AREA AND SPECTRUM ESTIMATES FOR STABLE MINIMAL SURFACES

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Abstract. This paper mainly concerns the area growth and bottom spectrum of complete stable minimal surfaces in a three-dimensional manifold with scalar curvature bounded from below. When the ambient manifold is the Euclidean space, by an elementary argument, it is shown directly from the stability inequality that the area of such minimal surfaces grows exactly as the Euclidean plane. Consequently, such minimal surfaces must be flat, a well-known result due to Fisher-Colbrie and Schoen as well as do Carmo and Peng. In the case of general ambient manifold, explicit area growth estimate is also derived. For the bottom spectrum, a self-contained argument involving positive Green's function is provided for its upper bound estimates. The argument extends to stable minimal hypersurfaces in a complete manifold of dimension up to six with sectional curvature bounded from below.

1. Introduction

This paper mainly concerns stable minimal surfaces in three-dimensional manifolds. Our goal is to derive geometric information of such surfaces under scalar curvature assumption on the ambient manifold. Recall that a minimal hypersurface Σ of a complete manifold M is said to be stable if it minimizes area up to the second order with respect to compactly supported variations. This is equivalent to the validity of the following stability inequality.

$$\int_\Sigma \left(|h|^2 + \mathrm{Ric}\,(\nu,\nu)\right)\phi^2 \leq \int_\Sigma |\nabla \phi|^2 \tag{1.1}$$

for any $\phi \in C_0^\infty(\Sigma)$, where h is the second fundamental form of Σ and $\mathrm{Ric}\,(\nu,\nu)$ the Ricci curvature of M evaluated at the unit normal vector ν to Σ. As is well-known, the stability inequality is equivalent to the existence of a positive solution u to the following equation.

$$\Delta u + \left(|h|^2 + \mathrm{Ric}\,(\nu,\nu)\right) u = 0$$

on Σ. Since the lift of u to the universal cover $\widetilde{\Sigma}$ of Σ remains a solution to the above equation, this shows that $\widetilde{\Sigma}$ is a stable minimal surface in the universal cover $\widetilde{M}$ of M as well. Also, observe that such u is positive superharmonic when M has nonnegative Ricci curvature. In particular, if Σ is parabolic, then function u must be a constant. Consequently, Σ is totally geodesic.

Historically, when the ambient three-dimensional manifold M has nonnegative scalar curvature, it was shown by Schoen and Yau in their pioneering work [25] that

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a compact stable minimal surface Σ must be of genus 0. Subsequently, it was proven by Fisher-Colbrie and Schoen [13] that a simply connected complete stable minimal surface Σ must be conformal to the Euclidean plane, hence parabolic. As pointed out above, this enabled them to conclude that such Σ is necessarily totally geodesic if the Ricci curvature of the ambient manifold M is nonnegative. In particular, they obtained the following theorem, which was also proved independently by do Carmo and Peng [11].

Theorem 1 (Fisher-Colbrie and Schoen, do Carmo and Peng)**.** *A complete stable minimal surface Σ in $\mathbb{R}^3$ must be flat.*

Later, an alternative proof of the above theorem was produced by Pogorelov [23] and Colding-Minicozzi [9] by establishing the following area estimate

$$A(r) \leq \frac{4}{3}\pi r^2$$

for all $r > 0$. Here, $A(r)$ denotes the area of the geodesic ball of radius r in Σ. It then follows [5] that Σ must be parabolic, hence totally geodesic and flat. In passing, we mention a recent exciting result by Chodosh and Li [7] that a three-dimensional complete stable minimal hypersurface in $\mathbb{R}^4$ must be flat as well.

One of our goals here is to improve the above area estimate to the sharp form of $A(r) \leq \pi r^2$ for all $r > 0$, thus bypassing the parabolicity consideration and leading directly to Theorem 1. Indeed, since the sectional curvature K_Σ of Σ is nonpositive and Σ can be assumed to be simply connected, one sees immediately from the area bound that Σ must be flat. In fact, our area estimate can be localized to stable minimal surfaces with boundary. In the following, we use $B_p(r)$ to denote the geodesic ball in Σ of radius r centered at point $p \in \Sigma$. Its area and the length of geodesic circle $\partial B_p(r)$ are denoted by $A(r)$ and $L(r)$, respectively.

Theorem 2. *Let Σ be a stable minimal surface in $\mathbb{R}^3$. Then there exists a universal constant R_0 such that for any geodesic ball $B_p(R)$ with no intersection with the boundary of Σ,*

$$L(r) \leq 2\pi r \left(1 + \frac{10}{\ln R}\right)$$

and

$$A(r) \leq \pi r^2 \left(1 + \frac{10}{\ln R}\right)$$

for all $r \leq \sqrt{R}$ and $R \geq R_0$. In particular, if Σ is complete, then $A(r) \leq \pi r^2$ for all $r > 0$. Consequently, Σ is flat.

More generally, we also obtain area estimate for Σ in terms of the lower bound of the scalar curvature S of the ambient manifold M.

Theorem 3. *Let $B_p(R)$ be a geodesic ball in a stable minimal surface Σ in a three dimensional manifold M. Assume that $B_p(R)$ does not intersect the boundary of Σ.*

- *If the scalar curvature S of M satisfies $S \geq -6$, then*

$$A(R) \leq C_1 e^{2R}$$

for some absolute constant $C_1 > 0$.

- *If the sectional curvature K of M satisfies $K \geq -1$, then*

$$A\,(R) \leq C_1\, e^{\frac{4}{\sqrt{7}}\,R}$$

for some absolute constant $C_1 > 0$.

Corollary 4. *Let Σ be a complete stable minimal surface in a three dimensional manifold M. Then there exists an absolute constant $C_1 > 0$ such that for all $R > 0$,*

$$A\,(R) \leq C_1\, e^{\beta\, R},$$

where $\beta = 2$ if the scalar curvature of M satisfies $S \geq -6$ and $\beta = \frac{4}{\sqrt{7}}$ if the sectional curvature of M satisfies $K \geq -1$.

It is unclear to us whether this estimate is sharp. It should be mentioned that, unlike the Euclidean case, there are infinitely many non-totally geodesic stable minimal surfaces in the hyperbolic space $\mathbb{H}^3$. Rotationally symmetric examples were constructed in [20] and [10].

Let us now briefly indicate the main ideas behind the proofs. Observe that for a geodesic ball $B_p(R)$ of a general surface N, if it has no intersection with either the boundary of N or the cut locus of p, then the Gauss-Bonnet formula gives

$$\begin{aligned}
\frac{d}{dr}L(r) &= \frac{d}{dr}\left(\int_{\partial B_p(r)} ds\right) \\
&= \int_{\partial B_p(r)} (\Delta r)\, ds \\
&= \int_{\partial B_p(r)} k_g ds \\
&= 2\pi\, \chi\,(B_p\,(r)) - \int_{B_p(r)} K_N dA \\
&\leq 2\pi - \int_{B_p(r)} K_N dA,
\end{aligned}$$

where k_g is the geodesic curvature of $\partial B_p\,(r)$ and K_N the sectional curvature of N. Note that we have used the fact that the Euler characteristic number $\chi\,(B_p\,(r)) \leq 1$.

In particular, for a complete stable minimal surface Σ in $\mathbb{R}^3$, since $K_\Sigma \leq 0$, by working with the universal cover of Σ if necessary, the inequality

$$\frac{d}{dr}L(r) \leq 2\pi - \int_{B_p(r)} K_\Sigma \tag{1.2}$$

is valid for all r. Combining it with the stability inequality (1.1) one concludes that

$$3\int_\Sigma (\phi')^2 + 4\int_\Sigma \phi\phi'' \leq 4\pi\phi^2\,(0) \tag{1.3}$$

holds for any compactly supported C^2 nonincreasing function $\phi = \phi\,(r)$ on $[0, \infty)$.

So far, we have followed the argument in [9, 8] closely. In fact, similar argument had been adopted earlier in [14], where it was shown that the area must be of quadratic growth for a finite index minimal surface in a three dimensional complete manifold with real analytic metric and nonnegative scalar curvature.

By choosing a linear cut-off function ϕ in (1.3), one concludes immediately that $A(r) \leq \frac{4}{3}\pi r^2$. For the desired conclusion that $A(r) \leq \pi r^2$, a different choice of ϕ is needed.

In a similar fashion, for the proof of Theorem 3, we only use the inequality (1.2) and the stability inequality (1.1). Of course, the preceding derivation shows that (1.2) holds for balls with no boundary points of Σ or the cut locus of p. It turns out that it remains true for any ball $B_p(r)$ in an arbitrary complete surface, possibly containing cut locus of p. This highly nontrivial result is due to Fiala [12] when the surface is the Euclidean plane endowed with a real analytic metric. Fiala's result was later extended by Hartman [15] to smooth metrics. In its full generality, the result is established by Shiohama and Tanaka [27, 28].

Our second goal concerns upper bound of the bottom spectrum of the Laplacian on stable minimal hypersurfaces. The bottom spectrum of the Laplacian on a complete manifold N, denoted by $\lambda_0(N)$, is an important geometric invariant and characterized as the optimal Poincaré inequality constant

$$\lambda_0\left(N\right)\int_N \phi^2 \leq \int_N \left|\nabla\phi\right|^2 \tag{1.4}$$

for all compactly supported smooth function ϕ.

According to [18], for any $p \in N$,

$$\lambda_0\left(N\right) \leq \frac{1}{4}\left(\liminf_{R\to\infty}\ \frac{\ln V_p(R)}{R}\right)^2, \tag{1.5}$$

where $V_p(R)$ denotes the volume of the geodesic ball $B_p(R)$ centered at point p of radius R. For a stable minimal surface Σ in three-dimensional complete manifolds M, applying Corollary 4 on the area estimate, one immediately obtains from (1.5) an upper bound estimate for $\lambda_0\left(\Sigma\right)$, a result previously proven by Bérard, Castillon, and Cavalcante (see Theorem 5.1 in [2]) by a different approach.

