GEOMETRY OF THREE-DIMENSIONAL MANIFOLDS WITH SCALAR CURVATURE LOWER BOUND

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Abstract. The paper concerns three-dimensional complete manifolds with scalar curvature bounded from below. One of the purposes is to establish a sharp comparison theorem for the bottom spectrum in the spirit of the classical result of Cheng. Another purpose is to derive volume and other geometric information in terms of the scalar curvature when the Ricci curvature is asymptotically nonnegative and the scalar curvature is positive. If the scalar curvature decays no faster than linearly, then the manifold does not admit any positive Green's function. When the scalar curvature is bounded from below by a positive constant, it is shown that the volume of unit balls must be bounded from above by the lower bound of the scalar curvature at infinity. In particular, in the case that the Ricci curvature is nonnegative, the volume of the manifold must be of linear growth. This answers a question of Gromov in the affirmative for dimension three. Volume estimates are also established for the case when scalar curvature decays polynomially.

1. Introduction

The paper concerns complete three dimensional manifolds with scalar curvature lower bound. Our aim is to derive both analytic and geometric information of such manifolds under suitable, yet necessary, topological assumptions. More specifically, results concerning bottom spectrum, volume growth and existence of positive Green's functions are established. This is achieved through working on the level sets of suitably chosen harmonic functions and by employing a technique introduced by Schoen and Yau [34] and further exploited in [19, 38, 4, 30].

Recall that the bottom spectrum $\lambda_1(M)$ of a complete manifold M is defined to be the smallest $\lambda \in \sigma(\Delta)$, the spectrum of the Laplacian of M. Alternatively, it is characterized as the optimal constant of the Poincaré inequality

$$\lambda_1(M) \int_M \varphi^2 \leq \int_M |\nabla \varphi|^2$$

for all smooth function φ with compact support.

A classical comparison result due to Cheng [7] provides a sharp upper bound for the bottom spectrum $\lambda_1(M)$ of an n-dimensional complete manifold M with its Ricci curvature $\mathrm{Ric} \geq -(n-1)K$ for some nonnegative constant K. Namely,

$$\lambda_1(M) \leq \frac{(n-1)^2}{4} K.$$

Naturally, one may wonder if the result remains true under a scalar curvature lower bound. Since the product manifold $\mathbb{H}^{n-2} \times \mathbb{S}^2(r)$ of the hyperbolic space $\mathbb{H}^{n-2}$ with the sphere $\mathbb{S}^2(r)$ of radius r has positive bottom spectrum, yet its scalar curvature can be made into an arbitrary positive constant by choosing r small, the answer is obviously negative in the case $n \geq 4$. It turns out the answer is negative for the three dimensional case as well. Indeed, consider the universal cover M of the connected sum $N = \left(\mathbb{S}^2 \times \mathbb{S}^1\right) \# \left(\mathbb{S}^2 \times \mathbb{S}^1\right)$. Then M carries a metric with positive scalar curvature as N does so [17, 35]. Note that the bottom spectrum of M is positive according to a result of Brooks [5]. This is because the first fundamental group of N is a free group on two generators and nonamenable.

Nonetheless, we have the following result concerning $\lambda_1(M)$.

Theorem 1.1. *Let (M, g) be a three-dimensional complete Riemannian manifold with scalar curvature $S \geq -6K$ on M for some nonnegative constant K. Suppose that M has finitely many ends and its first Betti number $b_1(M) < \infty$. Moreover, the Ricci curvature of M is bounded from below by a constant. Then the bottom spectrum of M satisfies*

$$\lambda_1(M) \leq K.$$

The estimate is obviously sharp for the hyperbolic space $\mathbb{H}^3$. The preceding example clearly demonstrates the necessity of the topological assumption that the number of ends of M is finite.

Our second purpose is to derive geometric information for three-dimensional manifolds with positive scalar curvature and asymptotically nonnegative Ricci curvature. This is partly motivated by the following well-known question posed by Yau [40].

Question 1.2. *Let (M^n, g) be a complete manifold with nonnegative Ricci curvature. Then for any point $p \in M$, do we have*

$$\limsup_{r\to\infty} r^{2-n} \int_{B_p(r)} S < \infty?$$

A related version due to Gromov [16] asks whether the volume growth is at most of order $n-2$ if the scalar curvature is bounded below by a positive constant. As a consequence of our more general results, the answer to Gromov's problem is positive in the case $n = 3$. We refer to [31, 39, 41] for recent progress on Yau's problem.

We begin with a result on the nonexistence of positive Green's functions.

Recall that a complete manifold M is said to have asymptotically nonnegative Ricci curvature if its Ricci curvature $\mathrm{Ric}(x) \geq -k\left(r(x)\right)$ for a nonnegative nonincreasing function $k(r)$ with $\int_0^\infty r\, k(r)\, dr < \infty$, where $r(x)$ is the geodesic distance from x to a fixed point $p \in M$. We have the following result.

Theorem 1.3. *Let (M, g) be a complete three-dimensional manifold with finite first Betti number and finitely many ends. Suppose that its Ricci curvature is asymptotically nonnegative. If the scalar curvature S is bounded below by*

$$\liminf_{x\to\infty} S(x)\, r(x) \geq C$$

for some positive constant C, then M is parabolic, that is, it does not admit any positive Green's function.

In particular, the manifold is parabolic if the scalar curvature is bounded from below by a positive constant. Our next result relates the lower bound of the scalar curvature with the volume of unit balls.

Theorem 1.4. *Let (M, g) be a complete three-dimensional manifold with finite first Betti number and finitely many ends. Suppose that its Ricci curvature is asymptotically nonnegative. Then there exists a constant C depending on k such that the scalar curvature S of M satisfies*

$$\liminf_{x\to\infty} S(x) \leq \frac{C}{\mathrm{V}_p(1)},$$

where $\mathrm{V}_p(1)$ denotes the volume of the unit ball $B_p(1)$.

In fact, volume information is also obtained under the more general assumption that $\liminf_{x\to\infty} S(x)\, r^{\alpha}(x) > 0$ for some $\alpha > 0$. We refer to Theorem 5.6 for the precise statement. It should be pointed out that it is necessary to assume that M has asymptotically nonnegative Ricci curvature for Theorem 1.4 to hold. Indeed, consider complete three-dimensional manifold (M, g) which is isometric to $[2,\infty)\times\mathbb{S}^2$ with the warped product metric $ds_M^2 = dt^2 + \frac{1}{\ln t} ds_{\mathbb{S}^2}^2$ outside a compact set. Then its scalar curvature satisfies $\liminf_{x\to\infty} S(x) = \infty$. Note that its Ricci curvature $\mathrm{Ric}(x) \geq -k(r(x))$ for $k(r) = \frac{C}{r^2 \ln r}$ when r is sufficiently large.

As a special case of the above results, we have the following conclusion. Recall by Cheeger-Gromoll splitting theorem [6], a complete manifold M^n with nonnegative Ricci curvature has at most two ends. Also, by [29], the first Betti number of such M^n is finite and bounded by its dimension n. Moreover, according to [27], M^n is parabolic if and only if

$$\int_1^\infty \frac{t}{\mathrm{V}_p(t)}\, dt = \infty,$$

where $\mathrm{V}_p(t)$ is the volume of geodesic ball $B_p(t)$.

Corollary 1.5. *Let (M, g) be a complete three-dimensional manifold with nonnegative Ricci curvature.*

- *If the scalar curvature is bounded below by*

  $$\liminf_{x\to\infty} S(x)\, r(x) \geq C$$

  for some positive constant C, where $r(x)$ is the distance function to a fixed point $p \in M$, then

  $$\int_1^\infty \frac{t}{\mathrm{V}_p(t)}\, dt = \infty.$$

- *There exists a universal constant C such that the scalar curvature S of M satisfies*

  $$\liminf_{x\to\infty} S(x) \leq \frac{C}{\mathrm{V}_p(1)}$$

  for every $p \in M$.
- *If the scalar curvature is bounded below by $S \geq 1$ on M, then there exists a universal constant $C > 0$ such that*

  $$\mathrm{V}_p(R) \leq C\, R$$

  for all $R > 0$ and $p \in M$.

- *If the scalar curvature*
$$S(x) \geq \frac{C}{r^{\alpha}(x)+1}$$
for some $\alpha \in [0,1]$, *then*
$$\mathrm{V}_p(R_i) \leq C\, R_i^{\alpha+1}$$
for a sequence $R_i \to \infty$.

We remark that the upper bound estimate of scalar curvature S in terms of the volume $\mathrm{V}_p(1)$ is sharp by considering the example of cylinder $\mathbb{R}\times\mathbb{S}^2(r)$, where $\mathbb{S}^2(r)$ is the sphere of radius r in Euclidean space $\mathbb{R}^3$. As mentioned already, the linear volume growth estimate in the corollary gives an affirmative answer in dimension three to a question of Gromov [16].

We now indicate some of the ideas involved in the proofs. Theorem 1.1 was previously proved in [30] under an additional assumption that the volume $\mathrm{V}_x(1)$ of unit balls $B_x(1)$ satisfies

$$\mathrm{V}_x(1) \geq C(\epsilon)\, \exp\left(-2\sqrt{K+\epsilon}\, r(x)\right)$$

for every $\epsilon > 0$ and all $x \in M$, where $r(x)$ is the geodesic distance from x to a fixed point p. This was used to ensure that the minimal positive Green's function $G(p,x)$ of M with fixed p goes to 0 at infinity in the case that $\lambda_1(M) > K$. Together with the topological assumptions, it was concluded that the regular level sets $l(t)$ of G are necessarily connected compact surfaces on each end. The proof then proceeded by integrating the following Bochner formula over the level sets $l(t)$

$$\Delta\left|\nabla G\right| = \left(\left|G_{ij}\right|^2 - \left|\nabla\left|\nabla G\right|\right|^2\right)\left|\nabla G\right|^{-1} + \mathrm{Ric}\left(\nabla G, \nabla G\right)\left|\nabla G\right|^{-1},$$

and applying the Gauss-Bonnet theorem, where the Ricci curvature term is rewritten into

$$\mathrm{Ric}\left(\nabla G, \nabla G\right)\left|\nabla G\right|^{-2} = \frac{1}{2}S - \frac{1}{2}S_{l(t)} + \frac{1}{\left|\nabla G\right|^2}\left(\left|\nabla\left|\nabla G\right|\right|^2 - \frac{1}{2}\left|\nabla^2 G\right|^2\right) \tag{1.1}$$

with $S_{l(t)}$ being the scalar curvature of the level set $l(t)$.

The idea of rewriting the Ricci curvature term as (1.1) has origin in Schoen and Yau [34], where they made the important observation that on a minimal surface N in a three-dimensional manifold M,

$$\mathrm{Ric}(\nu,\nu) = \frac{1}{2}S - \frac{1}{2}S_N - \frac{1}{2}\left|A\right|^2 \tag{1.2}$$

with ν, S_N and A being the unit normal vector, the scalar curvature and the second fundamental form of N, respectively. This observation enabled them to classify compact stable minimal surfaces in a three-dimensional manifold with nonnegative scalar curvature. In fact, it can be used to reprove the well-known classification by Fisher-Colbrie and Schoen [14] for complete stable minimal surfaces as well (see [26]). More generally, an identity of the nature (1.2) was derived for any surface N, not necessarily minimal, by Jezierski and Kijowski in [19]. It was applied to level sets of suitably chosen functions to prove both positive energy and positive mass results in [19, 18]. Recently, this identity was rediscovered by Stern [38] for the level sets of harmonic functions. In [4] it has been subsequently used to reprove the positive mass theorem of Schoen and Yau [36].

The lack of properness of the minimal positive Green's function creates serious difficulty for the above argument to work. To overcome this difficulty, we modify the function G by considering $u = G\,\psi$, where ψ is a smooth cut-off function. While this guarantees that the positive level sets of u are compact, the price we pay is that the function u is no longer harmonic. This creates many new technical issues. For one, it becomes unclear whether the intersection of level sets of u with each end is connected. To get around the issue, we consider only the component $L(t,\infty)$ of the super level set $\{u > t\}$ with the fixed pole $p \in L(t,\infty)$. It turns out that for each end E of M, the unbounded component of $E \setminus L(t,\infty)$ has exactly one component of the boundary of $L(t,\infty)$. This fact more or less suffices for our purpose, though there could be other components of the boundary of $L(t,\infty)$. Another issue is that the Bochner formula for u introduces many extra terms. Fortunately, those terms can be controlled with the help of the exponential decay estimate for Green's function from [23] together with a judicious choice of the cut-off function ψ based on a result from Schoen and Yau [37]. Actually, instead of G, we work with the so-called barrier function f, namely, a harmonic function defined on $M \setminus D$ for an arbitrarily large compact smooth domain D such that $f = 1$ on the boundary ∂D of D and $0 < f < 1$. The advantage is that it leads to a slightly stronger conclusion, that is, the smallest essential spectrum of M is bounded from above by K as well.

It remains to be seen whether Theorem 1.1 holds without assuming the Ricci curvature is bounded from below. For the aforementioned Cheng's estimate, there are rigidity results when $\lambda_1(M)$ attains its maximum value $\frac{(n-1)^2}{4}K$ (see [23, 24]). Analogously, one may ask what happens if $\lambda_1(M)$ attains its upper bound K in Theorem 1.1.

The proof of Theorem 1.3 is very much inspired by Theorem 1.1. However, the technical details differ. Again, we work with a barrier function f. Since f may not be proper, as above we consider $u = f\,\psi$ instead, where ψ is a smooth cut-off function on M. Unlike before, we no longer have the fact that the function f decays exponentially in the L^2 integral sense. To compensate, we choose a better behaved cut-off function ψ, which in turn depends on the construction of a well controlled distance-like function. That is where the stronger assumption on the Ricci curvature is used.

The proof of Theorem 1.4 relies on an elaboration of the following comparison theorem in [30].

Theorem 1.6. *Let (M,g) be a complete noncompact three-dimensional manifold with nonnegative scalar curvature. Assume that M has one end and its first Betti number $b_1(M) = 0$. If M is nonparabolic and the minimal positive Green's function $G(x) = G(p,x)$ satisfies $\lim_{x\to\infty} G(x) = 0$, then*

$$\frac{d}{dt}\left(\frac{1}{t}\int_{l(t)} |\nabla G|^2 - 4\pi t\right) \leq 0$$

for all $t > 0$. Moreover, equality holds for some $T > 0$ if and only if the super level set $\{x \in M, G(x) > T\}$ is isometric to a ball in the Euclidean space $\mathbb{R}^3$.

We point out that in [2] this type of result was applied to reprove the positive mass theorem [36]. Recently, using a variant of the above theorem, Chodosh and Li [9] have affirmed a conjecture of Schoen that a stable minimal hypersurface in Euclidean space $\mathbb{R}^4$ must be flat. Historically, more general versions of monotonicity formulas were established by Colding [10] and Colding-Minicozzi [12] for n-dimensional manifolds with nonnegative Ricci curvature and applied to the study of uniqueness of the tangent cones for Ricci flat manifolds with Euclidean volume growth [13]. We refer the readers to [11] for an exposition on monotonicity formulas in geometric analysis, and [1] for their applications to Willmore type inequalities.

The observation we make is that Theorem 1.6 can be localized to each end of M by working with the barrier function of an end. Recall by Li and Tam [20], each end E of a complete manifold admits a positive harmonic function u satisfying $u = 0$ on the boundary of E. Such u is bounded if and only if E is nonparabolic. On the other hand, by a result of Nakai [32], u can be chosen to be proper in the case E is parabolic. Applying the localized version of Theorem 1.6 to such u together with suitable estimates on u then yields Theorem 1.4.

Finally, we mention that Theorem 1.6 or its localized version to an end may also be used to show the nonexistence of proper positive Green's functions under a suitable assumption on scalar curvature. This is explicitly stated in Theorem 5.6. By [27], since the minimal positive Green's function must go to 0 at infinity for a complete nonparabolic manifold with nonnegative Ricci curvature, this provides an alternative proof to Theorem 1.3 in this special case.

The structure of the paper is as follows. After some preparations in Section 2, we complete the proof of Theorem 1.1 in Section 3. To address Theorem 1.3 and Theorem 1.4, we collect some preliminary facts in Section 4. Section 5 is devoted to the proofs of these theorems.

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2. Preliminaries when Ricci curvature bounded below

In this section we make some preparations for proving Theorem 1.1. First, we follow Theorem 4.2 in [37] and construct a smooth distance-like function on an arbitrary dimensional manifold M. Denote by $r(x)$ the distance to a fixed point $p \in M$.

Lemma 2.1. *Let (M, g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded from below by a constant. Then there exists a proper function $\rho \in C^\infty(M)$ such that*

$$\begin{aligned} \frac{1}{C} r &\leq \rho \leq Cr \\ |\nabla \rho| + |\Delta \rho| &\leq C \end{aligned} \tag{2.1}$$

and

$$|\nabla(\Delta \rho)| \leq C \left|\nabla^2 \rho\right| \tag{2.2}$$

for some constant $C > 0$.

