdots & -(\alpha_1-\beta_1\gamma_1)\cdots(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\frac{\gamma_n}{\delta_n} \\ \frac{1}{\delta_1} & -\frac{\theta_2+\gamma_2\beta_1}{\delta_2} & -\beta_1(\alpha_2-\beta_2\gamma_2)\frac{\gamma_3}{\delta_3} & & \cdots & -\beta_1(\alpha_2-\beta_2\gamma_2)\cdots(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\frac{\gamma_n}{\delta_n} \\ 0 & \frac{1}{\delta_2} & -\frac{\theta_3+\gamma_3\beta_2}{\delta_3} & & \ddots & -\beta_2(\alpha_3-\beta_3\gamma_3)\cdots(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\frac{\gamma_n}{\delta_n} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & -\beta_{n-1}(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\frac{\gamma_n}{\delta_n} \\ 0 & \cdots & 0 & & \frac{1}{\delta_{n-1}} & -\frac{\theta_n+\gamma_n\beta_{n-1}}{\delta_n} \end{bmatrix} \tag{3.10}$$

*is related to the system of polynomials $R$ via (1.1).*

THEOREM 3.6 (Quasiseparable generators $\Rightarrow$ recurrence relations coefficients). *Let $A$ be a $(H,1)$-semiseparable matrix specified by the generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$. Then the polynomial system $R$ corresponding to $A$ satisfies (3.1); specifically,*

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{p_{k+1}q_k} \begin{bmatrix} v_k & -g_{k+1} \\ h_k/b_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ u_k(x) r_{k-1}(x) \end{bmatrix}$$
$$\textit{with} \quad u_k(x) = (x - d_k) + \frac{g_k h_k}{b_k}, \qquad v_k = p_{k+1} b_{k+1} q_k - \frac{g_{k+1} h_k}{b_k}.$$

The proof follows in the same manner as that of Theorem 5.2, [BEGOT07].

**3.4. Signal flow graphs for classical filter structures. A review.** A major component of the paper is presented via signal flow graphs. Common in electrical engineering, control theory, etc., signal flow graphs represent realizations of systems as electronic devices. Signal flow graphs, although application–oriented, have been employed to answer purely mathematical questions, such as providing interpretations of classical algorithms such as those of Schur and Levinson, deriving fast algorithms, etc., see e.g. [BK86, BK87a, BK87b, LK84, LK86].

In this paper, we do not assume **any** familiarity with signal flow graphs, and the reader can consider them as a convenient way of visualizing recurrence relations. Writing an expression next to a transmission line indicates scaling the current value of that line by that expression, and drawing two arrows coming together indicates adding the two inputs to produce the output. The delay element, written as a block $\boxed{\text{x}}$ in the graphs, denotes a multiplication by $x = z^{-1}$.

For instance, the two-term recurrence relations (1.5) for the Szegö polynomials can be seen as a signal flow graph in Figure 3.1, and in Figure 3.2, the portions of the signal flow graphs corresponding to the Szegö polynomials $\{\phi_k^\#\}$ and the portions corresponding to auxiliary polynomials $\{\phi_k\}$ are highlighted.

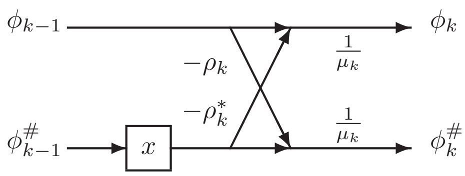



FIG. 3.1. *Signal flow graph representation of the recurrence relations (1.5) satisfied by the Szegö polynomials.*


$$\begin{bmatrix} \phi_k(x) \\ \phi_k^\#(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^\#(x) \end{bmatrix} \qquad \begin{bmatrix} \phi_k(x) \\ \phi_k^\#(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^\#(x) \end{bmatrix}$$

FIG. 3.2. *Portion of the signal flow graph representation of Figure 3.1 corresponding to (left) the Szegö polynomials $\Phi^\#$, and (right) the auxiliary polynomials $\Phi$.*

**3.5. $(H,1)$-semiseparable matrices. Signal flow graph characterization. Semiseparable filters.** The recurrence relations (3.1) can also be easily understood by the signal flow graph depicted in Figure 3.3. The figure depicts a feed-forward filter with transfer function given by the polynomial

$$P(z^{-1}) = P_0 r_0(z^{-1}) + P_1 r_1(z^{-1}) + \cdots + P_n r_n(z^{-1}). \tag{3.11}$$

The semiseparable filter structure is easily seen to be a generalization of the classical Markel-Grey filter depicted in Figure 1.1.

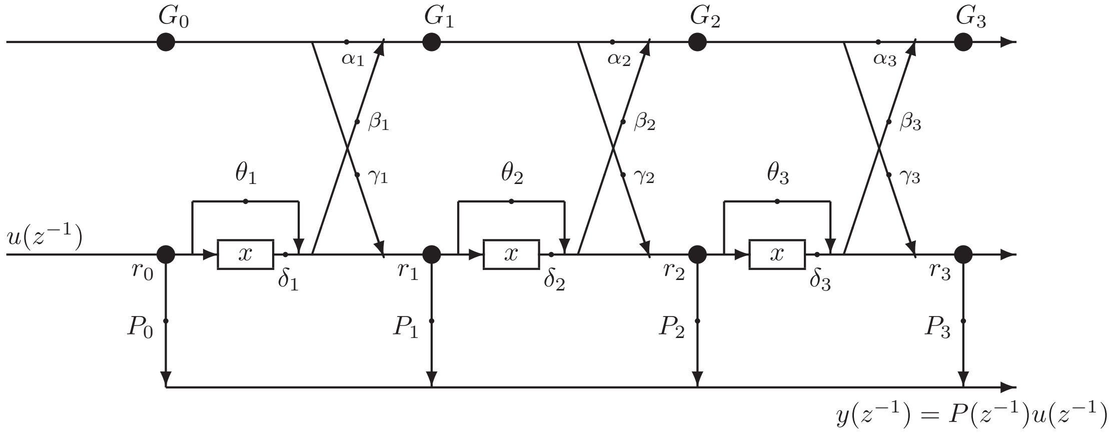
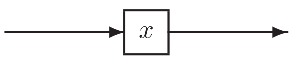
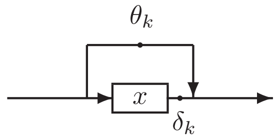



FIG. 3.3. *Semiseparable filter structure: Signal flow graph for polynomials R using Szegö-type recurrence relations (3.1).*

Specifically, the major difference between the Markel-Grey filter structure of Figure 1.1 and this new semiseparable filter structure is in the delay elements $\boxed{\mathrm{x}}$ denoting multiplication by $x = z^{-1}$. The Markel-Grey filter structure involves just the delay element, whereas the semiseparable filter structure involves a scaled delay element as well as a line circumventing the delay element.