Theorem 5 (Bérard, Castillon, and Cavalcante)**.** *Let Σ be a complete stable minimal surface in a three dimensional manifold M.*

- *If the scalar curvature of M satisfies $S \geq -6$, then*
$$\lambda_0\left(\Sigma\right) \leq 1.$$
- *If the sectional curvature of M satisfies $K \geq -1$, then*
$$\lambda_0\left(\Sigma\right) \leq \frac{4}{7}.$$

Previously, in [3], it was shown that for complete stable minimal surface Σ in $\mathbb{H}^3$, its bottom spectrum is at most $\frac{4}{3}$. It would be interesting to see if the improved upper bound of $\frac{4}{7}$ is sharp.

The proof in [2] and the one indicated above through the area estimates both rely on Fiala's inequality (1.2). To get around this rather difficult and dimension specific result, we provide yet another approach to the theorem. The argument instead follows the idea in [17, 21, 22] and involves the minimal positive Green's function G. Indeed, we take $\phi = \psi\left|\nabla G\right|^{1/2}$ as a test function in the Poincaré inequality (1.4), where ψ is a suitably chosen cut-off function. The Bochner formula for $|\nabla G|$ is then used to estimate the relevant terms, with the Ricci curvature term arising from the formula controlled by the stability inequality (1.1). This approach seems to be more direct and self-contained. Moreover, it generalizes to stable minimal hypersurfaces of dimension up to five.

Theorem 6. *Let Σ be a complete stable minimal hypersurface in $(n+1)$-dimensional manifold M with $n \leq 5$. If the sectional curvature of M satisfies $K \geq -\kappa$ for some nonnegative constant κ, then*

$$\lambda_0(\Sigma) \leq \frac{2n(n-1)^2}{6n-n^2-1}\,\kappa.$$

Presently, it is unclear to us how to derive an upper bound for $\lambda_0(\Sigma)$ when $n \geq 6$. It is also worth mentioning that for any complete m-dimensional minimal submanifold Σ in $\mathbb{H}^n$, according to [6], its bottom spectrum satisfies

$$\lambda_0(\Sigma) \geq \frac{(m-1)^2}{4}.$$

The paper is arranged as follows. Section 2 is devoted to the proofs of Theorem 2 and Theorem 3. The estimates for the bottom spectrum Theorem 5 and Theorem 6 are proved in Section 3.

We thank Marcos P. Cavalcante for his interest and for bringing the paper [2] to our attention. We would like to dedicate this work to Professor Peter Li on the occasion of his seventieth birthday. All of us have benefited enormously from his teaching and support over the years.

2. Area estimates

In this section, we prove both Theorem 2 and Theorem 3. We continue to assume that Σ is a stable minimal surface in three-dimensional manifold M. Recall the stability inequality.

$$\int_\Sigma \left(|h|^2 + \mathrm{Ric}(\nu,\nu)\right)\phi^2 \leq \int_\Sigma |\nabla\phi|^2 \tag{2.1}$$

for any $\phi \in C_0^\infty(\Sigma)$, where h is the second fundamental form of Σ and $\mathrm{Ric}(\nu,\nu)$ the Ricci curvature of M in the direction of the unit normal ν to Σ.

Fix $p \in \Sigma$. Let

$$r(x) = d_\Sigma(p,x)$$

be the intrinsic distance on Σ and

$$B_p(R) = \{x \in \Sigma : r(x) < R\}$$

the intrinsic geodesic ball of radius R in Σ. Denote with

$$\begin{aligned} L(r) &= \int_{\partial B_p(r)} ds \\ A(r) &= \int_{B_p(r)} dA \end{aligned}$$

the length of the geodesic circle $\partial B_p(r)$ and the area of $B_p(r)$, respectively.

Everywhere S, Ric and K denote the scalar, Ricci and sectional curvatures of M, respectively, while K_Σ denotes the Gauss curvature of Σ. For minimal surface Σ in M, the Gauss curvature equation gives

$$K_\Sigma = R_{1212} - \frac{1}{2}\left|h\right|^2, \tag{2.2}$$

where $\{e_1, e_2\}$ is a local orthonormal frame on Σ and R_{1212} the sectional curvature of M for the two-plane spanned by $\{e_1, e_2\}$. Note that since M is three dimensional,

$$R_{1212} + \mathrm{Ric}\left(\nu, \nu\right) = \frac{1}{2}S.$$

Therefore,

$$K_\Sigma = \frac{1}{2}S - \left(\mathrm{Ric}\left(\nu, \nu\right) + \frac{1}{2}\left|h\right|^2\right). \tag{2.3}$$

In the following, we present detailed proofs for Theorem 2 and Theorem 3 by assuming that the ball $B_p(R)$ contains no cut locus of p in Σ. This suffices for proving Theorem 2 in its full generality. It also proves Theorem 3 in the case that M has nonpositive sectional curvature. Indeed, by (2.2), the sectional curvature $K_\Sigma \leq 0$ when $R_{1212} \leq 0$. By working on the universal covering space of Σ if necessary, geodesic ball $B_p(R)$ is free of cut locus for all $R > 0$ as Σ is a Cartan-Hadamard manifold.

The same argument also applies to the general case of Theorem 3 by invoking Fiala's inequality (1.2) alluded in the introduction.

We start with the following lemma which has appeared in [14, 9] or Theorem 2.8 in [8], as well as [19, 4].

Lemma 7. *Let Σ be stable minimal surface in a three dimensional manifold M. Let $B_p(R)$ be a geodesic ball in Σ that does not intersect the cut locus of p in Σ or the boundary of Σ. Assume that $\phi = \phi(r)$ is a Lipschitz continuous, nonincreasing function on $[0, R]$ with $\phi(R) = 0$.*

- *If the scalar curvature of M satisfies $S \geq -6\alpha$ for some $\alpha \geq 0$, then*

$$-2\int_0^R \phi\left(r\right)\phi'\left(r\right)L'\left(r\right)dr \leq 2\pi\phi^2\left(0\right) + \int_{B_p(R)} \left(\phi'\right)^2 + 3\alpha\int_{B_p(R)} \phi^2. \tag{2.4}$$

- *If the sectional curvature of M satisfies $K \geq -\alpha$ for some $\alpha \geq 0$, then*

$$-4\int_0^R \phi\left(r\right)\phi'\left(r\right)L'\left(r\right)dr \leq 4\pi\phi^2\left(0\right) + \int_{B_p(R)} \left(\phi'\right)^2 + 4\alpha\int_{B_p(R)} \phi^2. \tag{2.5}$$

Proof. For any $0 < r < R$, by the Gauss-Bonnet formula we have

$$
\begin{aligned}
\frac{d}{dr}L(r) &= \frac{d}{dr}\left(\int_{\partial B_p(r)} ds\right) \\
&= \int_{\partial B_p(r)} (\Delta r)\, ds \\
&= \int_{\partial B_p(r)} k_g ds \\
&= 2\pi\chi\left(B_p\left(r\right)\right) - \int_{B_p(r)} K_\Sigma dA \\
&\leq 2\pi - \int_{B_p(r)} K_\Sigma dA,
\end{aligned}
\tag{2.6}
$$

where k_g is the geodesic curvature of $\partial B_p\left(r\right)$. Note that the Euler characteristic number $\chi\left(B_p\left(r\right)\right) \leq 1$.

(a) Assume first that $S \geq -6\alpha$. By (2.3) we have that

$$
K_\Sigma \geq -3\alpha - \left(\mathrm{Ric}\left(\nu,\nu\right) + \left|h\right|^2\right).
$$

Hence,

$$
\frac{d}{dr}\left(\int_{\partial B_p(r)} ds\right) \leq 2\pi + 3\alpha \int_{B_p(r)} dA + \int_{B_p(r)} \left(\mathrm{Ric}\left(\nu,\nu\right) + \left|h\right|^2\right).
$$

Equivalently,

$$
L'\left(r\right) \leq 2\pi + 3\alpha\, A\left(r\right) + \int_{B_p(r)} \left(\mathrm{Ric}\left(\nu,\nu\right) + \left|h\right|^2\right).
\tag{2.7}
$$

Multiply (2.7) by $-2\phi\left(r\right)\phi'\left(r\right) \geq 0$ and integrate from $r = 0$ to $r = R$. It follows that

$$
\begin{aligned}
&-2\int_0^R \phi\left(r\right)\phi'\left(r\right)L'\left(r\right)dr \\
\leq\ & 2\pi\phi^2\left(0\right) - 6\alpha\int_0^R \phi\left(r\right)\phi'\left(r\right)A\left(r\right)dr \\
&-2\int_0^R \phi\left(r\right)\phi'\left(r\right)\left(\int_{B_p(r)} \mathrm{Ric}\left(\nu,\nu\right) + \left|h\right|^2\right)dr.
\end{aligned}
\tag{2.8}
$$

Note that for any function $f\left(r\right)$ with $f\left(0\right) = 0$ we have

$$
-2\int_0^R \phi\left(r\right)\phi'\left(r\right)f\left(r\right)dr = \int_0^R f'\left(r\right)\phi^2\left(r\right)dr.
\tag{2.9}
$$

Applying this to $f\left(r\right) = \int_{B_p(r)} dA$ we conclude

$$
\begin{aligned}
-6\alpha\int_0^R \phi(r)\,\phi'(r)\,A(r)\,dr &= 3\alpha\int_0^R \phi^2(r)\,L(r)\,dr \\
&= 3\alpha\int_{B_p(R)}\phi^2,
\end{aligned}
\tag{2.10}
$$

where in the last line we have used the co-area formula. Similarly, for

$$
f(r)=\int_{B_p(r)}\left(\mathrm{Ric}(\nu,\nu)+|h|^2\right)
$$

we get

$$
\begin{aligned}
&-2\int_0^R \phi(r)\,\phi'(r)\left(\int_{B_p(r)}\mathrm{Ric}(\nu,\nu)+|h|^2\right) \\
&= \int_0^R \phi^2(r)\left(\int_{\partial B_p(r)}\mathrm{Ric}(\nu,\nu)+|h|^2\right)dr \\
&= \int_{B_p(R)}\left(\mathrm{Ric}(\nu,\nu)+|h|^2\right)\phi^2 \\
&\leq \int_{B_p(R)}(\phi')^2,
\end{aligned}
\tag{2.11}
$$

where in the last line we have used the stability inequality (2.1). Combining (2.10) and (2.11), we conclude from (2.8) that

$$
-2\int_0^R \phi(r)\,\phi'(r)\,L'(r)\,dr \leq 2\pi\phi^2(0)+3\alpha\int_{B_p(R)}\phi^2+\int_{B_p(R)}(\phi')^2,
$$

which is (2.4).

