COMPARISON THEOREMS FOR THREE-DIMENSIONAL MANIFOLDS WITH SCALAR CURVATURE BOUND

OVIDIU MUNTEANU AND JIAPING WANG

Abstract. Two sharp comparison results are derived for three-dimensional complete noncompact manifolds with scalar curvature bounded from below. The first one concerns the Green's function. When the scalar curvature is nonnegative, it states that the rate of decay of an energy quantity over the level set is strictly less than that of the Euclidean space unless the manifold itself is isometric to the Euclidean space. The result is in turn converted into a sharp area comparison for the level set of the Green's function when in addition the Ricci curvature of the manifold is assumed to be asymptotically nonnegative at infinity. The second result provides a sharp upper bound of the bottom spectrum in terms of the scalar curvature lower bound, in contrast to the classical result of Cheng which involves a Ricci curvature lower bound.

1. Introduction

The classical Laplacian comparison theorem [21, Chapter I] states that the Laplacian of a geodesic distance function is at most that of the corresponding space form under a Ricci curvature lower bound. It is obvious that such a result is no longer true under a scalar curvature lower bound. The purpose of this note is to search for suitable alternatives for complete three-dimensional manifolds.

Our first result deals with the case of nonnegative scalar curvature. Recall that a complete manifold is called nonparabolic if it admits a positive Green's function [11, Chapter 20]. It is well-known that in this case the minimal positive Green's function $G(x,y)$ may be obtained as the limit of the Dirichlet Green's function of a sequence of compact exhaustive domains of the manifold. Then

$$\Delta_x G(x,y) = -\delta(x,y),$$
$$G(x,y) = G(y,x) > 0$$

and

$$\liminf_{y\to\infty} G(x,y) = 0.$$

Throughout the paper we fix $p \in M$ and let

$$G(x) = G(p,x).$$

We also use the following notations to denote the level and sublevel sets of $G(x)$.

$$\begin{aligned} L(a,b) &= \{x \in M: \ a < G(x) < b\} \\ l(t) &= \{x \in M: \ G(x) = t\}. \end{aligned}$$

We have the following sharp comparison theorem concerning the minimal positive Green's function.

Theorem 1.1. *Let (M,g) be a complete noncompact three-dimensional manifold with nonnegative scalar curvature. Assume that M has one end and its first Betti number $b_1(M)=0$. If M is nonparabolic and the minimal positive Green's function $G(x)=G(p,x)$ satisfies $\lim_{x\to\infty} G(x)=0$, then*

$$\frac{d}{dt}\left(\frac{1}{t}\int_{l(t)}|\nabla G|^2-4\pi t\right)\leq 0$$

for all regular values $t>0$. Moreover, equality holds for some $T>0$ if and only if the super level set $\{x\in M: G(x)>T\}$ is isometric to a ball in the Euclidean space $\mathbb{R}^3$.

Some remarks are in order. First, the conclusion may be restated as

$$\frac{d}{dt}\left(\frac{1}{t}\int_{l(t)}|\nabla G|^2\right)\leq \frac{d}{dt}\left(\frac{1}{t}\int_{\bar{l}(t)}|\bar{\nabla}\bar{G}|^2\right)$$

for all $t>0$, where $\bar{G}(\bar{x})=\frac{1}{4\pi|\bar{x}|}$ is the Green's function of $\mathbb{R}^3$ and $\bar{l}(t)$ the level set of $\bar{G}$. As such, it may be viewed as a comparison of the decay rate concerning the energy quantity $\frac{1}{t}\int_{l(t)}|\nabla G|^2$ of M with that of $\mathbb{R}^3$. Second, the fact that $\lim_{x\to\infty} G(x)=0$ together with the topological information of M is to ensure that the level set $l(t)$ is compact and connected. Without those assumptions, one may work with the Dirichlet Green's function of an arbitrary bounded domain instead. The resulting conclusion now depends on the number of components of $l(t)$ as well. Third, it is unclear to us if an analogous conclusion holds in higher dimensions, though our proof is very dimension specific.

Under the additional assumption that the Ricci curvature is nonnegative at infinity, the preceding result may be converted into an area comparison theorem for the level sets of the Green's function.

Corollary 1.2. *Let (M,g) be a complete noncompact three-dimensional manifold with nonnegative scalar curvature and asymptotically nonnegative Ricci curvature, that is,*

$$\liminf_{x\to\infty} \mathrm{Ric}(x)\geq 0.$$

Assume that M has one end and its first Betti number $b_1(M)=0$. If M is nonparabolic and the minimal Green's function $G(x)=G(p,x)$ satisfies $\lim_{x\to\infty} G(x)=0$, then

$$\int_{l(t)}|\nabla G|^2\leq 4\pi t^2$$

and

$$\mathrm{Area}(l(t))\geq \frac{1}{4\pi t^2}$$

for all $t>0$. Moreover, if equality holds for some $T>0$, then (M,g) is isometric to $\mathbb{R}^3$.

Note that in the case that M has nonnegative Ricci curvature, its minimal positive Green's function G always satisfies $\lim_{x\to\infty} G(x)=0$ by [16] and the number of ends is necessarily one due to the Cheeger-Gromoll splitting theorem [3]. Therefore, Theorem 1.1 and Corollary 1.2 are both applicable to the universal cover of M.

Theorem 1.1 is motivated by the work of Colding [6] and Colding-Minicozzi [8], where monotonicity formulas for functionals of the form

$$w_p(r) = \frac{1}{r^{n-1}} \int_{b=r} |\nabla b|^p$$

are derived for n-dimensional manifolds with nonnegative Ricci curvature, where the function $b = G^{-\frac{1}{n-2}}$. So Theorem 1.1 concerns w_2 for dimension $n = 3$, while the exponent $p = 3$ in [6], and more generally $p \geq \frac{2n-3}{n-1}$ in [8], for all dimensions n. These monotonicity results have been applied to the study of uniqueness of the tangent cones for Ricci flat manifolds with Euclidean volume growth [9]. We refer the readers to [7] for an exposition on monotonicity formulas in geometric analysis, and [1] for their applications to Willmore-type inequalities.

Our second result concerns the bottom spectrum. Recall that the bottom spectrum $\lambda_1(M)$ is characterized by

$$\lambda_1(M) = \inf_{f \in C_0^\infty(M)} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

According to Cheng's theorem [4], for an n-dimensional complete manifold M,

$$\lambda_1(M) \leq \frac{(n-1)^2}{4} K$$

if the Ricci curvature of (M, g) satisfies $\mathrm{Ric} \geq -(n-1)K$ for some nonnegative constant K. By considering the example of the form $M = \mathbb{H}^2 \times \mathbb{S}^{n-2}(r)$, where $\mathbb{H}^2$ is the standard hyperbolic plane of sectional curvature -1 and $\mathbb{S}^{n-2}(r)$ the sphere of radius r in $\mathbb{R}^{n-1}$, one sees that a direct extension of Cheng's result to the scalar curvature lower bound is not possible for dimension $n \geq 4$. Indeed, by choosing r accordingly, the scalar curvature of M can be made as large as one desires, yet $\lambda_1(M) = \frac{1}{4}$. However, for $n = 3$, one does have the following theorem.