The reason for this can be seen from the recurrence relations defining the structures. From (1.5), the new polynomials involve $x\phi^{\#}_{k-1}(x)$, which is the origin of the sole delay element in the Markel-Grey filter. From (1.10), the new polynomials involve $(\delta_k x + \gamma_k)r_{k-1}(x)$, and multiplication of this linear term corresponds to the *generalized delay element*, or scaled delay element plus the outside line in the semiseparable filter. These differences are highlighted in Figure 3.4.




$$\begin{bmatrix} \phi_k(x) \\ \phi^{\#}_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho^*_k \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi^{\#}_{k-1}(x) \end{bmatrix} \qquad \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k) r_{k-1}(x) \end{bmatrix}$$

FIG. 3.4. *The delay element corresponding to multiplication by $x = z^{-1}$ found in the Markel-Grey filter (left), and the generalized delay element corresponding to multiplication by a linear polynomial in $x = z^{-1}$ found in the semiseparable filter structure (right).*

## 4. $(H,1)$-well-free matrices & three term recurrence relations (1.9).

In this section we present the subclass of $(H,1)$-quasiseparable matrices that correspond to polynomials satisfying the general three-term recurrence relations (1.9):

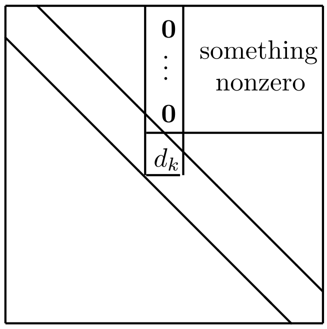
$$r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x) \tag{4.1}$$

which include real orthogonal polynomials (satisfying the special case (1.2)) and Szegö polynomials (satisfying the special case (1.7)), and hence is an interesting class to consider. We will show that (4.1) provides a full characterization for the class of matrices defined next. That is, we demonstrate the characterizations shown in the $(H,1)$-well-free row of Table 1.1.

DEFINITION 4.1 ($(H,1)$-well-free matrices).

- *An $n \times n$ matrix $A = (A_{i,j})$ is said to have a* ***well*** *in column $1 < k < n$ if $A_{i,k} = 0$ for $1 \leqslant i < k$ and there exists a pair $(i,j)$ with $1 \leqslant i < k$ and $k < j \leqslant n$ such that $A_{i,j} \neq 0$.*
- *A $(H,1)$-quasiseparable matrix is said to be $(H,1)$-**well-free** if none of its columns $k = 2, \ldots, n-1$ contain wells.*

In words, a matrix has a well in column $k$ if all entries above the main diagonal in the $k$-th column are zero, **except** if all entries in the upper-right block to the right of these zeros are also zeros, as shown in the following illustration:




**4.1.** $(H,1)$**-well-free matrices. Generator characterization.** As with $(H,1)$-semiseparable matrices above, the class of $(H,1)$-well-free matrices admit a nice characterization in terms of the generators of $(H,1)$-quasiseparable matrices.

LEMMA 4.2. *A $(H,1)$-quasiseparable matrix is $(H,1)$-well-free if and only if there exists a choice of generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ of the matrix such that $h_k \neq 0$ for $k = 2,\ldots,n$.*

*Proof.* Let $A = (A_{i,j})$ be a $(H,1)$-quasiseparable matrix with generator representation $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ as in Definition 1.3, and suppose first that $A$ is $(H,1)$-well-free. We will construct a set of generators $\{p_k, q_k, d_k, g'_k, b'_k, h'_k\}$ that generates $A$ and $h'_k \neq 0$ for each $k$. Define

$$g'_i = \begin{cases} g_i & \text{if } h_{i+1} \neq 0 \\ 0 & \text{if } h_{i+1} = 0 \end{cases}, \qquad b'_j = \begin{cases} b_j & \text{if } h_{j+1} \neq 0 \\ 0 & \text{if } h_{j+1} = 0 \end{cases}, \qquad h'_k = \begin{cases} h_k & \text{if } h_k \neq 0 \\ 1 & \text{if } h_k = 0 \end{cases},$$

for $i = 1,\ldots,n-1$, $j = 2,\ldots,n-1$, and $k = 2,\ldots,n$. Since $A$ is $(H,1)$-well-free, these modifications to the elements $g_k$ and $b_k$ do not change the resulting matrix, and thus the set $\{p_k, q_k, d_k, g'_k, b'_k, h'_k\}$ also generates $A$, and all $h$ elements are nonzero as desired.

On the other hand, suppose there exists a choice of generators such that $h_k \neq 0$ for $k \in [2,n]$. Then if an element $A_{i,j} = 0$ for $1 \leqslant i < j \leqslant n$, it follows that at least one element of $\{g_i, b_{i+1}, b_{i+1}, \ldots, b_{j-1}\}$ must equal zero. But each of these elements are also present in the expressions for $A_{i,k}$ for $k = j+1,\ldots,n$, and hence $A_{i,k} = 0$ for $k = j+1,\ldots,n$ as well. Thus $A$ cannot contain a well, and this completes the proof. □

Both of the motivating classes of matrices (tridiagonal and unitary Hessenberg) are subclasses of $(H,1)$-well-free matrices, as the next lemma shows. This fact is easily anticipated, as both tridiagonal matrices and unitary Hessenberg matrices correspond to systems of polynomials that satisfy special cases of the recurrence relations (4.1), and we claim (and prove below) that such polynomial systems correspond to $(H,1)$-well-free matrices.

COROLLARY 4.3. *The class of $(H,1)$-well-free matrices contains as subclasses both the class of tridiagonal matrices and the class of unitary Hessenberg matrices with $\rho_k \neq 0$ for each $k$.*

*Proof.* It suffices to produce a set of $(H,1)$-quasiseparable generators for each class such that $h_j \neq 0$, $(j = 2,\ldots,n)$. Such choices in each case can be given by

(4.2)

| Matrix class | $p_j$ | $q_i$ | $d_l$ | $g_i$ | $b_k$ | $h_j$ |
|---|---|---|---|---|---|---|
| Tridiagonal | 1 | $\frac{1}{\alpha_i}$ | $\frac{\delta_l}{\alpha_l}$ | $\frac{\gamma_{i+1}}{\alpha_{i+1}}$ | 0 | 1 |
| Unitary Hessenberg | 1 | $\mu_i$ | $-\rho^*_{l-1}\rho_l$ | $-\rho^*_{i-1}\mu_i$ | $\mu_k$ | $\rho_j$ |

for $i = 1,\ldots,n-1$, $j = 2,\ldots,n$, $k = 2,\ldots,n-1$, and $l = 1,\ldots,n$. □

**4.2.** $(H,1)$**-well-free matrices. Recurrence relation characterization.** In this section we present the principal result of the section: the characterization of $(H,1)$-well-free matrices as those corresponding to systems of polynomials satisfying the three-term recurrence relations (4.1).

THEOREM 4.4. *Let $R = \{r_0(x),\ldots,r_{n-1}(x)\}$ be a system of polynomials satisfying $\deg r_k = k$. Then $R$ satisfies the recurrence relations (4.1):*

$$r_k(x) = (\alpha_k x - \delta_k)\cdot r_{k-1}(x) - (\beta_k x + \gamma_k)\cdot r_{k-2}(x)$$

*if and only if there exists a $(H,1)$-well-free matrix $A$ corresponding to $R$ (i.e. related via (1.1)).*

*Proof.* Suppose first that the polynomial system $R$ satisfies the recurrence relations (4.1). We will show that the $(H,1)$-well-free matrix with generators

$$d_1 = \frac{\delta_1}{\alpha_1}, \quad d_k = \frac{\delta_k}{\alpha_k} + \frac{\beta_k}{\alpha_{k-1}\alpha_k}, \quad h_k = 1, \quad (k = 2, \ldots, n) \tag{4.3}$$

$$p_{k+1}q_k = \frac{1}{\alpha_k}, \quad g_k = \frac{d_k\beta_{k+1} + \gamma_{k+1}}{\alpha_{k+1}}, \quad (k = 1, \ldots, n-1), \quad b_k = \frac{\beta_{k+1}}{\alpha_{k+1}}, \quad (k = 2, \ldots, n-1) \tag{4.4}$$

is related to $R$ via (1.1). The proof is given by induction on $k$ that the choice of generators (4.3)-(4.4) results in the $n$-term recurrence relations (2.6), which gives the result by Lemma 2.1.

