GEOMETRY OF SHRINKING RICCI SOLITONS

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Abstract. The main purpose of this paper is to investigate the curvature behavior of four dimensional shrinking gradient Ricci solitons. For such a soliton M with bounded scalar curvature S, it is shown that the curvature operator Rm of M satisfies the estimate $|\mathrm{Rm}| \leq c\, S$ for some constant c. Moreover, the curvature operator Rm is asymptotically nonnegative at infinity and admits a lower bound $\mathrm{Rm} \geq -c\, (\ln(r+1))^{-1/4}$, where r is the distance function to a fixed point in M. As an application, we prove that if the scalar curvature converges to zero at infinity, the soliton is asymptotically conical.

As a separate issue, a diameter upper bound for compact shrinking gradient Ricci solitons of arbitrary dimension is derived in terms of the injectivity radius.

1. Introduction

This paper primarily concerns the geometry of the so-called shrinking gradient Ricci solitons. Recall that a complete manifold (M, g) is a gradient Ricci soliton if the equation

$$\mathrm{Ric} + \mathrm{Hess}\,(f) = \lambda\, g$$

holds for some function f and scalar λ. Here, Ric is the Ricci curvature of (M, g) and $\mathrm{Hess}\,(f)$ the Hessian of f. Note that if the potential function f is constant or the soliton is trivial, then the soliton equation simply says the Ricci curvature is constant. So Ricci solitons are natural generalization of Einstein manifolds. A soliton is called shrinking, steady and expanding, accordingly, if $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$. By scaling the metric g, one customarily assumes $\lambda \in \{-1/2, 0, 1/2\}$. Solitons may be regarded as self-similar solutions to the Ricci flows. As such, they are important in the singularity analysis of Ricci flows. Indeed, according to [11], the blow-ups around a type-I singularity point always converge to nontrivial gradient shrinking Ricci solitons. It is thus a central issue in the study of Ricci flows to understand and classify gradient Ricci solitons.

Aside from the Einstein manifolds, the Euclidean space $\mathbb{R}^n$ together with potential function $f(x) = \frac{\lambda}{2}\,|x|^2$ gives another important example of gradient Ricci solitons. In the case dimension $n = 2$, those are the only examples of gradient shrinking Ricci solitons [7]. For dimension $n = 3$, Perelman made the breakthrough in [25] and showed that a three dimensional non-collapsing shrinking gradient Ricci soliton with bounded curvature must be a quotient of the sphere $\mathbb{S}^3$, or $\mathbb{R}^3$, or $\mathbb{S}^2 \times \mathbb{R}$. His result played a crucial role in the affirmative resolution of the Poincaré conjecture. The extra conditions were later removed through the effort of Naber [23], Ni and Wallach [24], and Cao, Chen and Zhu [3]. We refer the readers to [1] for the classification of steady gradient Ricci solitons.

One salient feature of three dimensional shrinking Ricci solitons is that their curvature operator must be nonnegative [16]. This has been of great utility in Perelman's argument. Unfortunately, for dimension four or higher, this is no longer true as demonstrated by the example constructed in [13]. Also, the existence of examples (see [2] for a list) other than the aforementioned ones complicates the classification outlook.

The main purpose here is to investigate the curvature behavior of four dimensional shrinking gradient Ricci solitons. Our first result concerns the control of the curvature operator. Note that in the case of dimension three, the curvature operator, being nonnegative, is obviously bounded by the scalar curvature. In the case of dimension four, while the curvature operator no longer has a fixed sign, we show that such a conclusion still holds. In particular, it implies that the curvature operator must be bounded if the scalar curvature is so.

Theorem 1.1. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature S. Then there exists a constant $c > 0$ so that*

$$|\mathrm{Rm}| \leq c\, S \quad on\ M.$$

Our second result provides a lower bound for the curvature operator of a four dimensional shrinking Ricci soliton with bounded scalar curvature. It shows that the curvature operator becomes asymptotically nonnegative at infinity. The result may be viewed as an extension of Hamilton and Ivey curvature pinching estimate for the three dimension case. Note that Naber [23] has classified all four dimensional shrinking Ricci solitons with nonnegative and bounded curvature operator. In passing, we would also like to point out that Cao and Chen [4] have obtained some interesting classification results by imposing assumptions of different nature on the curvature tensor.

Theorem 1.2. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature. Then its curvature operator is bounded below by*

$$\mathrm{Rm} \geq -\left(\frac{c}{\ln(r+1)}\right)^{\frac{1}{4}},$$

where r is the distance function to a fixed point in M.

In both Theorem 1.1 and Theorem 1.2, the constant $c > 0$ depends only on the upper bound A of the scalar curvature on M, and on the geometry of the geodesic ball $B_p(r_0)$, where p is a minimum point of f and r_0 is determined by A.

We should point out that these conclusions are only effective for nontrivial solitons. In fact, the potential function f of the soliton is exploited in an essential way in our proofs by working on the level sets of f. Note that the level set is of three dimension. So its curvature tensor is determined by its Ricci curvature. This fact is crucial to our argument. It enables us to control the curvature tensor of the ambient manifold by its Ricci curvature, which leads to an estimate of the Ricci curvature by the scalar curvature. The fact that Ricci curvature controls the growth of the full curvature tensor is already known from the work of the first named author and Wang [22]. However, the argument and estimate there are global in nature, whereas the estimate here is valid in the pointwise sense, hence stronger.