Theorem 1.3. *Let (M, g) be a three-dimensional complete noncompact Riemannian manifold with scalar curvature $S \geq -6K$ on M for some nonnegative constant K. Suppose that M has finitely many ends and its first Betti number $b_1(M) < \infty$. Moreover, the Ricci curvature of M is bounded from below and the volume $V_x(1)$ of unit ball $B_x(1)$ satisfies*

$$V_x(1) \geq C(\epsilon) \exp\left(-2\sqrt{K+\epsilon}\, r(x)\right)$$

for every $\epsilon > 0$ and all $x \in M$, where $r(x)$ is the geodesic distance from x to a fixed point p. Then the bottom spectrum of the Laplacian satisfies

$$\lambda_1(M) \leq K.$$

Note that in the case that the Ricci curvature of M is bounded by $\mathrm{Ric} \geq -2K$, the Gromov-Bishop volume comparison theorem [11, Chapter 2] readily implies that volume lower bound for $V_x(1)$ holds. So, by considering the universal cover of M if necessary and modulo the topological assumption of finitely many ends, Theorem 1.3 provides a faithful generalization of Cheng's result to three-dimensional manifolds with only scalar curvature lower bound.

Corollary 1.4. *Let (M, g) be a three-dimensional complete noncompact Riemannian manifold with nonpositive sectional curvature. Assume that the scalar curvature S is bounded by*

$$S \geq -6K \quad on\ M$$

for some nonnegative constant K. Then

$$\lambda_1 (M) \leq K.$$

This is because Theorem 1.3 is applicable to the universal cover $\tilde{M}$ of M as $\tilde{M}$ is a Cartan-Hadamard manifold with bounded curvature. Since the bottom spectrum satisfies $\lambda_1(M) \leq \lambda_1(\tilde{M})$, the corollary follows.

Both Theorem 1.1 and Theorem 1.3 are proved by working with the minimal positive Green's function G of M. The idea of using Green's function to bound the bottom spectrum was introduced by the first author in [17]. Roughly speaking, one takes a test function $f = |\nabla G|^{\frac{1}{2}} \phi$ with a carefully chosen cut-off function ϕ which in turn depends on G. The proofs of both theorems hinge on manipulating the Bochner formula for the Green's function.

$$\Delta |\nabla G| = \left(|G_{ij}|^2 - |\nabla |\nabla G||^2 \right) |\nabla G|^{-1} + \mathrm{Ric}\left(\nabla G, \nabla G\right) |\nabla G|^{-1}.$$

A crucial point is to rewrite the Ricci curvature term by mimicking a trick which originated in the work of Schoen and Yau [18, 19, 20] on stable minimal surfaces in three-dimensional manifolds.

$$\mathrm{Ric}\left(\nabla G, \nabla G\right) |\nabla G|^{-2} = \frac{1}{2} S - \frac{1}{2} S_{l(r)} + \frac{1}{|\nabla G|^2} \left(|\nabla |\nabla G||^2 - \frac{1}{2} \left|\nabla^2 G\right|^2 \right),$$

where $S_{l(r)}$ denotes the scalar curvature of the level set $l\left(r\right)$. One can then proceed by integrating the formula over the level sets and applying the Gauss-Bonnet theorem. We note that this kind of idea has been exploited recently in [23, 2] as well.

In order to make the argument work, however, we need to ensure that the level sets $l(t)$ of G are compact with controlled number of components. This is where all the extra assumptions are used to show the following.

$$\lim_{x \to \infty} G\left(x\right) = 0, \tag{1.1}$$

$$\#\mathrm{Conn}\left(l\left(t\right)\right) \leq A \tag{1.2}$$

for all t, where A is a fixed constant and $\#\mathrm{Conn}\left(l(t)\right)$ the number of connected components of the level set $l(t)$.

Ideally, one would like to prove Theorem 1.3 under the sole assumption that the scalar curvature $S \geq -6K$. Another natural question is what happens if $\lambda_1(M)$ achieves its maximum value K in Theorem 1.3. In the case of the aforementioned Cheng's theorem, there are rigidity results [13, 14].

The study of scalar curvature has a long history with many significant results. The recent lecture notes [10] and the survey paper [22] are good sources for the state of the affairs and references.

The structure of the paper is at follows. After collecting some preliminary results in Section 2, we supply the proofs of Theorem 1.1 and Theorem 1.3 in Section 3 and Section 4, respectively.

2. Preliminaries

In this section we make some preparations for proving Theorem 1.1 and Theorem 1.3. Let us start with the following result. It relies on an idea from Schoen-Yau's work on minimal surfaces [18, 19, 20] and appears as Lemma 4.1 in [2]. We include details here for completeness.

Lemma 2.1. *Let (M,g) be a three-dimensional complete noncompact Riemannian manifold with scalar curvature S and u a harmonic function on M. Then on each regular level set $l(r)$ of u,*

$$\mathrm{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-2}=\frac{1}{2}S-\frac{1}{2}S_{l(r)}+\frac{1}{\left|\nabla u\right|^{2}}\left(\left|\nabla\left|\nabla u\right|\right|^{2}-\frac{1}{2}\left|\nabla^{2}u\right|^{2}\right),$$

where $S_{l(r)}$ denotes the scalar curvature of $l(r)$.

Proof. On a regular level set $l(r)$ of u, its unit normal vector is given by

$$e_1=\frac{\nabla u}{|\nabla u|}.$$

Choose $\{e_a\}_{a=2,3}$, unit vectors tangent to $l(r)$, such that $\{e_1,e_2,e_3\}$ forms a local orthonormal frame on M. Since u is harmonic, the second fundamental form and the mean curvature of $l(r)$ are given by

$$h_{ab}=\frac{u_{ab}}{|\nabla u|}\quad\text{and}\quad H=-\frac{u_{11}}{|\nabla u|},\quad\text{respectively.}$$

By the Gauss curvature equation, we have

$$S_{l(r)}=S-2R_{11}+H^2-|h|^2.$$

Therefore,

$$\begin{aligned}2\mathrm{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-2} &= 2R_{11}\\ &= S-S_{l(r)}+\frac{1}{\left|\nabla u\right|^{2}}\left(\left|u_{11}\right|^{2}-\left|u_{ab}\right|^{2}\right)\\ &= S-S_{l(r)}+\frac{1}{\left|\nabla u\right|^{2}}\left(2\left|\nabla\left|\nabla u\right|\right|^{2}-\left|u_{ij}\right|^{2}\right),\end{aligned}$$

where we have used the fact that

$$\left|\nabla\left|\nabla u\right|\right|^{2}=\left|u_{11}\right|^{2}+\left|u_{1a}\right|^{2}$$

and

$$\left|u_{ij}\right|^{2}=\left|u_{11}\right|^{2}+2\left|u_{1a}\right|^{2}+\left|u_{ab}\right|^{2}.$$

This proves the result. □

We will also use the following well known Kato inequality for harmonic functions.