For $k = 1$, the recurrence relations (4.1) result in

$$r_1(x) = (\alpha_1 x - \delta_1)r_0(x), \tag{4.5}$$

and using (4.3)-(4.4) we arrive at (2.6) for $k = 1$. For $k = 2$, inserting the relation

$$xr_0(x) = \frac{1}{\alpha_1}(r_1(x) + \delta_1 r_0(x))$$

from (4.5) into

$$r_2(x) = (\alpha_2 x - \delta_2)r_1(x) - (\beta_2 x + \gamma_2)r_0(x)$$

results in

$$r_2(x) = \left(\alpha_2 x - \left(\delta_2 + \frac{\beta_2}{\alpha_1}\right)\right) r_1(x) - \left(\beta_2 \frac{\delta_1}{\alpha_1} + \gamma_2\right) r_0(x),$$

and again using (4.3)-(4.4) this becomes

$$r_2(x) = \frac{1}{p_3 q_2}\left[(x - d_2)r_1(x) - g_1 r_0(x)\right]$$

which is (2.6) for $k = 2$.

Next suppose the choice of generators (4.3)-(4.4) results in (2.6) for some $k-1$ with $k \geqslant 3$. By adding and subtracting the quantity $\frac{\beta_k}{\alpha_{k-1}} r_{k-1}(x)$ to (4.1), we have

$$r_k(x) = \alpha_k \left[\left(x - \left(\frac{\delta_k}{\alpha_k} + \frac{\beta_k}{\alpha_k \alpha_{k-1}}\right)\right) r_{k-1}(x) + \left(\frac{\beta_k}{\alpha_k \alpha_{k-1}}\right) r_{k-1}(x) - \left(\frac{\beta_k}{\alpha_k}x + \frac{\gamma_k}{\alpha_k}\right) r_{k-2}(x)\right]. \tag{4.6}$$

By the inductive hypothesis,

$$\left(\frac{\beta_k}{\alpha_k \alpha_{k-1}}\right) r_{k-1}(x) - \left(\frac{\beta_k}{\alpha_k}x + \frac{\gamma_k}{\alpha_k}\right) r_{k-2}(x) = -\left(\frac{d_{k-1}\beta_k + \gamma_k}{\alpha_k}\right) r_{k-2}(x) - \sum_{j=0}^{k-3} g_{j+1} b^{\times}_{j+1,k-1} \frac{\beta_k}{\alpha_k} r_j(x),$$

and using (4.3)-(4.4), this is furthermore equal to

$$-g_{k-1}r_{k-2}(x) - \sum_{j=0}^{k-3} g_{j+1} b^{\times}_{j+1,k-1} b_{k-1} r_j(x) = \sum_{j=0}^{k-2} g_{j+1} b^{\times}_{j+1,k} r_j(x) \tag{4.7}$$

using the identities $b^{\times}_{k-1,k} = 1$ and $b^{\times}_{j+1,k-1} b_{k-1} = b^{\times}_{j+1,k}$. Inserting (4.7) into (4.6) and using (4.3) once more gives (2.6) for $k$. Thus by induction, $R$ is related to the $(H,1)$-quasiseparable matrix $A$ with the given generators. But (4.4) includes $h_k = 1$ for each $k$, and so by Lemma 4.2, $A$ is $(H,1)$-well-free.

Conversely, suppose there exists a $(H,1)$-well-free matrix $A$ given by its generators corresponding to $R$. By Lemma 4.2, since $A$ is $(H,1)$-well-free, the generators may be chosen such that $h_k \neq 0$ for $k = 2, \ldots, n$.

By Lemma 2.1, the polynomial system $R$ satisfies (2.6). Set $s_0(x) = r_0(x)$, and for $k = 1, \dots, n-1$, define $s_k(x)$ via the recurrence relations (4.1):

$$s_k(x) = (\alpha_k x - \delta_k) \cdot s_{k-1}(x) - (\beta_k x + \gamma_k) \cdot s_{k-2}(x) \tag{4.8}$$

with

$$\alpha_k = \frac{1}{p_{k+1}q_k}, \quad \delta_k = \frac{1}{p_{k+1}q_k}\left(d_k - \frac{p_k q_{k-1} h_k b_{k-1}}{h_{k-1}}\right) \tag{4.9}$$

$$\beta_k = \frac{1}{p_{k+1}q_k}\frac{h_k b_{k-1}}{h_{k-1}}, \quad \gamma_k = \frac{1}{p_{k+1}q_k}\frac{h_k}{h_{k-1}}\left(h_{k-1}g_{k-1} - d_{k-1}b_{k-1}\right), \tag{4.10}$$

We will show that the polynomial systems $R$ and $S$ coincide, so $r_k(x) = s_k(x)$ for $k = 0, \dots, n-1$, and hence $R$ satisfies the recurrence relations (4.1) as desired. We present this proof by induction on $k$. By definition, $r_0(x) = s_0(x)$, and by direct confirmation it is seen that $r_1(x) = s_1(x)$ as well.

Next suppose that the conclusion is true for each index less than or equal to $k-1$ for some $2 \leqslant k \leqslant n-1$; that is,

$$r_i(x) = s_i(x), \quad i = 0, 1, \dots, k-1. \tag{4.11}$$

Then using (2.6) for $k-1$ and (4.11), we have

$$\begin{aligned} x s_{k-2}(x) = {} & p_k q_{k-1} s_{k-1}(x) + d_{k-1} s_{k-2}(x) + g_{k-2} h_{k-1} s_{k-3}(x) + g_{k-3} b_{k-2} h_{k-1} s_{k-4}(x) \\ & + \cdots + g_2 b_3 \cdots b_{k-2} h_{k-1} s_1(x) + g_1 b_2 \cdots b_{k-2} h_{k-1} s_0(x) \end{aligned} \tag{4.12}$$

Next, the polynomial $s_k(x)$ satisfies the recurrence relations (4.8), and inserting (4.12) into (4.8) and using (4.11), we arrive at

$$s_k(x) = \frac{1}{p_{k+1}q_k}\left[(x - d_k) r_{k-1}(x) - \sum_{j=0}^{k-2} g_{j+1} b^{\times}_{j+1,k} h_k r_j(x)\right],$$

and hence $r_k(x) = s_k(x)$. This completes the proof. $\square$

**4.3.** $(H, 1)$**-well-free matrices. Conversions between recurrence relation coefficients &** $(H, 1)$**-quasiseparable generators.** In this section we conversions from recurrence relation coefficients to $(H, 1)$-quasiseparable generators and vice versa. Recall from above that both such representations are nonunique, and so what follows represents one possible conversion scheme. From the proof of Theorem 4.4, we have the following corollary.