The fact that the curvature operator of four dimensional shrinking gradient Ricci solitons enjoys similar control as in the dimension three case seems to provide a glimpse of hope for a possible classification of the solitons. In this sense, one goal here is to prove sharp decay estimates for the Riemann curvature tensor and its covariant derivatives, under the assumption that the scalar curvature converges to zero at infinity. This in particular enables us to conclude that such a soliton must in fact be smoothly asymptotic to a cone at infinity. Here, by a cone, we mean a manifold $[0,\infty)\times\Sigma$ endowed with Riemannian metric $g_c = dr^2 + r^2\, g_\Sigma$, where (Σ, g_Σ) is a closed $(n-1)$-dimensional Riemannian manifold. Denote $E_R = (R,\infty)\times\Sigma$ for $R \geq 0$ and define the dilation by λ to be the map $\rho_\lambda : E_0 \to E_0$ given by $\rho_\lambda(r,\sigma) = (\lambda\, r, \sigma)$.

We say that a Riemannian manifold (M,g) is C^k asymptotic to the cone (E_0, g_c) if, for some $R > 0$, there is a diffeomorphism $\Phi : E_R \to M \setminus \Omega$ such that $\lambda^{-2}\, \rho_\lambda^*\, \Phi^*\, g \to g_c$ as $\lambda \to \infty$ in $C^k_{loc}(E_0, g_c)$, where Ω is a compact subset of M.

We have the following result.

Theorem 1.3. *Let (M,g,f) be a complete four dimensional shrinking gradient Ricci soliton with scalar curvature converging to zero at infinity. Then there exists a cone E_0 such that (M,g) is C^k asymptotic to E_0 for all k.*

A recent result due to Kotschwar and L. Wang [20] states that two shrinking gradient Ricci solitons must be isometric if they are C^2 asymptotic to the same cone. Together with our result, this implies that the classification problem for four dimensional shrinking Ricci solitons with scalar curvature going to zero at infinity is reduced to the one for the limiting cones.

As a separate issue, we have also attempted to address the question whether the limit of compact shrinking gradient Ricci solitons remains compact. This question may be rephrased into one of obtaining a uniform upper bound for the diameter of such solitons. Note that in the opposite direction Futaki and Sano [14], see also an improvement in [15], have already established a universal diameter lower bound for (nontrivial) compact shrinking gradient Ricci solitons. It remains to be seen whether a universal diameter upper bound is available without any extra assumptions.

Theorem 1.4. *Let (M,g,f) be a compact gradient shrinking Ricci soliton of dimension n. Then the diameter of (M,g) has an upper bound of the form*

$$\operatorname{diam}(M) \leq c\,(n, \operatorname{inj}(M)),$$

where $\operatorname{inj}(M)$ *is the injectivity radius of* (M,g).

If one assumes in addition that the Ricci curvature of the soliton is nonnegative, then the conclusion follows from [12]. We also remark that the assumption on the injectivity radius seems to be natural in view of the non-collapsing result for Ricci flows proved by Perelman [25]. Combined with [5], our result implies an upper bound for the volume, depending on the injectivity radius alone.

The organization of the paper is as follows. In Section 2 we show that four dimensional shrinking Ricci solitons with bounded scalar curvature have bounded Riemann curvature tensor. The precise form stated in Theorem 1.1 is proved in Section 3, which builds upon and refines the techniques used in Section 2. These estimates are then used in Section 4 to establish the pinching estimate in Theorem 1.2. In Section 5 we have established the conical structure in Theorem 1.3. Finally, the diameter upper bound in Theorem 1.4 is proved in Section 6.

2. Curvature estimates

In this section, we show that the curvature operator of a four dimensional shrinking gradient Ricci soliton must be bounded if its scalar curvature is so. We first recall some general facts concerning shrinking gradient Ricci solitons which will be used throughout the paper. For (M^n, g, f) a shrinking gradient Ricci soliton, it is known [16] that $S + |\nabla f|^2 - f$ is constant on M, where S is the scalar curvature of M. So, by adding a constant to f if necessary, we may normalize the soliton such that

$$S + |\nabla f|^2 = f. \tag{2.1}$$

Also, by a result of Chen [6, 2], the scalar curvature $S > 0$ unless M is flat. So in the following, we will assume without loss of generality that $S > 0$. Tracing the soliton equation we get $\Delta f + S = \frac{n}{2}$. Combined with (2.1), this implies that

$$\Delta_f (f) = \frac{n}{2} - f. \tag{2.2}$$

Here, $\Delta_f = \Delta - \langle \nabla f, \nabla \rangle$ is the weighted Laplacian on M, which is self adjoint on the space of square integrable functions with respect to the weighted measure $e^{-f} dv$.

Concerning the potential function f, Cao and Zhou [5] have proved that

$$\left(\frac{1}{2} r(x) - c\right)^2 \leq f(x) \leq \left(\frac{1}{2} r(x) + c\right)^2 \tag{2.3}$$

for all $r(x) \geq r_0$. Here $r(x) := d(p, x)$ is the distance of x to p, a minimum point of f on M, which always exists. Both constants r_0 and c can be chosen to depend only on dimension n.

Throughout the paper, we denote

$$\begin{aligned} D(t) &= \{x \in M : f(x) \leq t\} \\ \Sigma(t) &= \partial D(t) = \{x \in M : f(x) = t\}. \end{aligned}$$

By (2.3), these are compact subsets of M.

We recall the following equations for curvatures. For proofs, one may consult [26].

$$\begin{aligned} \Delta_f S &= S - 2|\mathrm{Ric}|^2 \\ \Delta_f R_{ij} &= R_{ij} - 2R_{ikjl}R_{kl} \\ \Delta_f \mathrm{Rm} &= \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm} \\ \nabla_k R_{jk} &= R_{jk} f_k = \frac{1}{2} \nabla_j S \\ \nabla_l R_{ijkl} &= R_{ijkl} f_l = \nabla_j R_{ik} - \nabla_i R_{jk}. \end{aligned} \tag{2.4}$$

In this section we will assume

$$S \leq A \text{ on } M \tag{2.5}$$

for some constant $A > 0$. Obviously, there exists $r_0 > 0$, depending only on A, so that

$$|\nabla f| \geq \frac{1}{2}\sqrt{f} \geq 1 \text{ on } M \backslash D(r_0).$$

Our argument is based on the following important observation.