Lemma 2.2. *Let (M,g) be a three-dimensional complete noncompact Riemannian manifold and u a harmonic function on M. Then*

$$\left|\nabla^{2}u\right|^{2}\geq\frac{3}{2}\left|\nabla\left|\nabla u\right|\right|^{2}\quad\text{on } M.$$

Proof. It suffices to prove this at points where $|\nabla u| \neq 0$. Let

$$e_1 = \frac{\nabla u}{|\nabla u|}$$

and choose $\{e_a\}_{a=2,3}$ such that $\{e_1, e_2, e_3\}$ is a local orthonormal frame on M. Then

$$\begin{aligned}
|u_{ij}|^2 &= |u_{11}|^2 + 2|u_{1a}|^2 + |u_{ab}|^2 \\
&\geq |u_{11}|^2 + 2|u_{1a}|^2 + \frac{1}{2}|u_{22} + u_{33}|^2 \\
&= \frac{3}{2}|u_{11}|^2 + 2|u_{1a}|^2,
\end{aligned}$$

where in the last line we have used the fact that u is harmonic. Therefore,

$$\begin{aligned}
|u_{ij}|^2 &\geq \frac{3}{2}\left(|u_{11}|^2 + |u_{1a}|^2\right) \\
&= \frac{3}{2}|\nabla|\nabla u||^2.
\end{aligned}$$

□

We also need the following topological lemma concerning the number of components of the level sets of a proper Green's function. The proof is inspired by [12].

Lemma 2.3. *Let (M, g) be a complete noncompact Riemannian manifold with k ends and finite first Betti number $b_1(M)$. Assume that (M, g) is nonparabolic and its minimal positive Green's function G satisfies $\lim_{x\to\infty} G(x) = 0$. Then there exists $t_0 > 0$ such that the level set $l(t)$ of G has exactly k components for all $t \leq t_0$. In the case that M has only one end and the first Betti number $b_1(M) = 0$, the level set $l(t)$ is connected for all $t > 0$.*

Proof. We first claim that

$$L(a, \infty) \quad \text{is connected} \tag{2.1}$$

for all $a > 0$. Indeed, if this is not true, then there exists a connected component L_0 of $L(a, \infty)$ which does not contain the pole p of G. Moreover, L_0 is bounded by the fact that $\lim_{x\to\infty} G(x) = 0$. Then the harmonic function G on L_0 must achieve its maximum in the interior of L_0, which is a contradiction.

We also note that $L(0, a)$ contains no bounded components for any $a > 0$. Otherwise, on such a bounded component the function G would achieve its minimum at an interior point.

The fact that M has k ends means there exists a compact set $\Omega \subset M$ so that $M \setminus \Omega'$ has k unbounded connected components for any compact $\Omega \subset \Omega' \subset M$.

Since the first Betti number of M is finite, we may choose t_0 sufficiently small so that all the representatives of $H_1(M)$ lie in $L(t_0, \infty)$. As

$$\lim_{x\to\infty} G(x) = 0, \tag{2.2}$$

by arranging t_0 to be even smaller if necessary, we may assume that $L(0, t_0) \subset M \setminus \Omega$ and hence $L(0, t)$ has exactly k unbounded components for any $t \leq t_0$. By the fact that $L(0, t)$ contains no bounded components, it follows that $L(0, t)$ has exactly k components for any $t \leq t_0$.

From (2.2) and $l(t) \subset M \backslash \Omega$, we have that

$$l\left(t\right) \text{ has at least } k \text{ components,} \tag{2.3}$$

for all $t \leq t_0$.

We prove that in fact $l(t)$ cannot have more than k components. For $t < t_0$ and $\rho > 0$ such that $t + \rho < t_0$, since $M = L\left(t, \infty\right) \cup L\left(0, t+\rho\right)$, we have the Mayer-Vietoris sequence

$$\begin{aligned} &H_1\left(L\left(t,\infty\right)\right) \oplus H_1\left(L\left(0,t+\rho\right)\right) \to H_1\left(M\right) \to H_0\left(L\left(t,t+\rho\right)\right) \\ &\to H_0\left(L\left(t,\infty\right)\right) \oplus H_0\left(L\left(0,t+\rho\right)\right) \to H_0(M). \end{aligned}$$

Note that $H_1\left(L\left(0,t+\rho\right)\right)$ is trivial because all the representatives of $H_1\left(M\right)$ lie inside $L\left(t_0,\infty\right)$. In view of (2.1), we therefore conclude that

$$H_0\left(L\left(t,t+\rho\right)\right) = H_0\left(L\left(0,t+\rho\right)\right) = \mathbb{Z} \oplus .. \oplus \mathbb{Z}$$

with k summands. In other words, $L\left(t,t+\rho\right)$ has k components. Since $\rho > 0$ can be arbitrarily small, this proves that

$$l\left(t\right) \text{ has at most } k \text{ components} \tag{2.4}$$

for $t \leq t_0$. Therefore, by (2.4) and (2.3) we see that $l(t)$ has exactly k components for $t \leq t_0$.

In the special case that M has only one end and $b_1\left(M\right) = 0$, the number t_0 can be taken arbitrarily large. By (2.4), one concludes that $l\left(t\right)$ is connected for all $t > 0$. □

3. Nonnegative scalar curvature

In this section, we work with three-dimensional complete manifolds with nonnegative scalar curvature and prove Theorem 1.1 which is restated below.

Theorem 3.1. *Let (M, g) be a complete noncompact three-dimensional manifold with nonnegative scalar curvature. Assume that M has one end and its first Betti number $b_1\left(M\right) = 0$. If M is nonparabolic and the minimal positive Green's function $G\left(x\right) = G\left(p, x\right)$ satisfies $\lim_{x\to\infty} G(x) = 0$, then*

$$\frac{d}{dt}\left(\frac{1}{t}\int_{l(t)} \left|\nabla G\right|^2 - 4\pi t\right) \leq 0$$

for all regular values $t > 0$. Moreover, equality holds for some $T > 0$ if and only if the super level set $\{x \in M : G(x) > T\}$ is isometric to a ball in the Euclidean space $\mathbb{R}^3$.

Proof. Recall that

$$\begin{aligned} l\left(t\right) &= \left\{x \in M : G\left(x\right) = t\right\} \\ L\left(a,b\right) &= \left\{x \in M : a < G\left(x\right) < b\right\}. \end{aligned}$$

By the assumption that $\lim_{x\to\infty} G(x) = 0$, the level set $l\left(t\right)$ is compact for every $t > 0$. Moreover, since M is assumed to have one end and $b_1\left(M\right) = 0$, Lemma 2.3 implies that $l\left(t\right)$ is connected for all $t > 0$.