Corollary 4.5 (Recurrence relation coefficients $\Rightarrow$ quasiseparable generators). *Let $R$ be a system of polynomials satisfying the general three-term recurrence relations (4.1). Then the $(H, 1)$-well-free $A$ defined by*

$$\begin{bmatrix}
\frac{\delta_1}{\alpha_1} & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2} & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right) & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right)\left(\frac{\beta_4}{\alpha_4}\right) & \cdots & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right)\left(\frac{\beta_4}{\alpha_4}\right)\cdots\left(\frac{\beta_n}{\alpha_n}\right) \\
\frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} + \frac{\beta_2}{\alpha_1\alpha_2} & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3} & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3}\left(\frac{\beta_4}{\alpha_4}\right) & \cdots & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3}\left(\frac{\beta_4}{\alpha_4}\right)\cdots\left(\frac{\beta_n}{\alpha_n}\right) \\
 & \frac{1}{\alpha_2} & \frac{\delta_3}{\alpha_3} + \frac{\beta_3}{\alpha_2\alpha_3} & \frac{\left(\frac{\delta_3}{\alpha_3}+\frac{\beta_3}{\alpha_2\alpha_3}\right)\beta_4+\gamma_4}{\alpha_4} & & \\
 & & \frac{1}{\alpha_3} & \frac{\delta_4}{\alpha_4} + \frac{\beta_4}{\alpha_3\alpha_4} & & \vdots \\
 & & & \frac{1}{\alpha_4} & \ddots & \\
 & & & \ddots & \ddots & \frac{\left(\frac{\delta_{n-1}}{\alpha_{n-1}}+\frac{\beta_{n-1}}{\alpha_{n-2}\alpha_{n-1}}\right)\beta_n+\gamma_n}{\alpha_n} \\
 & & & & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n} + \frac{\beta_n}{\alpha_{n-1}\alpha_n}
\end{bmatrix} \tag{4.13}$$

*defined by the generators given in (4.3)-(4.4)is related to the system of polynomials R via (1.1).*

THEOREM 4.6 (Quasiseparable generators $\Rightarrow$ recurrence relations coefficients). *Let $A$ be a $(H,1)$-well-free matrix specified by the generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$. Then the polynomial system $R$ corresponding to $A$ satisfies (4.1) with*

$$\alpha_k = \frac{1}{p_{k+1}q_k}, \quad \delta_k = \frac{1}{p_{k+1}q_k}\left(d_k - \frac{p_k q_{k-1} b_{k-1} h_k}{h_{k-1}}\right)$$

$$\beta_k = \frac{1}{p_{k+1}q_k}\frac{h_k b_{k-1}}{h_{k-1}}, \quad \gamma_k = \frac{1}{p_{k+1}q_k}\frac{h_k}{h_{k-1}}\left(h_{k-1}g_{k-1} - d_{k-1}b_{k-1}\right)$$

The proof is a modification of that of Theorem 5.1, [BEGOT07].

**4.4. $(H,1)$-well-free matrices. Signal flow graph characterization. Well-free filters.** The recurrence relations (4.1) can also be easily understood by the signal flow graph depicted in Figure 4.1. As in the previous case, this is a feed-forward filter with polynomial transfer function given by (3.11):

$$P(z^{-1}) = P_0 r_0(z^{-1}) + P_1 r_1(z^{-1}) + \cdots + P_n r_n(z^{-1}).$$

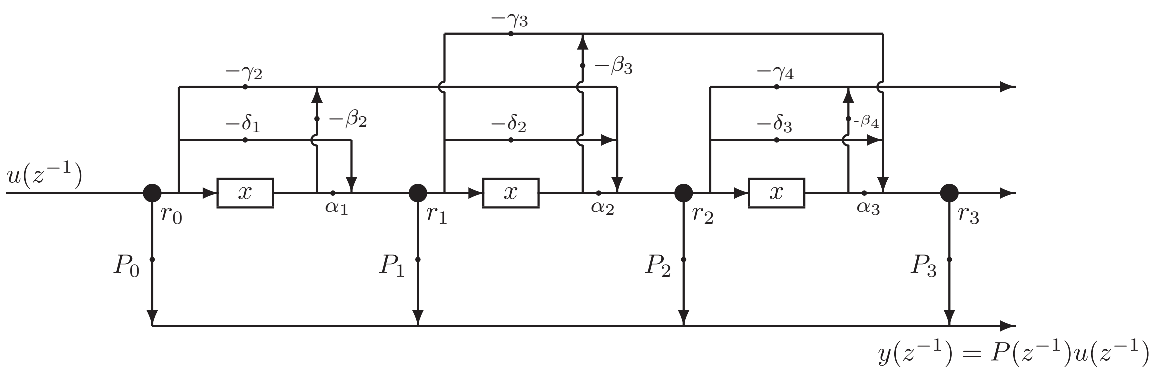



FIG. 4.1. *Well-free filter structure: Signal flow graph for polynomials $R$ using three-term recurrence relations (4.1).*

**5. Inclusions and intersections.** The goal of this section is to use the results of the previous sections to justify the relations between the classes of matrices presented in Figure 1.2.

**5.1. Subclasses of unitary Hessenberg matrices.** In Definition 1.1, we gave a definition splitting the class of unitary Hessenberg matrices into those whose corresponding Szegö polynomials satisfy three-term recurrence relations, called Geronimus-type unitary Hessenberg matrices, and those who do not, called non-Geronimus-type unitary Hessenberg matrices. We next give a definition to assist in providing the characterization of Geronimus-type unitary Hessenberg matrices.

DEFINITION 5.1. *A unitary Hessenberg matrix of the form (1.6) is called $m$-truncated provided the reflection coefficients satisfy*

$$\rho_2 \neq 0, \ldots, \rho_m \neq 0, \quad \rho_{m+1} = \cdots = \rho_n = 0.$$

*A set of Szegö polynomials corresponding to an $m$-truncated unitary Hessenberg matrix are called a set of $m$-truncated Szegö polynomials.*

The following theorem answers the question of which unitary Hessenberg matrices correspond to Szegö polynomials that satisfy three-term recurrence relations posed in the introduction.

THEOREM 5.2. *An $n \times n$ unitary Hessenberg matrix is Geronimus-type if and only if it is $m$-truncated for some $m \in [2, n]$. Furthermore, such an $m$-truncated system of Szegö polynomials satisfies the three-term* *recurrence relations*

$$(5.1)\qquad \phi_k^{\#}(x) = \begin{cases} \frac{1}{\mu_0} & k=0 \\ \frac{1}{\mu_1}(x\cdot\phi_0^{\#}(x) + \rho_1\rho_0^*\cdot\phi_0^{\#}(x)) & k=1 \\ \frac{1}{\mu_2}x\phi_1^{\#}(x) - \frac{\rho_2\mu_1}{\mu_2}\phi_0^{\#}(x) & k=2,\quad \rho_1=0 \\ \left[\frac{1}{\mu_2}\cdot x + \frac{\rho_2}{\rho_1}\frac{1}{\mu_2}\right]\phi_1^{\#}(x) - \frac{\rho_2}{\rho_1}\frac{\mu_1}{\mu_2}\cdot x\cdot\phi_0^{\#}(x) & k=2,\quad \rho_1\neq 0 \\ \left[\frac{1}{\mu_k}\cdot x + \frac{\rho_k}{\rho_{k-1}}\frac{1}{\mu_k}\right]\phi_{k-1}^{\#}(x) - \frac{\rho_k}{\rho_{k-1}}\frac{\mu_{k-1}}{\mu_k}\cdot x\cdot\phi_{k-2}^{\#}(x) & 2<k\leqslant m \\ x\cdot\phi_{k-1}^{\#}(x) & k>m \end{cases}$$

*Proof.* Let $H$ be unitary Hessenberg of the form (1.6). Then it is clear that for $1<k<n$, $H$ has a well in column $k$ if and only if $\rho_k = 0$ and there exists a $j>k$ such that $\rho_j \neq 0$. Thus $H$ is $m$-truncated Geronimus-type if and only if $H$ is $(H,1)$-well-free. By Theorem 4.4, this is equivalent to the corresponding system satisfying three-term recurrence relations. The justification for (5.1) is from Theorem 4.6. □

Notice that in (5.1) if $m=n$, then $\rho_k\neq 0$ for all $k$, and the result of this corollary is exactly three-term recurrence relations (1.7). So this result is a slight extension of the work of Geronimus in [G48].

**5.2. Bidiagonal-like matrices & almost factored polynomials.** The next definition and following discussion clarify questions about how the class of tridiagonal matrices intersects with the class of $(H,1)$-semiseparable matrices in Figure 1.2.