Proposition 2.1. *Let* (M, g, f) *be a four dimensional shrinking gradient Ricci soliton. Then for a universal constant* $c > 0$,

$$|\mathrm{Rm}| \leq c \left(\frac{|\nabla \mathrm{Ric}|}{\sqrt{f}} + \frac{|\mathrm{Ric}|^2 + 1}{f} + |\mathrm{Ric}| \right) \tag{2.6}$$

on $M \backslash D(r_0)$.

Proof. We work on $\Sigma := \Sigma(t)$, $t \geq r_0$. By the Gauss curvature equation, for an orthonormal frame $\{e_1, e_2, e_3\}$ tangent to Σ, the intrinsic Riemann curvature tensor R^Σ_{abcd} of Σ is given by

$$R^\Sigma_{abcd} = R_{abcd} + h_{ac}h_{bd} - h_{ad}h_{bc}, \tag{2.7}$$

where $R_{abcd} = \mathrm{Rm}(e_a, e_b, e_c, e_d)$ is the Riemann curvature tensor of M and h_{ab} the second fundamental form of Σ. In what follows, the indices $a, b, c, d \in \{1, 2, 3\}$ and $i, j, k, l \in \{1, 2, 3, 4\}$. Since $\Sigma = \{f = t\}$, we have

$$h_{ab} = \frac{f_{ab}}{|\nabla f|}.$$

Using the fact that $|\nabla f| \geq \frac{1}{2}\sqrt{f}$ on $M \backslash D(r_0)$ and $\mathrm{Ric}(e_a, e_b) + f_{ab} = \frac{1}{2}\delta_{ab}$, we have

$$|h_{ab}| \leq \frac{c}{\sqrt{f}} (|\mathrm{Ric}| + 1). \tag{2.8}$$

Since Σ is a three dimensional manifold, its Riemann curvature is determined by its Ricci curvature Ric^Σ.

$$\begin{aligned} R^\Sigma_{abcd} &= \left(R^\Sigma_{ac} g_{bd} - R^\Sigma_{ad} g_{bc} + R^\Sigma_{bd} g_{ac} - R^\Sigma_{bc} g_{ad} \right) \\ &\quad - \frac{S^\Sigma}{2} (g_{ac}g_{bd} - g_{ad}g_{bc}), \end{aligned} \tag{2.9}$$

where S^Σ is the scalar curvature of Σ and $R^\Sigma_{ab} := \mathrm{Ric}^\Sigma(e_a, e_b)$.

By tracing (2.7) we get

$$R^\Sigma_{ac} = R_{ac} - R_{a4c4} + H\, h_{ac} - h_{ab}\, h_{bc}, \tag{2.10}$$

where $R_{a4c4} = \mathrm{Rm}(e_a, \nu, e_c, \nu)$ with $\nu = \frac{\nabla f}{|\nabla f|}$ being the normal vector of Σ. Tracing this one more time, we have

$$S^\Sigma = S - 2R_{44} + H^2 - |h|^2.$$

Hence, by (2.8) and $S \leq 2\,|\mathrm{Ric}|$, we see that

$$\left| S^\Sigma \right| \leq c \left(\frac{|\mathrm{Ric}|^2 + 1}{f} + |\mathrm{Ric}| \right)$$

for some constant c. We now observe that by (2.4) we have an estimate

$$|R_{ijk4}| = \frac{1}{|\nabla f|} |R_{ijkl} f_l| \leq 4 \frac{|\nabla \mathrm{Ric}|}{\sqrt{f}}. \tag{2.11}$$

Using this in (2.10) implies that

$$\left|R_{ac}^{\Sigma}\right| \leq c\left(\frac{|\nabla \mathrm{Ric}|}{\sqrt{f}}+\frac{|\mathrm{Ric}|^{2}+1}{f}+|\mathrm{Ric}|\right).$$

Hence, we conclude from (2.9) and (2.7) that

$$\left|R_{abcd}\right| \leq c\left(\frac{|\nabla \mathrm{Ric}|}{\sqrt{f}}+\frac{|\mathrm{Ric}|^{2}+1}{f}+|\mathrm{Ric}|\right).$$

Together with (2.11), this proves the proposition. □

We now establish the following lemma. It is inspired by Hamilton's work in [17].

Lemma 2.2. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature. Then, for any $0<a<1$, the function $u:=|\mathrm{Ric}|^{2} S^{-a}$ satisfies the differential inequality*

$$\Delta_{f} u \geq\left(2 a-\frac{c}{1-a} \frac{S}{f}\right) u^{2} S^{a-1}-c\, u^{\frac{3}{2}} S^{\frac{a}{2}}-c\, u$$

on $M \backslash D\left(r_{0}\right)$, for some universal constant $c>0$.