Proof. Since the last estimate is not explicitly stated in Theorem 4.2 of [37], we sketch the proof below. For $\lambda > 0$, to be determined later, we solve the boundary value problem

$$\begin{aligned} \Delta h_R &= \lambda h_R \text{ in } B_p(R) \setminus B_p(1) \\ h_R &= 1 \text{ on } \partial B_p(1) \\ h_R &= 0 \text{ on } \partial B_p(R). \end{aligned}$$

The maximum principle implies that $0 < h_R < 1$ and that h_R is increasing in R. Letting $R \to \infty$, we obtain a positive function h on $M \setminus B_p(1)$ satisfying

$$\begin{aligned} \Delta h &= \lambda h \text{ in } M \setminus B_p(1) \\ h &= 1 \text{ on } \partial B_p(1). \end{aligned}$$

By Cheng-Yau's gradient estimate [8],

$$|\nabla \ln h| \leq C(\lambda) \tag{2.3}$$

as the Ricci curvature of M is bounded from below.

For any $\gamma > 0$, note that

$$\begin{aligned} \lambda \int_{B_p(R)\setminus B_p(1)} e^{\gamma r} h_R^2 &= \int_{B_p(R)\setminus B_p(1)} e^{\gamma r} h_R \Delta h_R \\ &= -\int_{B_p(R)\setminus B_p(1)} e^{\gamma r} |\nabla h_R|^2 \\ &\quad -\gamma \int_{B_p(R)\setminus B_p(1)} \langle \nabla h_R, \nabla r \rangle h_R e^{\gamma r} \\ &\quad -e^{\gamma} \int_{\partial B_p(1)} h_R \frac{\partial h_R}{\partial \nu} \\ &\leq \frac{\gamma^2}{4} \int_{B_p(R)\setminus B_p(1)} e^{\gamma r} h_R^2 - e^{\gamma} \int_{\partial B_p(1)} h_R \frac{\partial h_R}{\partial \nu}. \end{aligned}$$

Hence, setting $\gamma = \sqrt{\lambda}$ we conclude that

$$\int_{B_p(R)\setminus B_p(1)} e^{\sqrt{\lambda} r} h_R^2 \leq C$$

for all $R > 1$. This implies that

$$\int_M e^{\sqrt{\lambda} r} h^2 \leq C.$$

Together with (2.3) we get that

$$\begin{aligned} h^2(x) &\leq \frac{C}{\mathrm{V}_x(1)} \int_{B_x(1)} h^2 \\ &\leq \frac{C}{\mathrm{V}_x(1)} e^{-\sqrt{\lambda} r(x)}. \end{aligned}$$

Since the Ricci curvature is bounded below,

$$\frac{C}{\mathrm{V}_x(1)} \leq e^{Cr(x)}.$$

Hence, by choosing λ sufficiently large, we conclude that h decays exponentially and

$$h(x) \leq Ce^{-\frac{1}{4}\sqrt{\lambda}r(x)}. \tag{2.4}$$

The function ρ is essentially defined as $-\ln h$ with a slight modification. More precisely,

$$\rho(x) = (1-\eta(x))(-\ln h(x)) + \eta(x),$$

where η is a smooth cut-off function with $\eta = 1$ on $B_p(1)$ and $\eta = 0$ on $M \backslash B_p(2)$.
Now (2.1) follows from (2.3), (2.4) and the fact that $\Delta h = \lambda h$. Indeed,

$$\Delta\rho = -\lambda + |\nabla\rho|^2 \quad \text{on } M\backslash B_p(2). \tag{2.5}$$

Clearly, (2.5) implies (2.2). □

For given $R > 0$, let $\psi : (0,\infty) \to \mathbb{R}$ be a smooth function such that $\psi = 1$ on $(0, 2R)$ and $\psi = 0$ on $(3R,\infty)$ satisfying

$$|\psi'| \leq \frac{C}{R}, \quad |\psi''| \leq \frac{C}{R^2} \text{ and } |\psi'''| \leq \frac{C}{R^3}.$$

Composing it with ρ, we obtain a cut-off function

$$\psi(x) = \psi(\rho(x)).$$

Obviously,

$$\begin{aligned} \psi &= 1 \text{ on } D(2R) \\ \psi &= 0 \text{ on } M\backslash D(3R). \end{aligned}$$

Throughout this section,

$$D(t) = \{x \in M : \rho(x) < t\}. \tag{2.6}$$

Recall that a complete manifold (M, g) is nonparabolic if it admits a positive symmetric Green's function. Given a smooth connected bounded domain $D \subset M$, according to Li-Tam [20], there a exists a barrier function $f > 0$ on $M\backslash D$ satisfying

$$\begin{aligned} &\Delta f = 0 \quad \text{on } M\backslash D \\ &f = 1 \text{ on } \partial D \\ &\liminf_{x\to\infty} f(x) = 0. \end{aligned} \tag{2.7}$$

Moreover, such f can be obtained as a limit $f = \lim_{R_i\to\infty} f_i$, where each f_i is a harmonic function on $D(R_i) \setminus D$ with $f_i = 1$ on ∂D and $f_i = 0$ on $\partial D(R_i)$. It is easy to see that

$$\int_{M\backslash D} |\nabla f|^2 < \infty. \tag{2.8}$$

In particular, by [25], for all $0 < t \leq 1$ and $R \geq R_0$ with $D \subset D(R_0)$,

$$-\int_{\partial D(R)} \frac{\partial f}{\partial \nu} = \int_{\{f=t\}} |\nabla f| = C > 0,$$

where ν is the outward unit normal vector of $\partial D(R)$.

The following gradient estimate follows from [8] together with the standard partial differential equation theory.

Lemma 2.2. *Let (M, g) be an n-dimensional complete noncompact Riemannian manifold with Ricci curvature bounded below by a constant. Then*

$$|\nabla \ln f| \leq C \quad \text{on } M \backslash D.$$

We now consider the function u given by

$$u = f\psi. \tag{2.9}$$

We summarize some properties of u in the following result.

Lemma 2.3. *Let (M, g) be an n-dimensional complete noncompact Riemannian manifold with Ricci curvature bounded below by a constant. Then u is harmonic on $D(2R) \backslash D$ and*

$$\begin{aligned} |\nabla u| &\leq C\left(u + \frac{1}{R}\right) \\ |\Delta u| &\leq \frac{C}{R} \\ |\nabla(\Delta u)| &\leq \frac{C}{R}\left(\left|\nabla^2 \rho\right|^2 + \left|\nabla^2 f\right|^2 + 1\right) \end{aligned}$$

on $M \backslash D$.

Proof. By Lemma 2.1 we have that

$$|\nabla \psi| + |\Delta \psi| \leq \frac{C}{R} \tag{2.10}$$

and

$$|\nabla(\Delta \psi)| \leq \frac{C}{R}\left(\left|\nabla^2 \rho\right| + 1\right). \tag{2.11}$$

Indeed, the last estimate follows from

$$\begin{aligned} |\nabla(\Delta \psi)| &= \left|\nabla\left(\psi' \Delta \rho + \psi'' |\nabla \rho|^2\right)\right| \\ &\leq |\psi'| |\nabla(\Delta \rho)| + |\Delta \rho| |\nabla \psi'| + |\psi''| \left|\nabla |\nabla \rho|^2\right| + |\nabla \rho|^2 |\nabla \psi''| \\ &\leq \frac{C}{R}\left(\left|\nabla^2 \rho\right| + 1\right) \end{aligned}$$

by applying (2.1) and (2.2).

By (2.10) and Lemma 2.2 we immediately obtain

$$\begin{aligned} |\nabla u| &\leq |\nabla f| \psi + f |\nabla \psi| \\ &\leq Cu + \frac{C}{R} \end{aligned}$$

on $M\backslash D$, and

$$\begin{aligned} \Delta u &= 0 \text{ on } D(2R)\backslash D, \\ |\Delta u| &\leq \frac{C}{R} \text{ on } M\backslash D. \end{aligned}$$

Moreover, from (2.11), we have

$$\begin{aligned} |\nabla(\Delta u)| &\leq |\nabla(f\Delta\psi)| + 2|\nabla(\langle\nabla\psi,\nabla f\rangle)| \\ &\leq \frac{C}{R}\left(\left|\nabla^2\rho\right| + \left|\nabla^2 f\right| + 1\right) \\ &\leq \frac{C}{R}\left(\left|\nabla^2\rho\right|^2 + \left|\nabla^2 f\right|^2 + 1\right). \end{aligned}$$

This proves the result. □

We extend both f and u to M by setting $f=1$ and $u=1$ in D. For $0<t<1$, denote by $L(t,\infty)$ the connected component of the super level set $\{u>t\}$ that contains D. Note that all bounded components of $M\setminus D$ are contained in $L(t,\infty)$ as well. This is because $f=1$ there and each of the components intersects with D. Let

$$l(t) = \partial L(t,\infty). \tag{2.12}$$

Since u has compact support in M, the closure $\overline{L(t,\infty)}$ of $L(t,\infty)$ and its boundary $l(t)$ are compact in M.

We now assume that M has finite first Betti number and finitely many, say m, ends. Let the smooth connected bounded domain D be chosen large enough so that all representatives of the first homology group $H_1(M)$ are included in D. Moreover, $M\backslash D$ has exactly m unbounded connected components. Then the same holds true for $M\setminus\tilde{D}$, for any bounded domain $\tilde{D}$ with $D\subset\tilde{D}$.

We have the following estimate on the number of components of $l(t)$. The argument in [30] needs to be modified as u is no longer harmonic on M.

Lemma 2.4. *Let (M,g) be an n-dimensional complete Riemannian manifold with m ends and finite first Betti number $b_1(M)$. Assume that the Ricci curvature is bounded from below by a constant. Then for any $0<t<1$,*

$$l(t)\cap\overline{M_t} = \partial M_t \text{ has } m \text{ connected components} \tag{2.13}$$

and

$$l(t)\cap\left(M\backslash\overline{M_t}\right) \subset M\backslash D(2R), \tag{2.14}$$

where M_t is the union of all unbounded connected components of $M\backslash\overline{L(t,\infty)}$ and $l(t)$ is defined in (2.12).

Proof. Recall that all representatives of the first homology $H_1(M)$ lie in D and $M\setminus D$ has exactly m unbounded components. Therefore, as $D\subset L(t,\infty)$, $L(t,\infty)$ contains all representatives of $H_1(M)$ and M_t has m connected components for any $0<t<1$. Note also $\partial M_t\subset l(t)$.

For fixed $0<\delta<1$ such that $t+\delta<1$, define U to be the union of $L(t,\infty)$ with all bounded components of $M\backslash\overline{L(t+\delta,\infty)}$ and V the union of all unbounded components of $M \backslash \overline{L(t+\delta, \infty)}$. Since $M = U \cup V$, we have the following Mayer-Vietoris sequence

$$H_1(U) \oplus H_1(V) \xrightarrow{j_*} H_1(M) \xrightarrow{\partial} H_0(U \cap V) \xrightarrow{i_*} H_0(U) \oplus H_0(V) \xrightarrow{j'_*} H_0(M). \tag{2.15}$$

The map j_* is onto because all representatives of $H_1(M)$ lie inside U. The map j'_* is also onto. Note also that V has m components and U is connected. The latter is true because each component of $M \backslash \overline{L(t+\delta, \infty)}$ intersects with $L(t, \infty)$ as $\overline{L(t+\delta, \infty)} \subset L(t, \infty)$. In view of (2.15), we therefore obtain the short exact sequence

$$0 \to H_0(U \cap V) \to \mathbb{Z} \oplus .. \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

with $m+1$ summands. In conclusion,

$$H_0(U \cap V) = \mathbb{Z} \oplus .. \oplus \mathbb{Z}$$

with m summands. Since $\delta > 0$ can be arbitrarily small, this proves that

$$l(t) \cap \overline{M_t} \text{ has } m \text{ components}$$

for $0 < t < 1$. Hence, we have established (2.13).

For the second conclusion, write

$$M \backslash \overline{L(t, \infty)} = M_t \cup \Omega,$$

where Ω is the union of all bounded components of $M \backslash \overline{L(t, \infty)}$. Clearly, $\partial\Omega \subset l(t)$ and $l(t) \cap \left(M \backslash \overline{M_t}\right) \subset \partial\Omega$.

Note that $\partial M_t \subset l(t)$ and $u \leq f$. It follows that $f \geq t$ on $\partial M_t = \partial\left(M \backslash \overline{M_t}\right)$. Since the function f is superharmonic on the bounded set $M \backslash \overline{M_t}$, the strong maximum principle shows that $f > t$ in $M \backslash \overline{M_t}$. In particular, we have $f > t$ on $\overline{\Omega} \backslash \overline{M_t}$.

Now for $x_1 \in (\partial\Omega \cap D(2R)) \backslash \overline{M_t}$, we would have $u(x_1) > t$ as $f > t$ on $\overline{\Omega} \backslash \overline{M_t}$ and $u = f$ on $D(2R)$. This contradicts with $\partial\Omega \subset l(t)$.

Therefore, $(\partial\Omega \cap D(2R)) \backslash \overline{M_t} = \emptyset$. In conclusion, we have

$$l(t) \cap \left(M \backslash \overline{M_t}\right) \subset M \backslash D(2R).$$

This proves (2.14). $\square$

Let us point out that $\{u > t\}$ may have other connected components in addition to $L(t, \infty)$. We now estimate the size of those components that lie in a fixed $L(\varepsilon, \infty)$. Hence, for fixed $0 < \varepsilon < 1$, denote

$$\widetilde{L}(t, \infty) = (\{u > t\} \backslash L(t, \infty)) \cap L(\varepsilon, \infty) \tag{2.16}$$

the union of all such connected components in $L(\varepsilon, \infty)$ and its boundary

$$\widetilde{l}(t) = \partial \widetilde{L}(t, \infty). \tag{2.17}$$

The following result provides an area estimate for $\widetilde{l}(t)$.

Lemma 2.5. *Let (M,g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded from below by a constant. Then for any $0<\varepsilon<1$,*

$$\int_{\varepsilon}^{1}\operatorname{Area}\left(\widetilde{l}\,(t)\right)dt\leq\frac{C}{\sqrt{R\varepsilon^{3}}}\int_{L(\varepsilon,1)}u^{2},$$

where C is a constant independent of R and ε, and $\widetilde{l}\,(t)$ is defined in (2.17).

Proof. Applying the divergence theorem, for any regular set $\widetilde{l}\,(t)$ we have

$$\begin{aligned}\int_{\widetilde{l}(t)}|\nabla u| &= -\int_{\widetilde{L}(t,\infty)}\Delta u\\ &\leq \int_{\widetilde{L}(t,\infty)}|\Delta u|\\ &\leq \frac{C}{R}\operatorname{Vol}\left(\widetilde{L}\,(t,\infty)\right),\end{aligned}$$

where the last line follows from Lemma 2.3. Hence, for any $\varepsilon<t<1$,

$$\begin{aligned}\int_{\widetilde{l}(t)}|\nabla u| &\leq \frac{C}{R\varepsilon^{2}}\int_{\widetilde{L}(t,\infty)}u^{2}\\ &\leq \frac{C}{R\varepsilon^{2}}\int_{L(\varepsilon,1)}u^{2}.\end{aligned}\tag{2.18}$$

Since

$$1\leq\frac{1}{2}\sqrt{R\varepsilon^{3}}\frac{|\nabla u|}{u^{2}}+\frac{1}{2\sqrt{R\varepsilon^{3}}}\frac{u^{2}}{|\nabla u|},$$

it follows that

$$\operatorname{Area}\left(\widetilde{l}\,(t)\right)\leq\frac{1}{2}\sqrt{R\varepsilon^{3}}\int_{\widetilde{l}(t)}\frac{|\nabla u|}{u^{2}}+\frac{1}{2\sqrt{R\varepsilon^{3}}}\int_{\widetilde{l}(t)}\frac{u^{2}}{|\nabla u|}.\tag{2.19}$$

By (2.18) we have that

$$\begin{aligned}\int_{\widetilde{l}(t)}\frac{|\nabla u|}{u^{2}} &= \frac{1}{t^{2}}\int_{\widetilde{l}(t)}|\nabla u|\\ &\leq \frac{C}{R\varepsilon^{2}}\left(\int_{L(\varepsilon,1)}u^{2}\right)\frac{1}{t^{2}}.\end{aligned}$$

After integrating from $t=\varepsilon$ to $t=1$, we obtain

$$\int_{\varepsilon}^{1}\int_{\widetilde{l}(t)}\frac{|\nabla u|}{u^{2}}dt\leq\frac{C}{R\varepsilon^{3}}\left(\int_{L(\varepsilon,1)}u^{2}\right).\tag{2.20}$$

Using the co-area formula we estimate

$$\begin{aligned}\int_{\varepsilon}^{1}\int_{\widetilde{l}(t)}\frac{u^{2}}{|\nabla u|}dt &\leq \int_{\varepsilon}^{1}\int_{\{u=t\}\cap L(\varepsilon,1)}\frac{u^{2}}{|\nabla u|}dt\\ &= \int_{L(\varepsilon,1)}u^{2}.\end{aligned}\tag{2.21}$$

Now integrating (2.19) in t and applying (2.20) and (2.21) we arrive at

$$\int_{\varepsilon}^{1} \text{Area}\left(\widetilde{l}(t)\right) dt \leq \frac{C}{\sqrt{R\varepsilon^3}} \int_{L(\varepsilon,1)} u^2$$

as claimed. ☐

We now restrict to three-dimensional manifolds. The following result relies on an idea from [34] and follows as in [18] or Lemma 4.1 in [4]. For the sake of completeness, we include details here.