Definition 5.3 (Bidiagonal-like matrices). *A matrix $A$ is called bidiagonal-like if* **(i)** *it is strongly upper Hessenberg, and* **(ii)** *it is of the form $A = B + C$, with $B$ a lower bidiagonal matrix, and $C$ a matrix with at most one nonzero entry, and that entry is located in the first superdiagonal.*

So a matrix is bidiagonal-like if it is either lower bidiagonal or tridiagonal with only one nonzero entry above the diagonal. The following characterization of this class of matrix follows from the generator definition of quasiseparability (Definition 1.3).

Lemma 5.4. *A $(H,1)$-quasiseparable matrix is bidiagonal-like if and only if there is an index $k$ such that for any choice of generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$, $g_i h_j = 0$ for all $i\in[1,n-1]$ and $j\in[2,n]$ except possibly when $i=k$ and $j=k+1$.*

We next present a characterization of this class of matrices in terms the recurrence relations satisfied by the corresponding system of polynomials. The justification follows from Theorem 4.4 (since bidiagonal-like matrices are tridiagonal and hence $(H,1)$-well-free by Corollary 4.3) and applying Lemma 5.4.

Theorem 5.5. *Let $R=\{r_0(x),\ldots,r_{n-1}(x)\}$ be a system of polynomials satisfying* $\deg r_k = k$. *Then for some index $j\in[1,n-1]$, $R$ satisfies the recurrence relations*

$$(5.2)\qquad r_k(x) = \begin{cases} (\alpha_k x - \delta_k)\cdot r_{k-1}(x) & k\neq j \\ ((\alpha_{k-1}x-\delta_{k-1})(\alpha_k x-\delta_k)-\gamma_k)\cdot r_{k-2}(x) & k=j \end{cases}$$

*if and only if there exists a bidiagonal-like matrix $A$ corresponding to $R$ (i.e. related via (1.1)).*

We refer to such polynomial systems in Figure 1.2 as *almost-factored polynomials*, since each polynomial with the exception of at most one is a linear factor times the previous polynomial. The theorem shows that the polynomials $r_k(x)$ are of the form

$$r_k(x) = \begin{cases} r_0\cdot\displaystyle\prod_{i=1}^{k}(\alpha_i x-\delta_i) & k<j \\ r_0\cdot\displaystyle\prod_{\substack{i=1\\ i\neq j-1,j}}^{k}(\alpha_i x-\delta_i)\cdot((\alpha_{j-1}x-\delta_{j-1})(\alpha_j x-\delta_j)-\gamma_j) & k\geqslant j \end{cases},$$

so the polynomials with $k<j$ are of the factored form, but the polynomial $r_j(x)$ introduces an unfactored quadratic term (provided $\gamma_j\neq 0$, which is not required, only possible), so all subsequent polynomials each contains at most one unfactored quadratic term of the form $(\alpha_{j-1}x-\delta_{j-1})(\alpha_j x-\delta_j)-\gamma_j$.

**5.3. Inclusions and intersections.** In this section we use the generator characterization results of the previous two sections to present a theorem to demonstrate the relationships between the various described classes. That is, the next theorem collects together some results given above and some new results to complete the justification of Figure 1.2.

THEOREM 5.6.

(i) *The set of tridiagonal matrices is contained in the set of* $(H, 1)$*-well-free matrices.*

(ii) *The set of unitary Hessenberg matrices is contained in the set of* $(H, 1)$*-semiseparable matrices.*

(iii) *The sets of* $(H, 1)$*-well-free matrices and* $(H, 1)$*-semiseparable matrices have a nontrivial intersection, properly containing the set of Geronimus-type unitary Hessenberg matrices.*

(iv) *The sets of irreducible tridiagonal matrices and* $(H, 1)$*-semiseparable matrices and are mutually disjoint (for matrices larger than* $2 \times 2$*).*

(v) *The sets of non-Geronimus-type unitary Hessenberg matrices and* $(H, 1)$*-well-free matrices are mutually disjoint.*

(vi) *The intersection of the sets of tridiagonal matrices and* $(H, 1)$*-semiseparable matrices is exactly the set of bidiagonal-like matrices.*

(vii) *The intersection of the sets of tridiagonal matrices and unitary Hessenberg matrices is the set of* $2$*-truncated unitary Hessenberg matrices.*

(viii) *For the sets of* $(H, 1)$*-well-free matrices and* $(H, 1)$*-semiseparable, neither set contains the other.*

*Proof.* (i) This is stated and proved as Corollary 4.3.

(ii) This is stated and proved as Corollary 3.3.

(iii) From Corollary 3.3, any unitary Hessenberg matrix $H$ of the form (1.6) is $(H, 1)$-semiseparable. It was shown in Theorem 5.2 that unitary Hessenberg matrices are also $(H, 1)$-well-free if and only if they are Geronimus-type unitary Hessenberg, hence the intersection of these classes contains the class of Geronimus-type unitary Hessenberg matrices. To show this intersection properly contains Geronimus-type unitary Hessenberg matrices, note that scaling entries of the matrix $H$ of (1.6) with a nonzero constant can destroy the unitary property without affecting the argument above.

(iv) From Definition 1.3, the entries on the first superdiagonal of a matrix $A$ are $g_k h_{k+1}$, $k = 1, \ldots, n-1$. Since the matrix is strongly Hessenberg, it follows that $A$ is irreducible if and only if $g_i \neq 0, h_j \neq 0$ for $i = 1, \ldots, n-1$ and $j = 2, \ldots, n$. But then since $A$ is tridiagonal, $b_k = 0$ for $k = 2, \ldots, n-1$ for any choice of generators, and hence $A$ is not $(H, 1)$-semiseparable by Lemma 3.2.

(v) This is also a part of Corollary 4.3.

(vi) It is easily observed that bidiagonal-like matrices are tridiagonal, and satisfy Definition 3.1, and hence the set of bidiagonal-like matrices is contained in the intersection of the sets of $(H, 1)$-semiseparable matrices and tridiagonal matrices.

Let a matrix $A$ be both tridiagonal and $(H, 1)$-semiseparable, and denote by $k$ the number of nonzero entries in the first superdiagonal of $A$. Due to the zeros above this superdiagonal, it follows that any matrix $A_U$ of Definition 3.1 must satisfy $\operatorname{rank}(A_U) \geqslant k$. Thus $k \leqslant 1$ is equivalent to $A$ being $(H, 1)$-semiseparable, which coincides with Definition 5.3.

(vii) From Theorem 5.2, it follows that unitary Hessenberg matrices are $(H, 1)$-well-free if and only if they are $m$-truncated for some $m \in [2, n]$. From (vi), it then suffices to show that the intersection of the sets of $m$-truncated unitary Hessenberg matrices and bidiagonal-like matrices is the set of 2-truncated unitary Hessenberg matrices. This follows immediately from their respective definitions.

(viii) From (iv), irreducible tridiagonal matrices are $(H, 1)$-well-free but not $(H, 1)$-semiseparable. From (v), non-Geronimus-type unitary Hessenberg matrices are $(H, 1)$-semiseparable but not $(H, 1)$-well-free. Thus neither class contains the other.
□

**5.4. Why there are no Szegö-type recurrence relations for real orthogonal polynomials.** We can now provide an explanation in these terms for why no two-term recurrence relations of the form (3.1) exist for real orthogonal polynomials. From Theorem 5.6, we have the fact that irreducible tridiagonal matrices are not $(H, 1)$-semiseparable. Then, Theorem 3.4 states that it is exactly the class of $(H, 1)$-semiseparable matrices that satisfy two-term recurrence relations of the form (3.1). Since real orthogonal polynomials correspond to irreducible tridiagonal matrices, the claim follows.