Proof. Note that by Proposition 2.1 and (2.4)

$$\begin{aligned}
\Delta_{f}|\mathrm{Ric}|^{2} & \geq 2|\nabla \mathrm{Ric}|^{2}-c|\mathrm{Rm}||\mathrm{Ric}|^{2} \\
& \geq 2|\nabla \mathrm{Ric}|^{2}-\frac{c}{\sqrt{f}}|\nabla \mathrm{Ric}||\mathrm{Ric}|^{2} \\
& \quad -\frac{c}{f}|\mathrm{Ric}|^{4}-c|\mathrm{Ric}|^{3}-\frac{c}{f}|\mathrm{Ric}|^{2} .
\end{aligned} \tag{2.12}$$

For $1>a>0$ direct computation gives

$$\begin{aligned}
& \Delta_{f}\left(|\mathrm{Ric}|^{2} S^{-a}\right) \\
= & S^{-a} \Delta_{f}\left(|\mathrm{Ric}|^{2}\right)+|\mathrm{Ric}|^{2} \Delta_{f}\left(S^{-a}\right)+2\left\langle\nabla S^{-a}, \nabla|\mathrm{Ric}|^{2}\right\rangle \\
= & S^{-a} \Delta_{f}\left(|\mathrm{Ric}|^{2}\right)+2\left\langle\nabla S^{-a}, \nabla|\mathrm{Ric}|^{2}\right\rangle \\
+ & |\mathrm{Ric}|^{2}\left(-a S^{-a}+2 a|\mathrm{Ric}|^{2} S^{-a-1}+a(a+1)|\nabla S|^{2} S^{-a-2}\right) .
\end{aligned} \tag{2.13}$$

We can estimate

$$\begin{aligned}
2\left\langle\nabla S^{-a}, \nabla|\mathrm{Ric}|^{2}\right\rangle & \geq -4 a|\nabla \mathrm{Ric}||\nabla S| S^{-a-1}|\mathrm{Ric}| \\
& \geq -a(a+1)|\nabla S|^{2} S^{-a-2}|\mathrm{Ric}|^{2} \\
& \quad -\frac{4 a}{a+1}|\nabla \mathrm{Ric}|^{2} S^{-a} .
\end{aligned}$$

Plugging this in (2.13) and combining with (2.12) shows that

$$\begin{aligned}
\Delta_{f}\left(|\mathrm{Ric}|^{2} S^{-a}\right) \geq & \frac{2(1-a)}{1+a}|\nabla \mathrm{Ric}|^{2} S^{-a}-\frac{c}{\sqrt{f}}|\nabla \mathrm{Ric}||\mathrm{Ric}|^{2} S^{-a}-\frac{c}{f}|\mathrm{Ric}|^{4} S^{-a} \\
& -c|\mathrm{Ric}|^{3} S^{-a}-\frac{c}{f}|\mathrm{Ric}|^{2} S^{-a}-a|\mathrm{Ric}|^{2} S^{-a}+2 a|\mathrm{Ric}|^{4} S^{-a-1} \\
\geq & \left(2 a-\frac{c}{1-a} \frac{S}{f}\right)|\mathrm{Ric}|^{4} S^{-a-1}-c|\mathrm{Ric}|^{3} S^{-a}-c|\mathrm{Ric}|^{2} S^{-a} .
\end{aligned}$$

In the last line, we have used that

$$\frac{c}{\sqrt{f}}\left|\nabla \mathrm{Ric}\right|\left|\mathrm{Ric}\right|^2 S^{-a} \leq \frac{2\left(1-a\right)}{1+a}\left|\nabla \mathrm{Ric}\right|^2 S^{-a} + \frac{1+a}{8\left(1-a\right)}\frac{c^2}{f}\left|\mathrm{Ric}\right|^4 S^{-a}.$$

It follows that

$$\Delta_f u \geq \left(2a - \frac{c}{1-a}\frac{S}{f}\right) u^2 S^{a-1} - c\, u^{\frac{3}{2}} S^{\frac{a}{2}} - c\, u.$$

This proves the result. □

Proposition 2.3. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature $S \leq A$. Then there exist r_1 and C depending only on A so that*

$$\left|\mathrm{Ric}\right|^2 \leq C \quad \text{on } M\backslash D\left(r_1\right).$$

Proof. We use Lemma 2.2 with $a := \frac{1}{2}$ and obtain on $M\backslash D\left(r_0\right)$

$$\Delta_f u \geq \left(1 - \frac{c}{f}\right) u^2 S^{-\frac{1}{2}} - c\, u^{\frac{3}{2}} - c\, u,$$

where $u := \left|\mathrm{Ric}\right|^2 S^{-\frac{1}{2}}$ and $c > 0$ depends only on A. There exists $r_1 > 0$, depending only on A, so that $1 - \frac{c}{f} \geq \frac{1}{2}$. This implies that

$$\Delta_f u \geq \frac{1}{2} u^2 S^{-\frac{1}{2}} - c\, u^{\frac{3}{2}} - c\, u \quad \text{on } M\backslash D\left(r_1\right).$$

For $R > 2r_1$, let ϕ be a smooth non-negative function defined on the real line so that $\phi\left(t\right) = 1$ for $R \leq t \leq 2R$ and $\phi\left(t\right) = 0$ for $t \leq \frac{R}{2}$ and for $t \geq 3R$. We may choose ϕ so that

$$t^2\left(\left|\phi'\right|^2\left(t\right) + \left|\phi''\right|\left(t\right)\right) \leq c.$$

We use $\phi\left(f\left(x\right)\right)$ as a cut-off function with support in $D\left(3R\right)\backslash D\left(\frac{R}{2}\right)$. Note that

$$\begin{aligned} \left|\nabla\phi\right| &\leq \frac{c}{\sqrt{R}} \\ \left|\Delta_f \phi\right| &\leq c \end{aligned}$$

for a universal constant $c > 0$. Here, for the second inequality we have used (2.2) that $\Delta_f\left(f\right) = 2 - f$.