Lemma 2.6. *Let (M, g) be a three-dimensional complete Riemannian manifold with scalar curvature S and u a smooth function on M. Then on each regular level set $\{u = t\}$ of u,*

$$\begin{aligned}\text{Ric}(\nabla u, \nabla u) |\nabla u|^{-2} \quad &= \quad \frac{1}{2} S - \frac{1}{2} S_t + \frac{1}{2} \frac{1}{|\nabla u|^2} \left(|\nabla |\nabla u||^2 - \left|\nabla^2 u\right|^2 \right) \\ &\quad + \frac{1}{2} \frac{1}{|\nabla u|^2} \left((\Delta u)^2 - 2 \frac{\langle \nabla |\nabla u|, \nabla u \rangle}{|\nabla u|} \Delta u + |\nabla |\nabla u||^2 \right),\end{aligned}$$

where S_t denotes the scalar curvature of the surface $\{u = t\}$.

Proof. On a regular level set $\{u = t\}$ of u, its unit normal vector is given by

$$e_1 = \frac{\nabla u}{|\nabla u|}.$$

Choose unit vectors $\{e_2, e_3\}$ tangent to $\{u = t\}$ such that $\{e_1, e_2, e_3\}$ forms a local orthonormal frame on M. The second fundamental form and the mean curvature of $\{u = t\}$ are then given by

$$h_{ab} = \frac{u_{ab}}{|\nabla u|} \quad \text{and } H = \frac{\Delta u - u_{11}}{|\nabla u|}, \quad \text{respectively,}$$

where indices a and b range from 2 to 3. By the Gauss curvature equation, we have

$$S_t = S - 2R_{11} + H^2 - |h|^2.$$

Therefore,

$$\begin{aligned}&2\text{Ric}(\nabla u, \nabla u) |\nabla u|^{-2} \\ = \quad &S - S_t + \frac{1}{|\nabla u|^2} \left((\Delta u - u_{11})^2 - |u_{ab}|^2 \right) \\ = \quad &S - S_t + \frac{1}{|\nabla u|^2} \left((\Delta u)^2 - 2u_{11} (\Delta u) + 2 |\nabla |\nabla u||^2 - |u_{ij}|^2 \right),\end{aligned}$$

where we have used the fact that

$$|\nabla |\nabla u||^2 = |u_{11}|^2 + |u_{1a}|^2$$

and

$$|u_{ij}|^2 = |u_{11}|^2 + 2 |u_{1a}|^2 + |u_{ab}|^2.$$

This proves the result. ☐

Finally, we recall the following well known Kato inequality for harmonic functions (cf. [30]).

Lemma 2.7. *Let (M,g) be a three-dimensional Riemannian manifold and f a harmonic function on $M\backslash D$. Then*

$$\left|\nabla^2 f\right|^2 \geq \frac{3}{2}\left|\nabla\left|\nabla f\right|\right|^2 \quad \text{on } M\backslash D.$$

3. Upper bound of bottom spectrum

We now turn to the proof of Theorem 1.1 which is restated below. We continue to follow the notations from Section 2.

Theorem 3.1. *Let (M,g) be a three-dimensional complete noncompact Riemannian manifold with scalar curvature $S \geq -6K$ on M for some nonnegative constant K. Suppose that M has finitely many ends and its first Betti number $b_1(M) < \infty$. Moreover, the Ricci curvature of M is bounded from below. Then the bottom spectrum of the Laplacian satisfies*

$$\lambda_1(M) \leq K.$$

Proof. For given small $0 < \varepsilon < 1$, let $R > R_0$ and

$$R > \frac{1}{\varepsilon^4}, \tag{3.1}$$

where R_0 is chosen to be large enough so that $D \subset D(R_0)$ with D being a smooth connected bounded domain that contains all representatives of $H_1(M)$ and that $M\backslash D$ has the maximal number m of ends, while $D(t)$ is as defined in (2.6).

In the following, we use C to denote a positive constant that is independent of R and ε, while its value may change from line to line.

To prove the theorem, we may assume that $\lambda_1(M) > 0$. In particular, (M,g) is nonparabolic. Let f be the Li-Tam barrier function defined by (2.7). Set

$$u = f\psi$$

as in (2.9). Everywhere we denote with

$$L(a,b) = L(a,\infty) \cap \{a < u < b\}.$$

Recall that from [23],

$$\int_{M\backslash B_p(r)} f^2(x)\,dx \leq C\,e^{-2\sqrt{\lambda_1(M)}r}$$

for all r with $D \subset B_p(r)$. Therefore,

$$\int_{M\backslash D} f^2(x)\,dx \leq C$$

and by (2.1),

$$\int_{M\backslash D(r)} f^2(x)\,dx \leq C\,e^{-\frac{1}{C}r} \tag{3.2}$$

for all $r > R_0$.

Using (3.2) and that $f \geq u$, we get

$$\begin{aligned} \operatorname{Vol}(L(\varepsilon,1)) &\leq \frac{1}{\varepsilon^2}\int_{L(\varepsilon,1)} u^2 \\ &\leq \frac{1}{\varepsilon^2}\int_{M\setminus D} f^2 \\ &\leq \frac{C}{\varepsilon^2}. \end{aligned} \tag{3.3}$$

Similarly,

$$\begin{aligned} \operatorname{Vol}((M\setminus D(R))\cap L(\varepsilon,1)) &\leq \frac{1}{\varepsilon^2}\int_{(M\setminus D(R))\cap L(\varepsilon,1)} u^2(x)\,dx \\ &\leq \frac{1}{\varepsilon^2}\int_{M\setminus D(R)} f^2(x)\,dx \\ &\leq \frac{C}{\varepsilon^2}e^{-\frac{1}{C}R} \\ &\leq C, \end{aligned} \tag{3.4}$$

where the last line follows from (3.1).

By Lemma 2.3 and (3.3), for $\varepsilon < t < 1$,

$$\begin{aligned} \int_{L(t,1)} |\Delta u| &\leq \frac{C}{R}\operatorname{Vol}(L(\varepsilon,1)) \\ &\leq \frac{C}{R\varepsilon^2} \\ &\leq C, \end{aligned}$$

where the last line is from (3.1). On the other hand, the divergence theorem implies that

$$\begin{aligned} \int_{L(t,1)} \Delta u &= -\int_{\partial D} \frac{\partial f}{\partial \nu} - \int_{l(t)} |\nabla u| \\ &= C - \int_{l(t)} |\nabla u|, \end{aligned}$$

where we have used that $u = f$ on $D(2R)\setminus D$ and ν is the outward unit normal vector to ∂D. In conclusion,

$$\int_{l(t)} |\nabla u| \leq C \tag{3.5}$$

for all $\varepsilon < t < 1$.

Let

$$\phi(x) = \begin{cases} \phi(u(x)) & \text{on } L(\varepsilon,\infty) \\ 0 & \text{on } M\setminus L(\varepsilon,\infty). \end{cases}$$

Here, the function $\phi(t)$ is smooth, $\phi(t) = 1$ for $2\varepsilon \leq t \leq 1$ and $\phi(t) = 0$ for $t < \varepsilon$. Moreover,

$$|\phi'|(t) \leq \frac{C}{\varepsilon} \text{ and } |\phi''|(t) \leq \frac{C}{\varepsilon^2} \text{ for } \varepsilon < t < 2\varepsilon. \tag{3.6}$$

Clearly, by the definition, the function $\phi(x)$ satisfies $\phi = 1$ on $L(2\varepsilon, \infty)$ and $\phi = 0$ on $M \setminus L(\varepsilon, \infty)$. Furthermore,

$$|\nabla \phi| \leq C. \tag{3.7}$$

Indeed, $\nabla \phi = 0$ except for $L(\varepsilon, 2\varepsilon)$. On $L(\varepsilon, 2\varepsilon)$, according to Lemma 2.3 and (3.1),

$$\begin{aligned} |\nabla u| &\leq C\, u + \frac{C}{R} \\ &\leq C\, \varepsilon \end{aligned} \tag{3.8}$$

and

$$|\nabla \phi| \leq \frac{C}{\varepsilon} |\nabla u| \leq C.$$

We first prove the following inequality.

Claim 1: $\displaystyle\frac{3}{4}\int_{D(R)\setminus D} |\nabla |\nabla u||^2 |\nabla u|^{-1} \phi^2 \leq 3K \int_{D\left(\frac{1}{2}R\right)\setminus D} |\nabla u| \phi^2 + C.$

According to the Bochner formula,

$$\frac{1}{2}\Delta |\nabla u|^2 = |u_{ij}|^2 + \langle \nabla \Delta u, \nabla u\rangle + \operatorname{Ric}(\nabla u, \nabla u)$$

on $M\setminus D$. Note that

$$\langle \nabla \Delta u, \nabla u \rangle \geq -|\nabla(\Delta u)| |\nabla u|.$$

Therefore,

$$\Delta |\nabla u| \geq \left(|u_{ij}|^2 - |\nabla |\nabla u||^2\right) |\nabla u|^{-1} + \operatorname{Ric}(\nabla u, \nabla u) |\nabla u|^{-1} - |\nabla(\Delta u)| \tag{3.9}$$

holds on $M\setminus D$ whenever $|\nabla u| \neq 0$.

From (3.9), we have

$$\begin{aligned} &\int_{M\setminus D} (\Delta |\nabla u|)\, \phi^2 \\ \geq &\int_{M\setminus D} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) |\nabla u|^{-1} \phi^2 \\ &- \int_{M\setminus D} |\nabla(\Delta u)|\, \phi^2. \end{aligned} \tag{3.10}$$

We estimate the last term. According to Lemma 2.3,

$$\int_{M\setminus D} |\nabla(\Delta u)|\, \phi^2 \leq \frac{C}{R}\int_{M\setminus D} \left(\left|\nabla^2 \rho\right|^2 + \left|\nabla^2 f\right|^2 + 1\right) \phi^2. \tag{3.11}$$

Integrating by parts gives

$$\begin{aligned}
(3.12)\qquad & \int_{M\setminus D}\left|\nabla^2\rho\right|^2\phi^2 \\
= & -\int_{M\setminus D}\left\langle\nabla\left(\Delta\rho\right),\nabla\rho\right\rangle\phi^2-\int_{M\setminus D}\mathrm{Ric}\left(\nabla\rho,\nabla\rho\right)\phi^2 \\
& -2\int_{M\setminus D}\rho_{ij}\rho_i\phi_j\phi-\int_{\partial D}\rho_{ij}\rho_i\nu_j\phi^2.
\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}
-\int_{M\setminus D}\left\langle\nabla\left(\Delta\rho\right),\nabla\rho\right\rangle\phi^2 &\le C\int_{M\setminus D}\left|\nabla^2\rho\right|\phi^2 \\
&\le \frac{1}{4}\int_{M\setminus D}\left|\nabla^2\rho\right|^2\phi^2+C\int_{M\setminus D}\phi^2 \\
&\le \frac{1}{4}\int_{M\setminus D}\left|\nabla^2\rho\right|^2\phi^2+\frac{C}{\varepsilon^2},
\end{aligned}$$

where in the last line we have used (3.3). Since Ricci curvature is bounded from below by a constant, we have

$$\begin{aligned}
-\int_{M\setminus D}\mathrm{Ric}\left(\nabla\rho,\nabla\rho\right)\phi^2 &\le C\int_{M\setminus D}\left|\nabla\rho\right|^2\phi^2 \\
&\le C\int_{M\setminus D}\phi^2 \\
&\le \frac{C}{\varepsilon^2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
-2\int_{M\setminus D}\rho_{ij}\rho_i\phi_j\phi &\le 2\int_{M\setminus D}\left|\nabla^2\rho\right|\left|\nabla\phi\right|\phi \\
&\le \frac{1}{4}\int_{M\setminus D}\left|\nabla^2\rho\right|^2\phi^2+4\int_{L(\varepsilon,1)}\left|\nabla\phi\right|^2.
\end{aligned}$$

Using (3.3) and (3.7) we have

$$\int_{L(\varepsilon,1)}\left|\nabla\phi\right|^2\le\frac{C}{\varepsilon^2}.$$

This shows that

$$-2\int_{M\setminus D}\rho_{ij}\rho_i\phi_j\phi\le\frac{1}{4}\int_{M\setminus D}\left|\nabla^2\rho\right|^2\phi^2+\frac{C}{\varepsilon^2}.$$

Certainly,

$$\left|\int_{\partial D}\rho_{ij}\rho_i\nu_j\phi^2\right|\le C.$$

Plugging these estimates into (3.12) implies that

$$\int_{M\setminus D}\left|\nabla^2\rho\right|^2\phi^2\le\frac{C}{\varepsilon^2}.$$

Similarly, we see that

$$\int_{M\setminus D}\left|\nabla^2 f\right|^2\phi^2\leq C.$$

Indeed, as f is harmonic on $M\setminus D$,

$$\begin{aligned}&\int_{M\setminus D}\left|\nabla^2 f\right|^2\phi^2\\ =\;& -\int_{M\setminus D}\mathrm{Ric}\left(\nabla f,\nabla f\right)\phi^2\\ &-2\int_{M\setminus D} f_{ij}f_i\phi_j\phi-\int_{\partial D} f_{ij}f_i\nu_j\phi^2.\end{aligned}$$

Again, as the Ricci curvature is bounded from below by a constant, we have

$$\begin{aligned}-\int_{M\setminus D}\mathrm{Ric}\left(\nabla f,\nabla f\right)\phi^2 &\leq C\int_{M\setminus D}\left|\nabla f\right|^2\\ &\leq C.\end{aligned}$$

Also,

$$\begin{aligned}-2\int_{M\setminus D} f_{ij}f_i\phi_j\phi &\leq 2\int_{M\setminus D}\left|\nabla^2 f\right|\left|\nabla f\right|\left|\nabla\phi\right|\phi\\ &\leq \frac{1}{4}\int_{M\setminus D}\left|\nabla^2 f\right|^2\phi^2+C\int_{M\setminus D}\left|\nabla f\right|^2\\ &\leq \frac{1}{4}\int_{M\setminus D}\left|\nabla^2 f\right|^2\phi^2+C.\end{aligned}$$

This shows that

$$\int_{M\setminus D}\left|\nabla^2 f\right|^2\phi^2\leq C.$$

We conclude from (3.11) and (3.1) that

$$\begin{aligned}\int_{M\setminus D}\left|\nabla\left(\Delta u\right)\right|\phi^2 &\leq \frac{C}{R\varepsilon^2}\\ &\leq C.\end{aligned}$$

Thus, (3.10) becomes

$$\begin{aligned}&\int_{M\setminus D}\left(\Delta\left|\nabla u\right|\right)\phi^2\\ \geq\;&\int_{M\setminus D}\Big(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2+\mathrm{Ric}\left(\nabla u,\nabla u\right)\Big)\left|\nabla u\right|^{-1}\phi^2-C.\end{aligned}\tag{3.13}$$

Using the co-area formula, we have

$$\begin{aligned}
&\int_{M\setminus D}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-1}\phi^2\\
= &\int_\varepsilon^1\phi^2(t)\int_{\{u=t\}}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt\\
= &\int_\varepsilon^1\phi^2(t)\int_{l(t)}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt\\
&+\int_\varepsilon^1\phi^2(t)\int_{\widetilde{l}(t)}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt,
\end{aligned}$$

where $l(t)$ and $\widetilde{l}(t)$ were defined in (2.12) and (2.17), respectively.