Consider the function

$$w(t)=\int_{l(t)}|\nabla G|^2.$$

Whenever $l(t)$ is regular, its mean curvature H is given by

$$H_{l(t)}=\frac{\sum_a G_{aa}}{|\nabla G|}=-\frac{G_{11}}{|\nabla G|}=-\frac{\langle\nabla|\nabla G|,\nabla G\rangle}{|\nabla G|^2},$$

where $e_1=\frac{\nabla G}{|\nabla G|}$ and $\{e_a\}_{a=2,3}$ are unit tangent vectors on $l(t)$ such that $\{e_1,e_2,e_3\}$ is a local orthonormal frame on M. It follows that

$$\begin{aligned}\frac{dw}{dt}(t) &= \int_{l(t)}\left(\frac{\left\langle\nabla|\nabla G|^2,\nabla G\right\rangle}{|\nabla G|^2}+\frac{H_{l(t)}}{|\nabla G|}|\nabla G|^2\right)\\ &= \int_{l(t)}\frac{\langle\nabla|\nabla G|,\nabla G\rangle}{|\nabla G|}.\end{aligned}$$

Multiplying the equation by t^{-2} we get

$$t^{-2}\frac{dw}{dt}(t)=\int_{l(t)}\frac{\langle\nabla|\nabla G|,\nabla G\rangle}{|\nabla G|}G^{-2}. \tag{3.1}$$

On the other hand, by Green's identity we have

$$\begin{aligned}&\int_{L(t,T)}\left(G^{-2}\Delta|\nabla G|-|\nabla G|\Delta G^{-2}\right)\\ =&\int_{l(T)}\left(G^{-2}\frac{\langle\nabla|\nabla G|,\nabla G\rangle}{|\nabla G|}-|\nabla G|\frac{\left\langle\nabla G^{-2},\nabla G\right\rangle}{|\nabla G|}\right)\\ &-\int_{l(t)}\left(G^{-2}\frac{\langle\nabla|\nabla G|,\nabla G\rangle}{|\nabla G|}-|\nabla G|\frac{\left\langle\nabla G^{-2},\nabla G\right\rangle}{|\nabla G|}\right).\end{aligned} \tag{3.2}$$

Since G is harmonic on $L(t,T)$,

$$|\nabla G|\Delta G^{-2}=6G^{-4}|\nabla G|^3.$$

Recall that $G(x)$ is the limit of Dirichlet Green's functions $G_i(x)$ of a compact exhaustion of M, and the convergence is uniform on compact subsets of $M\backslash\{p\}$, where $p\in M$ is the pole of $G(x)$. Therefore, we note that as $x\to p$,

$$\begin{aligned}G(x) &= \frac{1}{4\pi}(1+o(1))\frac{1}{r(x)}\\ |\nabla G|(x) &= \frac{1}{4\pi}(1+o(1))\frac{1}{r^2(x)}\\ \left\langle\nabla|\nabla G|,\frac{\nabla G}{|\nabla G|}\right\rangle &= \frac{1}{2\pi}(1+o(1))\frac{1}{r^3(x)}.\end{aligned} \tag{3.3}$$

Hence, it follows that

$$\lim_{T\to\infty}\int_{l(T)}G^{-2}\frac{\langle\nabla|\nabla G|,\nabla G\rangle}{|\nabla G|}=0 \text{ and } \lim_{T\to\infty}\int_{l(T)}|\nabla G|\frac{\left\langle\nabla G^{-2},\nabla G\right\rangle}{|\nabla G|}=0.$$

In conclusion, (3.2) implies that

$$\left|\int_{L(t,\infty)}\left(G^{-2}\Delta\left|\nabla G\right|-\left|\nabla G\right|\Delta G^{-2}\right)\right|<\infty,$$

and the following identity holds

$$\begin{aligned}&\int_{l(t)}\left(G^{-2}\frac{\left\langle\nabla\left|\nabla G\right|,\nabla G\right\rangle}{\left|\nabla G\right|}-\left|\nabla G\right|\frac{\left\langle\nabla G^{-2},\nabla G\right\rangle}{\left|\nabla G\right|}\right)\\=&-\int_{L(t,\infty)}\left(G^{-2}\Delta\left|\nabla G\right|-6G^{-4}\left|\nabla G\right|^{3}\right).\end{aligned}$$

Together with (3.1), we conclude that

$$\begin{aligned}t^{-2}\frac{dw}{dt}(t) &= \int_{l(t)}G^{-2}\frac{\left\langle\nabla\left|\nabla G\right|,\nabla G\right\rangle}{\left|\nabla G\right|}\\&= -2\int_{l(t)}G^{-3}\left|\nabla G\right|^{2}-\int_{L(t,\infty)}G^{-2}\Delta\left|\nabla G\right|\\&\quad+6\int_{L(t,\infty)}G^{-4}\left|\nabla G\right|^{3}.\end{aligned}\tag{3.4}$$

Note that by the co-area formula,

$$\begin{aligned}\int_{L(t,\infty)}G^{-4}\left|\nabla G\right|^{3} &= \int_{t}^{\infty}r^{-4}\int_{l(r)}\left|\nabla G\right|^{2}\\&= \int_{t}^{\infty}r^{-4}w(r)\,dr.\end{aligned}$$

Hence, (3.4) can be written as

$$\begin{aligned}t^{-2}\frac{dw}{dt}(t) &= -2t^{-3}w(t)+6\int_{t}^{\infty}r^{-4}w(r)\,dr\\&\quad-\int_{L(t,\infty)}G^{-2}\Delta\left|\nabla G\right|.\end{aligned}\tag{3.5}$$

We now estimate the last term. Using the Bochner formula

$$\Delta\left|\nabla G\right|=\left(\left|G_{ij}\right|^{2}-\left|\nabla\left|\nabla G\right|\right|^{2}\right)\left|\nabla G\right|^{-1}+\operatorname{Ric}\left(\nabla G,\nabla G\right)\left|\nabla G\right|^{-1},$$

we have

$$\begin{aligned}\int_{l(r)}\left|\nabla G\right|^{-1}\Delta\left|\nabla G\right| &= \int_{l(r)}\left(\left|G_{ij}\right|^{2}-\left|\nabla\left|\nabla G\right|\right|^{2}\right)\left|\nabla G\right|^{-2}\\&\quad+\int_{l(r)}\operatorname{Ric}\left(\nabla G,\nabla G\right)\left|\nabla G\right|^{-2}.\end{aligned}$$

Applying Lemma 2.1 to G gives

$$\operatorname{Ric}\left(\nabla G,\nabla G\right)\left|\nabla G\right|^{-2}=\frac{1}{2}S-\frac{1}{2}S_{l(r)}+\left(\left|\nabla\left|\nabla G\right|\right|^{2}-\frac{1}{2}\left|G_{ij}\right|^{2}\right)\left|\nabla G\right|^{-2},$$

where $S_{l(r)}$ is the scalar curvature of $l\left(r\right).$ We therefore conclude that

$$\int_{l(r)} \left|\nabla G\right|^{-1} \Delta \left|\nabla G\right| = \frac{1}{2}\int_{l(r)} \left(\left|G_{ij}\right|^{2}\left|\nabla G\right|^{-2} + S - S_{l(r)}\right). \tag{3.6}$$

Note that by Lemma 2.2,

$$\left|G_{ij}\right|^{2} \geq \frac{3}{2}\left|\nabla\left|\nabla G\right|\right|^{2}.$$

Also, since $l\left(r\right)$ is compact and connected for any $r>0$, the Gauss-Bonnet theorem implies that

$$\int_{l(r)} S_{l(r)} = 4\pi\chi\left(l\left(r\right)\right) \leq 8\pi$$

whenever r is a regular value of G. Therefore, on any regular level set $l(r)$, one obtains from (3.6) that

$$\int_{l(r)} \left|\nabla G\right|^{-1} \Delta\left|\nabla G\right| \geq \frac{3}{4}\int_{l(r)} \left|\nabla\left|\nabla G\right|\right|^{2}\left|\nabla G\right|^{-2} - 4\pi. \tag{3.7}$$