**6. $(H,1)$-quasiseparable matrices & [EGO05]-type recurrence relations (1.11).** We next present a theorem characterizing the matrices corresponding to systems of polynomials satisfying the [EGO05]-type two-term recurrence relations (1.11). A motivation for considering this class is that all previous classes have limitations, and none of the recurrence relations are general enough to include all special cases. For instance, the Szegö-type two-term recurrence relations (1.10) do not apply to Chebyshev polynomials, and the more general three-term recurrence relations (1.9) do not apply to non-Geronimus-type Szegö polynomials. Correspondingly, all filter structures have limitations as well. For instance, the Markel-Grey-type semiseparable filter structure of Figure 3.3 cannot be used to realize the Chebyshev polynomials, and the well-free filter structure of Figure 4.1 cannot be used to realize non-Geronimus-type Szegö polynomials.

In this section we present recurrence relations that apply to all special cases considered in this paper. That is, we give a form of recurrence relations satisfied by both Chebyshev polynomials and non-Geronimus-type Szegö polynomials. Using these recurrence relations, we propose a new quasiseparable filter structure that can be used to realize both Chebyshev polynomials, non-Geronimus-type Szegö polynomials, and all special cases considered in the paper.

THEOREM 6.1. *Let $R = \{r_0(x), \ldots, r_{n-1}(x)\}$ be a system of polynomials satisfying* $\deg r_k = k$. *Then $R$ satisfies the recurrence relations (1.11):*

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix} \tag{6.1}$$

*if and only if there exists a $(H,1)$-quasiseparable matrix $A$ corresponding to $R$ (i.e. related via (1.1)).*

*Proof.* Suppose $R$ satisfies the [EGO05]-type recurrence relations (6.1). We will show that the $(H,1)$-quasiseparable matrix $A$ with generators

$$d_k = -\tfrac{\theta_k}{\delta_k}, \quad (k=1,\ldots,n), \qquad p_{k+1}q_k = \tfrac{1}{\delta_k}, \quad (k=1,\ldots,n-1)$$
$$g_k = \beta_k, \quad (k=1,\ldots,n-1), \qquad b_k = \alpha_k, \quad (k=2,\ldots,n-1), \qquad h_k = -\tfrac{\gamma_k}{\delta_k}, \quad (k=2,\ldots,n)$$

corresponds to the polynomial system $R$. Inserting the specified choice of generators into the general $n$-term recurrence relations (2.6), we arrive at

$$\begin{aligned} r_k(x) = {} & (\delta_k x + \theta_k) r_{k-1}(x) + \gamma_k \beta_{k-1} r_{k-2}(x) + \gamma_k \alpha_{k-1} \beta_{k-2} r_{k-3}(x) \\ & + \gamma_k \alpha_{k-1} \alpha_{k-2} \beta_{k-3} r_{k-4}(x) + \cdots + \gamma_k \alpha_{k-1} \cdots \alpha_2 \beta_1 r_0(x) \end{aligned} \tag{6.2}$$

It suffices to show that the polynomial system satisfying the two-term recurrence relations also satisfies these $n$-term recurrence relations. Beginning with

$$r_k(x) = \gamma_k G_{k-1}(x) + (\delta_k x + \theta_k) r_{k-1}(x) \tag{6.3}$$

and using the relation $G_{k-1}(x) = \alpha_{k-1} G_{k-2}(x) + \beta_{k-1} r_{k-2}(x)$, (6.3) becomes

$$r_k(x) = \gamma_k \alpha_{k-1} G_{k-2}(x) + \gamma_k \beta_{k-1} r_{k-2}(x) + (\delta_k x + \theta_k) r_{k-1}(x)$$

and continuing this procedure to obtain $n$-term recurrence relations. It can easily be checked that this procedure yields exactly (6.2).

Suppose $A$ is a $(H,1)$-semiseparable matrix related to $R$ via (1.1) given by its generators. We show that $R$ satisfies the recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{p_{k+1}q_k} \begin{bmatrix} q_k p_k b_k & -q_k g_k \\ p_k h_k & x - d_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}. \tag{6.4}$$

The recurrence relations (6.4) define a system of polynomials which satisfy the $n$-term recurrence relations

$$r_k(x) = (\alpha_k x - a_{k-1,k}) \cdot r_{k-1}(x) - a_{k-2,k} \cdot r_{k-2}(x) - \ldots - a_{0,k} \cdot r_0(x) \tag{6.5}$$

for some coefficients $\alpha_k, a_{k-1,k}, \ldots, a_{0,k}$. The proof is presented by showing that these $n$-term recurrence relations in fact coincide exactly with those of Lemma 2.1, so these coefficients coincide with those of the $n$-term recurrence relations of the polynomials $R$. Using relations for $r_k(x)$ and $F_{k-1}(x)$ from (6.4), we have

$$r_k(x) = \frac{1}{p_{k+1}q_k} \left[ (x - d_k) r_{k-1}(x) - g_{k-1} h_k r_{k-2}(x) + h_k p_{k-1} b_{k-1} F_{k-2}(x) + r_0(x) \right]. \tag{6.6}$$

Notice that again using (6.4) to eliminate $F_{k-2}(x)$ from the equation (6.6) will result in an expression for $r_k(x)$ in terms of $r_{k-1}(x)$, $r_{k-2}(x)$, $r_{k-3}(x)$, $F_{k-3}(x)$, and $r_0(x)$ without modifying the coefficients of $r_{k-1}(x)$, $r_{k-2}(x)$, or $r_0(x)$. Again applying (6.4) to eliminate $F_{k-3}(x)$ results in an expression in terms of $r_{k-1}(x)$, $r_{k-2}(x)$, $r_{k-3}(x)$, $r_{k-4}(x)$, $F_{k-4}(x)$, and $r_0(x)$ without modifying the coefficients of $r_{k-1}(x)$, $r_{k-2}(x)$, $r_{k-3}(x)$, or $r_0(x)$. Continuing in this way, the $n$-term recurrence relations of the form (6.5) are obtained without modifying the coefficients of the previous ones.

Suppose that for some $0 < j < k-1$ the expression for $r_k(x)$ is of the form

$$r_k(x) = \frac{1}{p_{k+1}q_k}\left[(x-d_k)r_{k-1}(x) - g_{k-1}h_k r_{k-2}(x) - \cdots - g_{j+1}b^{\times}_{j+1,k}h_k r_j(x) + p_{j+1}b^{\times}_{j,k}h_k F_j(x) + r_0(x)\right]. \tag{6.7}$$

Using (6.4) for $F_j(x)$ gives the relation

$$F_j(x) = \frac{1}{p_{j+1}q_j}\left(q_j p_j b_j F_{j-1}(x) - q_j g_j r_{j-1}(x)\right) \tag{6.8}$$

Inserting (6.8) into (6.7) gives

$$r_k(x) = \frac{1}{p_{k+1}q_k}\left[(x-d_k)r_{k-1}(x) - g_{k-1}h_k r_{k-2}(x) - \cdots - g_j b^{\times}_{j,k}h_k r_{j-1}(x) + p_j b^{\times}_{j-1,k}h_k F_{j-1}(x) + r_0(x)\right]. \tag{6.9}$$

Therefore since (6.6) is the case of (6.7) for $j = k-2$, (6.7) is true for each $j = k-2, k-3, \ldots, 0$, and for $j = 0$, using the fact that $F_0 = 0$ we have

$$r_k(x) = \frac{1}{p_{k+1}q_k}\left[(x-d_k)r_{k-1}(x) - g_{k-1}h_k r_{k-2}(x) - \cdots - g_1 b^{\times}_{1,k}h_k r_0(x) + r_0(x)\right] \tag{6.10}$$

Since these coefficients coincide with those in Lemma 2.1 that are satisfied by the polynomial system $R$, the polynomials given by (6.4) must coincide with these polynomials. This proves the theorem. □

**6.1. $(H,1)$-quasiseparable matrices. Conversions between recurrence relation coefficients & $(H,1)$-quasiseparable generators.** In Theorem 6.1, the following conversion was proved.