By Lemma 2.5 and Kato inequality we have

$$\begin{aligned}
&\int_\varepsilon^1\phi^2(t)\int_{\widetilde{l}(t)}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt\\
\geq &\int_\varepsilon^1\phi^2(t)\int_{\widetilde{l}(t)}\operatorname{Ric}(\nabla u,\nabla u)|\nabla u|^{-2}\,dt\\
\geq &-C\int_\varepsilon^1\operatorname{Area}\left(\widetilde{l}(t)\right)dt\\
\geq &-\frac{C}{\sqrt{R\varepsilon^3}}\int_{L(\varepsilon,1)}u^2.
\end{aligned}$$

Hence, according to (3.2) and (3.1), it follows that

$$\begin{aligned}
&\int_{M\setminus D}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-1}\phi^2\\
\geq &\int_\varepsilon^1\phi^2(t)\int_{l(t)}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt\\
&-C.
\end{aligned}$$

Rewrite it into

$$\begin{aligned}
(3.14)\quad &\int_{M\setminus D}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-1}\phi^2\\
\geq &\int_\varepsilon^1\phi^2(t)\int_{l(t)\cap\overline{M_t}}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt\\
&+\int_\varepsilon^1\phi^2(t)\int_{l(t)\cap\left(M\setminus\overline{M_t}\right)}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\operatorname{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt\\
&-C.
\end{aligned}$$

Recall that from Lemma 2.4,

$$l(t)\cap\left(M\setminus\overline{M_t}\right)\subset M\setminus D(R).$$

Since

$$\operatorname{Ric}(\nabla u,\nabla u)\geq -C|\nabla u|^2,$$

together with the Kato inequality $|u_{ij}|^2 \geq |\nabla |\nabla u||^2$, it follows that

$$
\begin{aligned}
& \int_\varepsilon^1 \phi^2(t) \int_{l(t)\cap\left(M\setminus\overline{M_t}\right)} \left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt \\
\geq & -C\int_\varepsilon^1 \phi^2(t)\int_{l(t)\cap\left(M\setminus\overline{M_t}\right)} dt \\
\geq & -C\int_\varepsilon^1 \phi^2(t)\int_{l(t)\cap(M\setminus D(R))} dt \\
\geq & -C\int_{M\setminus D(R)} |\nabla u|\,\phi^2 \\
\geq & -C\int_{M\setminus D(R)} \phi^2.
\end{aligned}
$$

By (3.4) we have that $\int_{M\setminus D(R)}\phi^2 \leq C$. In conclusion,

$$
\int_\varepsilon^1 \phi^2(t)\int_{l(t)\cap\left(M\setminus\overline{M_t}\right)} \left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt \geq -C
$$

and (3.14) becomes

$$
\begin{aligned}
& \int_{M\setminus D} \left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-1}\phi^2 \\
\geq & \int_\varepsilon^1 \phi^2(t)\int_{l(t)\cap\overline{M_t}} \left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2}\,dt - C.
\end{aligned}
\tag{3.15}
$$

By Lemma 2.6,

$$
\begin{aligned}
\mathrm{Ric}(\nabla u,\nabla u)|\nabla u|^{-2} = & \frac{1}{2}S - \frac{1}{2}S_t + \frac{1}{2}\frac{1}{|\nabla u|^2}\left(|\nabla|\nabla u||^2 - |u_{ij}|^2\right) \\
& + \frac{1}{2}\frac{1}{|\nabla u|^2}\left((\Delta u)^2 - 2\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}\Delta u + |\nabla|\nabla u||^2\right)
\end{aligned}
$$

on any regular level set $l(t)$. It follows that

$$
\begin{aligned}
& \left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-2} \\
= & \frac{1}{2}S - \frac{1}{2}S_t + \frac{1}{2}\left(|u_{ij}|^2 - |\nabla|\nabla u||^2\right)|\nabla u|^{-2} \\
& + \frac{1}{2}\left((\Delta u)^2 - 2\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}\Delta u + |\nabla|\nabla u||^2\right)|\nabla u|^{-2}.
\end{aligned}
\tag{3.16}
$$

According to the Gauss-Bonnet theorem, on each regular connected component $l_k(t)$ of $l(t)$,

$$
\int_{l_k(t)} S_t = 4\pi\chi(l_k(t)) \leq 8\pi.
$$

Invoking Lemma 2.4 that $l(t)\cap\overline{M_t} = \partial M_t$ has m connected components, one has

$$\int_{l(t)\cap\overline{M_t}} S_t \leq 8\pi m \leq C$$

for all regular value t of u with $0 < t < 1$. Therefore, (3.16) together with the assumption $S \geq -6K$ implies that

$$\begin{aligned}
&\int_{l(t)\cap\overline{M_t}} \left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right)|\nabla u|^{-2} \\
\geq\ & \frac{1}{2}\int_{l(t)\cap\overline{M_t}} \left(|u_{ij}|^2 - |\nabla|\nabla u||^2\right)|\nabla u|^{-2} \\
&+\frac{1}{2}\int_{l(t)\cap\overline{M_t}} \left((\Delta u)^2 - 2\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}\Delta u + |\nabla|\nabla u||^2\right)|\nabla u|^{-2} \\
&-3K\operatorname{Area}\left(l(t)\cap\overline{M_t}\right) - C.
\end{aligned}$$

So (3.15) becomes

$$\begin{aligned}
(3.17)\quad &\int_{M\setminus D} \left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right)|\nabla u|^{-1}\phi^2 \\
\geq\ & \frac{1}{2}\int_{\varepsilon}^{1}\phi^2(t)\int_{l(t)\cap\overline{M_t}} \left(|u_{ij}|^2 - |\nabla|\nabla u||^2\right)|\nabla u|^{-2}\,dt \\
&+\frac{1}{2}\int_{\varepsilon}^{1}\phi^2(t)\int_{l(t)\cap\overline{M_t}} \left((\Delta u)^2 - 2\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}\Delta u + |\nabla|\nabla u||^2\right)|\nabla u|^{-2}\,dt \\
&-3K\int_{\varepsilon}^{1}\phi^2(t)\operatorname{Area}\left(l(t)\cap\overline{M_t}\right)dt - C\int_{\varepsilon}^{1}\phi^2(t)\,dt - C.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\varepsilon}^{1}\phi^2(t)\operatorname{Area}\left(l(t)\cap\overline{M_t}\right)dt &\leq \int_{\varepsilon}^{1}\phi^2(t)\operatorname{Area}(l(t))\,dt \\
&= \int_{M\setminus D}|\nabla u|\,\phi^2
\end{aligned}$$

and that $\int_{\varepsilon}^{1}\phi^2(t)\,dt = C$. Moreover, by Lemma 2.3 and (3.4),

$$\begin{aligned}
\int_{M\setminus D}|\nabla u|\,\phi^2 &= \int_{D(R)\setminus D}|\nabla u|\,\phi^2 + \int_{M\setminus D(R)}|\nabla u|\,\phi^2 \\
&\leq \int_{D(R)\setminus D}|\nabla u|\,\phi^2 + C.
\end{aligned}$$

Plugging into (3.17) we get

$$
\begin{aligned}
(3.18)\quad & \int_{M\setminus D}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2+\mathrm{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-1}\phi^2 \\
\geq\ & \frac{1}{2}\int_{\varepsilon}^{1}\phi^2\left(t\right)\int_{l(t)\cap\overline{M_t}}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2\right)\left|\nabla u\right|^{-2}dt \\
& +\frac{1}{2}\int_{\varepsilon}^{1}\phi^2\left(t\right)\int_{l(t)\cap\overline{M_t}}\left(\left(\Delta u\right)^2-2\frac{\left\langle\nabla\left|\nabla u\right|,\nabla u\right\rangle}{\left|\nabla u\right|}\Delta u+\left|\nabla\left|\nabla u\right|\right|^2\right)\left|\nabla u\right|^{-2}dt \\
& -3K\int_{D(R)\setminus D}\left|\nabla u\right|\phi^2-C.
\end{aligned}
$$

Using the Kato inequality

$$
\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2\geq 0
$$

and Lemma 2.4 that

$$
l\left(t\right)\cap D\left(R\right)\subset\overline{M_t},
$$

the first term on the right hand side can be estimated as

$$
\begin{aligned}
& \frac{1}{2}\int_{\varepsilon}^{1}\phi^2\left(t\right)\int_{l(t)\cap\overline{M_t}}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2\right)\left|\nabla u\right|^{-2}dt \\
\geq\ & \frac{1}{2}\int_{\varepsilon}^{1}\phi^2\left(t\right)\int_{l(t)\cap D(R)}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2\right)\left|\nabla u\right|^{-2}dt \\
=\ & \frac{1}{2}\int_{D(R)\setminus D}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2\right)\left|\nabla u\right|^{-1}\phi^2.
\end{aligned}
$$

Since $u=f$ on $D\left(R\right)$,

$$
\Delta u=0\ \text{ and }\ \left|u_{ij}\right|^2\geq\frac{3}{2}\left|\nabla\left|\nabla u\right|\right|^2\quad\text{in }D\left(R\right)\setminus D
$$

by Lemma 2.7. It follows from above that

$$
\begin{aligned}
(3.19)\quad & \frac{1}{2}\int_{\varepsilon}^{1}\phi^2\left(t\right)\int_{l(t)\cap\overline{M_t}}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2\right)\left|\nabla u\right|^{-2}dt \\
\geq\ & \frac{1}{4}\int_{D(R)\setminus D}\left|\nabla\left|\nabla u\right|\right|^2\left|\nabla u\right|^{-1}\phi^2.
\end{aligned}
$$

Note that

$$
\left(\Delta u\right)^2-2\frac{\left\langle\nabla\left|\nabla u\right|,\nabla u\right\rangle}{\left|\nabla u\right|}\Delta u+\left|\nabla\left|\nabla u\right|\right|^2\geq 0.
$$

By Lemma 2.4, the second term is bounded by

$$
\begin{aligned}
(3.20)\quad & \frac{1}{2}\int_{\varepsilon}^{1}\phi^{2}(t)\int_{l(t)\cap\overline{M_t}}\left((\Delta u)^2-2\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}\Delta u+|\nabla|\nabla u||^2\right)|\nabla u|^{-2}\,dt\\
\geq\quad & \frac{1}{2}\int_{\varepsilon}^{1}\phi^{2}(t)\int_{l(t)\cap D(R)}\left((\Delta u)^2-2\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}\Delta u+|\nabla|\nabla u||^2\right)|\nabla u|^{-2}\,dt\\
=\quad & \frac{1}{2}\int_{D(R)\setminus D}\left((\Delta u)^2-2\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}\Delta u+|\nabla|\nabla u||^2\right)|\nabla u|^{-1}\phi^2\\
=\quad & \frac{1}{2}\int_{D(R)\setminus D}|\nabla|\nabla u||^2|\nabla u|^{-1}\phi^2,
\end{aligned}
$$

where we have used that $\Delta u=0$ on $D(R)\setminus D$ in the last line.

In view of (3.19) and (3.20), (3.18) becomes

$$
\begin{aligned}
(3.21)\qquad & \int_{M\setminus D}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\right)|\nabla u|^{-1}\phi^2\\
\geq\quad & \frac{3}{4}\int_{D(R)\setminus D}|\nabla|\nabla u||^2|\nabla u|^{-1}\phi^2-3K\int_{D(R)\setminus D}|\nabla u|\,\phi^2-C.
\end{aligned}
$$

Plugging into (3.13) yields

$$
\begin{aligned}
(3.22)\qquad \int_{M\setminus D}(\Delta|\nabla u|)\,\phi^2 \quad\geq\quad & \frac{3}{4}\int_{D(R)\setminus D}|\nabla|\nabla u||^2|\nabla u|^{-1}\phi^2\\
& -3K\int_{D(R)\setminus D}|\nabla u|\,\phi^2-C.
\end{aligned}
$$

To estimate the left hand side, we integrate by parts and get

$$
(3.23)\qquad \int_{M\setminus D}(\Delta|\nabla u|)\,\phi^2=\int_{M\setminus D}|\nabla u|\,\Delta\phi^2-\int_{\partial D}\phi^2\,|\nabla u|_{\nu}\,.
$$

Since $u=f$ on $D(2R)\setminus D$, we have $|\nabla u|_{\nu}=|\nabla f|_{\nu}$, where ν is the unit outward normal to ∂D. Hence,

$$
\int_{\partial D}\phi^2\,|\nabla u|_{\nu}=\int_{\partial D}\phi^2\,|\nabla f|_{\nu}=C.
$$

To estimate the first term in (3.23), note that $\Delta\phi^2=0$ on $M\setminus L(\varepsilon,2\varepsilon)$. On $L(\varepsilon,2\varepsilon)$, by (3.6) and Lemma 2.3, we have

$$
\begin{aligned}
\left|\Delta\phi^2\right| \quad&\leq\quad \frac{C}{\varepsilon^2}\left(|\Delta u|+|\nabla u|^2\right)\\
&\leq\quad \frac{C}{R\varepsilon^2}+\frac{C}{\varepsilon^2}|\nabla u|^2\\
&\leq\quad C\varepsilon^2+\frac{C}{\varepsilon}|\nabla u|\,,
\end{aligned}
$$

where in the last line we have used (3.1) and (3.8). In view of (3.3), we get

$$\begin{aligned}\int_{M\setminus D}|\nabla u|\left|\Delta\phi^2\right| &\leq C\varepsilon^2\mathrm{Vol}\left(L\left(\varepsilon,1\right)\right)+\frac{C}{\varepsilon}\int_{L(\varepsilon,2\varepsilon)}|\nabla u|^2\\ &\leq C+\frac{C}{\varepsilon}\int_{L(\varepsilon,2\varepsilon)}|\nabla u|^2.\end{aligned}$$

The co-area formula and (3.5) imply that

$$\begin{aligned}\frac{1}{\varepsilon}\int_{L(\varepsilon,2\varepsilon)}|\nabla u|^2 &= \frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon}\int_{l(t)}|\nabla u|\,dt\\ &\leq C.\end{aligned}$$

Hence,

$$\int_{M\setminus D}|\nabla u|\left|\Delta\phi^2\right|\leq C \tag{3.24}$$

and (3.23) becomes

$$\int_{M\setminus D}\left(\Delta\left|\nabla u\right|\right)\phi^2\leq C.$$

Note that the same argument also shows

$$\int_{M\setminus D}|\nabla u|\left|\nabla\phi\right|^2\leq C. \tag{3.25}$$

We therefore conclude from (3.22) that

$$\frac{3}{4}\int_{D(R)\setminus D}\left|\nabla\left|\nabla u\right|\right|^2|\nabla u|^{-1}\phi^2\leq 3K\int_{D(R)\setminus D}|\nabla u|\,\phi^2+C. \tag{3.26}$$

To complete the proof of **Claim 1**, observe that

$$\int_{D(R)\setminus D}|\nabla u|\,\phi^2=\int_{D\left(\frac{1}{2}R\right)\setminus D}|\nabla u|\,\phi^2+\int_{D(R)\setminus D\left(\frac{1}{2}R\right)}|\nabla u|\,\phi^2$$

and that by (3.4),

$$\int_{D(R)\setminus D\left(\frac{1}{2}R\right)}|\nabla u|\,\phi^2\leq C.$$

Therefore,

$$\int_{D(R)\setminus D}|\nabla u|\,\phi^2\leq\int_{D\left(\frac{1}{2}R\right)\setminus D}|\nabla u|\,\phi^2+C$$

and (3.26) becomes

$$\frac{3}{4}\int_{D(R)\setminus D}\left|\nabla\left|\nabla u\right|\right|^2|\nabla u|^{-1}\phi^2\leq 3K\int_{D\left(\frac{1}{2}R\right)\setminus D}|\nabla u|\,\phi^2+C. \tag{3.27}$$

This verifies **Claim 1**.

We now establish the following claim.

Claim 2: $\lambda_1(M) \int_{D\left(\frac{1}{2}R\right)\setminus D} |\nabla u| \, \phi^2 \leq \frac{1}{4} \int_{D(R)\setminus D} |\nabla |\nabla u||^2 \, |\nabla u|^{-1} \, \phi^2 + C.$

Let $\sigma : (0, \infty) \to \mathbb{R}$ be a smooth cut-off function so that $\sigma(t) = 1$ for $t \leq \frac{1}{2}R$ and $\sigma(t) = 0$ for $t \geq R$. Moreover, it satisfies

$$|\sigma'| + |\sigma''| \leq \frac{C}{R}.$$

Denote

$$\sigma(x) = \sigma(\rho(x)).$$

By Lemma 2.1 we have

$$|\nabla \sigma| + |\Delta \sigma| \leq \frac{C}{R}. \tag{3.28}$$

Let ξ be a Lipschitz cut-off such that $\xi = 0$ on $D(R_0)$ and $\xi = 1$ on $M \setminus D(R_0 + 1)$.

Since the function $|\nabla u| \, \phi^2 (\xi\sigma)^2$ has compact support in M, we have

$$\begin{aligned} \lambda_1(M) \int_M |\nabla u| \, \phi^2 (\xi\sigma)^2 &\leq \int_M \left| \nabla \left(|\nabla u|^{\frac{1}{2}} \phi\xi\sigma \right) \right|^2 \\ &= \frac{1}{4} \int_M |\nabla |\nabla u||^2 \, |\nabla u|^{-1} \, \phi^2 \xi^2 \sigma^2 \\ &\quad + \int_M |\nabla u| \, |\nabla (\phi\xi\sigma)|^2 \\ &\quad + \frac{1}{2} \int_M \left\langle \nabla |\nabla u|, \nabla (\phi\xi\sigma)^2 \right\rangle. \end{aligned} \tag{3.29}$$

By (3.28), (3.3), and (3.25), the second last term is bounded by

$$\begin{aligned} \int_M |\nabla u| \, |\nabla (\phi\xi\sigma)|^2 &\leq \frac{C}{R^2} \int_{M\setminus D} |\nabla u| \, \phi^2 \\ &\quad + \int_{M\setminus D} |\nabla u| \, |\nabla \phi|^2 \\ &\quad + \int_{M\setminus D} |\nabla u| \, |\nabla \xi|^2 \\ &\leq \frac{C}{R^2} \mathrm{Vol}(L(\varepsilon, 1)) + C \\ &\quad + \int_{D(R_0+1)\setminus D(R_0)} |\nabla f| \, |\nabla \xi|^2 \\ &\leq C. \end{aligned}$$

For the last term in (3.29), we first perform integration by parts and get

$$
\begin{aligned}
\frac{1}{2}\int_M \left\langle \nabla\left|\nabla u\right|, \nabla\left(\phi\xi\sigma\right)^2\right\rangle &= \frac{1}{2}\int_M \left\langle \nabla\left|\nabla u\right|, \nabla\sigma^2\right\rangle \xi^2\phi^2 \\
&\quad +\frac{1}{2}\int_M \left\langle \nabla\left|\nabla u\right|, \nabla\phi^2\right\rangle \xi^2\sigma^2 + C \\
&= -\frac{1}{2}\int_M \left|\nabla u\right|\left(\Delta\sigma^2\right)\xi^2\phi^2 \\
&\quad -\frac{1}{2}\int_M \left|\nabla u\right|\left(\Delta\phi^2\right)\xi^2\sigma^2 \\
&\quad -\int_M \left|\nabla u\right|\left\langle \nabla\sigma^2, \nabla\phi^2\right\rangle \xi^2 + C.
\end{aligned}
$$

By (3.24),

$$
\int_M \left|\nabla u\right|\left|\Delta\phi^2\right|\xi^2\sigma^2 \leq C.
$$

Moreover, by (3.28) and (3.3),

$$
\begin{aligned}
-\frac{1}{2}\int_M \left|\nabla u\right|\left(\Delta\sigma^2\right)\xi^2\phi^2 &\leq \int_M \left|\nabla u\right|\left(\sigma\left|\Delta\sigma\right| + \left|\nabla\sigma\right|^2\right)\xi^2\phi^2 \\
&\leq \frac{C}{R}\int_{M\setminus D}\left|\nabla u\right|\phi^2 \\
&\leq \frac{C}{R}\operatorname{Vol}\left(L(\varepsilon,1)\right) \\
&\leq C.
\end{aligned}
$$

From (3.7) it is easy to see that

$$
\begin{aligned}
-\int_M \left|\nabla u\right|\left\langle \nabla\sigma^2, \nabla\phi^2\right\rangle \xi^2 &\leq \frac{C}{R}\int_{M\setminus D}\left|\nabla u\right|\phi \\
&\leq \frac{C}{R}\operatorname{Vol}\left(L(\varepsilon,1)\right) \\
&\leq C.
\end{aligned}
$$

In conclusion,

$$
\frac{1}{2}\int_M \left\langle \nabla\left|\nabla u\right|, \nabla\left(\phi\xi\sigma\right)^2\right\rangle \leq C.
$$

Plugging these estimates into (3.29) we arrive at

$$
\lambda_1\left(M\right)\int_M \left|\nabla u\right|\phi^2\xi^2\sigma^2 \leq \frac{1}{4}\int_M \left|\nabla\left|\nabla u\right|\right|^2\left|\nabla u\right|^{-1}\phi^2\xi^2\sigma^2 + C.
$$

In view of the definition of σ and ξ, we have

$$
\lambda_1\left(M\right)\int_{D\left(\frac{1}{2}R\right)\setminus D(R_0+1)}\left|\nabla u\right|\phi^2 \leq \frac{1}{4}\int_{D(R)\setminus D}\left|\nabla\left|\nabla u\right|\right|^2\left|\nabla u\right|^{-1}\phi^2 + C. \tag{3.30}
$$

Since $u = f$ on $D\left(R_0+1\right)\setminus D$,

$$\int_{D(R_0+1)\setminus D} |\nabla u| \, \phi^2 \leq \int_{D(R_0+1)\setminus D} |\nabla f| = C.$$

Claim 2 now follows from (3.30).