Observe from (3.1) that

$$\begin{aligned} \left|w'\left(r\right)\right| &= \left|\int_{l(r)} \frac{\left\langle \nabla\left|\nabla G\right|, \nabla G\right\rangle}{\left|\nabla G\right|}\right| \\ &\leq \int_{l(r)} \left|\nabla\left|\nabla G\right|\right| \\ &\leq \left(\int_{l(r)} \left|\nabla\left|\nabla G\right|\right|^{2}\left|\nabla G\right|^{-2}\right)^{\frac{1}{2}} \left(\int_{l(r)} \left|\nabla G\right|^{2}\right)^{\frac{1}{2}}, \end{aligned} \tag{3.8}$$

which says that

$$\int_{l(r)} \left|\nabla\left|\nabla G\right|\right|^{2}\left|\nabla G\right|^{-2} \geq \frac{\left(w'\right)^{2}}{w}\left(r\right).$$

Combining with (3.7) we conclude that

$$\int_{l(r)} \left|\nabla G\right|^{-1} \Delta\left|\nabla G\right| \geq \frac{3}{4}\frac{\left(w'\right)^{2}}{w}\left(r\right) - 4\pi. \tag{3.9}$$

By (3.9) and the co-area formula it follows that

$$\begin{aligned} -\int_{L(t,\infty)} G^{-2}\Delta\left|\nabla G\right| &= -\int_{t}^{\infty} r^{-2}\int_{l(r)} \left|\nabla G\right|^{-1}\Delta\left|\nabla G\right| \\ &\leq -\frac{3}{4}\int_{t}^{\infty} r^{-2}\frac{\left(w'\right)^{2}}{w}\left(r\right)dr + \frac{4\pi}{t}. \end{aligned} \tag{3.10}$$

From the elementary inequality

$$
\begin{aligned}
0 &\leq w\left(\frac{w'}{w}-\frac{2}{r}\right)^2 \\
&= \frac{(w')^2}{w}-\frac{4}{r}w'+\frac{4}{r^2}w,
\end{aligned} \tag{3.11}
$$

one sees that

$$
-\frac{3}{4}\int_t^\infty r^{-2}\frac{(w')^2}{w}(r)\,dr \leq -3\int_t^\infty r^{-3}w'(r)\,dr+3\int_t^\infty r^{-4}w(r)\,dr.
$$

Furthermore, integrating by parts implies that

$$
\begin{aligned}
-3\int_t^\infty r^{-3}w'(r)\,dr &= -3r^{-3}w(r)\,|_t^\infty - 9\int_t^\infty r^{-4}w(r)\,dr \\
&= 3t^{-3}w(t)-9\int_t^\infty r^{-4}w(r)\,dr.
\end{aligned}
$$

In conclusion,

$$
-\frac{3}{4}\int_t^\infty r^{-2}\frac{(w')^2}{w}(r)\,dr \leq 3t^{-3}w(t)-6\int_t^\infty r^{-4}w(r)\,dr. \tag{3.12}
$$

Plugging (3.12) into (3.10) we get

$$
-\int_{L(t,\infty)} G^{-2}\Delta|\nabla G| \leq 3t^{-3}w(t)-6\int_t^\infty r^{-4}w(r)\,dr+\frac{4\pi}{t}. \tag{3.13}
$$

Hence, by (3.13) and (3.5) we obtain

$$
t^{-2}\frac{dw}{dt}(t) \leq t^{-3}w(t)+\frac{4\pi}{t},
$$

or equivalently,

$$
\frac{d}{dt}\left(\frac{1}{t}w(t)-4\pi t\right) \leq 0. \tag{3.14}
$$

Finally, if

$$
\frac{d}{dt}\left(\frac{1}{t}w(t)-4\pi t\right)=0 \text{ for } t=T,
$$

then all the inequalities in Lemma 2.2, (3.11) and (3.8) become equality on $L(T,\infty)$. That is, for any regular value $t>T$ of G it follows that the Hessian of G is diagonal in the frame $\{e_1=\frac{\nabla G}{|\nabla G|}, e_2, e_3\}$, and

$$
G_{11}=|\nabla|\nabla G||. \tag{3.15}
$$

Furthermore, there exists $\lambda(t)\in\mathbb{R}$ so that

$$
|\nabla|\nabla G||=\lambda(t)|\nabla G| \text{ on } l(t) \tag{3.16}
$$

and

$$
\frac{dw}{dt}(t)=\frac{2}{t}w(t). \tag{3.17}
$$

By (3.8), (3.15) and (3.16) we find that $\frac{dw}{dt} = \lambda(t)$, so (3.17) implies

$$t\frac{d\lambda}{dt}(t) = \lambda(t). \tag{3.18}$$

Along the gradient flow $\frac{d\Phi}{dt} = \frac{\nabla G}{|\nabla G|^2}$ we have $\frac{d|\nabla G|}{dt} = \lambda(t)$, therefore by (3.18) we conclude that all values $t > T$ are regular. Furthermore, (3.18) and (3.3) imply that $\lambda(t) = 8\pi t$, for all $t > T$.

It follows from above that $\nabla\left(|\nabla G| - 4\pi G^2\right) = 0$, from which we deduce that $|\nabla G| = 4\pi G^2$.

In particular, the Hessian of G on $L(T,\infty)$ must be of the form

$$\begin{aligned} G_{11} &= 32\pi^2 G^3 \\ G_{22} &= -16\pi^2 G^3 \\ G_{33} &= -16\pi^2 G^3 \\ G_{ij} &= 0 \quad \text{otherwise} \end{aligned}$$

and $|\nabla G| = 4\pi G^2$. Now consider the function $f(x) = \frac{1}{4\pi\, G(x)}$. Then $|\nabla f| = 1$ and the Hessian of the function $\frac{1}{2} f^2$ is the identity matrix. This immediately implies that $L(T,\infty)$ is isometric to the ball $B_0(\frac{1}{4\pi T})$ in $\mathbb{R}^3$. □

We conclude this section with the following corollary.

Corollary 3.2. *Let (M,g) be a complete noncompact three-dimensional manifold with nonnegative scalar curvature and asymptotically nonnegative Ricci curvature, that is,*

$$\liminf_{x\to\infty} \mathrm{Ric}\,(x) \geq 0. \tag{3.19}$$

Assume that M has one end and its first Betti number $b_1(M) = 0$. If M is nonparabolic and the minimal Green's function $G(x) = G(p,x)$ satisfies $\lim_{x\to\infty} G(x) = 0$, then

$$\int_{l(t)} |\nabla G|^2 \leq 4\pi t^2$$

and

$$\mathrm{Area}\,(l(t)) \geq \frac{1}{4\pi t^2}$$

for all $t > 0$. Moreover, if equality holds for some $T > 0$, then (M,g) is isometric to $\mathbb{R}^3$.