COROLLARY 6.2 (Recurrence relation coefficients ⇒ quasiseparable generators). *Let $R$ be a system of polynomials satisfying the [EGO05]-type two-term recurrence relations (6.1). Then the $(H,1)$-quasiseparable matrix $A$ defined by*

$$\begin{bmatrix} -\frac{\theta_1}{\delta_1} & -\beta_1(\frac{\gamma_2}{\delta_2}) & -\beta_1\alpha_2(\frac{\gamma_3}{\delta_3}) & -\beta_1\alpha_2\alpha_3(\frac{\gamma_4}{\delta_4}) & \cdots & -\beta_1\alpha_2\alpha_3\alpha_4\cdots\alpha_{n-1}(\frac{\gamma_n}{\delta_n}) \\ \frac{1}{\delta_1} & -\frac{\theta_2}{\delta_2} & -\beta_2(\frac{\gamma_3}{\delta_3}) & -\beta_2\alpha_3(\frac{\gamma_4}{\delta_4}) & \cdots & -\beta_2\alpha_3\alpha_4\cdots\alpha_{n-1}(\frac{\gamma_n}{\delta_n}) \\ 0 & \frac{1}{\delta_2} & -\frac{\theta_3}{\delta_3} & -\beta_3(\frac{\gamma_4}{\delta_4}) & \ddots & -\beta_3\alpha_4\cdots\alpha_{n-1}(\frac{\gamma_n}{\delta_n}) \\ 0 & 0 & \frac{1}{\delta_3} & -\frac{\theta_4}{\delta_4} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -\beta_{n-1}(\frac{\gamma_n}{\delta_n}) \\ 0 & \cdots & 0 & 0 & \frac{1}{\delta_{n-1}} & -\frac{\theta_n}{\delta_n} \end{bmatrix} \tag{6.11}$$

*is related to the system of polynomials $R$ via (1.1).*

THEOREM 6.3 (Quasiseparable generators ⇒ recurrence relations coefficients). *Let $A$ be a $(H,1)$-quasiseparable matrix specified by the generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$. Then the polynomial system $R$ corresponding to $A$ satisfies (6.1); specifically,*

$$\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{p_{k+1}q_k}\begin{bmatrix} q_k p_k b_k & -q_k g_k \\ p_k h_k & x - d_k \end{bmatrix}\begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}. \tag{6.12}$$

The proof follows in the same manner as that of Theorem 5.3, [BEGOT07].

**6.2. $(H,1)$-quasiseparable matrices. Signal flow graph characterization. Quasiseparable filters.** The recurrence relations (6.1) can also be easily understood by the signal flow graph depicted in Figure 6.1. Again, it depicts a feed-forward filter with transfer function (3.11).

This quasiseparable filter structure has the property of capturing the recurrence relations of both real orthogonal polynomials and Szegö polynomials. That is, it is a single filter structure that can implement any of these classical polynomial systems, as well as the more general case of $(H,1)$-quasiseparable systems.

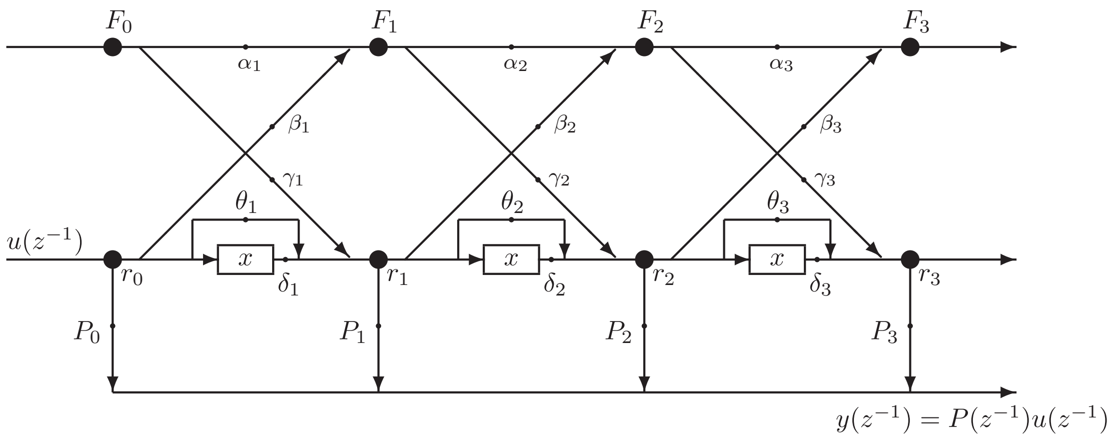



FIG. 6.1. *Quasiseparable filter structure: Signal flow graph for polynomials R using [EGO05]-type recurrence relations (6.1).*

Notice that the cross–lines are located *around* the delay elements as opposed to *in between* the delay elements as in the semiseparable filter structure. Algebraically, this corresponds to the polynomials $r_k$ having different degrees than their related auxiliary polynomials $F_k$ in the [EGO05]-type two-term recurrence relations (6.1), as opposed to them having the same degrees as in the Szegö-type two-term recurrence relations (3.1).

**7. Special cases of the new filter structures.** As was shown in Section 5.4, no two-term recurrence relations of the form (3.1) exist for real orthogonal polynomials.

In terms of the filter structures, this means that filter structures like the Markel-Grey filter (Figure 1.1) and its generalization the semiseparable filter (Figure 3.3) can not realize any system of real orthogonal polynomials. Because it is more general, the quasiseparable filter structure can realize both real orthogonal polynomials and Szegö polynomials (by implementing either the two-term (1.5) or three-term (1.7) recurrence relations for the Szegö polynomials) as special cases.

In what follows we present recurrence relation coefficients (that is, sets of $\{\alpha_k, \beta_k, \gamma_k, \delta_k, \theta_k\}$) for relations of the form (6.1) corresponding to the classical cases of real orthogonal polynomials and Szegö polynomials.

**7.1. Real orthogonal polynomials & the new quasiseparable filter structure.** As discussed above, real orthogonal polynomials satisfy three-term recurrence relations of the form

$$r_k(x) = (\widetilde{\alpha}_k x - \widetilde{\delta}_k) r_{k-1}(x) - \widetilde{\gamma_k} \cdot r_{k-2}(x), \quad \widetilde{\alpha}_k \neq 0, \widetilde{\gamma}_k \neq 0. \tag{7.1}$$

In order to specify the quasiseparable filter structure of Figure 6.1 to this case, we use the generators for tridiagonal matrices of (4.2):

| Matrix class | $p_k$ | $q_k$ | $d_k$ | $g_k$ | $b_k$ | $h_k$ |
|---|---|---|---|---|---|---|
| Tridiagonal | 1 | $\frac{1}{\widetilde{\alpha}_k}$ | $\frac{\widetilde{\delta}_k}{\widetilde{\alpha}_k}$ | $\frac{\widetilde{\gamma}_{k+1}}{\widetilde{\alpha}_{k+1}}$ | 0 | 1 |

as well as Theorem 6.3 to obtain

(7.2)

| Coefficients of (6.1) | $\alpha_k$ | $\beta_k$ | $\gamma_k$ | $\delta_k$ | $\theta_k$ |
|---|---|---|---|---|---|
| Coefficients of (7.1) | 0 | $-\frac{\widetilde{\gamma}_{k+1}}{\widetilde{\alpha}_{k+1}}$ | $\widetilde{\alpha}_k$ | $\widetilde{\alpha}_k$ | $-\widetilde{\delta}_k$ |

,

which are recurrence relation coefficients for (6.1) for the special case of real orthogonal polynomials in terms of the recurrence relation coefficients of the form (7.1). Inserting these into (6.1), we have

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\widetilde{\gamma}_{k+1}}{\widetilde{\alpha}_{k+1}} \\ \widetilde{\alpha}_k & \widetilde{\alpha}_k x - \widetilde{\delta}_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix},$$

and using $G_{k-1}(x) = -\frac{\widetilde{\gamma}_k}{\widetilde{\alpha}_k} r_{k-2}(x)$, we see this is identical to the three-term recurrence relations (7.1). That is, the restriction $\alpha_k = 0$ in (6.1) implies the recurrence relations of the form (7.1).