Combining **Claim 1** and **Claim 2**, or (3.30) and (3.27), we conclude that

$$\lambda_1(M) \int_{D\left(\frac{1}{2}R\right)\setminus D} |\nabla u| \, \phi^2 \leq K \int_{D\left(\frac{1}{2}R\right)\setminus D} |\nabla u| \, \phi^2 + C. \tag{3.31}$$

We now show $\lambda_1(M) \leq K$. Assume by contradiction that

$$\lambda_1(M) > K. \tag{3.32}$$

We then infer from (3.31) that

$$\int_{D\left(\frac{1}{2}R\right)\cap L(2\varepsilon,1)} |\nabla u| \leq C \tag{3.33}$$

for any $\varepsilon > 0$ and $R > \max\{R_0, \frac{1}{\varepsilon^4}\}$ with the constant C independent of ε and R.

For arbitrary large r such that $D \subset B_p(r)$, let

$$t(r) = \min_{x\in\partial B_p(r)} f(x).$$

The maximum principle implies that

$$B_p(r) \subset \{f > t(r)\}.$$

So for $\varepsilon < \frac{1}{2}t(r)$ and all R sufficiently large,

$$B_p(r) \setminus D \subset D\left(\frac{1}{2}R\right) \cap L(2\varepsilon, 1).$$

It follows from (3.33) that, as $u = f$ on $B_p(r)\setminus D$,

$$\int_{B_p(r)\setminus D} |\nabla f| \leq C.$$

Since r is arbitrary, this proves

$$\int_{M\setminus D} |\nabla f| \leq C. \tag{3.34}$$

According to [25],

$$\int_{\{f=t\}} |\nabla f| = C. \tag{3.35}$$

Applying (3.35) and the co-area formula we get

$$\begin{aligned} \int_{\{\delta<f<1\}} |\nabla f|^2 f^{-1} &= C\int_\delta^1 \frac{1}{t}dt \\ &= -C\ln\delta. \end{aligned}$$

In view of Lemma 2.2 and (3.34), we conclude

$$-C\ln\delta=\int_{\{\delta<f<1\}}|\nabla f|^2 f^{-1}\leq C\int_{M\setminus D}|\nabla f|\leq C$$

for any $\delta>0$. This is a contradiction. Therefore, (3.32) does not hold. This proves the theorem. □

4. Preliminaries for asymptotically nonnegative Ricci curvature

In this section, in anticipation of the applications in the next section, we establish some preliminary results for manifolds with asymptotically nonnegative Ricci curvature. Mainly, we show the existence of a distance-like function that behaves better than the function ρ constructed in Section 2. As before, we use $r(x)=d(p,x)$ to denote the distance from x to a fixed point $p\in M$.

Lemma 4.1. *Let (M,g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by*

$$\mathrm{Ric}\geq-\frac{C}{r^2+1}.$$

Then there exists a smooth proper function η such that

$$\begin{aligned}\frac{1}{C}\ln(r+2) &\leq \eta\leq C\ln(r+2)\\ |\nabla\eta| &\leq \frac{C}{r+1}\\ |\Delta\eta| &\leq \frac{C}{(r+1)^2}\end{aligned}\tag{4.1}$$

and

$$|\nabla(\Delta\eta)|\leq\frac{C}{(r+1)^3}+\frac{C}{r+1}\left|\nabla^2\eta\right|\tag{4.2}$$

for some constant $C>0$.

Proof. The argument is similar to that of Lemma 2.1 with the details spelled out in [3] except for (4.2). For completeness, we include them here. For $\lambda>0$ to be determined later, we solve the boundary value problem.

$$\begin{aligned}\Delta h_R &= \frac{\lambda}{\rho^2+1}h_R \ \text{ in } D(R)\setminus D(1)\\ h_R &= 1 \ \text{ on } \partial D(1)\\ h_R &= 0 \ \text{ on } \partial D(R),\end{aligned}$$

where ρ is defined in Lemma 2.1 and $D(R)$ in (2.6).

The maximum principle implies that $0<h_R<1$ and that h_R is increasing in R. Letting $R\to\infty$, we obtain a positive function h on $M\setminus D(1)$ satisfying

$$\begin{aligned}\Delta h &= \frac{\lambda}{\rho^2+1}h \ \text{ in } M\setminus D(1)\\ h &= 1 \ \text{ on } \partial D(1).\end{aligned}\tag{4.3}$$

By Cheng-Yau's gradient estimate [8], it is easy to see that

$$|\nabla \ln h| \leq \frac{C(\lambda)}{\rho+1}. \tag{4.4}$$

For any $m>0$, note that

$$\begin{aligned}
\frac{\lambda}{2}\int_{D(R)\setminus D(1)} \rho^{m-2}h_R^2 &\leq \int_{D(R)\setminus D(1)} \frac{\lambda}{\rho^2+1}\rho^m h_R^2 \\
&= \int_{D(R)\setminus D(1)} \rho^m h_R \Delta h_R \\
&= -\int_{D(R)\setminus D(1)} \rho^m \left|\nabla h_R\right|^2 \\
&\quad -m\int_{D(R)\setminus D(1)} \left\langle \nabla h_R, \nabla \rho\right\rangle h_R \rho^{m-1} \\
&\quad - \int_{\partial D(1)} h_R \frac{\partial h_R}{\partial \nu}.
\end{aligned}$$

Since

$$\begin{aligned}
m\int_{D(R)\setminus D(1)} \left|\left\langle \nabla h_R, \nabla \rho\right\rangle\right| h_R \rho^{m-1} &\leq Cm\int_{D(R)\setminus D(1)} \left|\nabla h_R\right| h_R \rho^{m-1} \\
&\leq \int_{D(R)\setminus D(1)} \rho^m \left|\nabla h_R\right|^2 \\
&\quad +Cm^2\int_{D(R)\setminus D(1)} \rho^{m-2}h_R^2,
\end{aligned}$$

it follows that

$$\int_{D(R)\setminus D(1)} \rho^{m-2}h_R^2 \leq C$$

by taking $\lambda > C\, m^2$ for some large C. This shows that

$$\int_M \rho^{m-2}h^2 \leq C.$$

Together with (4.4) and Lemma 2.1 we get that

$$\begin{aligned}
h^2(x) &\leq \frac{C}{\mathrm{V}_x(1)}\int_{B_x(1)} h^2 \\
&\leq \frac{C}{\mathrm{V}_x(1)}\frac{1}{R^{m-2}}
\end{aligned}$$

for $x\in \partial B_p(R)$.

The assumption that Ricci curvature is bounded below by $\mathrm{Ric} \geq -\frac{C}{r^2+1}$ implies that (cf. [3])

$$\frac{C}{\mathrm{V}_x(1)} \leq R^C.$$

By arranging m to be sufficiently large, we conclude that h decays polynomially and that

$$h(x) \leq \rho^{-\frac{m}{2}}(x) \quad \text{on } M \backslash D(1). \tag{4.5}$$

Now the desired function η is defined as

$$\eta(x) = (1 - \xi(x))(-\ln h(x)) + \xi(x),$$

where ξ is a smooth cut-off function with $\xi = 1$ on $D(1)$ and $\xi = 0$ on $M \backslash D(2)$.

By (4.5) we have that

$$\eta \geq \frac{1}{C} \ln(r+2).$$

On the other hand, (4.4) implies that

$$\eta \leq C \ln(r+2)$$

and

$$|\nabla \eta| \leq \frac{C}{r+1}.$$

Furthermore, together with (4.3), we get

$$|\Delta \eta| \leq \frac{C}{(r+1)^2}.$$

Finally, using (4.3) again that

$$\Delta \eta = -\frac{\lambda}{\rho^2 + 1} + |\nabla \eta|^2 \text{ on } M \backslash D(2),$$

we conclude

$$|\nabla(\Delta \eta)| \leq \frac{C}{(r+1)^3} + \frac{C}{r+1} \left|\nabla^2 \eta\right|.$$

This proves the result. □

For the remaining part of the paper η, rather than ρ, will be our choice of smooth proper function. We henceforth denote with

$$D(t) = \{x \in M : \eta(x) < t\}. \tag{4.6}$$

As in Section 2, we use η to construct a cut-off function on M. For given $R > 0$, let $\psi : (0, \infty) \to \mathbb{R}$ be a smooth function such that $\psi = 1$ on $(0, \ln R)$ and $\psi = 0$ on $(2 \ln R, \infty)$, and

$$|\psi'| \leq \frac{C}{\ln R}, \quad |\psi''| \leq \frac{C}{\ln^2 R} \text{ and } |\psi'''| \leq \frac{C}{\ln^3 R}.$$

Composing it with η, we obtain a cut-off function

$$\psi(x) = \psi(\eta(x)).$$

Obviously,

$$\begin{aligned} \psi &= 1 \text{ on } D(\ln R) \\ \psi &= 0 \text{ on } M \backslash D(2 \ln R). \end{aligned}$$

In the following we assume that (M,g) is nonparabolic and let f be a barrier function on $M\setminus D$ as constructed in Section 2, where D is a smooth connected bounded domain in M. Again, we extend f to M by setting $f=1$ on D. The next result is well-known [8].

Lemma 4.2. *Let (M,g) be an n-dimensional complete noncompact Riemannian manifold with Ricci curvature bounded below by*

$$\mathrm{Ric}\geq -\frac{C}{r^2+1}.$$

Then

$$|\nabla \ln f|\leq \frac{C}{r} \quad \text{on } M\backslash D.$$

We now consider the function u given by

$$u=f\psi. \tag{4.7}$$

Lemma 4.3. *Let (M,g) be an n-dimensional complete noncompact Riemannian manifold with Ricci curvature bounded below by*

$$\mathrm{Ric}\geq -\frac{C}{r^2+1}.$$

Then u is harmonic on $D(\ln R)\backslash D$ and on $M\backslash D$,

$$\begin{aligned}
|\nabla u| &\leq u\,|\nabla \ln f|+\frac{C}{r}\frac{1}{\ln R}\\
|\Delta u| &\leq \frac{C}{r^2\ln R}+\frac{C}{\ln R}\,|\nabla f|^2\\
|\nabla(\Delta u)| &\leq \frac{Cr}{\ln R}\left(\left|\nabla^2\eta\right|^2+\left|\nabla^2 f\right|^2\right)+\frac{C}{r^3\ln R}+\frac{C}{\ln R}\,|\nabla f|^2.
\end{aligned}$$

Proof. By Lemma 4.1,

$$\begin{aligned}
|\nabla\psi| &\leq \frac{C}{r\ln R},\\
|\Delta\psi| &\leq \frac{C}{r^2\ln R}
\end{aligned}$$

and

$$\begin{aligned}
|\nabla(\Delta\psi)| &= \left|\nabla\left(\psi'\Delta\eta+\psi''\,|\nabla\eta|^2\right)\right|\\
&\leq |\psi'|\,|\nabla(\Delta\eta)|+|\Delta\eta|\,|\nabla\psi'|+|\psi''|\left|\nabla\,|\nabla\eta|^2\right|+|\nabla\eta|^2\,|\nabla\psi''|\\
&\leq \frac{C}{\ln R}\left(\frac{1}{r}\left|\nabla^2\eta\right|+\frac{1}{r^3}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
|\nabla u| &\leq |\nabla f|\,\psi+f\,|\nabla\psi|\\
&\leq u\,|\nabla\ln f|+\frac{C}{r\ln R}
\end{aligned}$$

on $M\backslash D$. Moreover, $\Delta u=0$ on $D(\ln R)\backslash D$ as $u=f$ in $D(\ln R)$. Since

$$\Delta u=f\Delta\psi+2\langle\nabla f,\nabla\psi\rangle,$$

we get from above that

$$\begin{aligned}
|\Delta u| &\leq \frac{C}{r^2 \ln R} + \frac{C}{r \ln R} |\nabla f| \\
&\leq \frac{C}{r^2 \ln R} + \frac{C}{\ln R} |\nabla f|^2 .
\end{aligned}$$

Finally,

$$\begin{aligned}
|\nabla (\Delta u)| &\leq |\nabla (f \Delta \psi)| + 2 |\nabla (\langle \nabla \psi, \nabla f \rangle)| \\
&\leq |\nabla f| |\Delta \psi| + f |\nabla (\Delta \psi)| + 2 \left|\nabla^2 f\right| |\nabla \psi| \\
&\quad +2 \left|\nabla^2 \psi\right| |\nabla f| \\
&\leq \frac{C}{r \ln R} \left(\left|\nabla^2 \eta\right| + \left|\nabla^2 f\right| + \frac{1}{r^2} \right) \\
&\quad + \frac{C}{r^2 \ln R} |\nabla f| + \frac{C}{\ln R} |\nabla f| \left|\nabla^2 \eta\right| ,
\end{aligned}$$

where we have used that

$$\begin{aligned}
\left|\nabla^2 \psi\right| &\leq |\psi'| \left|\nabla^2 \eta\right| + |\psi''| |\nabla \eta|^2 \\
&\leq \frac{C}{\ln R} \left|\nabla^2 \eta\right| + \frac{C}{r^2 \ln^2 R}.
\end{aligned}$$

Consequently,

$$|\nabla (\Delta u)| \leq \frac{Cr}{\ln R} \left(\left|\nabla^2 \eta\right|^2 + \left|\nabla^2 f\right|^2 \right) + \frac{C}{r^3 \ln R} + \frac{C}{\ln R} |\nabla f|^2 .$$

This proves the lemma. □

In the case that M has finite first Betti number and finitely many, say m, ends, we choose D to be large enough so that all representatives of the first homology group $H_1(M)$ are included in D. Moreover, $M \backslash D$ has exactly m unbounded connected components. Let $L(t, \infty)$ be the connected component of the super-level set $\{u > t\}$ that contains D and

$$l(t) = \partial L(t, \infty) .$$

The same argument as Lemma 2.4 gives the following.

Lemma 4.4. *Let (M, g) be an n-dimensional complete Riemannian manifold with m ends and finite first Betti number $b_1(M)$. Assume that the Ricci curvature is bounded below by*

$$\mathrm{Ric} \geq -\frac{C}{r^2 + 1}.$$

Then for all $0 < t < 1$,

$$l(t) \cap \overline{M_t} = \partial M_t \text{ has } m \text{ connected components}$$

and

$$l(t) \cap \left(M \backslash \overline{M_t} \right) \subset M \backslash D(\ln R) ,$$

where M_t is the union of all unbounded connected components of $M \backslash \overline{L(t, \infty)}$.

5. Positive scalar curvature

In this section, we study the geometry of three-dimensional complete manifolds with positive scalar curvature and prove Theorem 1.3 and Theorem 1.4. We continue to follow the notations from Section 4.