A direct computation shows that the function $G := u\phi^2$ satisfies

$$\begin{aligned} \phi^2 \Delta_f G &= \phi^2 \Delta_f\left(u\phi^2\right) \\ &= \phi^4\left(\Delta_f u\right) + \phi^2 u\left(\Delta_f \phi^2\right) + 2\phi^2\left\langle \nabla u, \nabla\phi^2\right\rangle \\ &\geq \frac{1}{2} u^2 S^{-\frac{1}{2}}\phi^4 - cu^{\frac{3}{2}}\phi^3 - cu\phi^2 + 2\left\langle \nabla\left(u\phi^2\right), \nabla\phi^2\right\rangle \\ &\geq \frac{1}{2} G^2 S^{-\frac{1}{2}} - cG^{\frac{3}{2}} - cG + 2\left\langle \nabla G, \nabla\phi^2\right\rangle. \end{aligned}$$

The maximum principle implies that $G \leq c$, where the constant c depends only on A. Hence, on $D\left(2R\right)\backslash D\left(R\right)$,

$$\left|\mathrm{Ric}\right|^2 = GS^{\frac{1}{2}} \leq c.$$

This proves the proposition. □

We now prove that the curvature of four dimensional shrinking gradient Ricci solitons with bounded scalar curvature is bounded.

Theorem 2.4. *Let (M,g,f) be a four dimensional shrinking Ricci soliton with bounded scalar curvature $S\leq A$. Then the Riemann curvature tensor and its covariant derivatives are bounded in norm as well. More precisely, there exists $r_1>0$ depending only on A so that for any $k\geq 1$,*

$$\sup_{M\setminus D(r_1)}\left(|\mathrm{Rm}|+\left|\nabla^k\mathrm{Rm}\right|\right)\leq C_k,$$

where $C_k>0$ is a constant depending only on A and k.

Proof. We first show that

$$|\mathrm{Rm}|\leq c \quad \text{on } M\setminus D\left(r_1\right). \tag{2.14}$$

Using (2.4) and the Kato inequality, one sees that

$$\Delta_f\left|\mathrm{Rm}\right|\geq -c\left|\mathrm{Rm}\right|^2,$$

for a universal constant c. Rewrite this into

$$\Delta_f\left|\mathrm{Rm}\right|\geq \left|\mathrm{Rm}\right|^2-(c+1)\left|\mathrm{Rm}\right|^2. \tag{2.15}$$

By Proposition 2.3 and Proposition 2.1, we have

$$\left|\mathrm{Rm}\right|^2\leq c\left(\frac{1}{f}\left|\nabla\mathrm{Ric}\right|^2+1\right)$$

on $M\setminus D\left(r_1\right)$, where $c>0$ depends only on A. Plugging into (2.15), we conclude

$$\Delta_f\left|\mathrm{Rm}\right|\geq \left|\mathrm{Rm}\right|^2-\frac{c}{f}\left|\nabla\mathrm{Ric}\right|^2-c. \tag{2.16}$$

On the other hand, we know from (2.12) that

$$\Delta_f\left|\mathrm{Ric}\right|^2\geq\left|\nabla\mathrm{Ric}\right|^2-c. \tag{2.17}$$

Therefore, combining (2.16) and (2.17), we obtain

$$\begin{aligned}\Delta_f\left(\left|\mathrm{Rm}\right|+\left|\mathrm{Ric}\right|^2\right) &\geq \left|\mathrm{Rm}\right|^2-c\\ &\geq \frac{1}{2}\left(\left|\mathrm{Rm}\right|+\left|\mathrm{Ric}\right|^2\right)^2-c.\end{aligned}$$

In other words, the function

$$v:=\left|\mathrm{Rm}\right|+\left|\mathrm{Ric}\right|^2$$

satisfies the following differential inequality on $M\setminus D\left(r_1\right)$

$$\Delta_f v\geq\frac{1}{2}v^2-c,$$

for some constant depending only on A. Arguing as in Proposition 2.3, we conclude that $v\leq c$ on $M\setminus D\left(r_1\right)$, for a constant c depending only on A. This implies (2.14).

Now we use Shi's derivative estimates [27] to prove that

$$\left|\nabla\mathrm{Rm}\right|\leq c\ \text{ on } M\setminus D\left(r_1\right), \tag{2.18}$$

for some constant $c>0$ depending only on A. Note that

$$\begin{aligned}\Delta_f\left|\nabla\mathrm{Rm}\right|^2 &\geq 2\left|\nabla^2\mathrm{Rm}\right|^2-c\left|\nabla\mathrm{Rm}\right|^2\left|\mathrm{Rm}\right|\\ &\geq 2\left|\nabla\left|\nabla\mathrm{Rm}\right|\right|^2-c\left|\nabla\mathrm{Rm}\right|^2,\end{aligned}$$

where in the last line we have used 2.14. This implies

$$\Delta_f |\nabla \mathrm{Rm}| \geq -c |\nabla \mathrm{Rm}| \quad \text{on} \quad M \backslash D(r_1).$$

We also know from (2.4) that

$$\begin{aligned}\Delta_f |\mathrm{Rm}|^2 &\geq 2 |\nabla \mathrm{Rm}|^2 - c |\mathrm{Rm}|^3 \\ &\geq 2 |\nabla \mathrm{Rm}|^2 - c.\end{aligned}$$

This implies that there exists a constant $c > 0$, depending only on A, so that on $M \backslash D(r_1)$,

$$\Delta_f \left(|\nabla \mathrm{Rm}| + |\mathrm{Rm}|^2 \right) \geq \left(|\nabla \mathrm{Rm}| + |\mathrm{Rm}|^2 \right)^2 - c.$$

Now a maximum principle argument as in Proposition 2.3 shows that $|\nabla \mathrm{Rm}| + |\mathrm{Rm}|^2$ is bounded on $M \backslash D(r_1)$ by a constant depending only on A. So (2.18) follows. A similar argument can be used to obtain estimates for higher order derivatives. For details, see e.g. [8]. □

Theorem 2.4 has some interesting applications to compactness properties of the set of four dimensional shrinking Ricci solitons. Indeed, as a consequence of Haslhofer and Müller's work in [18], see also [19] for recent progress, the set of four dimensional shrinkers with entropy bounded below and with Euler characteristic and scalar curvature bounded above is precompact in orbifold Cheeger-Gromov sense. Theorem 2.4 shows moreover that, for the limit of a sequence of such shrinkers, the possible orbifold points must be contained within a fixed compact set.