Proof. Let us note that by Theorem 3.1 we have

$$\frac{1}{t}w(t) - 4\pi t \leq \frac{1}{\delta}w(\delta) - 4\pi\delta \tag{3.20}$$

for all $0 < \delta < t$. Now the gradient estimate in [5] together with the assumption (3.19) implies that for any $\varepsilon > 0$ there exists sufficiently small $\delta > 0$ such that

$$|\nabla \ln G| \leq \varepsilon \quad \text{on } L(0,\delta). \tag{3.21}$$

Therefore,

$$\frac{1}{\delta}w(\delta)=\frac{1}{\delta}\int_{l(\delta)}|\nabla G|^2\le\varepsilon,$$

where we have used the fact that

$$\int_{l(\delta)}|\nabla G|=1. \tag{3.22}$$

This shows that the right hand side of (3.20) goes to 0 as $\delta\to 0$. Hence, $w(t)\le 4\pi t^2$, or

$$\int_{l(t)}|\nabla G|^2\le 4\pi t^2$$

for all $t>0$.

We now derive a sharp area estimate for the level sets of the Green's function. Indeed,

$$1=\int_{l(t)}|\nabla G|\le\left(\int_{l(t)}|\nabla G|^2\right)^{\frac{1}{2}}(\mathrm{Area}\,(l\,(t)))^{\frac{1}{2}}.$$

It follows that

$$\mathrm{Area}\,(l\,(t))\ge\frac{1}{4\pi t^2}$$

for all $t>0$.

Moreover, if there exists $T>0$ such that

$$\int_{l(T)}|\nabla G|^2=4\pi T^2,$$

then (3.20) implies that

$$\int_{l(t)}|\nabla G|^2=4\pi t^2,$$

for all $0<t<T$. Hence, we have equality in Theorem 3.1 for all $t<T$. This implies that (M,g) is isometric to $\mathbb{R}^3$. □

We note that under the hypothesis of Corollary 3.2 if

$$\limsup_{t\to\infty}\left(\frac{1}{t}\int_{l(t)}|\nabla G|^2-4\pi t\right)\ge 0, \tag{3.23}$$

then (M,g) is isometric to $\mathbb{R}^3$. Indeed, by (3.20) and (3.23) we have

$$0\le\frac{1}{\delta}\int_{l(\delta)}|\nabla G|^2-4\pi\delta$$

for any $\delta>0$. Hence, equality must hold in Corollary 3.2 and (M,g) is isometric to the Euclidean space $\mathbb{R}^3$.

4. Negative scalar lower bound

We now turn to the proof of Theorem 1.3. We start by establishing some lemmas under the assumption that (M,g) admits a positive Green's function G satisfying

$$\lim_{x\to\infty} G(x) = 0, \tag{4.1}$$

$$\#\mathrm{Conn}\,(l(t)) \leq A \tag{4.2}$$

for all $t \le t_0$, where t_0 and $A>0$ are fixed constants, and $\#\mathrm{Conn}\,(l(t))$ denotes the number of connected components of the level set $l(t)$ of G.

Lemma 4.1. *Let (M,g) be a three-dimensional complete noncompact Riemannian manifold satisfying (4.1) and (4.2). Assume that the Ricci curvature of M is bounded from below and the scalar curvature S is bounded by*

$$S \geq -6K \quad \text{on } M.$$

Then for any $\varepsilon>0$,

$$\int_{L(\varepsilon,t_0)} |\nabla|\nabla G||^2 |\nabla G|^{-1} \leq 4K \int_{L\left(\frac{1}{2}\varepsilon,2t_0\right)} |\nabla G| + \frac{32\pi}{3} t_0 A + C,$$

where C is a constant depending only on t_0 and the Ricci curvature lower bound of M, but not ε.

Proof. According to the Bochner formula,

$$\begin{aligned} \frac{1}{2}\Delta|\nabla G|^2 &= |G_{ij}|^2 + \langle \nabla\Delta G, \nabla G\rangle + \mathrm{Ric}(\nabla G,\nabla G) \\ &= |G_{ij}|^2 + \mathrm{Ric}(\nabla G,\nabla G) \end{aligned}$$

on $M\backslash\{p\}$. Therefore,

$$\Delta|\nabla G| = \left(|G_{ij}|^2 - |\nabla|\nabla G||^2\right)|\nabla G|^{-1} + \mathrm{Ric}(\nabla G,\nabla G)\,|\nabla G|^{-1} \tag{4.3}$$

holds on $M\backslash\{p\}$ whenever $|\nabla G|\neq 0$.

Fix $0<\varepsilon<t_0<\infty$ and let ϕ be the Lipschitz function with support in $L\left(\frac{1}{2}\varepsilon, 2t_0\right)$ defined by

$$\phi = \begin{cases} 1 & \text{on } L(\varepsilon,t_0) \\ \frac{\ln G - \ln\left(\frac{1}{2}\varepsilon\right)}{\ln 2} & \text{on } L\left(\frac{1}{2}\varepsilon,\varepsilon\right) \\ \frac{\ln(2t_0)-\ln G}{\ln 2} & \text{on } L(t_0,2t_0) \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

By the co-area formula, we have

$$\begin{aligned} &\int_M \left(|G_{ij}|^2 - |\nabla|\nabla G||^2 + \mathrm{Ric}(\nabla G,\nabla G)\right)|\nabla G|^{-1}\phi^2 \\ = &\int_{\frac{1}{2}\varepsilon}^{2t_0} \phi^2(r) \int_{l(r)} \left(|G_{ij}|^2 - |\nabla|\nabla G||^2 + \mathrm{Ric}(\nabla G,\nabla G)\right)|\nabla G|^{-2}\,dr. \end{aligned}$$

However, Lemma 2.1 says that

$$\begin{aligned} & \left(|G_{ij}|^2 - |\nabla |\nabla G||^2 + \mathrm{Ric}(\nabla G, \nabla G)\right) |\nabla G|^{-2} \\ = \; & \frac{1}{2} S - \frac{1}{2} S_{l(r)} + \frac{1}{2} |G_{ij}|^2 |\nabla G|^{-2}. \end{aligned}$$

Applying Lemma 2.2 for the last term, we conclude that

$$\begin{aligned} & \left(|G_{ij}|^2 - |\nabla |\nabla G||^2 + \mathrm{Ric}(\nabla G, \nabla G)\right) |\nabla G|^{-2} \\ \geq \; & \frac{1}{2} S - \frac{1}{2} S_{l(r)} + \frac{3}{4} |\nabla |\nabla G||^2 |\nabla G|^{-2}. \end{aligned} \tag{4.5}$$

According to the Gauss-Bonnet theorem, on each regular connected component $l_k(r)$ of $l(r)$,

$$\int_{l_k(r)} S_{l(r)} = 4\pi \chi (l_k(r)) \leq 8\pi$$

as $l_k(r)$ is compact by (4.1). Since by hypothesis (4.2) there are at most A connected components of $l(r)$, it follows that

$$\int_{l(r)} S_{l(r)} \leq 8\pi A$$

for all regular value r with $r \leq t_0$. Therefore, using that $S \geq -6K$, we conclude from (4.5) that

$$\begin{aligned} & \int_{l(r)} \left(|G_{ij}|^2 - |\nabla |\nabla G||^2 + \mathrm{Ric}(\nabla G, \nabla G)\right) |\nabla G|^{-2} \\ \geq \; & \int_{l(r)} \left(\frac{1}{2} S - \frac{1}{2} S_{l(r)} + \frac{3}{4} |\nabla |\nabla G||^2 |\nabla G|^{-2}\right) \\ \geq \; & -3K \mathrm{Area}(l(r)) + \frac{3}{4} \int_{l(r)} |\nabla |\nabla G||^2 |\nabla G|^{-2} - 4\pi A. \end{aligned}$$