Recall that an n-dimensional manifold M has asymptotically nonnegative Ricci curvature if its Ricci curvature is bounded by

$$\mathrm{Ric}(x) \geq -k\left(r(x)\right)$$

for a continuous nonincreasing function $k : [0, \infty) \to [0, \infty)$ with

$$\int_0^\infty r k\left(r\right) dr < \infty.$$

It is well known [15] that M has at most Euclidean area growth, that is, the area $A_p(t)$ of geodesic sphere $\partial B_p(t)$ satisfies

$$\mathrm{A}_p\left(t\right) \leq C\, t^{n-1} \tag{5.1}$$

for all $t > 0$. It is also clear that

$$\mathrm{Ric} \geq -\frac{C}{r^2+1}.$$

So the results from Section 4 are applicable to M.

In passing, we mention that Li and Tam [20] have shown that M^n has finitely many ends under the stronger assumption that $\int_0^\infty r^{n-1}\, k\left(r\right) dr < \infty$.

We also note that according to Lemma 4.1,

$$D\left(2\ln R\right) \subset B_p\left(R^C\right) \tag{5.2}$$

for some $C > 0$, where $D(t)$ was defined in (4.6).

We are now ready to prove Theorem 1.3 which is restated below.

Theorem 5.1. *Let (M, g) be a three-dimensional complete noncompact Riemannian manifold with asymptotically nonnegative Ricci curvature. Suppose that its scalar curvature satisfies*

$$\liminf_{x\to\infty} S\left(x\right) r\left(x\right) \geq C_0$$

for some $C_0 > 0$ and that M has finitely many ends and finite first Betti number $b_1(M)$. Then (M, g) is parabolic.

Proof. Assume by contradiction that (M, g) is nonparabolic. Let f be a barrier function on $M \backslash D$, where the smooth connected compact domain D contains all representatives of $H_1\left(M\right)$ and $M \backslash D$ has m ends. Note that the assumption on scalar curvature implies

$$S \geq \frac{1}{2}\frac{C_0}{r} \quad \text{on } M\backslash D. \tag{5.3}$$

For given small $0 < \varepsilon < 1$, consider R large so that

$$\ln R > \frac{1}{\varepsilon^2}. \tag{5.4}$$

In the following, constant $C>0$ is independent of R and ε. But its value may change from line to line.

Define the function

$$u=f\psi$$

as in (4.7). Then by the Bochner formula and the inequality

$$\langle \nabla \Delta u, \nabla u\rangle \geq -\left|\nabla\left(\Delta u\right)\right|\left|\nabla u\right|,$$

$$\Delta\left|\nabla u\right| \geq \left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2\right)\left|\nabla u\right|^{-1}+\operatorname{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-1}-\left|\nabla\left(\Delta u\right)\right|$$

holds on $M\backslash D$ whenever $\left|\nabla u\right|\neq 0$.

Let $\phi\left(x\right)$ be the smooth cut-off function from the proof of Theorem 3.1, that is,

$$\phi(x)=\begin{cases} \phi\left(u\left(x\right)\right) & \text{on } L\left(\varepsilon,\infty\right) \\ 0 & \text{on } M\backslash L\left(\varepsilon,\infty\right)\end{cases}$$

Here, the function $\phi\left(t\right)$ is smooth, $\phi\left(t\right)=1$ for $2\varepsilon\leq t\leq 1$ and $\phi\left(t\right)=0$ for $t<\varepsilon$. Moreover,

$$\left|\phi'\right|\left(t\right)\leq\frac{C}{\varepsilon}\ \text{ and }\ \left|\phi''\right|\left(t\right)\leq\frac{C}{\varepsilon^2}\ \text{ for } \varepsilon<t<2\varepsilon. \tag{5.5}$$

Clearly, by definition, the function $\phi\left(x\right)$ satisfies $\phi=1$ on $L\left(2\varepsilon,\infty\right)$ and $\phi=0$ on $M\setminus L\left(\varepsilon,\infty\right)$.

It follows that

$$\begin{aligned} &\int_{M\backslash D}\left(\Delta\left|\nabla u\right|\right)\phi^2 \\ \geq\ &\int_{M\backslash D}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2+\operatorname{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-1}\phi^2 \\ &-\int_{M\backslash D}\left|\nabla\left(\Delta u\right)\right|\phi^2. \end{aligned} \tag{5.6}$$

To estimate the last term, we use Lemma 4.3 to obtain

$$\begin{aligned} \int_{M\backslash D}\left|\nabla\left(\Delta u\right)\right|\phi^2 \ \leq\ &\frac{C}{\ln R}\int_{M\backslash D}\left|\nabla^2\eta\right|^2 r\phi^2 \\ &+\frac{C}{\ln R}\int_{M\backslash D}\left|\nabla^2 f\right|^2 r\phi^2 \\ &+\frac{C}{\ln R}\int_{D(2\ln R)\backslash D}\frac{1}{r^3} \\ &+\frac{C}{\ln R}\int_{M\backslash D}\left|\nabla f\right|^2. \end{aligned} \tag{5.7}$$

By (2.8), the last term is bounded. By (5.1) and (5.2) we have

$$\begin{aligned} \frac{C}{\ln R}\int_{D(2\ln R)\setminus D}\frac{1}{r^3} &\leq \frac{C}{\ln R}\int_{B_p(R^C)}\frac{1}{(r+1)^3} \\ &\leq \frac{C}{\ln R}\int_0^{R^C}\frac{r^2}{(r+1)^3}\,dr \\ &\leq C. \end{aligned} \tag{5.8}$$

Now we deal with the first term in (5.7). For convenience, we extend ϕ everywhere on M by setting $\phi(x)=1$ for $x\in D$. Integrating by parts gives

$$\begin{aligned} \int_M \left|\nabla^2\eta\right|^2(r+1)\,\phi^2 &= -\int_M \langle\nabla(\Delta\eta),\nabla\eta\rangle(r+1)\,\phi^2 \\ &\quad -\int_M \operatorname{Ric}(\nabla\eta,\nabla\eta)(r+1)\,\phi^2 \\ &\quad -\int_M \eta_{ij}\eta_i r_j\phi^2 \\ &\quad -2\int_M \eta_{ij}\eta_i\phi_j(r+1)\phi. \end{aligned} \tag{5.9}$$

By Lemma 4.1

$$\begin{aligned} -\int_M \langle\nabla(\Delta\eta),\nabla\eta\rangle(r+1)\,\phi^2 &\leq C\int_M\frac{1}{r+1}\left|\nabla^2\eta\right|\phi^2 + C\int_{D(2\ln R)}\frac{1}{(r+1)^3} \\ &\leq \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + C\int_{D(2\ln R)}\frac{1}{(r+1)^3} \\ &\leq \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + C\ln R, \end{aligned}$$

where in the last line we have used (5.8). According to the Ricci lower bound, we have

$$\begin{aligned} -\int_M \operatorname{Ric}(\nabla\eta,\nabla\eta)(r+1)\,\phi^2 &\leq C\int_M\frac{1}{r+1}\left|\nabla\eta\right|^2 \\ &\leq C\int_{D(2\ln R)}\frac{1}{(r+1)^3} \\ &\leq C\ln R. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} -\int_M \eta_{ij}\eta_i r_j\phi^2 &\leq \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + \int_{D(2\ln R)}\frac{1}{(r+1)^3} \\ &\leq \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + C\ln R. \end{aligned}$$

Finally, we estimate the last term in (5.9) by

$$\begin{aligned} -2\int_M \eta_{ij}\eta_i\phi_j\phi(r+1) &\leq \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + 4\int_M |\nabla\eta|^2|\nabla\phi|^2(r+1) \\ &\leq \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + C\int_M\frac{1}{r+1}|\nabla\phi|^2. \end{aligned}$$

Lemma 4.3 and Lemma 4.2 imply that on $L(\varepsilon, 2\varepsilon)$,

$$
\begin{aligned}
|\nabla u| &\leq \frac{C}{r \ln R}+2\varepsilon|\nabla \ln f| \\
&\leq \frac{C}{r \ln R}+\frac{C\varepsilon}{r} \\
&\leq \frac{C\varepsilon}{r}.
\end{aligned} \tag{5.10}
$$

Using the definition of ϕ we see that

$$
\begin{aligned}
\int_M \frac{1}{r+1}|\nabla \phi|^2 &\leq \frac{C}{\varepsilon^2} \int_{M \setminus D} \frac{1}{r}|\nabla u|^2 \\
&\leq \int_{D(2 \ln R)} \frac{1}{(r+1)^3} \\
&\leq C \ln R.
\end{aligned}
$$

We have proved that

$$
-2 \int_M \eta_{ij} \eta_i \phi_j \phi(r+1) \leq \frac{1}{4} \int_M (r+1)\left|\nabla^2 \eta\right|^2 \phi^2+C \ln R.
$$

Plugging these estimates into (5.9) implies that

$$
\frac{C}{\ln R} \int_M\left|\nabla^2 \eta\right|^2(r+1) \phi^2 \leq C. \tag{5.11}
$$

Since f is harmonic on $M \setminus D$, we similarly have

$$
\begin{aligned}
\int_{M \setminus D}\left|\nabla^2 f\right|^2 r \phi^2 = & -\int_{M \setminus D} \operatorname{Ric}(\nabla f, \nabla f) r \phi^2 \\
& -\int_{M \setminus D} f_{ij} f_i r_j \phi^2-2 \int_{M \setminus D} f_{ij} f_i \phi_j r \phi \\
& -\int_{\partial D} f_{ij} f_i \nu_j r \phi^2 \\
\leq & \frac{1}{2} \int_{M \setminus D}\left|\nabla^2 f\right|^2 r \phi^2+C \int_{M \setminus D}|\nabla f|^2|\nabla \phi|^2 r+C.
\end{aligned}
$$

However, by (5.10) and (2.8) we have that

$$
\begin{aligned}
\int_{M \setminus D}|\nabla f|^2|\nabla \phi|^2 r &\leq C \int_{M \setminus D} \frac{1}{r}|\nabla f|^2 \\
&\leq C.
\end{aligned}
$$

In conclusion,

$$
\frac{C}{\ln R} \int_{M \setminus D}\left|\nabla^2 f\right|^2(r+1) \phi^2 \leq C. \tag{5.12}
$$

Plugging (5.8), (5.11), and (5.12) into (5.7) implies that

$$
\int_{M \setminus B_p(R_0)}|\nabla(\Delta u)| \phi^2 \leq C. \tag{5.13}
$$

Consequently, (5.6) becomes

$$\begin{aligned}(5.14)\qquad &\int_{M\setminus D}(\Delta|\nabla u|)\,\phi^2\\ \geq\ &\int_{M\setminus D}\Big(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\Big)|\nabla u|^{-1}\phi^2-C.\end{aligned}$$

By the co-area formula, we have

$$\begin{aligned}&\int_{M\setminus D}\Big(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\Big)|\nabla u|^{-1}\phi^2\\ =\ &\int_\varepsilon^1\phi^2(t)\int_{\{u=t\}}\Big(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\Big)|\nabla u|^{-2}\,dt\\ =\ &\int_\varepsilon^1\phi^2(t)\int_{l(t)}\Big(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\Big)|\nabla u|^{-2}\,dt\\ &+\int_\varepsilon^1\phi^2(t)\int_{\widetilde{l}(t)}\Big(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\Big)|\nabla u|^{-2}\,dt,\end{aligned}$$

for $l(t)$ and $\widetilde{l}(t)$ defined in (2.12) and (2.17). By the Kato inequality and Ricci curvature bound $\mathrm{Ric}(\nabla u,\nabla u)\geq-\frac{C}{(r+1)^2}|\nabla u|^2$ we have

$$\begin{aligned}&\int_\varepsilon^1\phi^2(t)\int_{\widetilde{l}(t)}\Big(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\Big)|\nabla u|^{-2}\,dt\\ \geq\ &\int_\varepsilon^1\phi^2(t)\int_{\widetilde{l}(t)}\mathrm{Ric}(\nabla u,\nabla u)|\nabla u|^{-2}\,dt\\ \geq\ &-C\int_\varepsilon^1\phi^2(t)\int_{\{u=t\}}\frac{1}{(r+1)^2}dt\\ \geq\ &-C\int_{L(\varepsilon,1)}\frac{1}{(r+1)^2}|\nabla u|,\end{aligned}$$

where the last line follows from the co-area formula. However, Lemma 4.3, (2.8) and (5.8) imply that

$$\begin{aligned}(5.15)\qquad\int_{M\setminus D}\frac{1}{r^2}|\nabla u|\ \leq\ &\int_{M\setminus D}\frac{1}{(r+1)^2}|\nabla f|\\ &+\frac{1}{\ln R}\int_{D(2\ln R)}\frac{1}{(r+1)^3}\\ \leq\ &\int_{M\setminus D}\frac{1}{(r+1)^4}+\int_{M\setminus D}|\nabla f|^2+C\\ \leq\ &C.\end{aligned}$$

In conclusion,

$$\begin{aligned}&\int_\varepsilon^1\phi^2(t)\int_{\widetilde{l}(t)}\Big(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\Big)|\nabla u|^{-2}\,dt\\ \geq\ &-C.\end{aligned}$$

Thus, the following holds

$$\begin{aligned} & \int_{M\setminus D}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}+\operatorname{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-1}\phi^{2} \\ \geq & \int_{\varepsilon}^{1}\phi^{2}\left(t\right)\int_{l(t)}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}+\operatorname{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-2}dt \\ & -C. \end{aligned}$$

Rewrite it into

$$\begin{aligned} (5.16) \quad & \int_{M\setminus D}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}+\operatorname{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-1}\phi^{2} \\ = & \int_{\varepsilon}^{1}\phi^{2}\left(t\right)\int_{l(t)\cap\overline{M_t}}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}+\operatorname{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-2}dt \\ & +\int_{\varepsilon}^{1}\phi^{2}\left(t\right)\int_{l(t)\cap\left(M\setminus\overline{M_t}\right)}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}+\operatorname{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-2}dt. \end{aligned}$$

Recall that from Lemma 4.4,

$$l\left(t\right)\cap\left(M\setminus\overline{M_t}\right)\subset M\setminus D\left(\ln R\right).$$

Since Ricci curvature is bounded below by

$$\operatorname{Ric}\left(\nabla u,\nabla u\right)\geq-\frac{C}{\left(r+1\right)^{2}}\left|\nabla u\right|^{2},$$

together with the Kato inequality $\left|u_{ij}\right|^{2}\geq\left|\nabla\left|\nabla u\right|\right|^{2}$, it follows that

$$\begin{aligned} & \int_{\varepsilon}^{1}\phi^{2}\left(t\right)\int_{l(t)\cap\left(M\setminus\overline{M_t}\right)}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}+\operatorname{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-2}dt \\ \geq & -C\int_{\varepsilon}^{1}\int_{l(t)\cap\left(M\setminus\overline{M_t}\right)}\frac{1}{r^{2}}dt \\ \geq & -C\int_{\varepsilon}^{1}\int_{l(t)\cap\left(M\setminus D(\ln R)\right)}\frac{1}{r^{2}}dt \\ \geq & -C\int_{M\setminus D(\ln R)}\frac{1}{r^{2}}\left|\nabla u\right| \\ \geq & -C, \end{aligned}$$

where the last line follows from (5.15). In conclusion,

$$\int_{\varepsilon}^{1}\phi^{2}\left(t\right)\int_{l(t)\cap\left(M\setminus\overline{M_t}\right)}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}+\operatorname{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-2}dt\geq-C$$

and (5.16) becomes

$$\begin{aligned}(5.17)\qquad &\int_M \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) |\nabla u|^{-1} \phi^2 \\ \geq\quad &\int_\varepsilon^1 \phi^2(t) \int_{l(t)\cap\overline{M_t}} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) |\nabla u|^{-2}\, dt - C.\end{aligned}$$

By Lemma 2.6,

$$\operatorname{Ric}(\nabla u, \nabla u) |\nabla u|^{-2} \geq \frac{1}{2}S - \frac{1}{2}S_t + \frac{1}{2}\frac{1}{|\nabla u|^2}\left(|\nabla |\nabla u||^2 - |u_{ij}|^2\right)$$

on any regular level set $l(t)$. It follows that

$$(5.18)\qquad \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) |\nabla u|^{-2} \geq \frac{1}{2}S - \frac{1}{2}S_t.$$

Note that by the Gauss-Bonnet theorem, one has

$$\int_{l(t)\cap\overline{M_t}} S_t \leq 8\pi m \leq C$$

for all regular value t with $0 < t < 1$ as Lemma 4.4 says that $l(t) \cap \overline{M_t} = \partial M_t$ has m connected components. Therefore, we conclude from (5.18) that

$$\int_{l(t)\cap\overline{M_t}} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) |\nabla u|^{-2} \geq \frac{1}{2}\int_{l(t)\cap\overline{M_t}} S - C.$$

Together with (5.17), this implies that

$$\begin{aligned}(5.19)\qquad &\int_{M\setminus D} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) |\nabla u|^{-1} \phi^2 \\ \geq\quad &\frac{1}{2}\int_\varepsilon^1 \phi^2(t) \int_{l(t)\cap\overline{M_t}} S dt - C.\end{aligned}$$

Recall that by Lemma 4.4,

$$l(t) \cap D(\ln R) \subset \overline{M_t}.$$

According to (5.3), $S > 0$ on $M\setminus D$. Hence,

$$\begin{aligned}\frac{1}{2}\int_\varepsilon^1 \phi^2(t) \int_{l(t)\cap\overline{M_t}} S dt \quad &\geq \quad \frac{1}{2}\int_\varepsilon^1 \phi^2(t) \int_{l(t)\cap D(\ln R)} S dt \\ &\geq \quad \frac{1}{2}\int_{D(\ln R)\cap L(\varepsilon,1)} S |\nabla u| \phi^2.\end{aligned}$$