(b) Assume now that $K\geq-\alpha$. Then according to (2.2),

$$
-2K_\Sigma\leq 2\alpha+|h|^2.
$$

Plugging into (2.6) gives

$$
2L'(r)\leq 4\pi+2\alpha\,A(r)+\int_{B_p(r)}|h|^2.
$$

Multiplying by $-2\phi(r)\,\phi'(r)\geq 0$ and integrating from $r=0$ to $r=R$ we obtain

$$
\begin{aligned}
-4\int_0^R \phi(r)\,\phi'(r)\,L'(r)\,dr \leq\ & 4\pi\phi^2(0) \\
& -4\alpha\int_0^R \phi(r)\,\phi'(r)\,A(r)\,dr \\
& -2\int_0^R \phi(r)\,\phi'(r)\left(\int_{B_p(r)}|h|^2\right)dr.
\end{aligned}
\tag{2.12}
$$

Since $\mathrm{Ric}(\nu,\nu)\geq-2\alpha$, it follows from (2.11) that

$$-2\int_0^R \phi(r)\,\phi'(r)\left(\int_{B_p(r)} |h|^2\right) \leq \int_\Sigma (\phi')^2 + 2\alpha \int_\Sigma \phi^2.$$

Together with (2.10), we conclude from (2.12) that

$$-4\int_0^R \phi(r)\,\phi'(r)\,L'(r)\,dr \leq 4\pi\,\phi^2(0) + \int_{B_p(R)} (\phi')^2 + 4\alpha \int_{B_p(R)} \phi^2.$$

This proves (2.5). □

Corollary 8. *Let Σ be stable minimal surface in a three dimensional manifold M. Let $B_p(R)$ be a geodesic ball in Σ that does not intersect the cut locus of p in Σ or the boundary of Σ. Assume that $\phi = \phi(r)$ is a C^2 nonincreasing function on $[0, R]$ with $\phi(R) = 0$.*

- *If the scalar curvature of M satisfies $S \geq -6\alpha$ for some $\alpha \geq 0$, then*
$$\int_{B_p(R)} (\phi')^2 + 2\int_{B_p(R)} \phi\phi'' \leq 2\pi\phi^2(0) + 3\alpha \int_{B_p(R)} \phi^2.$$
- *If the sectional curvature of M satisfies $K \geq -\alpha$ for some $\alpha \geq 0$, then*
$$3\int_{B_p(R)} (\phi')^2 + 4\int_{B_p(R)} \phi\phi'' \leq 4\pi\phi^2(0) + 4\alpha \int_{B_p(R)} \phi^2.$$

Proof. Integrating by parts we have

$$\begin{aligned} -2\int_0^R \phi(r)\,\phi'(r)\,L'(r)\,dr &= 2\int_0^R \left(\phi\phi'' + (\phi')^2\right) L(r)\,dr \\ &= 2\int_{B_p(R)} \left(\phi\phi'' + (\phi')^2\right). \end{aligned}$$

By (2.4) and (2.5), the desired conclusions follow. □

We now use the lemma and the corollary to establish area estimates for stable minimal surfaces. We start with the case $M = \mathbb{R}^3$.

Theorem 9. *Let Σ be a stable minimal surface in $\mathbb{R}^3$. Then there exists a universal constant R_0 such that for geodesic ball $B_p(R)$ with no intersection with the boundary of Σ or the cut locus of p,*

$$L(r) \leq 2\pi r\left(1 + \frac{10}{\ln R}\right)$$

and

$$A(r) \leq \pi\, r^2\left(1 + \frac{10}{\ln R}\right)$$

for all $r \leq \sqrt{R}$ and $R \geq R_0$. In particular, if Σ is complete, then $A(r) \leq \pi\, r^2$ for all $r > 0$. Consequently, Σ is flat.

Proof. By Corollary 8, the inequality

$$3\int_{B_p(R)} (\phi')^2 + 4\int_{B_p(R)} \phi\phi'' \leq 4\pi\phi^2(0)$$

holds for any C^2 nonincreasing function $\phi = \phi(r)$ on $[0, R]$ with $\phi(R) = 0$. Set

$$\phi(r) = \ln(R+1) - \ln(r+1).$$

The inequality becomes

$$3\int_{B_p(R)} \frac{1}{(r+1)^2} + 4\int_{B_p(R)} \frac{\ln(R+1) - \ln(r+1)}{(r+1)^2} \leq 4\pi \ln^2(R+1).$$

In particular,

$$\int_{B_p(R)} \frac{\ln(R+1) - \ln(r+1)}{(r+1)^2} \leq \pi \ln^2(R+1). \tag{2.13}$$

Since $K_\Sigma \leq 0$ on $B_p(R)$, the Hessian comparison theorem (see Theorem 1.1 in [26]) implies that

$$2\pi \leq \frac{L(r)}{r} \leq \frac{L(R)}{R} \tag{2.14}$$

for all $0 < r < R$. Assume by contradiction that

$$\frac{L(r)}{r} \geq 2\pi\left(1 + \frac{10}{\ln(R+1)}\right) \tag{2.15}$$

for all $r \in \left[\sqrt{R}, R\right]$. According to (2.13) we have

$$\begin{aligned} \pi \ln^2(R+1) &\geq \int_0^R \frac{\ln(R+1) - \ln(r+1)}{(r+1)^2} L(r)\, dr \\ &= \int_0^R \frac{\ln(R+1) - \ln(r+1)}{(r+1)^2} (L(r) - 2\pi r)\, dr \\ &\quad + 2\pi \int_0^R \frac{\ln(R+1) - \ln(r+1)}{(r+1)^2} r\, dr. \end{aligned}$$

Hence, (2.14) and (2.15) imply that

$$\begin{aligned} \pi \ln^2(R+1) &\geq 2\pi \int_0^R \frac{\ln(R+1) - \ln(r+1)}{(r+1)^2} r\, dr \\ &\quad + \frac{20\pi}{\ln(R+1)} \int_{\sqrt{R}}^R \frac{\ln(R+1) - \ln(r+1)}{(r+1)^2} r dr. \end{aligned} \tag{2.16}$$

The first term can be computed as

$$\begin{aligned} & 2\pi \int_0^R \frac{\ln(R+1) - \ln(r+1)}{(r+1)^2} r dr \\ = & -2\pi \ln(R+1) + 2\pi \int_0^R \frac{1}{r+1}\left(\ln(r+1) + \frac{1}{r+1}\right) dr \\ = & -2\pi \ln(R+1) + \pi \ln^2(R+1) + 2\pi - \frac{2\pi}{R+1}. \end{aligned}$$

Plugging this into (2.16) we get

$$2\pi \ln (R+1) \geq \frac{20\pi}{\ln (R+1)} \int_{\sqrt{R}}^{R} \frac{\ln (R+1) - \ln (r+1)}{(r+1)^2} r\, dr. \tag{2.17}$$

However, integration by parts gives

$$\begin{aligned} & \int_{\sqrt{R}}^{R} \frac{\ln (R+1) - \ln (r+1)}{(r+1)^2} r dr \\ = \; & -\ln \left(\frac{R+1}{\sqrt{R}+1} \right) \left(\ln \left(\sqrt{R}+1 \right) + \frac{1}{\sqrt{R}+1} \right) \\ & + \frac{1}{2} \ln^2 (R+1) - \frac{1}{2} \ln^2 \left(\sqrt{R}+1 \right) \\ & + \frac{1}{\sqrt{R}+1} - \frac{1}{R+1} \\ \geq \; & \frac{1}{9} \ln^2 (R+1) \end{aligned}$$

for all $R > R_0$ large enough. In view of (2.17), this yields a contradiction. In conclusion, (2.15) is false. In other words, there exists $r_0 \in \left[\sqrt{R}, R \right]$ such that

$$\frac{L(r_0)}{r_0} \leq 2\pi \left(1 + \frac{10}{\ln (R+1)} \right).$$

Now for $r < \sqrt{R}$, by (2.14) we have

$$\begin{aligned} \frac{L(r)}{r} & \leq \frac{L(r_0)}{r_0} \\ & \leq 2\pi \left(1 + \frac{10}{\ln R} \right). \end{aligned}$$

This proves the length estimate. The area estimate then follows immediately.

Finally, if Σ is complete, then the length and area estimates are applicable for all R on the universal cover of Σ. It follows that

$$L(r) \leq 2\pi\, r$$

and

$$A(r) \leq \pi\, r^2$$

for all $r > 0$. This implies that Σ is flat as $K_\Sigma \leq 0$. □

We now turn to the case of more general ambient manifolds.

Theorem 10. *Let $B_p(R)$ be a geodesic ball in a stable minimal surface Σ in a three dimensional manifold M. Assume that $B_p(R)$ does not intersect the boundary of Σ or the cut locus of p in Σ.*

- *If the scalar curvature S of M satisfies $S \geq -6$, then*

$$A(R) \leq C_1\, e^{2\,R}$$

for some absolute constant $C_1 > 0$.

- *If the sectional curvature K of M satisfies $K \geq -1$, then*

$$A(R) \leq C_1 \, e^{\frac{4}{\sqrt{7}} R}$$

for some absolute constant $C_1 > 0$.

Proof. Since the arguments are similar for both cases, we supply details for the second case and only provide a sketch for the first case.

So we assume $K \geq -1$. By (2.5) from Lemma 7, for any Lipschitz continuous, nonincreasing function $\phi = \phi(r)$ on $[0, t]$ with $\phi(t) = 0$ we have

$$-4 \int_0^t \phi(r) \phi'(r) L'(r) \, dr \leq 4\pi \phi^2(0) + \int_{B_p(t)} (\phi')^2 + 4 \int_{B_p(t)} \phi^2 \tag{2.18}$$

for all $0 < t < R$.