3. Improved curvature estimates

Our goal in this section is to prove Theorem 1.1. This is done by a careful refinement of the techniques of the previous section. We begin with an improved estimate for Ricci curvature. From now on $r_0 > 0$ denotes a radius depending only on A, the upper bound of S on M. Unless otherwise specified, $c > 0$ is a constant depending only on A and the geometry of $D(r_0)$. These constants may change from line to line.

Proposition 3.1. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature $S \leq A$. There exists $c > 0$ so that*

$$\sup_M \frac{|\mathrm{Ric}|^2}{S} \leq c.$$

Proof. We use Lemma 2.2 that

$$\Delta_f u \geq \left(2a - \frac{c}{1-a} \frac{S}{f} \right) u^2 S^{a-1} - c\, u^{\frac{3}{2}} S^{\frac{a}{2}} - c\, u$$

on $M \backslash D(r_0)$, where $u := |\mathrm{Ric}|^2 S^{-a}$ and $0 < a < 1$.

For $R > 2r_0$, let ϕ be a smooth non-negative function defined on the real line so that $\phi(t) = 1$ for $R \leq t \leq 2R$ and $\phi(t) = 0$ for $t \leq \frac{R}{2}$ and for $t \geq 3R$. We may choose ϕ so that

$$t^2 \left(|\phi'|^2 (t) + |\phi''| (t) \right) \leq c.$$

We use $\phi(f(x))$ as a cut-off function with support in $D(3R)\setminus D\left(\frac{R}{2}\right)$. Note that $|\nabla\phi| \leq \frac{c}{\sqrt{R}}$ and $|\Delta_f \phi| \leq c$, for a universal constant $c > 0$. Direct computation gives

$$\begin{aligned}\phi^2 \Delta_f \left(u\phi^2\right) &= \phi^4 \left(\Delta_f u\right) + \phi^2 u \left(\Delta_f \phi^2\right) + 2\phi^2 \left\langle \nabla u, \nabla \phi^2 \right\rangle \\ &\geq \left(2a - \frac{c}{1-a}\frac{S}{f}\right) u^2 S^{a-1}\phi^4 - cu^{\frac{3}{2}} S^{\frac{a}{2}} \phi^3 \\ &\quad -cu\phi^2 + 2\left\langle \nabla \left(u\phi^2\right), \nabla\phi^2 \right\rangle.\end{aligned}$$

Since ϕ has support in $D(3R)\setminus D\left(\frac{R}{2}\right)$, we know that $f \geq \frac{1}{2}R$ on the support of ϕ. Hence, we may choose $a := 1 - \frac{C}{R}$ with a sufficiently large constant $C > 0$ such that

$$2a - \frac{c}{1-a}\frac{S}{f} \geq 1$$

on the support of ϕ. As a result, the function $G := u\phi^2$ satisfies

$$\phi^2 \Delta_f G \geq S^{a-1} G^2 - cG^{\frac{3}{2}} - cG + 2\left\langle \nabla G, \nabla \phi^2 \right\rangle.$$

Since $a < 1$ and $S^{a-1} \geq A^{a-1}$, the maximum principle implies that there exists $c > 0$ so that $G \leq c$ on $D(2R)\setminus D(R)$.

Hence, on $D(2R)\setminus D(R)$,

$$\frac{|\mathrm{Ric}|^2}{S} = GS^{a-1} \leq cS^{a-1}.$$

Let us recall a result in [9] that there exists a constant $c > 0$ so that $Sf \geq c$ on M. In our context, this constant has the dependency as stated in the conclusion of the proposition.

Since $a-1 = -\frac{C}{R}$ and $S \geq \frac{c}{R}$ on $D(2R)$, it follows that $S^{a-1} \leq c$ on $D(2R)\setminus D(R)$. Therefore,

$$\frac{|\mathrm{Ric}|^2}{S} \leq c$$

on $D(2R)\setminus D(R)$. Since R is arbitrary, this proves the result. □

We now extend this result to the full curvature tensor.

Proposition 3.2. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature $S \leq A$. There exists $c > 0$ so that*

$$\sup_M \frac{|\mathrm{Rm}|^2}{S} \leq c.$$

Proof. According to Proposition 2.1,

$$\begin{aligned}|\mathrm{Rm}|^2 &\leq c\left(\frac{|\nabla \mathrm{Ric}|^2}{f} + \frac{c}{f^2} + |\mathrm{Ric}|^2\right) \\ &\leq c\left(\frac{1}{f} + S\right) \\ &\leq cS.\end{aligned} \tag{3.1}$$

In the second and third line above we have used Proposition 3.1, Theorem 2.4 and the fact that $\frac{1}{f} \leq cS$ from [9], respectively. This proves the proposition. □

We continue with a similar estimate for the covariant derivative of curvature.