Consequently, this implies that

$$\begin{aligned} & \int_M \left(|G_{ij}|^2 - |\nabla |\nabla G||^2 + \mathrm{Ric}(\nabla G, \nabla G)\right) |\nabla G|^{-1} \phi^2 \\ \geq \; & \int_{\frac{1}{2}\varepsilon}^{2t_0} \phi^2(r) \left(-3K \mathrm{Area}(l(r)) + \frac{3}{4} \int_{l(r)} |\nabla |\nabla G||^2 |\nabla G|^{-2}\right) dr - 8\pi t_0 A \\ = \; & -3K \int_M |\nabla G| \phi^2 + \frac{3}{4} \int_M |\nabla |\nabla G||^2 |\nabla G|^{-1} \phi^2 - 8\pi t_0 A. \end{aligned}$$

Together with the Bochner formula (4.3), we arrive at

$$-3K \int_M |\nabla G| \phi^2 + \frac{3}{4} \int_M |\nabla |\nabla G||^2 |\nabla G|^{-1} \phi^2 \leq \int_M \phi^2 \Delta |\nabla G| + 8\pi t_0 A. \tag{4.6}$$

To estimate the right hand side, we make use of the gradient estimate for positive harmonic functions. Choose $R_0 > 0$ so that $L(0, 2t_0) \subset M \backslash B(p, R_0)$, where p is the pole of G. Since the Ricci curvature is bounded from below on M, applying Cheng-Yau's gradient estimate [5] to the positive harmonic function G on $M\backslash B\left(p,\frac{R_0}{2}\right)$, one concludes that

$$|\nabla \ln G| \leq \Lambda \text{ on } M\backslash B\left(p,R_0\right), \tag{4.7}$$

where the constant Λ depends only on R_0 and Ricci curvature lower bound of M.

Integration by parts implies that

$$\begin{aligned} \int_M \phi^2 \Delta |\nabla G| &= -\int_M \left\langle \nabla\phi^2, \nabla|\nabla G|\right\rangle \\ &= -\int_{L\left(\frac{1}{2}\varepsilon,\varepsilon\right)} \left\langle \nabla\phi^2, \nabla|\nabla G|\right\rangle \\ &\quad -\int_{L(t_0,2t_0)} \left\langle \nabla\phi^2, \nabla|\nabla G|\right\rangle. \end{aligned}$$

Further integration by parts on each term leads to

$$\begin{aligned} -\int_{L\left(\frac{1}{2}\varepsilon,\varepsilon\right)} \left\langle \nabla\phi^2, \nabla|\nabla G|\right\rangle &= \int_{L\left(\frac{1}{2}\varepsilon,\varepsilon\right)} |\nabla G|\,\Delta\phi^2 - \int_{l(\varepsilon)} \left\langle \nabla\phi^2, \nabla G\right\rangle \\ -\int_{L(t_0,2t_0)} \left\langle \nabla\phi^2, \nabla|\nabla G|\right\rangle &= \int_{L(t_0,2t_0)} |\nabla G|\,\Delta\phi^2 + \int_{l(t_0)} \left\langle \nabla\phi^2, \nabla G\right\rangle. \end{aligned}$$

Noting that G is harmonic, we get that on $L\left(\frac{1}{2}\varepsilon,\varepsilon\right)$

$$\begin{aligned} \Delta\phi^2 &= 2\phi\Delta\phi + 2|\nabla\phi|^2 \\ &= 2\left(-\frac{1}{\ln 2}\phi + \frac{1}{(\ln 2)^2}\right)|\nabla \ln G|^2. \end{aligned}$$

Similarly, on $L\left(t_0,2t_0\right)$,

$$\Delta\phi^2 = 2\left(\frac{1}{\ln 2}\phi + \frac{1}{(\ln 2)^2}\right)|\nabla \ln G|^2.$$

In both cases, in view of (4.7), we have

$$\begin{aligned} \left|\Delta\phi^2\right| &\leq c\,|\nabla \ln G|^2 \\ &\leq c\Lambda\,|\nabla \ln G|, \end{aligned}$$

where c is a universal constant. Hence, by the co-area formula and the fact that

$$\int_{l(r)} |\nabla G| = 1, \tag{4.8}$$

we get

$$\begin{aligned}\left|\int_{L\left(\frac{1}{2}\varepsilon,\varepsilon\right)}|\nabla G|\,\Delta\phi^2\right| &\leq c\Lambda\int_{L\left(\frac{1}{2}\varepsilon,\varepsilon\right)}|\nabla G|^2\,G^{-1}\\ &= c\Lambda\int_{\frac{1}{2}\varepsilon}^{\varepsilon}\frac{1}{r}dr\\ &= c\Lambda\ln 2.\end{aligned}$$

The other term is estimated as follows.

$$\begin{aligned}\left|\int_{l(\varepsilon)}\left\langle\nabla\phi^2,\nabla G\right\rangle\right| &\leq \frac{2}{\ln 2}\int_{l(\varepsilon)}|\nabla G|^2\,G^{-1}\\ &\leq c\Lambda.\end{aligned}$$

Similarly,

$$\left|\int_{L(t_0,2t_0)}|\nabla G|\,\Delta\phi^2\right| + \left|\int_{l(t_0)}\left\langle\nabla\phi^2,\nabla G\right\rangle\right| \leq c\Lambda.$$

In conclusion, we have shown that

$$\left|\int_M \phi^2\Delta\,|\nabla G|\right| \leq c\Lambda \quad\text{and}\quad \int_M |\nabla\phi|^2\,|\nabla G| \leq c\Lambda. \tag{4.9}$$

Plugging into (4.6) implies that

$$\frac{3}{4}\int_M |\nabla\,|\nabla G||^2\,|\nabla G|^{-1}\,\phi^2 \leq 3K\int_M |\nabla G|\,\phi^2 + c\Lambda + 8\pi t_0 A,$$

which is what to be proved. □

Lemma 4.2. *Let (M,g) be a three-dimensional complete noncompact Riemannian manifold satisfying (4.1) and (4.2). Assume that the Ricci curvature of M is bounded from below and the scalar curvature S is bounded by*

$$S \geq -6K \quad \text{on } M.$$

Then

$$\lambda_1(M) \leq K.$$

Proof. According to Lemma 4.1,

$$\frac{1}{4}\int_M |\nabla\,|\nabla G||^2\,|\nabla G|^{-1}\,\phi^2 \leq K\int_M |\nabla G|\,\phi^2 + c\Lambda + \frac{8\pi}{3}t_0 A, \tag{4.10}$$

where ϕ is given in (4.4).