For convenience, denote with

$$a = \frac{4}{\sqrt{7}}. \tag{2.19}$$

Let

$$\phi(r) = e^{-\frac{a}{2} r} \psi(r),$$

where ψ is a nonincreasing Lipschitz function such that $\psi(t) = 0$. We have

$$\begin{aligned} (\phi')^2 &= \frac{a^2}{4} e^{-ar} \psi^2 + e^{-ar} (\psi')^2 - a e^{-ar} \psi \psi' \\ -4\phi\phi' &= 2a e^{-ar} \psi^2 - 4 e^{-ar} \psi \psi'. \end{aligned}$$

Therefore,

$$\begin{aligned} -4 \int_0^t \phi(r) \phi'(r) L'(r) \, dr &= 2a \int_0^t e^{-ar} \psi^2(r) L'(r) \, dr \\ &\quad -4 \int_0^t e^{-ar} \psi(r) \psi'(r) L'(r) \, dr. \end{aligned}$$

After integration by parts, the first term on the right side becomes

$$\begin{aligned} 2a \int_0^t e^{-ar} \psi^2(r) L'(r) \, dr &= \int_0^t \left(2a^2 e^{-ar} \psi^2(r) - 4a \psi(r) \psi'(r) \right) L(r) \, dr \\ &= \int_{B_p(t)} \left(2a^2 \psi^2 - 4a \psi \psi' \right) e^{-ar}. \end{aligned}$$

Plugging these identities into (2.18), we get

$$\begin{aligned} -4 \int_0^t e^{-ar} \psi(r) \psi'(r) L'(r) \, dr &\leq 4\pi \psi^2(0) + \int_{B_p(t)} (\psi')^2 e^{-ar} \\ &\quad + 3a \int_{B_p(t)} \psi \psi' e^{-ar}. \end{aligned} \tag{2.20}$$

For arbitrary η with $0 < \eta < R$ and $\eta \leq t$, let

$$\psi(r)=\begin{cases}1 & \text{for } r\le t-\eta\\ \frac{t-r}{\eta} & \text{for } r\in(t-\eta,t)\\ 0 & \text{for } r\ge t\end{cases} \tag{2.21}$$

It follows from (2.20) that

$$\frac{4}{\eta}\int_{t-\eta}^{t} e^{-ar}\psi(r)\,L'(r)\,dr \le 4\pi+\frac{1}{\eta^2}\int_{B_p(t)\setminus B_p(t-\eta)} e^{-ar} -\frac{3a}{\eta}\int_{B_p(t)\setminus B_p(t-\eta)}\psi e^{-ar}. \tag{2.22}$$

The term on the left hand side, after integrating by parts, becomes

$$\frac{4}{\eta}\int_{t-\eta}^{t} e^{-ar}\psi(r)\,L'(r)\,dr = -\frac{4}{\eta}L(t-\eta)\,e^{-a(t-\eta)}+\frac{4a}{\eta}\int_{B_p(t)\setminus B_p(t-\eta)}\psi e^{-ar} +\frac{4}{\eta^2}\int_{B_p(t)\setminus B_p(t-\eta)} e^{-ar}.$$

Therefore, combining with (2.22), we have

$$\frac{3}{\eta^2}\int_{B_p(t)\setminus B_p(t-\eta)} e^{-ar}+\frac{7a}{\eta}\int_{B_p(t)\setminus B_p(t-\eta)}\psi e^{-ar}\le 4\pi+\frac{4}{\eta}L(t-\eta)\,e^{-a(t-\eta)} \tag{2.23}$$

for any $0<\eta\le t<R$. In particular, by setting $\eta=t$, it follows that

$$\frac{3}{t^2}\int_{B_p(t)} e^{-ar}\le 4\pi.$$

Clearly, this implies that

$$A(t)\le\frac{4\pi}{3}\,t^2\,e^{at} \tag{2.24}$$

for all $t\le R$.

Claim: There exists an absolute constant $\Lambda>0$ such that for all τ and s satisfying $0<2\tau\le s<R-3\tau$,

$$\int_{B_p(s)\setminus B_p(s-\tau)} e^{-ar}\le\Lambda\tau+\frac{\Lambda}{\tau}\int_{B_p(s-\tau)\setminus B_p(s-2\tau)} e^{-ar}. \tag{2.25}$$

Indeed, let $\eta=4\tau$ and $T=s-\frac{3\tau}{2}$. Then

$$\frac{\eta}{8}\le T<R-\frac{9\eta}{8}.$$

The mean value theorem implies that there exists $\xi\in\left(T-\frac{\eta}{8},T+\frac{\eta}{8}\right)$ such that

$$\int_{B_p\left(T+\frac{\eta}{8}\right)\setminus B_p\left(T-\frac{\eta}{8}\right)} e^{-ar}=\frac{\eta}{4}L(\xi)\,e^{-a\xi}. \tag{2.26}$$

Denote with

$$t=\xi+\eta\in\left(T+\frac{7\eta}{8},T+\frac{9\eta}{8}\right). \tag{2.27}$$

By (2.21) and (2.27) we get that

$$\begin{aligned}\int_{B_p(t)\setminus B_p(t-\eta)}\psi e^{-ar} &\geq \frac{1}{2}\int_{B_p\left(t-\frac{\eta}{2}\right)\setminus B_p(t-\eta)} e^{-ar}\\ &\geq \frac{1}{2}\int_{B_p\left(T+\frac{3\eta}{8}\right)\setminus B_p\left(T+\frac{\eta}{8}\right)} e^{-ar}.\end{aligned}$$

Together with (2.23) and (2.26) we conclude

$$\begin{aligned}\frac{7a}{2\eta}\int_{B_p\left(T+\frac{3\eta}{8}\right)\setminus B_p\left(T+\frac{\eta}{8}\right)} e^{-ar} &\leq \frac{7a}{\eta}\int_{B_p(t)\setminus B_p(t-\eta)}\psi e^{-ar}\\ &\leq 4\pi+\frac{4}{\eta}L\left(t-\eta\right)e^{-a(t-\eta)}\\ &= 4\pi+\frac{16}{\eta^2}\int_{B_p\left(T+\frac{\eta}{8}\right)\setminus B_p\left(T-\frac{\eta}{8}\right)} e^{-ar}.\end{aligned}$$

Therefore, there exists $\Lambda>0$ such that

$$\int_{B_p\left(T+\frac{3\eta}{8}\right)\setminus B_p\left(T+\frac{\eta}{8}\right)} e^{-ar}\leq\Lambda\eta+\frac{\Lambda}{\eta}\int_{B_p\left(T+\frac{\eta}{8}\right)\setminus B_p\left(T-\frac{\eta}{8}\right)} e^{-ar}. \tag{2.28}$$

Substituting $\eta=4\tau$ and $s=T+\frac{3\tau}{2}$ in (2.28) implies the claim.

For $s\geq 6\Lambda$, letting $\tau=2\Lambda$ and iterating (2.25) m times with $m=\left[\frac{s}{2\Lambda}\right]-2$ we get

$$\begin{aligned}\int_{B_p(s)\setminus B_p(s-2\Lambda)} e^{-ar} &\leq 2\Lambda^2\sum_{k=0}^{m-1}\frac{1}{2^k}+\frac{1}{2^m}\int_{B_p(6\Lambda)} e^{-ar}\\ &\leq C_2\end{aligned} \tag{2.29}$$

by invoking (2.24) for the last term. This holds for any $6\Lambda\leq s\leq R-6\Lambda$.

To finish, we apply the mean value theorem to conclude that there exists $\xi\in(R-16\Lambda,R-14\Lambda)$ such that

$$\int_{B_p(R-14\Lambda)\setminus B_p(R-16\Lambda)} e^{-ar}=2\Lambda L\left(\xi\right)e^{-a\xi}. \tag{2.30}$$

Applying (2.29) with $s=R-14\Lambda$, we get from (2.30) that

$$L\left(R-\eta\right)e^{-a(R-\eta)}\leq C_3$$

for some constant C_3, where

$$\eta=R-\xi\in\left(14\Lambda,16\Lambda\right).$$

By (2.23),

$$\begin{aligned}\frac{3}{\eta^2}\int_{B_p(R)\setminus B_p(R-\eta)} e^{-ar} &\leq 4\pi+\frac{4}{\eta}L\left(R-\eta\right)e^{-a(R-\eta)}\\ &\leq C_4.\end{aligned}$$

This implies that

$$\int_{B_p(R)\setminus B_p(R-\eta)} dA \leq C\, e^{a\,R}$$

for some $\eta\in\left(14\Lambda,16\Lambda\right)$. In particular,

$$\int_{B_p(R)\setminus B_p(R-14\Lambda)} dA \leq C\, e^{a\,R}. \tag{2.31}$$

Applying (2.31) with R replaced by $R-14k\Lambda$, $k=1,2,\cdots,n$ and $n=\left[\frac{R}{14\Lambda}\right]-1$, we arrive at

$$\begin{aligned}A(R) &\leq \sum_{k=0}^{n}\int_{B_p(R-14k\Lambda)\setminus B_p(R-14(k+1)\Lambda)} dA + A(14\Lambda)\\ &\leq C\sum_{k=0}^{n} e^{a\,(R-14k\Lambda)}+C\\ &\leq C\,e^{a\,R}.\end{aligned}$$

This proves the area estimate for the second case.

Let us now sketch the argument for the first case. Assume now that $S\geq -6$. Then by (2.4),

$$-2\int_0^t \phi\left(r\right)\phi'\left(r\right)L'\left(r\right)dr\leq 2\pi\phi^2\left(0\right)+\int_{B_p(t)}\left(\phi'\right)^2+3\int_{B_p(t)}\phi^2.$$

We set

$$\phi\left(r\right)=e^{-r}\psi\left(r\right),$$

for ψ nonincreasing Lipschitz on $[0,t]$ so that $\psi(t)=0$, and get that

$$-2\int_0^t \psi\left(r\right)\psi'\left(r\right)L'\left(r\right)e^{-2r}\leq 2\pi+2\int_{B_p(t)}\psi\psi' e^{-2r}+\int_{B_p(t)}\left(\psi'\right)^2 e^{-2r}.$$

Taking ψ as defined in (2.21) and integrating by parts imply that

$$\frac{1}{\eta^2}\int_{B_p(t)\setminus B_p(t-\eta)} e^{-2r}+\frac{6}{\eta}\int_{B_p(t)\setminus B_p(t-\eta)}\psi e^{-2r}\leq 2\pi+\frac{2}{\eta}L\left(t-\eta\right)e^{-2(t-\eta)}$$

The rest of the argument follows verbatim. □

We point out that the same proof applies to the general case of Theorem 3 as well, that is, Theorem 10 continues to hold for arbitrary ball $B_p(R)$ in Σ. Indeed, Lemma 7 is valid for arbitrary ball $B_p(R)$, even if it contains cut locus of p, by invoking Fiala's inequality (1.2) established through the work of Fiala [12], Hartman [15], and Shiohama and Tanaka [27, 28]. A priori, the length function $L(r)$ of the geodesic circle $\partial B_p(r)$ is only defined for almost all r. Their work implies that the function $L(r)$ can be extended to all $r\geq 0$. So extended function, denoted by L again, may not be continuous. But it satisfies $L(r^+) = L(r)$ and $L(r^-) \geq L(r)$ for all $r > 0$. Moreover, it can be written as $L(r) = L_1(r) + L_2(r)$, where $L_1(r)$ is absolutely continuous on any finite interval and $L_2(r)$ is nonincreasing. At the points where $L(r)$ is differentiable, its derivative satisfies the Fiala inequality

$$\frac{d}{dr}L(r) \leq 2\pi - \int_{B_p(r)} K_\Sigma.$$

Although it is not needed here, we remark that Corollary 8 is also valid on arbitrary ball $B_p(R)$ for all C^2 function $\phi = \phi(r)$ with $\phi'(r) \leq 0$, $\phi''(r) \geq 0$ for all $r \in [0, R]$ and $\phi(R) = 0$. We refer to [1] for a proof.