Correspondingly, in terms of the related matrices, taking $\alpha_k = 0$ in the matrix (6.11) results in a tridiagonal matrix, in fact exactly (1.3) with tildes after applying the conversions (7.2). This is to be expected as such tridiagonal matrices are related to the real orthogonal polynomials being considered.

The corresponding reduction in terms of signal flow graphs is given in Figure 7.1 (a specification of Figure 6.1). Notice how the 0 in the upper transmission line removes the dependence on previous auxiliary polynomials, and so the upper line serves only to recall the previous polynomial $r_{k-2}$, resulting in three-term recurrence relations.

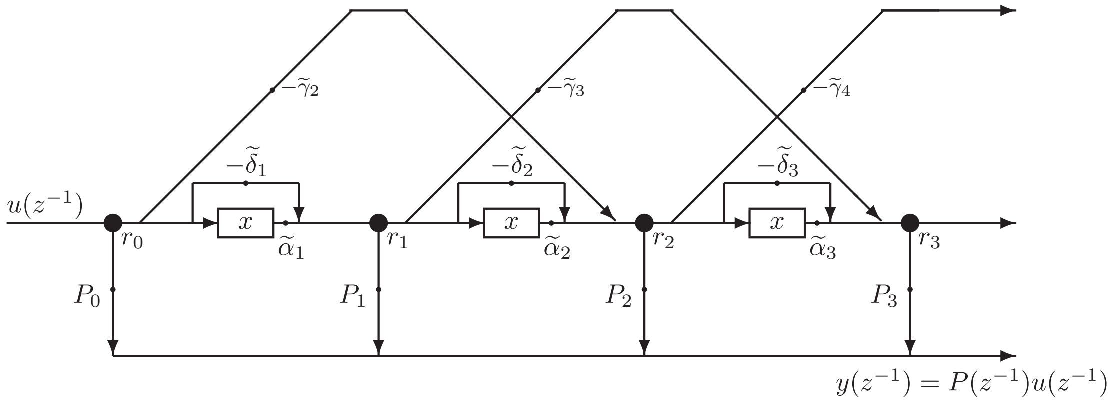



FIG. 7.1. *Quasiseparable filter structure for the real orthogonal case in terms of the recurrence relation coefficients (7.1).*

**7.2. Szegö polynomials & the new quasiseparable filter structure.** Using the generators for unitary Hessenberg matrices of (4.2):

| Matrix class | $p_k$ | $q_k$ | $d_k$ | $g_k$ | $b_k$ | $h_k$ |
|---|---|---|---|---|---|---|
| Unitary Hessenberg | 1 | $\mu_i$ | $-\rho_{l-1}^*\rho_l$ | $-\rho_{i-1}^*\mu_i$ | $\mu_k$ | $\rho_j$ |

together with Theorem 6.3, we obtain

(7.3)

| Coefficients of (6.1) | $\alpha_k$ | $\beta_k$ | $\gamma_k$ | $\delta_k$ | $\theta_k$ |
|---|---|---|---|---|---|
| Coefficients of (1.5) | $\mu_k$ | $\rho_{k-1}^*\mu_k$ | $\frac{\rho_k}{\mu_k}$ | $\frac{1}{\mu_k}$ | $\frac{\rho_{k-1}^*\rho_k}{\mu_k}$ |

,

which give the following new two-term recurrence relations for the Szegö polynomials:

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \mu_k & \rho_{k-1}^*\mu_k \\ \frac{\rho_k}{\mu_k} & \frac{1}{\mu_k}x - \frac{\rho_{k-1}^*\rho_k}{\mu_k} \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}.$$

The fact that the $x$ term appears in only the $(2,2)$ entry of the transfer matrix here and not both the $(1,2)$ and $(2,2)$ entries as in the classical two-term recurrence relations (1.5) shows that the auxiliary polynomials do not have the same degrees as the corresponding Szegö polynomials. Hence these recurrence relations are not a simple modification of (1.5).

The corresponding filter is shown in Figure 7.2. As mentioned above, there is a difference in the locations of the cross–lines (*around* the delay elements as opposed to *in between* the delay elements) means that the quasiseparable filter structure is not a generalization of the Markel-Grey filter structure, it is in fact a new filter structure and hence gives a new realization of the Szegö polynomials.

**8. Conclusions.** In this paper we provided several characterizations of $(H,1)$-quasiseparable matrices and the subclasses of $(H,1)$-semiseparable matrices and $(H,1)$-well-free matrices in terms of recurrence relations on their corresponding systems of polynomials, restrictions on their quasiseparable generators, and filter structures of their corresponding signal flow graphs.

Three new filter structures were introduced, well-free filters, semiseparable filters, a generalization of the well-known Markel-Grey filters, and quasiseparable filters. It was shown that a nice property of the quasiseparable filters is that they realize all of the considered classes of polynomials, including real orthogonal polynomials (via their three-term recurrence relations) and Szegö polynomials (via either two- or three-term recurrence relations).

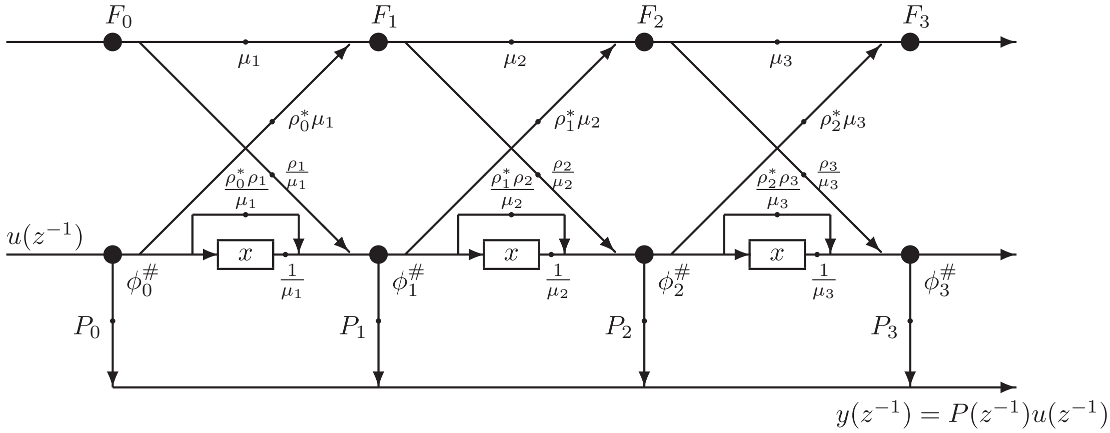



FIG. 7.2. *Quasiseparable filter structure for the Szegö case in terms of reflection coefficients in (1.5) & (1.7).*

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