Proposition 3.3. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature. Then there exists a constant $c > 0$ so that*

$$\sup_M \frac{|\nabla \mathrm{Rm}|^2}{S} \leq c.$$

Proof. Let us first prove the following inequality,

$$\Delta_f |\nabla \mathrm{Rm}|^2 \geq 2 \left|\nabla^2 \mathrm{Rm}\right|^2 + 3 |\nabla \mathrm{Rm}|^2 - c |\mathrm{Rm}| |\nabla \mathrm{Rm}|^2 . \tag{3.2}$$

We have that

$$\Delta_f |\nabla \mathrm{Rm}|^2 = 2 \left|\nabla^2 \mathrm{Rm}\right|^2 + 2 \langle \Delta_f (\nabla \mathrm{Rm}), \nabla \mathrm{Rm}\rangle .$$

Now we compute

$$\begin{aligned}
\Delta_f (\nabla_q \mathrm{Rm}) &= \nabla_p \nabla_p \nabla_q R_{ijkl} - \nabla_p (\nabla_q R_{ijkl}) f_p \\
&= \nabla_p \nabla_q \nabla_p R_{ijkl} - \nabla_q (\nabla_p R_{ijkl}) f_p + \mathrm{Rm} * \nabla \mathrm{Rm} \\
&= \nabla_q \nabla_p \nabla_p R_{ijkl} - \nabla_q (\nabla_p R_{ijkl} f_p) + f_{pq} (\nabla_p R_{ijkl}) + \mathrm{Rm} * \nabla \mathrm{Rm} \\
&= \nabla_q (\Delta_f R_{ijkl}) + \frac{1}{2} \nabla_q R_{ijkl} + \mathrm{Rm} * \nabla \mathrm{Rm} \\
&= \frac{3}{2} \nabla_q R_{ijkl} + \mathrm{Rm} * \nabla \mathrm{Rm}.
\end{aligned}$$

In these equalities we used the Ricci identities, the formulas $R_{ijkl} f_l = \nabla_j R_{ik} - \nabla_i R_{jk}$ and $\Delta_f \mathrm{Rm} = \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}$ from (2.4). Hence, (3.2) is proved.

Using (3.2) we get

$$\begin{aligned}
\Delta_f \left(|\nabla \mathrm{Rm}|^2 S^{-1}\right) &\geq S^{-1} \left(2 \left|\nabla^2 \mathrm{Rm}\right|^2 + 3 |\nabla \mathrm{Rm}|^2 - c |\mathrm{Rm}| |\nabla \mathrm{Rm}|^2\right) \\
&\quad + |\nabla \mathrm{Rm}|^2 \left(-S^{-1} + 2 |\mathrm{Ric}|^2 S^{-2} + 2 |\nabla S|^2 S^{-3}\right) \\
&\quad - 4 |\nabla S| \left|\nabla^2 \mathrm{Rm}\right| S^{-2} |\nabla \mathrm{Rm}| \\
&\geq 2 |\nabla \mathrm{Rm}|^2 S^{-1} - c |\mathrm{Rm}| |\nabla \mathrm{Rm}|^2 S^{-1}.
\end{aligned} \tag{3.3}$$

To derive the last line of (3.3) we have used that

$$\begin{aligned}
2 \left|\left\langle \nabla |\nabla \mathrm{Rm}|^2 , \nabla S^{-1}\right\rangle\right| &\leq 4 \left|\nabla^2 \mathrm{Rm}\right| |\nabla \mathrm{Rm}| S^{-2} |\nabla S| \\
&\leq 2 \left|\nabla^2 \mathrm{Rm}\right|^2 S^{-1} + 2 |\nabla S|^2 S^{-3} |\nabla \mathrm{Rm}|^2 .
\end{aligned}$$

Using Proposition 3.2 and (2.18) we can bound

$$\begin{aligned}
c |\mathrm{Rm}| |\nabla \mathrm{Rm}|^2 S^{-1} &\leq c |\nabla \mathrm{Rm}| S^{-\frac{1}{2}} \\
&\leq |\nabla \mathrm{Rm}|^2 S^{-1} + c.
\end{aligned}$$

Therefore, the function

$$w := |\nabla \mathrm{Rm}|^2 S^{-1} - c$$

satisfies

$$\Delta_f w \geq w. \tag{3.4}$$

Our goal is to show that w must be bounded above. We use the maximum principle again.

Let $\psi(t) = \frac{R-t}{R}$ on $[0,R]$ and $\psi = 0$ for $t \geq R$. Then $\psi(f)$ as a cutoff function on M satisfies

$$\begin{aligned} |\nabla \psi| &= \frac{|\nabla f|}{R} \\ \Delta_f \psi &= \frac{1}{R}(f-2). \end{aligned} \tag{3.5}$$

Therefore, for $G := \psi^2 w$, using (3.4), we have

$$\Delta_f G \geq \left(1 + \psi^{-1}\frac{2}{R}(f-2) - 6\psi^{-2}|\nabla \psi|^2\right) G + 4\psi^{-1}\langle \nabla G, \nabla \psi \rangle. \tag{3.6}$$

Suppose that $G(q) > 0$ at the maximum point q of G. Then (3.6) implies that

$$\frac{2}{R}(f-2)\psi \leq 6|\nabla \psi|^2 \leq 6\frac{1}{R^2}f. \tag{3.7}$$

If $q \in D(r_0)$, then

$$\begin{aligned} \sup_{D\left(\frac{R}{2}\right)} \left(|\nabla \mathrm{Rm}|^2 S^{-1}\right) &\leq c + 4 \sup_{D\left(\frac{R}{2}\right)} G \\ &\leq c + 4 \sup_{D(r_0)} G \\ &\leq c. \end{aligned}$$

On the other hand, if $q \in M \backslash D(r_0)$, then $f(q) - 2 \geq \frac{1}{2} f(q)$. By (3.7), $\psi(q) R \leq 6$. This shows that $f(q) \geq R - 6$. Therefore,

$$\begin{aligned} \frac{1}{4}\sup_{D\left(\frac{R}{2}\right)} \left(|\nabla \mathrm{Rm}|^2 S^{-1} - c\right) &\leq \sup_{D\left(\frac{R}{2}\right)} G \\ &\leq G(q) \\ &\leq \frac{36}{R^2} \sup_{D(R)} \left(|\nabla \mathrm{Rm}|^2 S^{-1}\right) \\ &\leq \frac{c}{R}, \end{aligned}$$

where in the last line we have used (2.18) and that $Sf \geq c > 0$ by [9].