3. Bottom spectrum estimates

In this section, we consider upper bounds for the bottom spectrum of complete stable minimal hypersurfaces. The bottom spectrum of the Laplacian on a complete manifold N, denoted by $\lambda_0(N)$, is an important geometric invariant and characterized as the optimal Poincaré inequality constant or

$$\lambda_0(N) = \inf_{\phi \in C_0^\infty(N)} \frac{\int_N |\nabla\phi|^2}{\int_N \phi^2}.$$

According to [18], for any $p \in N$,

$$\lambda_0(N) \leq \frac{1}{4}\left(\liminf_{R\to\infty} \frac{\ln V_p(R)}{R}\right)^2,$$

where $V_p(R)$ denotes the volume of the geodesic ball $B_p(R)$ centered at point p of radius R. As an immediate corollary to the area estimate Corollary 4, one obtains the following result which is due to Bérard, Castillon and Cavalcante [2].

Theorem 11 (Bérard, Castillon and Cavalcante). *Let Σ be a complete noncompact stable minimal surface in a three dimensional manifold M.*

(a) If the scalar curvature S of M satisfies $S \geq -6$, then

$$\lambda_0(\Sigma) \leq 1.$$

(b) If the sectional curvature K of M satisfies $K \geq -1$, then

$$\lambda_0(\Sigma) \leq \frac{4}{7}.$$

We now provide a different argument for this result. Recall that an $n-$dimensional manifold Σ is called nonparabolic if it admits a positive symmetric Green's function. It is well-known that this is the case if $\lambda_0(\Sigma) > 0$ (see [16]). Let $G(p, x)$ be the minimal positive Green's function. Then $G(p, x) = G(x, p) > 0$,

$$\Delta_x G(p, x) = -\delta(p, x)$$

and

$$G(p, x) = \lim_{i\to\infty} G_i(p, x),$$

where $G_i(p, x)$ is the Dirichlet Green's function of a compact exhaustion Ω_i of Σ. For fixed point p, we denote $G(x) = G(p, x)$. It follows from the construction that $\max_{\partial B_p(r)} G$ is a strictly decreasing function in r and that

$$\int_{\Sigma\setminus B_p(1)} |\nabla G|^2 < \infty. \tag{3.1}$$

Since G is harmonic away from the pole p, the Kato inequality implies

$$|G_{ij}|^2 \geq \frac{n}{n-1} |\nabla |\nabla G||^2 .$$

By the Bochner formula we have

$$\Delta |\nabla G| \geq \frac{1}{n-1} |\nabla |\nabla G||^2 |\nabla G|^{-1} + \mathrm{Ric}^{\Sigma} (\nabla G, \nabla G) |\nabla G|^{-1} \tag{3.2}$$

on $\Sigma \setminus \{p\}$. Similarly, for $v = \ln G$, using

$$\Delta v = - |\nabla v|^2$$

and

$$|\nabla v| \, v_{11} = \langle \nabla |\nabla v| , \nabla v \rangle ,$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame on Σ with $e_1 = \frac{\nabla}{|\nabla v|}$, one easily sees from the Bochner formula together with the standard manipulation that

$$\begin{aligned} \frac{1}{2}\Delta |\nabla v|^2 \quad \geq \quad & \frac{n}{n-1} |\nabla |\nabla v||^2 + \frac{1}{n-1} |\nabla v|^4 \\ & -\frac{n-2}{n-1} \left\langle \nabla |\nabla v|^2 , \nabla v \right\rangle + \mathrm{Ric}^{\Sigma} (\nabla v, \nabla v) \end{aligned} \tag{3.3}$$

on $\Sigma \setminus \{p\}$.

Denote with

$$\begin{aligned} L(a,b) &= \{x \in \Sigma : a < G(x) < b\} \\ l(t) &= \{x \in \Sigma : G(x) = t\} . \end{aligned}$$

Then $L(\alpha, \infty) \subset B_p(1)$ for $\alpha = \max_{\partial B_p(1)} G$. According to Lemma 5.1 in [17],

$$\begin{aligned} \int_{l(t)} |\nabla G| &= 1 \\ \int_{L(a,b)} |\nabla G|^2 f(G) &= \int_a^b f(t)dt \end{aligned} \tag{3.4}$$

for any integrable function f provided that $\lambda_0(\Sigma) > 0$.

We need the following integral gradient estimate.

Lemma 12. *Let M be a three dimensional manifold with scalar curvature bounded below. For a complete stable minimal surface Σ in M with $\lambda_0(\Sigma) > 0$, its minimal positive Green's function G satisfies*

$$\int_{\Sigma\setminus B_p(1)} \frac{|\nabla G|^4}{G^3 \ln^{2q} (1 + G^{-1})} < \infty$$

for any $q > \frac{1}{2}$.

Proof. Let $v=\ln G$. Then by (3.3),

$$\frac{1}{2}\Delta\left|\nabla v\right|^{2}\geq\left|\nabla v\right|^{4}+2\left|\nabla\left|\nabla v\right|\right|^{2}+K_{\Sigma}\left|\nabla v\right|^{2}.$$

on $\Sigma\setminus\{p\}$. According to (2.3),

$$\begin{aligned}K_{\Sigma}\left|\nabla v\right|^{2} &= \frac{1}{2}S\left|\nabla v\right|^{2}-\left(\operatorname{Ric}(\nu,\nu)+\frac{1}{2}\left|h\right|^{2}\right)\left|\nabla v\right|^{2}\\ &\geq -C\left|\nabla v\right|^{2}-\left(\operatorname{Ric}(\nu,\nu)+\left|h\right|^{2}\right)\left|\nabla v\right|^{2}.\end{aligned}$$

Thus, for any cut-off function ϕ, we have

$$\begin{aligned}\int_{\Sigma}\left|\nabla v\right|^{4}\phi^{2} &\leq \frac{1}{2}\int_{\Sigma}\phi^{2}\,\Delta\left|\nabla v\right|^{2}+C\int_{\Sigma}\phi^{2}\left|\nabla v\right|^{2}\\ &\quad -2\int_{\Sigma}\phi^{2}\left|\nabla\left|\nabla v\right|\right|^{2}\\ &\quad +\int_{\Sigma}\left(\operatorname{Ric}(\nu,\nu)+\left|h\right|^{2}\right)\left|\nabla v\right|^{2}\phi^{2}.\end{aligned}\tag{3.5}$$

By the stability inequality (2.1), the last term is estimated as

$$\begin{aligned}&\int_{\Sigma}\left(\operatorname{Ric}(\nu,\nu)+\left|h\right|^{2}\right)\left|\nabla v\right|^{2}\phi^{2}\\ \leq&\int_{\Sigma}\left|\nabla\left(\left|\nabla v\right|\phi\right)\right|^{2}\\ =&\int_{\Sigma}\left|\nabla\phi\right|^{2}\left|\nabla v\right|^{2}-\int_{\Sigma}\phi^{2}\left|\nabla v\right|\,\Delta\left|\nabla v\right|.\end{aligned}$$

Hence, (3.5) becomes

$$\int_{\Sigma}\left|\nabla v\right|^{4}\phi^{2}\leq\int_{\Sigma}\left(C\,\phi^{2}+\left|\nabla\phi\right|^{2}\right)\left|\nabla v\right|^{2}.$$

For $\frac{1}{2}<q<1$, let $\phi=\psi\,f(G)$, where ψ is a cut-off function such that $\psi=0$ on $B_p(1)\cup\left(M\setminus B_p(2R)\right)$, $\psi=1$ on $B_p(R)\setminus B_p(2)$, and

$$f\left(G\right)=\frac{G^{\frac{1}{2}}}{\ln^{q}\left(A\,G^{-1}\right)}$$

with $A=e^{4}\alpha$, $\alpha=\max_{\partial B_p(1)}G$. Direct calculations imply

$$\int_{\Sigma}\left|\nabla\phi\right|^{2}\left|\nabla v\right|^{2}\leq 4\int_{\Sigma}\left|\nabla\psi\right|^{2}f^{2}\left|\nabla v\right|^{2}+\frac{3}{4}\int_{\Sigma}\left|\nabla v\right|^{4}\phi^{2}.$$

Therefore,

$$\begin{aligned}
\int_\Sigma |\nabla v|^4 \phi^2 &\leq C \int_\Sigma \left(\psi^2 + |\nabla \psi|^2\right) f^2 |\nabla v|^2 \\
&\leq C \int_{\Sigma \setminus B_p(1)} \frac{|\nabla G|^2}{G \left(\ln \frac{A}{G}\right)^{2q}} \\
&\leq C \int_{L(0,\alpha)} \frac{|\nabla G|^2}{G \left(\ln \frac{A}{G}\right)^{2q}} \\
&= C \int_0^\alpha \frac{1}{t \left(\ln \frac{A}{t}\right)^{2q}} dt \\
&\leq C.
\end{aligned}$$

Letting $R \to \infty$, we conclude

$$\int_{\Sigma \setminus B_p(2)} \frac{|\nabla G|^4}{G^3 \ln^{2q}\left(AG^{-1}\right)} \leq C. \tag{3.6}$$

This proves the result. □

Note that by the Cauchy-Schwarz inequality it follows that

$$\begin{aligned}
\left(\int_{\Sigma \setminus B_p(1)} \frac{|\nabla G|^3}{G^2 \ln^{2q}\left(1+G^{-1}\right)} \right)^2 &\leq \left(\int_{\Sigma \setminus B_p(1)} \frac{|\nabla G|^4}{G^3 \ln^{2q}\left(1+G^{-1}\right)} \right) \\
&\quad \times \left(\int_{\Sigma \setminus B_p(1)} \frac{|\nabla G|^2}{G \ln^{2q}\left(1+G^{-1}\right)} \right) \\
&\leq C \left(\int_{\Sigma \setminus B_p(1)} \frac{|\nabla G|^2}{G \ln^{2q}\left(1+G^{-1}\right)} \right).
\end{aligned}$$

Again, the last integral is finite for $q > \frac{1}{2}$ by (3.4) and the co-area formula. Therefore,

$$\int_{\Sigma \setminus B_p(1)} \frac{|\nabla G|^3}{G^2 \ln^{2q}\left(1+G^{-1}\right)} < \infty. \tag{3.7}$$

We are now ready to prove Theorem 11.