We now bound the left hand side from below using the bottom spectrum $\lambda_1(M)$. Note that

$$\begin{aligned}\lambda_1(M)\int_M |\nabla G|\,\phi^2 &= \lambda_1(M)\int_M \left(|\nabla G|^{\frac{1}{2}}\,\phi\right)^2\\ &\leq \int_M \left|\nabla\left(|\nabla G|^{\frac{1}{2}}\,\phi\right)\right|^2.\end{aligned}$$

Expanding the right side, we get

$$\begin{aligned}\int_M \left|\nabla\left(|\nabla G|^{\frac{1}{2}}\,\phi\right)\right|^2 &= \frac{1}{4}\int_M |\nabla|\nabla G||^2\,|\nabla G|^{-1}\,\phi^2 + \int_M |\nabla G|\,|\nabla\phi|^2 \\ &\quad + \frac{1}{2}\int_M \left\langle \nabla|\nabla G|\,,\nabla\phi^2\right\rangle.\end{aligned}$$

By (4.9) we have

$$\int_M |\nabla G|\,|\nabla\phi|^2 + \frac{1}{2}\int_M \left\langle \nabla|\nabla G|\,,\nabla\phi^2\right\rangle \le c\Lambda.$$

Therefore,

$$\int_M \left|\nabla\left(|\nabla G|^{\frac{1}{2}}\,\phi\right)\right|^2 \le \frac{1}{4}\int_M |\nabla|\nabla G||^2\,|\nabla G|^{-1}\,\phi^2 + c\Lambda.$$

In conclusion,

$$\begin{aligned}\lambda_1(M)\int_M |\nabla G|\,\phi^2 &\le \int_M \left|\nabla\left(|\nabla G|^{\frac{1}{2}}\,\phi\right)\right|^2 \\ &\le \frac{1}{4}\int_M |\nabla|\nabla G||^2\,|\nabla G|^{-1}\,\phi^2 + c\Lambda.\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11) we arrive at the following inequality.

$$\lambda_1(M)\int_M |\nabla G|\,\phi^2 \le K\int_M |\nabla G|\,\phi^2 + c\Lambda + \frac{8\pi}{3}t_0 A, \tag{4.12}$$

where c is a universal constant. Assume by contradiction that

$$\lambda_1(M) > K. \tag{4.13}$$

Then in view of the definition of ϕ in (4.4) and (4.12) we conclude that

$$(\lambda_1(M) - K)\int_{L(\varepsilon,t_0)} |\nabla G| \le c\,(\Lambda + t_0 A). \tag{4.14}$$

However, applying (4.8) and the co-area formula, we have

$$\begin{aligned}\int_{L(\varepsilon,t_0)} |\nabla G|^2\,G^{-1} &= \int_\varepsilon^{t_0}\frac{1}{r}\left(\int_{l(r)}|\nabla G|\right)dr \\ &= \ln\left(t_0\varepsilon^{-1}\right).\end{aligned}$$

On the other hand, Cheng-Yau's gradient estimate shows that

$$\int_{L(\varepsilon,t_0)} |\nabla G|^2\,G^{-1} \le \Lambda\int_{L(\varepsilon,t_0)}|\nabla G|\,.$$

We thus conclude that

$$\int_{L(\varepsilon,t_0)} |\nabla G| \ge \Lambda^{-1}\ln\left(t_0\varepsilon^{-1}\right). \tag{4.15}$$

Finally, we infer from (4.15) and (4.14) that

$$\lambda_1(M) - K \le \frac{c\Lambda(\Lambda + t_0 A)}{\ln(t_0 \varepsilon^{-1})}. \tag{4.16}$$

Note that c is a universal constant and that both A and Λ are independent of ε. Taking $\varepsilon \to 0$ leads to a contradiction. Therefore, we have

$$\lambda_1(M) \le K.$$

□

We are now ready to prove Theorem 1.3. The only thing left to do is to verify that both (4.1) and (4.2) hold.

Theorem 4.3. *Let (M, g) be a three-dimensional complete noncompact Riemannian manifold with scalar curvature $S \ge -6K$ on M for some nonnegative constant K. Suppose that M has finitely many ends and its first Betti number $b_1(M) < \infty$. Moreover, the Ricci curvature of M is bounded below and the volume $V_x(1)$ of unit ball $B_x(1)$ satisfies*

$$V_x(1) \ge C(\epsilon) \exp\left(-2\sqrt{K+\epsilon}\, r(x)\right)$$

for every $\epsilon > 0$ and all $x \in M$, where $r(x)$ is the geodesic distance from x to a fixed point p. Then the bottom spectrum of the Laplacian satisfies

$$\lambda_1(M) \le K.$$

Proof. We prove the result by contradiction. Suppose that $\lambda_1(M) > K$. Then M is necessarily nonparabolic as $\lambda_1(M) > 0$, see [11, Chapter 22]. Let G be the minimal positive Green's function. We first show that $\lim_{x\to\infty} G(x) = 0$, that is, (4.1) holds.

According to a result of Li and the second author [15],

$$\int_{B_p(R+1)\setminus B_p(R-1)} G^2 \le C\, e^{-2\sqrt{\lambda_1(M)}R} \int_{B_p(2)\setminus B_p(1)} G^2 \tag{4.17}$$

for all $R > 4$, where $C > 0$ is a constant depending only on $\lambda_1(M)$. Since the Ricci curvature is bounded from below, by (4.7), the mean value inequality holds for the function G. That is, for any $x \in M \setminus B_p(2)$,

$$G^2(x) \le \frac{C}{V_x(1)} \int_{B(x,1)} G^2 \tag{4.18}$$

for some constant C only depending on the Ricci curvature lower bound. Now fix $\epsilon > 0$ such that $\lambda_1(M) \ge K + 2\epsilon$. Combining (4.17) and (4.18), we conclude that

$$G^2(x) \le \frac{C}{V_x(1)}\, e^{-2\sqrt{K+2\epsilon}\, r(x)} \int_{B_p(2)\setminus B_p(1)} G^2, \tag{4.19}$$

where $r(x)$ is the geodesic distance from x to point p. Using the assumption that

$$V_x(1) \ge C(\epsilon) \exp\left(-2\sqrt{K+\epsilon}\, r(x)\right),$$

one sees immediately from (4.19) that $\lim_{x\to\infty} G(x) = 0$.

The fact that G satisfies (4.2) follows directly from Lemma 2.3. Therefore, Lemma 4.2 is applicable and $\lambda_1(M) \le K$. □

We now draw a corollary.

Corollary 4.4. *Let (M, g) be a three-dimensional complete noncompact Riemannian manifold with nonpositive sectional curvature. Assume that the scalar curvature S is bounded by*

$$S \geq -6K \quad \text{on } M$$

for some nonnegative constant K. Then

$$\lambda_1(M) \leq K.$$

Proof. This is because Theorem 4.3 is applicable to the universal cover $\tilde{M}$ of M as $\tilde{M}$ is a Cartan-Hadamard manifold with bounded curvature. Indeed, being diffeomorphic to $\mathbb{R}^3$, $\tilde{M}$ has first Betti number 0 and exactly one end. Also, the volume $V_x(1)$ of unit balls $B_x(1)$ is at least that of the unit ball in $\mathbb{R}^3$ by the volume comparison theorem. Therefore, $\lambda_1(\tilde{M}) \leq K$. However, the bottom spectrum satisfies $\lambda_1(M) \leq \lambda_1(\tilde{M})$, so the corollary follows. □

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Department of Mathematics, University of Connecticut, Storrs, CT 06268, USA
E-mail address: ovidiu.munteanu@uconn.edu

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: jiaping@math.umn.edu