In conclusion,

$$\begin{aligned}&\int_{M\setminus D} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) |\nabla u|^{-1} \phi^2 \\ \geq\quad &\frac{1}{2}\int_{D(\ln R)\cap L(\varepsilon,1)} S |\nabla u| \phi^2 - C.\end{aligned}$$

Plugging into (5.14) yields

$$\int_{M\setminus D}(\Delta|\nabla u|)\,\phi^2 \geq \frac{1}{2}\int_{D(\ln R)\cap L(\varepsilon,1)} S\,|\nabla u|\,\phi^2 - C. \tag{5.20}$$

To estimate the left hand side, we integrate by parts to get

$$\begin{aligned}\int_{M\setminus D}(\Delta|\nabla u|)\,\phi^2 &= \int_{M\setminus D}|\nabla u|\,\Delta\phi^2 - \int_{\partial D}|\nabla u|_\nu \\ &\leq \frac{C}{\varepsilon}\int_{L(\varepsilon,2\varepsilon)}|\nabla u|\;|\Delta u| + \frac{C}{\varepsilon^2}\int_{L(\varepsilon,2\varepsilon)}|\nabla u|^3 + C.\end{aligned}$$

First, by (5.10), (5.8), and Lemma 4.3 we have

$$\begin{aligned}\frac{C}{\varepsilon}\int_{L(\varepsilon,2\varepsilon)}|\nabla u|\;|\Delta u| &\leq \frac{C}{\ln R}\int_{D(2\ln R)\setminus D}\left(\frac{1}{r^3}+|\nabla f|^2\right) \\ &\leq C.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\frac{C}{\varepsilon^2}\int_{L(\varepsilon,2\varepsilon)}|\nabla u|^3 &\leq C\int_{D(2\ln R)\setminus D}\frac{1}{r^2}\,|\nabla u| \\ &\leq \frac{C}{\ln R}\int_{D(2\ln R)\setminus D}\frac{1}{r^3} + C\int_{D(2\ln R)\setminus D}\frac{1}{r^2}\,|\nabla f| \\ &\leq C + \int_{M\setminus D}|\nabla f|^2 \\ &\leq C,\end{aligned}$$

where we have applied (5.10) in the first line, and used Lemma 4.3 and that $u\,|\nabla \ln f| \leq |\nabla f|$ in the second line. In conclusion,

$$\int_{M\setminus D}(\Delta|\nabla u|)\,\phi^2 \leq C.$$

Plugging this into (5.20) yields

$$\int_{(D(\ln R)\setminus D)\cap L(2\varepsilon,1)} S\,|\nabla u| \leq C$$

for any R such that $\ln R > \frac{1}{\varepsilon^2}$. As in the proof of Theorem 3.1, for arbitrary large r such that $D\subset B_p(r)$, let

$$t\,(r) = \min_{x\in\partial B_p(r)} f\,(x)\,.$$

Then for $\varepsilon < \frac{1}{2}t\,(r)$ and all R sufficiently large,

$$B_p\,(r)\setminus D \subset D\,(\ln R)\cap L\,(2\varepsilon,1)\,.$$

It follows that

$$\int_{B_p(r)\setminus D} S\;|\nabla f| \leq C.$$

Since r is arbitrary, this proves

$$\int_{M\backslash D} S\,|\nabla f| \leq C. \tag{5.21}$$

On the other hand, since $\Delta f = 0$ on $B_p(r)\backslash B_p(R_0)$ for $R_0 < r$ with $D \subset B_p(R_0)$,

$$\int_{\partial B_p(r)} \frac{\partial f}{\partial r} = \int_{\partial B_p(R_0)} \frac{\partial f}{\partial r} = -C_1 < 0.$$

Therefore, using (5.3) we have

$$\begin{aligned}
\int_{B_p(T)\backslash B_p(R_0)} S|\nabla f| &= \int_{R_0}^{T}\int_{\partial B_p(r)} S|\nabla f| dr \\
&\geq \frac{C_0}{2}\int_{R_0}^{T}\frac{1}{r}\left(\int_{\partial B_p(r)}|\nabla f|\right)dr \\
&\geq \frac{C_0}{2}\int_{R_0}^{T}\frac{1}{r}\left|\int_{\partial B_p(r)}\frac{\partial f}{\partial r}\right|dr \\
&= \frac{1}{2}C_0C_1\ln\frac{T}{R_0}.
\end{aligned}$$

Together with (5.21) we have

$$\frac{1}{2}C_0C_1\ln\frac{T}{R_0} \leq C$$

for any $T > R_0$. This contradiction shows that (M,g) is parabolic. $\square$

The proof in fact shows that Theorem 5.1 holds true provided the scalar curvature $S(x) \geq \chi(r(x))$ for some nonnegative function $\chi(r)$ with $\int_0^\infty \chi(r)\,dr = \infty$.

We now turn to the proof of Theorem 1.4. Assume from now on that E is a parabolic end of (M,g) with respect to a smooth connected bounded domain D. A result of Nakai [32] says that the barrier function may be chosen to be proper on E, that is, there exists positive harmonic function u with

$$u = 1 \text{ on } \partial E$$

and

$$\lim_{x\to E(\infty)} u(x) = \infty. \tag{5.22}$$

Denote by

$$\begin{aligned}
L(a,b) &= \{x\in E : a < u(x) < b\} \\
l(t) &= \{x \in E : u(x) = t\}.
\end{aligned}$$

When (M,g) has finitely many ends and finite first Betti number, assume that D is chosen such that all representatives of $H_1(M)$ lie in D and $M\setminus D$ has maximal number of ends. The following result follows as Lemma 2.4, and is essentially contained in [30, 21].

Lemma 5.2. *Let (M,g) be a parabolic complete manifold with finite first Betti number and finite number of ends. Then $l(t)$ is connected for all $t\geq 1$.*

We first establish a localized version of Theorem 1.6 on the end E. For any regular level set $l(t)$ of u, let

$$w(t)=\int_{l(t)}|\nabla u|^{2}. \tag{5.23}$$

For $\alpha\in\mathbb{R}$, define

$$\begin{aligned} H_{\alpha}(t) &= t^{\alpha}\frac{dw}{dt}(t)-(\alpha+3)\,t^{\alpha-1}w(t)+\frac{4\pi}{\alpha+1}t^{\alpha+1} \\ &= t^{\alpha}\int_{l(t)}\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}-(\alpha+3)\,t^{\alpha-1}\int_{l(t)}|\nabla u|^{2} \\ &\quad +\frac{4\pi}{\alpha+1}t^{\alpha+1}, \end{aligned} \tag{5.24}$$

where the last term should be replaced by $4\pi\ln t$ when $\alpha=-1$.

Lemma 5.3. *Let (M,g) be a parabolic complete three-dimensional manifold with finite first Betti number and finite number of ends. Then*

$$H_{\alpha}(T)\geq H_{\alpha}(s)-\left(\alpha^{2}+2\alpha\right)\int_{s}^{T}w(t)\,t^{\alpha-2}dt+\frac{1}{2}\int_{L(s,T)}u^{\alpha}S|\nabla u|$$

for all $1\leq s<T<\infty$.

Proof. For regular values $1\leq s<T<\infty$ we have

$$\begin{aligned} &\left(T^{\alpha}\frac{dw}{dT}(T)-\alpha T^{\alpha-1}w(T)\right)-\left(s^{\alpha}\frac{dw}{ds}(s)-\alpha s^{\alpha-1}w(s)\right) \\ =\;&\left(\int_{l(T)}u^{\alpha}\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}-\int_{l(s)}u^{\alpha}\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}\right) \\ &-\left(\int_{l(T)}|\nabla u|\frac{\langle\nabla u^{\alpha},\nabla u\rangle}{|\nabla u|}-\int_{l(s)}|\nabla u|\frac{\langle\nabla u^{\alpha},\nabla u\rangle}{|\nabla u|}\right) \\ =\;&\int_{L(s,T)}\left(u^{\alpha}\Delta|\nabla u|-|\nabla u|\,\Delta u^{\alpha}\right) \\ =\;&\int_{L(s,T)}\frac{u^{\alpha}}{|\nabla u|}\left(|u_{ij}|^{2}-|\nabla|\nabla u||^{2}+\mathrm{Ric}(\nabla u,\nabla u)\right) \\ &-\alpha(\alpha-1)\int_{L(s,T)}|\nabla u|^{3}u^{\alpha-2}. \end{aligned}$$

Note that the term $\frac{1}{|\nabla u|}|\mathrm{Ric}(\nabla u,\nabla u)|\leq|\mathrm{Ric}|\,|\nabla u|$ is integrable even if u has critical points in $L(s,T)$. The same can be concluded for $\frac{1}{|\nabla u|}\left(|u_{ij}|^{2}-|\nabla|\nabla u||^{2}\right)$ via a classical regularization procedure [28] from the above identity by noticing that it is nonnegative due to the Kato inequality.

Using the co-area formula to rewrite the last term

$$\int_{L(s,T)}|\nabla u|^{3}u^{\alpha-2}=\int_{s}^{T}t^{\alpha-2}w(t)\,dt,$$

we conclude

$$\begin{aligned}(5.25)\qquad & \left(T^{\alpha}\frac{dw}{dT}(T)-\alpha T^{\alpha-1}w(T)\right)-\left(s^{\alpha}\frac{dw}{ds}(s)-\alpha s^{\alpha-1}w(s)\right)\\ = & \int_{L(s,T)}\frac{u^{\alpha}}{|\nabla u|}\left(|u_{ij}|^{2}-|\nabla|\nabla u||^{2}+\mathrm{Ric}(\nabla u,\nabla u)\right)\\ & -\alpha(\alpha-1)\int_{s}^{T}t^{\alpha-2}w(t)\,dt.\end{aligned}$$

On the other hand, by Lemma 2.6,

$$\begin{aligned}(5.26)\qquad & \int_{l(t)}\frac{1}{|\nabla u|^{2}}\left(|u_{ij}|^{2}-|\nabla|\nabla u||^{2}+\mathrm{Ric}(\nabla u,\nabla u)\right)\\ = & \frac{1}{2}\int_{l(t)}\left(S-S_{t}+\frac{1}{|\nabla u|^{2}}|u_{ij}|^{2}\right)\end{aligned}$$

on any regular level set $l(t)$, where S_t is the scalar curvature of $l(t)$. Recall that by Lemma 5.2 $l(t)$ is connected for all $t \geq 1$. So the Gauss-Bonnet theorem and the Kato inequality Lemma 2.7 imply that

$$(5.27)\quad \frac{1}{2}\int_{l(t)}\left(S-S_{t}+\frac{1}{|\nabla u|^{2}}|u_{ij}|^{2}\right)\geq\frac{1}{2}\int_{l(t)}S-4\pi+\frac{3}{4}\int_{l(t)}\frac{1}{|\nabla u|^{2}}|\nabla|\nabla u||^{2}.$$

Observe that

$$|w'(t)|\leq\int_{l(t)}|\nabla|\nabla u||\leq\left(\int_{l(t)}\frac{1}{|\nabla u|^{2}}|\nabla|\nabla u||^{2}\right)^{\frac{1}{2}}\left(\int_{l(t)}|\nabla u|^{2}\right)^{\frac{1}{2}},$$

which implies

$$\int_{l(t)}\frac{1}{|\nabla u|^{2}}|\nabla|\nabla u||^{2}\geq\frac{(w')^{2}(t)}{w(t)}.$$

Together with the elementary inequality

$$\frac{(w')^{2}}{w}(t)\geq\frac{4}{t}w'(t)-\frac{4}{t^{2}}w(t),$$

(5.27) becomes

$$\frac{1}{2}\int_{l(t)}\left(S-S_{t}+\frac{1}{|\nabla u|^{2}}|u_{ij}|^{2}\right)\geq\frac{1}{2}\int_{l(t)}S-4\pi+\frac{3}{t}w'(t)-\frac{3}{t^{2}}w(t).$$

Hence, by (5.26) we get

$$\begin{aligned}& \int_{l(t)}\frac{1}{|\nabla u|^{2}}\left(|u_{ij}|^{2}-|\nabla|\nabla u||^{2}+\mathrm{Ric}(\nabla u,\nabla u)\right)\\ \geq & \frac{1}{2}\int_{l(t)}S-4\pi+\frac{3}{t}w'(t)-\frac{3}{t^{2}}w(t)\end{aligned}$$

for any regular level set $l(t)$. We now use it and the co-area formula to conclude

$$\begin{aligned}&\int_{L(s,T)}\frac{u^{\alpha}}{|\nabla u|}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\right)\\ \geq\;&\frac{1}{2}\int_{L(s,T)}u^{\alpha}S|\nabla u|+\int_s^T\left(-4\pi+\frac{3}{t}w'(t)-\frac{3}{t^2}w(t)\right)t^{\alpha}dt.\end{aligned}$$

Integrating by parts and noting that w is Lipschitz, we obtain

$$\begin{aligned}&\int_{L(s,T)}\frac{u^{\alpha}}{|\nabla u|}\left(|u_{ij}|^2-|\nabla|\nabla u||^2+\mathrm{Ric}(\nabla u,\nabla u)\right)\\ \geq\;&\frac{1}{2}\int_{L(s,T)}u^{\alpha}S|\nabla u|-\frac{4\pi}{\alpha+1}\left(T^{\alpha+1}-s^{\alpha+1}\right)\\ +\;&3T^{\alpha-1}w(T)-3s^{\alpha-1}w(s)-3\alpha\int_s^T t^{\alpha-2}w(t)\,dt.\end{aligned}$$

Plugging this into (5.25) yields

$$\begin{aligned}&\left(T^{\alpha}\frac{dw}{dT}(T)-\alpha T^{\alpha-1}w(T)\right)-\left(s^{\alpha}\frac{dw}{ds}(s)-\alpha s^{\alpha-1}w(s)\right)\\ \geq\;&\frac{1}{2}\int_{L(s,T)}u^{\alpha}S|\nabla u|-\frac{4\pi}{\alpha+1}\left(T^{\alpha+1}-s^{\alpha+1}\right)+3T^{\alpha-1}w(T)-3s^{\alpha-1}w(s)\\ &-\alpha(\alpha+2)\int_s^T t^{\alpha-2}w(t)\,dt\end{aligned}$$

for any $1\leq s<T<\infty$. This proves the result. □

We use the monotonicity in Lemma 5.3 to prove the following integral estimate.

Corollary 5.4. *Let (M,g) be a parabolic complete three-dimensional manifold with finite first Betti number and finite number of ends. Assume the barrier u defined on the end E has bounded gradient,*

$$\sup_E|\nabla u|<\infty.$$

Then there exists a constant $\Upsilon>0$ such that

$$\int_{L(1,t)}S|\nabla u|\leq 8\pi t+\Upsilon,$$

for all $t>1$.

Proof. Applying Lemma 5.3 with $\alpha=0$ to u we get

$$\frac{1}{2}\int_{L(1,t)}S|\nabla u|\leq\frac{dw}{dt}-3\frac{w(t)}{t}+4\pi t+\Upsilon_0\tag{5.28}$$

for all $t>1$, where

$$\Upsilon_0=-\left(\frac{dw}{dt}|_{t=1}-3w(1)+4\pi\right).$$

Since u is harmonic, we know that $\int_{l(t)}|\nabla u|$ is constant in t. Together with the assumption that u has bounded gradient, we obtain

$$\begin{aligned} w(t) &= \int_{l(t)} |\nabla u|^2 \\ &\leq \Upsilon_1 \int_{l(t)} |\nabla u| \\ &= \Upsilon_2. \end{aligned}$$

So, for every $t > 1$ there exists $t_i \in (t, t+1)$

$$\frac{dw}{dt}(t_i) \leq \Upsilon_2.$$

Then (5.28) implies

$$\begin{aligned} \int_{L(1,t)} S|\nabla u| &\leq \int_{L(1,t_i)} S|\nabla u| \\ &\leq 8\pi t + \Upsilon, \end{aligned} \tag{5.29}$$

where

$$\Upsilon = 2\Upsilon_0 + 2\Upsilon_2 + 8\pi.$$

□

Recall that, as u is harmonic, $\int_{l(t)} |\nabla u|$ is constant in t. Let us normalize u from here on such that

$$\int_{l(t)} |\nabla u| = 1 \tag{5.30}$$

for all $t \geq 1$.

Lemma 5.5. *Let (M, g) be a parabolic complete three-dimensional Riemannian manifold. Assume that the Ricci curvature of M is bounded from below by*

$$\mathrm{Ric} \geq -k(r(x))$$

for a continuous nonincreasing function $k(r)$ satisfying $\int_0^\infty rk(r)\,dr < \infty$. Normalize the barrier u as in (5.30). Then

$$M(R) = \sup_{B_p(2R)\setminus B_p(R)} |\nabla u|$$

satisfies

$$M^2(R) \leq CR\left(\inf_{x\in B_p(2R)\setminus B_p(R)} \mathrm{V}_x\left(\frac{R}{2}\right)\right)^{-1}\left(M\left(\frac{R}{2}\right) + M(R) + M(2R)\right)$$

for a constant C depending only on k and all $R > 2R_0$ large enough.