Proof of Theorem 11. Without loss of generality we may assume that $\lambda_0(\Sigma) > 0$. Then Σ is nonparabolic. Let $G(p,x)$ be the minimal positive Green's function of Σ with a pole at $p \in \Sigma$. For simplicity, we denote this function by $G(x)$. Note that G is harmonic on $\Sigma \setminus \{p\}$.

Let

$$L(a,b) = \{x \in \Sigma : a < G(x) < b\}.$$

For $\varepsilon > 0$ small enough, define function χ by

$$\chi(x) = \begin{cases} 1 & \text{on } L(\varepsilon, \infty) \\ \frac{\ln G(x) - \ln\left(\varepsilon^2\right)}{-\ln \varepsilon} & \text{on } L\left(\varepsilon^2, \varepsilon\right) \\ 0 & \text{on } L(0, \varepsilon) \end{cases} \tag{3.8}$$

Since $G(x)$ may not converge to zero as $x \to \infty$, the function χ may not have compact support. So we consider the cut-off function

$$\varphi = \chi\psi,$$

where

$$\psi(x) = \begin{cases} 0 & \text{on } B_p(1) \\ r(x) - 1 & \text{on } B_p(2) \setminus B_p(1) \\ 1 & \text{on } B_p(R) \setminus B_p(2) \\ R + 1 - r(x) & \text{on } B_p(R+1) \setminus B_p(R) \\ 0 & \text{on } \Sigma \setminus B_p(R+1) \end{cases} \tag{3.9}$$

Setting

$$\phi = |\nabla G|^{\frac{1}{2}} \varphi$$

in the Poincaré inequality, we have

$$\lambda_0(\Sigma) \int_\Sigma |\nabla G| \varphi^2 \le \int_\Sigma \left| \nabla \left(|\nabla G|^{\frac{1}{2}} \varphi \right) \right|^2.$$

Expanding the right side, we get

$$\begin{aligned} \lambda_0(\Sigma) \int_\Sigma |\nabla G| \varphi^2 &\le \frac{1}{4} \int_\Sigma |\nabla |\nabla G||^2 |\nabla G|^{-1} \varphi^2 \\ &\quad + \int_\Sigma |\nabla G| |\nabla \varphi|^2 \\ &\quad + \frac{1}{2} \int_\Sigma \left\langle \nabla \varphi^2, \nabla |\nabla G| \right\rangle \\ &\le \left(\frac{1}{4} + \delta \right) \int_\Sigma |\nabla |\nabla G||^2 |\nabla G|^{-1} \varphi^2 \\ &\quad + C(\delta) \int_\Sigma |\nabla G| |\nabla \varphi|^2 \end{aligned} \tag{3.10}$$

for any $\delta > 0$. We now estimate the first term on the right hand side. Note that by (3.2),

$$\Delta |\nabla G| \ge |\nabla |\nabla G||^2 |\nabla G|^{-1} + K_\Sigma |\nabla G| \tag{3.11}$$

on $\Sigma \setminus \{p\}$ whenever $|\nabla G| \neq 0$.

In the case that $S \ge -6$, by (2.3) it follows that

$$K_\Sigma |\nabla G| \ge -3 |\nabla G| - \left(\operatorname{Ric}(\nu, \nu) + |h|^2 \right) |\nabla G|.$$

Hence, (3.11) becomes

$$\begin{aligned} \Delta |\nabla G| &\ge |\nabla |\nabla G||^2 |\nabla G|^{-1} - 3 |\nabla G| \\ &\quad - \left(\operatorname{Ric}(\nu, \nu) + |h|^2 \right) |\nabla G|. \end{aligned} \tag{3.12}$$

Integrating by parts we have

$$\int_\Sigma \varphi^2 \Delta |\nabla G| = -\int_\Sigma \left\langle \nabla |\nabla G|, \nabla \varphi^2 \right\rangle.$$

Therefore, (3.12) implies

$$\begin{aligned}(3.13)\qquad \int_\Sigma \left|\nabla\left|\nabla G\right|\right|^2 \left|\nabla G\right|^{-1}\varphi^2 \;\le\;& 3\int_\Sigma \left|\nabla G\right|\varphi^2 \\ &+\int_\Sigma \left(\operatorname{Ric}(\nu,\nu)+|h|^2\right)\left|\nabla G\right|\varphi^2 \\ &-\int_\Sigma \left\langle \nabla\left|\nabla G\right|,\nabla\varphi^2\right\rangle .\end{aligned}$$

Using the stability inequality (2.1) we have that

$$\begin{aligned}\int_\Sigma \left(\operatorname{Ric}(\nu,\nu)+|h|^2\right)\left|\nabla G\right|\varphi^2 \;\le\;& \int_\Sigma \left|\nabla\left(\left|\nabla G\right|^{\frac12}\varphi\right)\right|^2 \\ \le\;& \frac14\int_\Sigma \left|\nabla\left|\nabla G\right|\right|^2\left|\nabla G\right|^{-1}\varphi^2 \\ &+\int_\Sigma \left|\nabla G\right|\left|\nabla\varphi\right|^2 \\ &+\frac12\int_\Sigma \left\langle \nabla\varphi^2,\nabla\left|\nabla G\right|\right\rangle .\end{aligned}$$

Combining with (3.13) we obtain that

$$\begin{aligned}\int_\Sigma \left|\nabla\left|\nabla G\right|\right|^2\left|\nabla G\right|^{-1}\varphi^2 \;\le\;& 4\int_\Sigma \left|\nabla G\right|\varphi^2+\frac43\int_\Sigma \left|\nabla G\right|\left|\nabla\varphi\right|^2 \\ &-\frac23\int_\Sigma \left\langle \nabla\varphi^2,\nabla\left|\nabla G\right|\right\rangle \\ \le\;& 4\int_\Sigma \left|\nabla G\right|\varphi^2+C(\delta)\int_\Sigma \left|\nabla G\right|\left|\nabla\varphi\right|^2 \\ &+\delta\int_\Sigma \left|\nabla\left|\nabla G\right|\right|^2\left|\nabla G\right|^{-1}\varphi^2.\end{aligned}$$

Therefore,

$$\int_\Sigma \left|\nabla\left|\nabla G\right|\right|^2\left|\nabla G\right|^{-1}\varphi^2\le\frac{4}{1-\delta}\int_\Sigma \left|\nabla G\right|\varphi^2+C(\delta)\int_\Sigma \left|\nabla G\right|\left|\nabla\varphi\right|^2 .$$

Plugging into (3.10) then yields that

$$(3.14)\qquad \left(\lambda_0(\Sigma)-\frac{1+4\delta}{1-\delta}\right)\int_\Sigma \left|\nabla G\right|\varphi^2\le C(\delta)\int_\Sigma \left|\nabla G\right|\left|\nabla\varphi\right|^2 .$$

We now estimate the right hand side. Obviously,

$$(3.15)\qquad \int_\Sigma \left|\nabla G\right|\left|\nabla\varphi\right|^2\le 2\int_\Sigma \left|\nabla G\right|\left|\nabla\chi\right|^2\psi^2+2\int_\Sigma \left|\nabla G\right|\left|\nabla\psi\right|^2\chi^2.$$

The first term of the right hand side is bounded by

$$\begin{aligned}\int_\Sigma |\nabla G| \left|\nabla \chi\right|^2 \psi^2 &\leq \frac{1}{(\ln\varepsilon)^2} \int_{L(\varepsilon^2,\varepsilon)} \frac{|\nabla G|^3}{G^2} \\ &\leq C\, \frac{1}{(\ln\varepsilon)^2}\, \ln^{2q}\left(1+\varepsilon^{-2}\right) \\ &\leq C\, \ln^{2q-2}\left(1+\varepsilon^{-1}\right),\end{aligned}$$

where we have used (3.7) in the second line. The second term of (3.15) is estimated as

$$\int_\Sigma |\nabla G| \left|\nabla \psi\right|^2 \chi^2 \leq \left(\int_{B_p(R+1)\setminus B_p(R)} |\nabla G|^2\right)^{\frac{1}{2}} \left(\int_{(B_p(R+1)\setminus B_p(R))\cap L(\varepsilon^2,\infty)} \chi^2\right)^{\frac{1}{2}} + C.$$

However, the integral estimate in [17] says that

$$\int_{\Sigma\setminus B_p(R)} G^2 \leq C\, e^{-2\sqrt{\lambda_0(\Sigma)}R}$$

and

$$\int_{\Sigma\setminus B_p(R)} |\nabla G|^2 \leq C\, e^{-2\sqrt{\lambda_0(\Sigma)}R}.$$

In particular,

$$\begin{aligned}\int_{(B_p(R+1)\setminus B_p(R))\cap L(\varepsilon^2,\infty)} \chi^2 &\leq \frac{1}{\varepsilon^4} \int_{\Sigma\setminus B_p(R)} G^2 \\ &\leq \frac{C}{\varepsilon^4} e^{-2\sqrt{\lambda_0(\Sigma)}\,R}.\end{aligned}$$

Hence,

$$\int_\Sigma |\nabla G| \left|\nabla \psi\right|^2 \chi^2 \leq \frac{C}{\varepsilon^2} e^{-2\sqrt{\lambda_0(\Sigma)}\,R} + C.$$

In conclusion, (3.15) becomes

$$\int_\Sigma |\nabla G| \left|\nabla \varphi\right|^2 \leq C\, \ln^{2q-2}\left(1+\varepsilon^{-1}\right) + \frac{C}{\varepsilon^2} e^{-2\sqrt{\lambda_0(\Sigma)}\,R} + C.$$

Plugging into (3.14), we arrive at

$$\left(\lambda_0\left(\Sigma\right) - \frac{1+4\delta}{1-\delta}\right) \int_\Sigma |\nabla G|\, \varphi^2 \leq C(\delta)\, \left(\ln^{2q-2}\left(1+\varepsilon^{-1}\right) + \varepsilon^{-2}\, e^{-2\sqrt{\lambda_0(\Sigma)}\,R} + C\right)$$

By first letting $R \to \infty$ and then $\varepsilon \to 0$, as $q < 1$, we conclude that

$$\left(\lambda_0\left(\Sigma\right) - \frac{1+4\delta}{1-\delta}\right) \int_{\Sigma\setminus B_p(2)} |\nabla G| \leq C(\delta). \tag{3.16}$$

However, using

$$\int_{\partial B_p(r)} \frac{\partial G}{\partial r} = -1,$$

we have

$$\int_{\Sigma\setminus B_p(2)} |\nabla G| \geq -\int_2^\infty dr \int_{\partial B_p(r)} \frac{\partial G}{\partial r} = \infty.$$

By (3.16), it follows that

$$\lambda_0(\Sigma) \leq \frac{1+4\delta}{1-\delta}$$

for any $\delta > 0$. Therefore, $\lambda_0(\Sigma) \leq 1$.

Now we consider the case that $K \geq -1$. By (2.2) we have

$$K_\Sigma |\nabla G| \geq -|\nabla G| - \frac{1}{2}|h|^2 |\nabla G|.$$

Therefore, (3.11) becomes

$$\Delta |\nabla G| \geq |\nabla |\nabla G||^2 |\nabla G|^{-1} - |\nabla G| - \frac{1}{2}|h|^2 |\nabla G|. \tag{3.17}$$

However, by the stability inequality (2.1),

$$\begin{aligned}
\int_\Sigma \left(|h|^2 + \mathrm{Ric}(\nu,\nu)\right)|\nabla G|\varphi^2 \quad \leq \quad & \frac{1}{4}\int_\Sigma |\nabla|\nabla G||^2 |\nabla G|^{-1}\varphi^2 \\
& + \frac{1}{2}\int_\Sigma \left\langle \nabla|\nabla G|, \nabla\varphi^2\right\rangle \\
& + \int_\Sigma |\nabla G| |\nabla\varphi|^2.
\end{aligned}$$

Hence, using that $\mathrm{Ric}(\nu,\nu) \geq -2$, we conclude

$$\begin{aligned}
\frac{1}{2}\int_\Sigma |h|^2 |\nabla G|\varphi^2 \quad \leq \quad & \frac{1}{8}\int_\Sigma |\nabla|\nabla G||^2 |\nabla G|^{-1}\varphi^2 + \int_\Sigma |\nabla G|\varphi^2 \\
& + \frac{1}{4}\int_\Sigma \left\langle \nabla|\nabla G|, \nabla\varphi^2\right\rangle + \frac{1}{2}\int_\Sigma |\nabla G||\nabla\varphi|^2.
\end{aligned} \tag{3.18}$$

By (3.17) and (3.18) it follows that for any small $\delta > 0$,

$$\begin{aligned}
\int_\Sigma |\nabla|\nabla G||^2 |\nabla G|^{-1}\varphi^2 \quad \leq \quad & \frac{1}{2}\int_\Sigma |h|^2|\nabla G|\varphi^2 + \int_\Sigma |\nabla G|\varphi^2 - \int_\Sigma \left\langle \nabla|\nabla G|, \nabla\varphi^2\right\rangle \\
\leq \quad & \left(\frac{1}{8}+\delta\right)\int_\Sigma |\nabla|\nabla G||^2 |\nabla G|^{-1}\varphi^2 + 2\int_\Sigma |\nabla G|\varphi^2 \\
& + C(\delta)\int_\Sigma |\nabla G||\nabla\varphi|^2.
\end{aligned}$$

In conclusion,

$$\int_\Sigma |\nabla|\nabla G||^2 |\nabla G|^{-1}\varphi^2 \leq \frac{16}{7-8\delta}\int_\Sigma |\nabla G|\varphi^2 + C(\delta)\int_\Sigma |\nabla G||\nabla\varphi|^2.$$

Plugging into (3.10), we have

$$\left(\lambda_0\left(\Sigma\right)-\frac{4+16\delta}{7-8\delta}\right)\int_\Sigma \left|\nabla G\right|\varphi^2\le C(\delta)\int_\Sigma \left|\nabla G\right|\left|\nabla\varphi\right|^2.$$

Similarly, we can conclude that

$$\lambda_0\left(\Sigma\right)\le\frac{4+16\delta}{7-8\delta}.$$

Now letting $\delta\to 0$, we have $\lambda_0\left(\Sigma\right)\le\frac{4}{7}$. □

The preceding argument can be generalized to stable minimal hypersurfaces of dimension up to five. Let (M,g) be an $(n+1)$-dimensional complete Riemannian manifold with its sectional curvature bounded below by

$$K\ge -1.$$

Let $\Sigma\subset M$ be a stable minimal hypersurface in M. Then the stability inequality (2.1) implies that

$$\int_\Sigma \left|h\right|^2\phi^2\le\int_\Sigma\left|\nabla\phi\right|^2+n\int_\Sigma\phi^2. \tag{3.19}$$

For a local orthonormal frame $\{e_1,\cdots,e_n\}$ of Σ, by the Gauss curvature equations,

$$\begin{aligned} R^\Sigma_{aa} &= \sum_c R_{acac}-\sum_c\left|h_{ac}\right|^2 \\ &\ge -(n-1)-\frac{n-1}{n}\left|h\right|^2 \end{aligned} \tag{3.20}$$

for indices $1\le a$, $c\le n$, where in the last line we have used that Σ is minimal. The argument in Lemma 12 can be carried over to prove the following.

Lemma 13. *Let M be an $(n+1)$-dimensional complete manifold with sectional curvature bounded below and $n\le 5$. For a complete stable minimal hypersurface Σ in M with $\lambda_0(\Sigma)>0$, its minimal positive Green's function $G\left(x\right)=G\left(p,x\right)$ satisfies*

$$\int_{\Sigma\setminus B_p(1)}\frac{\left|\nabla G\right|^4}{G^3\ln^{2q}\left(1+G^{-1}\right)}<\infty$$

for any $q>\frac{1}{2}$.

Proof. Let $v=\ln G$. Then, according to (3.3),

$$\begin{aligned} \frac{1}{2}\Delta\left|\nabla v\right|^2 &\ge \frac{1}{n-1}\left|\nabla v\right|^4+\frac{n}{n-1}\left|\nabla\left|\nabla v\right|\right|^2 \\ &\quad -\frac{n-2}{n-1}\left\langle\nabla\left|\nabla v\right|^2,\nabla v\right\rangle+\mathrm{Ric}^\Sigma\left(\nabla v,\nabla v\right) \end{aligned}$$

on $\Sigma\setminus\{p\}$. Note that by (3.20),

$$\mathrm{Ric}^\Sigma(\nabla v,\nabla v)\ge -C\left|\nabla v\right|^2-\frac{n-1}{n}\left(\mathrm{Ric}\left(\nu,\nu\right)+\left|h\right|^2\right)\left|\nabla v\right|^2$$

as the sectional curvature of M is bounded from below. Thus, for any cut-off function ϕ, noting that

$$\int_\Sigma \left(\mathrm{Ric}\,(\nu,\nu)+|h|^2\right)|\nabla v|^2\,\phi^2 \le \int_\Sigma |\nabla\left(|\nabla v|\,\phi\right)|^2$$

by the stability inequality (2.1), we have

$$\begin{aligned}
(3.21)\qquad & \frac{1}{n-1}\int_\Sigma |\nabla v|^4\,\phi^2 \\
\le\ & \frac{1}{2}\int_\Sigma \phi^2\,\Delta|\nabla v|^2 + C\int_\Sigma \phi^2\,|\nabla v|^2 - \frac{n}{n-1}\int_\Sigma \phi^2\,|\nabla|\nabla v||^2 \\
& +\frac{n-2}{n-1}\int_\Sigma \phi^2\,\left\langle \nabla|\nabla v|^2,\nabla v\right\rangle + \frac{n-1}{n}\int_\Sigma |\nabla\left(|\nabla v|\,\phi\right)|^2 \\
=\ & -\frac{1}{n}\int_\Sigma \phi\,\left\langle \nabla\phi,\nabla|\nabla v|^2\right\rangle + C\int_\Sigma \phi^2\,|\nabla v|^2 - \frac{2n-1}{n(n-1)}\int_\Sigma \phi^2\,|\nabla|\nabla v||^2 \\
& +\frac{n-2}{n-1}\int_\Sigma \phi^2\,\left\langle \nabla|\nabla v|^2,\nabla v\right\rangle + \frac{n-1}{n}\int_\Sigma |\nabla v|^2\,|\nabla\phi|^2.
\end{aligned}$$

Let $\phi = G^{\frac{1}{2}}\,\eta$, where η is a cut-off function on Σ with $\eta = 0$ on $B_p(1)$. Note that for $\delta > 0$,

$$\int_\Sigma |\nabla v|^2\,|\nabla\phi|^2 \le C(\delta)\int_\Sigma |\nabla v|^2\,|\nabla\eta|^2\,G + \frac{1+\delta}{4}\int_\Sigma \phi^2\,|\nabla v|^4.$$