Proof. We follow an argument in [33]. Pick $x \in \partial B_p(t)$, where $R \leq t \leq 2R$, such that

$$M(R) = |\nabla u|(x).$$

On $B_x\left(\frac{R}{2}\right)$ we have

$$\mathrm{Ric} \geq -\frac{C}{R^2}$$

and

$$\Delta|\nabla u| \geq -\frac{C}{R^2}|\nabla u|.$$

Applying the mean value inequality from [22] we get

$$|\nabla u|^2 (x) \leq \frac{C}{\mathrm{V}_x\left(\frac{R}{2}\right)} \int_{B_x\left(\frac{R}{2}\right)} |\nabla u|^2 .$$

This shows that

$$M^2(R) \leq \frac{C}{\mathrm{V}_x\left(\frac{R}{2}\right)} \int_{L(\alpha_x,\beta_x)} |\nabla u|^2 , \tag{5.31}$$

where

$$\alpha_x = \min_{B_x\left(\frac{R}{2}\right)} u \text{ and } \beta_x = \max_{B_x\left(\frac{R}{2}\right)} u.$$

By the co-area formula and (5.30) we have

$$\begin{aligned} \int_{L(\alpha_x,\beta_x)} |\nabla u|^2 &= \int_{\alpha_x}^{\beta_x} \int_{l(t)} |\nabla u|\, dt \\ &= \beta_x - \alpha_x \\ &\leq R \sup_{B_x\left(\frac{R}{2}\right)} |\nabla u| . \end{aligned}$$

Since $x \in B_p(2R) \backslash B_p(R)$, it follows that

$$\int_{L(\alpha_x,\beta_x)} |\nabla u|^2 \leq R\left(M\left(\frac{R}{2}\right) + M(R) + M(2R)\right).$$

Together with (5.31) we conclude that

$$M^2(R) \leq \frac{CR}{\mathrm{V}_x\left(\frac{R}{2}\right)} \left(M\left(\frac{R}{2}\right) + M(R) + M(2R)\right)$$

for some $x \in B_p(2R) \backslash B_p(R)$ and all $R > 2R_0$. This proves the result. $\square$

We are now ready to prove Theorem 1.4. A manifold (M, g) is said to satisfy the volume doubling property if there exists a constant $\nu > 0$ such that

$$\frac{\mathrm{V}_x(2r)}{\mathrm{V}_x(r)} \leq \nu \tag{5.32}$$

for any $x \in M$ and any $r > 0$.

Theorem 5.6. *Let (M, g) be a three-dimensional complete noncompact Riemannian manifold with finitely many ends and finite first Betti number $b_1(M) < \infty$. Assume that the Ricci curvature of M is bounded from below by*

$$\mathrm{Ric}(x) \geq -k(r(x))$$

for a continuous nonincreasing function $k(r)$ satisfying $\int_0^\infty rk(r)\, dr < \infty$. Then

(a) there exists a constant $C_0 > 0$, depending only on k, such that

$$\liminf_{x\to\infty} S(x) \leq \frac{C_0}{\mathrm{V}_p(1)},$$

where $\mathrm{V}_p(1)$ is the volume of the geodesic ball $B_p(1)$.

(b) there exists a constant $\bar{C}_0 > 0$, depending only on k and ν, such that

$$\liminf_{x\to\infty} \left(S(x)\, r^{\alpha}(x)\right) \leq \bar{C}_0 \limsup_{R\to\infty} \frac{R^{\alpha+1}}{\mathrm{V}_p(R)}$$

for any $0 \leq \alpha \leq 1$ provided that (M, g) satisfies the volume doubling property (5.32).

Proof. We first prove (a). Let us assume by contradiction that

$$\liminf_{x\to\infty} S(x) > \frac{\Gamma}{\mathrm{V}_p(1)}, \tag{5.33}$$

where $\Gamma > 0$ is a large enough constant to be specified later. In particular,

$$S \geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p(1)} \quad \text{on } M\backslash B_p(R_0) \tag{5.34}$$

for some large enough R_0. Below, C denotes a constant that depends only on k.

By Theorem 5.1 we know that M is parabolic. By localizing a barrier function to an end as above, we may assume without loss of generality that M has only one end. According to Nakai [32], there exists a proper harmonic function $u : M\backslash D \to (0, \infty)$ with $u = 1$ on ∂D. We normalize u as in (5.30).

Since M has asymptotically nonnegative Ricci curvature, a volume comparison result in [21] implies

$$\mathrm{V}_x\left(\frac{R}{2}\right) \geq \frac{1}{C} R\mathrm{V}_p(1)$$

for $x \in M$ with $r(x) = R$. Choosing $R_0 > 0$ sufficiently large enough with $D \subset B_p(R_0)$ and applying Lemma 5.5 we get

$$M^2(R) \leq \frac{C_1}{\mathrm{V}_p(1)}\left(M\left(\frac{R}{2}\right) + M(R) + M(2R)\right) \tag{5.35}$$

for all $R > 2R_0$. On the other hand, by Lemma 4.2,

$$|\nabla \ln u| \leq \frac{C}{r} \quad \text{on } M\backslash B_p(2R_0).$$

It follows that

$$u \leq r^C \sup_{\partial B_p(2R_0)} u \quad \text{on } M\backslash B_p(2R_0)$$

and

$$|\nabla u| \leq r^C \sup_{\partial B_p(2R_0)} u \;\; \text{on } M\backslash B_p(2R_0).$$

In particular, there exist constant C_2 with

$$C_2 > 2C_1$$

and $R_1 > 2R_0$ sufficiently large such that

$$M(R) \leq \frac{1}{\mathrm{V}_p(1)} R^{\frac{C_2}{2}} \tag{5.36}$$

for all $R \geq R_1$.

We now claim that for $m \geq 1$,

$$M(R) \leq \frac{2^{C_2}}{\mathrm{V}_p(1)} R^{\frac{C_2}{2^m}} \tag{5.37}$$

for any $R \geq 2^m R_1$.

By (5.36) we see that (5.37) holds for $m = 1$. We assume by induction that it holds for m and prove it for $m+1$. In other words, we aim to prove that

$$M(R) \leq \frac{2^{C_2}}{\mathrm{V}_p(1)} R^{\frac{C_2}{2^{m+1}}} \tag{5.38}$$

for $R \geq 2^{m+1} R_1$. By the induction hypothesis (5.37) we get that

$$\begin{aligned} M\left(\frac{R}{2}\right) &\leq \frac{2^{C_2}}{\mathrm{V}_p(1)} \left(\frac{R}{2}\right)^{\frac{C_2}{2^m}}, \\ M(R) &\leq \frac{2^{C_2}}{\mathrm{V}_p(1)} R^{\frac{C_2}{2^m}}, \\ M(2R) &\leq \frac{2^{C_2}}{\mathrm{V}_p(1)} (2R)^{\frac{C_2}{2^m}}. \end{aligned}$$

Using (5.35) we therefore obtain

$$\begin{aligned} M^2(R) &\leq \frac{2^{C_2} C_1}{(\mathrm{V}_p(1))^2} \left(\left(\frac{1}{2}\right)^{\frac{C_2}{2^m}} + 1 + 2^{\frac{C_2}{2^m}} \right) R^{\frac{C_2}{2^m}} \\ &\leq \frac{2^{2C_2}}{(\mathrm{V}_p(1))^2} R^{\frac{C_2}{2^m}}, \end{aligned}$$

where the second line follows from $C_2 > 2C_1$. In conclusion,

$$M(R) \leq \frac{2^{C_2}}{\mathrm{V}_p(1)} R^{\frac{C_2}{2^{m+1}}}$$

and (5.38) is proved. Hence, (5.37) holds for any $m \geq 1$ and $R \geq 2^m R_1$. By taking $2^m = [\ln R]$, this readily implies that

$$M(R) \leq \frac{C}{\mathrm{V}_p(1)}$$

for all $R \geq R_1^2$. In conclusion,

$$\sup_{M \setminus B_p(R_2)} |\nabla u| \leq \frac{C}{\mathrm{V}_p(1)} \tag{5.39}$$

for sufficiently large $R_2 > R_1$. In particular,

$$u(x) \leq \frac{C}{\mathrm{V}_p(1)} r(x) \quad \text{on } M \setminus B_p(R_2), \tag{5.40}$$

by assuming R_2 is large enough. By (5.40) we conclude that

$$B_p(R) \setminus B_p(R_2) \subset L\left(1, \frac{C}{\mathrm{V}_p(1)} R\right).$$

Since u has bounded gradient by (5.39), according to Corollary 5.4 we have

$$\begin{aligned} \int_{B_p(R)\setminus B_p(R_2)} S\,|\nabla u| &\leq \int_{L\left(1,\frac{C}{\mathrm{V}_p(1)}R\right)} S\,|\nabla u| \\ &\leq \frac{C}{\mathrm{V}_p\,(1)}R+\Upsilon \end{aligned} \tag{5.41}$$

for all $R>R_2$, where Υ is a constant.

Now by (5.34),

$$S\geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p\,(1)}\quad \text{on } B_p\,(R)\setminus B_p\,(R_2)$$

and by (5.30),

$$\int_{\partial B_p(r)} \frac{\partial u}{\partial r}=1 \tag{5.42}$$

for all $r>R_3$. The co-area formula then implies

$$\begin{aligned} \int_{B_p(R)\setminus B_p(R_2)} S\,|\nabla u| &\geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p\,(1)}\int_{B_p(R)\setminus B_p(R_2)}|\nabla u| \\ &\geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p\,(1)}\int_{R_2}^{R}\left(\int_{\partial B_p(r)}\frac{\partial u}{\partial r}\right)dr \\ &\geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p\,(1)}\,(R-R_2)\,. \end{aligned}$$

From (5.41) we get

$$\frac{\Gamma-C}{\mathrm{V}_p\,(1)}R\leq \Upsilon_1,$$

for some constant $\Upsilon_1>0$. Since $R>R_3$ is arbitrary, this forces $\Gamma\leq C$ and part (a) of the theorem is proved.

We now prove part (b). Define

$$A=\liminf_{x\to\infty}\left(S\,(x)\,r^{\alpha}\,(x)\right) \tag{5.43}$$

if the right hand side is finite. Otherwise, take $A>0$ to be arbitrary. Since there is nothing to prove if $A=0$, we assume $A>0$. In particular, by Theorem 5.1, (M,g) is parabolic. As before, we may assume that M has one end and that $u:M\setminus D\to(0,\infty)$ with $u=1$ on ∂D is a proper barrier function satisfying (5.22).

We normalize u to satisfy (5.30) and assume by contradiction that

$$A>\Gamma\limsup_{R\to\infty}\frac{R^{\alpha+1}}{\mathrm{V}_p\,(R)}, \tag{5.44}$$

for some large constant Γ to be specified later.

Recall Lemma 5.5 that

$$M^2\,(R)\leq \frac{C_1R}{\mathrm{V}_x\left(\frac{R}{2}\right)}\left(M\left(\frac{R}{2}\right)+M\,(R)+M\,(2R)\right) \tag{5.45}$$

for some $x\in B_p\,(2R)\setminus B_p\,(R)$ and all $R>2R_0$ with the constant C_1 depending only on k.

Denote

$$\delta=\frac{2C_1 A\nu^3}{\Gamma}. \tag{5.46}$$

Note that (5.44) implies

$$\frac{R^{\alpha+1}}{\mathrm{V}_p\left(R\right)}\leq\frac{\delta}{C_1\nu^3}$$

for all $R>2R_0$ large enough. By (5.32) we have

$$\frac{\mathrm{V}_p\left(R\right)}{\mathrm{V}_x\left(\frac{R}{2}\right)}\leq\frac{\mathrm{V}_x\left(3R\right)}{\mathrm{V}_x\left(\frac{R}{2}\right)}\leq\nu^3.$$

Therefore,

$$\frac{R^{\alpha+1}}{\mathrm{V}_x\left(\frac{R}{2}\right)}\leq\frac{\delta}{C_1}$$

for all $R>2R_0$ and (5.45) becomes

$$M^2\left(R\right)\leq\frac{\delta}{R^\alpha}\left(M\left(\frac{R}{2}\right)+M\left(R\right)+M\left(2R\right)\right). \tag{5.47}$$

By Lemma 4.2, there exists a constant $C_2>0$ depending only on k so that

$$M\left(R\right)\leq\delta R^{\frac{C_2}{2}-\alpha}$$

for all $R\geq R_1$ large enough. A similar induction argument as above then shows that

$$M\left(R\right)\leq 2^{C_2}\delta R^{\frac{C_2}{2^m}-\alpha}$$

for all $R\geq 2^m R_1$. Assuming $R\geq R_2$ is even larger if necessary, we may take $m=[\ln R]$ and conclude that

$$M\left(R\right)\leq\frac{C\delta}{R^\alpha}.$$

In other words,

$$\left|\nabla u\right|\leq\frac{C\delta}{r^\alpha}\quad\text{on } M\backslash B_p\left(R_2\right), \tag{5.48}$$

where C depends only on k. For $0\leq\alpha<1$, integrating (5.48) along minimal geodesics we obtain

$$u\leq\frac{C\delta}{1-\alpha}r^{1-\alpha}\quad\text{on } M\backslash B_p\left(R_3\right)$$

for some R_3 sufficiently large, which proves that

$$\begin{aligned}B_p\left(R\right)\backslash B_p\left(R_3\right) &\subset L\left(1,\frac{C\delta}{1-\alpha}R^{1-\alpha}\right)\\ &= L\left(1,\frac{CA}{\left(1-\alpha\right)\Gamma}R^{1-\alpha}\right),\end{aligned}$$

where in the last line we have used (5.46). The gradient of u is bounded, therefore, by Corollary 5.4

$$\begin{aligned} \int_{B_p(R)\setminus B_p(R_3)} S|\nabla u| &\leq \int_{L\left(1,\frac{CA}{(1-\alpha)\Gamma}\right)} S\,|\nabla u| \\ &\leq \frac{CA}{(1-\alpha)\,\Gamma} R^{1-\alpha} + \Upsilon, \end{aligned} \tag{5.49}$$

for some constant Υ. However, using (5.43) that

$$S \geq \frac{A}{2} r^{-\alpha} \quad \text{on } M\setminus B_p\left(R_3\right)$$

and (5.42) we obtain from the co-area formula that

$$\begin{aligned} \int_{B_p(R)\setminus B_p(R_3)} S|\nabla u| &\geq \int_{R_3}^{R} \left(\frac{A}{2} r^{-\alpha}\right)\left(\int_{\partial B_p(r)} \frac{\partial u}{\partial r}\right) dr \\ &= \frac{A}{2\,(1-\alpha)}\left(R^{1-\alpha} - R_3^{1-\alpha}\right). \end{aligned} \tag{5.50}$$

In conclusion, from (5.49) and (5.50),

$$\frac{CA}{(1-\alpha)\,\Gamma} R^{1-\alpha} + \Upsilon \geq \frac{A}{2\,(1-\alpha)}\left(R^{1-\alpha} - R_3^{1-\alpha}\right)$$

for all $R > R_3$ with the constant $C > 0$ depending only on k and ν. This shows that $\Gamma \leq C$ when $0 \leq \alpha < 1$. A similar argument also works for $\alpha = 1$. This proves the result. □

The volume doubling property (5.32) obviously holds for a manifold with non-negative Ricci curvature. So the above theorem implies that

$$\mathrm{V}_p\left(R_i\right) \leq C\, R_i^{\alpha+1}$$

for a sequence $R_i \to \infty$ if the scalar curvature

$$S\left(x\right) \geq \frac{C}{r^{\alpha}\left(x\right) + 1}$$

for some $\alpha \in [0,1]$. In the special case that $\alpha = 0$ or $S \geq C > 0$, a standard volume comparison argument then implies $\mathrm{V}_p(R) \leq c\,R$. This fact also follows from

$$\liminf_{x\to\infty} S\left(x\right) < \frac{C_0}{\mathrm{V}_p\left(1\right)}$$

by a scaling argument.

Finally, we point out that Lemma 5.3 also holds for positive harmonic functions converging to zero at infinity along an end and apply it to rule out the existence of proper positive Green's functions under a suitable scalar curvature assumption. One may wish to compare the following result with Theorem 1.3.

Theorem 5.7. *For complete three-dimensional manifold M with Ricci curvature bounded from below by a constant and scalar curvature satisfying*

$$\liminf_{x\to\infty} \left(S\left(x\right) r\left(x\right)\right) \geq C_0, \tag{5.51}$$

if it has finitely many ends and finite first Betti number, then it does not admit any positive Green's function G satisfying

$$\lim_{x\to\infty} G(x) = 0.$$

Proof. The proof is by contradiction and uses an analogous monotonicity formula as Lemma 5.3 for

$$w(t) = \int_{l(t)} |\nabla G|^2 .$$

In particular,

$$\begin{aligned} & \frac{dw}{dt}(t) - 3\frac{w(t)}{t} + 4\pi t \\ \geq \quad & \frac{dw}{ds}(s) - 3\frac{w(s)}{s} + 4\pi s + \frac{1}{2}\int_{L(s,t)} S|\nabla G| \end{aligned}$$

for all $0 < s < t \leq t_0$, where $t_0 > 0$ is a fixed small number. Let $s \to 0$. Lemma 2.2 implies that

$$\begin{aligned} \frac{w(s)}{s} & \leq C \\ \liminf_{s\to 0} \frac{dw}{ds}(s) & \leq C. \end{aligned}$$

Hence, we conclude that

$$\int_M S|\nabla G| \leq C.$$

As in Theorem 5.1, this leads to a contradiction by (5.51). □

References

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