Therefore, we conclude from (3.21) that

$$\begin{aligned}
(3.22)\qquad & \left(\frac{1}{n-1}-\frac{1+\delta}{4}\,\frac{n-1}{n}\right)\int_\Sigma |\nabla v|^4\,\phi^2 \\
\le\ & C(\delta)\int_\Sigma \left(\eta^2+|\nabla\eta|^2\right)G\,|\nabla v|^2 - \frac{1}{2n}\int_\Sigma \left\langle \nabla(G\eta^2),\nabla|\nabla v|^2\right\rangle \\
& -\frac{2n-1}{n(n-1)}\int_\Sigma \phi^2\,|\nabla|\nabla v||^2 + \frac{n-2}{n-1}\int_\Sigma \eta^2\,\left\langle \nabla|\nabla v|^2,\nabla G\right\rangle \\
=\ & C(\delta)\int_\Sigma \left(\eta^2+|\nabla\eta|^2\right)G\,|\nabla v|^2 - \frac{1}{2n}\int_\Sigma \left\langle \nabla\eta^2,\nabla|\nabla v|^2\right\rangle G \\
& -\frac{2n-1}{n(n-1)}\int_\Sigma \phi^2\,|\nabla|\nabla v||^2 + \left(\frac{n-2}{n-1}-\frac{1}{2n}\right)\int_\Sigma \eta^2\,\left\langle \nabla|\nabla v|^2,\nabla G\right\rangle.
\end{aligned}$$

However, as G is harmonic on $\Sigma\setminus\{p\}$,

$$\int_\Sigma \eta^2\,\left\langle \nabla|\nabla v|^2,\nabla G\right\rangle = -\int_\Sigma \left\langle\nabla\eta^2,\nabla G\right\rangle\,|\nabla v|^2.$$

Plugging into (3.22), we get

$$\begin{aligned}
(3.23) \quad & \left(\frac{1}{n-1}-\frac{1+\delta}{4}\,\frac{n-1}{n}\right)\int_\Sigma |\nabla v|^4\ \phi^2 \\
\leq\ & C(\delta)\int_\Sigma \left(\eta^2+|\nabla\eta|^2\right) G\ |\nabla v|^2-\frac{2}{n}\int_\Sigma \langle\nabla\eta,\nabla|\nabla v|\rangle\ G\eta\,|\nabla v| \\
& -\frac{2n-1}{n(n-1)}\int_\Sigma \eta^2 G\ |\nabla|\nabla v||^2-\left(\frac{n-2}{n-1}-\frac{1}{2n}\right)\int_\Sigma \left\langle\nabla\eta^2,\nabla G\right\rangle|\nabla v|^2. \\
\leq\ & C(\delta)\int_\Sigma\left(\eta^2+|\nabla\eta|^2\right)G\ |\nabla v|^2+\delta\int_\Sigma|\nabla v|^4\ \phi^2.
\end{aligned}$$

Therefore,

$$\left(\frac{1}{n-1}-\frac{1+\delta}{4}\,\frac{n-1}{n}-\delta\right)\int_\Sigma|\nabla v|^4\ \phi^2\leq C(\delta)\int_\Sigma\left(\eta^2+|\nabla\eta|^2\right)G\ |\nabla v|^2.$$

Since $n\leq 5$, one may choose $\delta=\delta(n)>0$ such that $\frac{1}{n-1}-\frac{1+\delta}{4}\,\frac{n-1}{n}-\delta>0$. In conclusion, there exists an absolute constant $\Gamma>0$ such that

$$\int_\Sigma|\nabla v|^4\ \phi^2\leq\Gamma\int_\Sigma\left(\eta^2+|\nabla\eta|^2\right)G\ |\nabla v|^2 \tag{3.24}$$

for any cut-off function η satisfying $\eta=0$ on $B_p(1)$.

For $\frac{1}{2}<q<1$, let $\eta=\psi\,w(G)$, where ψ is a cut-off function such that $\psi=0$ on $B_p(1)\cup(M\setminus B_p(2R))$, $\psi=1$ on $B_p(R)\setminus B_p(2)$, and

$$w(G)=\frac{1}{\ln^q\left(A\,G^{-1}\right)}$$

with $A=e^{2\sqrt{\Gamma}}\alpha$, $\alpha=\max_{\partial B_p(1)}G$. Direct calculations imply

$$\int_\Sigma\eta^2\,G\ |\nabla v|^2\leq\int_{L(0,\alpha)}\frac{|\nabla G|^2}{G\ln^{2q}\left(A\,G^{-1}\right)}\leq C$$

and

$$\begin{aligned}
\int_\Sigma|\nabla\eta|^2\ G\ |\nabla v|^2\ \leq\ & 2\int_\Sigma|\nabla\psi|^2\ \frac{|\nabla G|^2}{G\ln^{2q}\left(A\,G^{-1}\right)} \\
& +2\int_\Sigma\psi^2\,|\nabla w|^2\ G\ |\nabla v|^2 \\
\leq\ & C+\frac{1}{2\Gamma}\int_\Sigma\phi^2\ |\nabla v|^4.
\end{aligned}$$

Together with (3.24), we arrive at

$$\int_\Sigma|\nabla v|^4\ \phi^2\leq C+\frac{1}{2}\int_\Sigma\phi^2\ |\nabla v|^4.$$

In other words,

$$\int_\Sigma|\nabla v|^4\ \phi^2\leq C.$$

Finally, letting $R\to\infty$, one concludes that

$$\int_{\Sigma\setminus B_p(2)} \frac{|\nabla G|^4}{G^3 \ln^{2q}\left(A\, G^{-1}\right)} < \infty.$$

This proves the desired result. □

We are now ready to prove the following spectral estimate.

Theorem 14. *Let Σ be a complete stable minimal hypersurface in $(n+1)$-dimensional manifold M with $n \leq 5$. If the sectional curvature of M satisfies $K \geq -\kappa$ for some nonnegative constant κ, then*

$$\lambda_0\left(\Sigma\right) \leq \frac{2n(n-1)^2}{6n-n^2-1}\, \kappa.$$

Proof. Without loss of generality we assume that $\kappa = 1$ and $\lambda_0\left(\Sigma\right) > 0$. In particular, Σ is nonparabolic. For fixed $\varepsilon > 0$ small enough, define χ and ψ by (3.8) and (3.9), respectively, and let

$$\varphi = \chi\psi.$$

Setting

$$\phi = |\nabla G|^{\frac{1}{2}}\, \varphi$$

in the Poincaré inequality and expanding the right side, we get

$$\lambda_0\left(\Sigma\right) \int_\Sigma |\nabla G|\, \varphi^2 \leq \left(\frac{1}{4}+\delta\right) \int_\Sigma |\nabla\, |\nabla G||^2\, |\nabla G|^{-1}\, \varphi^2 + C(\delta) \int_\Sigma |\nabla G|\, |\nabla \varphi|^2 \tag{3.25}$$

for any $\delta > 0$. We now estimate the first term on the right hand side. Note that by (3.2) and (3.20),

$$\begin{aligned} \Delta\, |\nabla G| &\geq \frac{1}{n-1}\, |\nabla\, |\nabla G||^2\, |\nabla G|^{-1} + \mathrm{Ric}^\Sigma\left(\nabla G, \nabla G\right) |\nabla G|^{-1} \\ &\geq \frac{1}{n-1}\, |\nabla\, |\nabla G||^2\, |\nabla G|^{-1} - \left((n-1) + \frac{n-1}{n}\, |h|^2\right) |\nabla G|\,. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{n-1} \int_\Sigma |\nabla\, |\nabla G||^2\, |\nabla G|^{-1}\, \varphi^2 \quad &\leq \quad (n-1) \int_\Sigma |\nabla G|\, \varphi^2 \\ &\quad + \frac{n-1}{n} \int_\Sigma |h|^2\, |\nabla G|\, \varphi^2 \\ &\quad - \int_\Sigma \left\langle \nabla\, |\nabla G|\,, \nabla \varphi^2 \right\rangle. \end{aligned} \tag{3.26}$$

Using the stability inequality (3.19) we have that

$$\begin{aligned} \int_\Sigma |h|^2\, |\nabla G|\, \varphi^2 \quad &\leq \quad \int_\Sigma \left|\nabla \left(|\nabla G|^{\frac{1}{2}}\, \varphi\right)\right|^2 + n \int_\Sigma |\nabla G|\, \varphi^2 \\ &\leq \quad \frac{1}{4} \int_\Sigma |\nabla\, |\nabla G||^2\, |\nabla G|^{-1}\, \varphi^2 + \int_\Sigma |\nabla G|\, |\nabla \varphi|^2 \\ &\quad + \frac{1}{2} \int_\Sigma \left\langle \nabla \varphi^2, \nabla\, |\nabla G| \right\rangle + n \int_\Sigma |\nabla G|\, \varphi^2. \end{aligned}$$

Combining with (3.26) we obtain that

$$\begin{aligned}\left(\frac{1}{n-1}-\frac{n-1}{4n}\right)\int_\Sigma \left|\nabla\left|\nabla G\right|\right|^2\left|\nabla G\right|^{-1}\varphi^2 &\leq 2(n-1)\int_\Sigma\left|\nabla G\right|\varphi^2+C\int_\Sigma\left|\nabla G\right|\left|\nabla\varphi\right|^2\\ &\quad +C\int_\Sigma\left|\left\langle\nabla\varphi^2,\nabla\left|\nabla G\right|\right\rangle\right|\\ &\leq 2(n-1)\int_\Sigma\left|\nabla G\right|\varphi^2+C(\delta)\int_\Sigma\left|\nabla G\right|\left|\nabla\varphi\right|^2\\ &\quad +\delta\int_\Sigma\left|\nabla\left|\nabla G\right|\right|^2\left|\nabla G\right|^{-1}\varphi^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_\Sigma\left|\nabla\left|\nabla G\right|\right|^2\left|\nabla G\right|^{-1}\varphi^2 &\leq \frac{8n(n-1)^2}{6n-n^2-1-4n(n-1)\delta}\int_\Sigma\left|\nabla G\right|\varphi^2\\ &\quad +C(\delta)\int_\Sigma\left|\nabla G\right|\left|\nabla\varphi\right|^2.\end{aligned}$$

Plugging into (3.25) then yields that

$$\left(\lambda_0(\Sigma)-\left(\frac{1}{4}+\delta\right)\frac{8n(n-1)^2}{6n-n^2-1-4n(n-1)\delta}\right)\int_\Sigma\left|\nabla G\right|\varphi^2\leq C(\delta)\int_\Sigma\left|\nabla G\right|\left|\nabla\varphi\right|^2.$$

Using Lemma 13, one concludes as before that

$$\lambda_0(\Sigma)\leq\frac{2n(n-1)^2}{6n-n^2-1}.$$

This proves the result. □

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