In conclusion, we have proved that if $G(q) > 0$, then

$$\sup_{D\left(\frac{R}{2}\right)} \left(|\nabla \mathrm{Rm}|^2 S^{-1}\right) \leq c.$$

On the other hand, if at the maximum point q of G we have $G(q) \leq 0$, then w is nonpositive on $D(R)$, which again implies

$$\sup_{D\left(\frac{R}{2}\right)} \left(|\nabla \mathrm{Rm}|^2 S^{-1}\right) \leq c.$$

This proves the proposition. □

We now wish to establish a gradient estimate for the scalar curvature. This will be improved later.

Lemma 3.4. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature $S \leq A$. Then there exists a constant $c > 0$ so that*

$$|\nabla \ln S|^2 \leq c \ln(f+2) \text{ on } M.$$

Proof. We adopt an argument in [21]. Let $h := \frac{1}{\epsilon} S^{\epsilon}$ with $\epsilon > 0$ small to be determined later. Then a direct computation gives

$$\Delta_f h = \epsilon h - 2\epsilon \left|\mathrm{Ric}\right|^2 S^{-1} h + (\epsilon - 1) S^{\epsilon-2} \left|\nabla S\right|^2 .$$

Let us denote $\sigma := \left|\nabla h\right|^2 = S^{2\epsilon-2} \left|\nabla S\right|^2$. The Bochner formula asserts that

$$\begin{aligned}\frac{1}{2}\Delta_f \sigma &= \left|\mathrm{Hess}\,(h)\right|^2 + \langle \nabla h, \nabla (\Delta_f h)\rangle + \mathrm{Ric}_f (\nabla h, \nabla h)\\ &\geq \langle \nabla h, \nabla (\Delta_f h)\rangle\\ &\geq (\epsilon - 1)\left\langle \nabla h, \nabla \left(S^{\epsilon-2}\left|\nabla S\right|^2\right)\right\rangle - 2\epsilon \left\langle \nabla h, \nabla\left(\left|\mathrm{Ric}\right|^2 S^{-1} h\right)\right\rangle .\end{aligned}$$

Note that

$$\begin{aligned}\left\langle \nabla h, \nabla \left(S^{\epsilon-2}\left|\nabla S\right|^2\right)\right\rangle &= \left\langle \nabla h, \nabla\left(S^{-\epsilon}\sigma\right)\right\rangle\\ &= -\epsilon \langle \nabla h, \nabla S\rangle S^{-\epsilon-1}\sigma + S^{-\epsilon}\langle \nabla h, \nabla \sigma\rangle\\ &= -\epsilon \left|\nabla h\right|^2 S^{-2\epsilon}\sigma + S^{-\epsilon}\langle \nabla h, \nabla\sigma\rangle .\end{aligned}$$

Furthermore, we have

$$\begin{aligned}-\epsilon\left\langle \nabla h, \nabla\left(\left|\mathrm{Ric}\right|^2 S^{-1} h\right)\right\rangle &\geq -2\epsilon \left|\nabla \mathrm{Ric}\right| \left|\mathrm{Ric}\right| h \left|\nabla h\right| S^{-1} - \epsilon \left|\mathrm{Ric}\right|^2 S^{-1}\left|\nabla h\right|^2\\ &\geq -c\epsilon h\left|\nabla h\right| - c\epsilon\sigma\\ &\geq -c - c\sigma,\end{aligned}$$

where in the second line we used Proposition 3.1 and Proposition 3.3 to bound $\left|\nabla\mathrm{Ric}\right| \left|\mathrm{Ric}\right| S^{-1} \leq c$, and in the last line we used $\epsilon h = S^{\epsilon} \leq c$.

Consequently,

$$\frac{1}{2}\Delta_f \sigma \geq \epsilon (1 - \epsilon) S^{-2\epsilon}\sigma^2 + (\epsilon - 1) S^{-\epsilon}\langle \nabla h, \nabla\sigma\rangle - c\sigma - c. \tag{3.8}$$

Let ϕ be a smooth non-negative function defined on the real line so that $\phi(t) = 1$ for $0 \leq t \leq R$ and $\phi(t) = 0$ for $t \geq 2R$. We may choose ϕ so that

$$t^2\left(\left|\phi'\right|^2 (t) + \left|\phi''\right|(t)\right) \leq c.$$

We use $\phi(f(x))$ as a cut-off function with support in $D(2R)$. Note that we have $\left|\nabla\phi\right| \leq \frac{c}{\sqrt{R}}$ and $\left|\Delta_f \phi\right| \leq c$ for a universal constant $c > 0$.

Let $G := \phi^2\sigma$. From (3.8), we find that

$$\begin{aligned}\frac{1}{2}\phi^2\Delta_f G &= \frac{1}{2}\phi^4 (\Delta_f\sigma) + \frac{1}{2} G\left(\Delta_f \phi^2\right) + \phi^2\left\langle \nabla\phi^2, \nabla\sigma\right\rangle\\ &\geq \epsilon(1-\epsilon) S^{-2\epsilon}\, G^2 + (\epsilon - 1) S^{-\epsilon}\langle \nabla h, \nabla G\rangle \phi^2 - (\epsilon - 1) S^{-\epsilon}\left\langle \nabla h, \nabla\phi^2\right\rangle G\\ &\quad - cG - c + \left\langle \nabla\phi^2, \nabla G\right\rangle .\end{aligned}$$

At the maximum point of G we have

$$\begin{aligned}\epsilon G^2 &\leq c\, G^{\frac{3}{2}}\left|\nabla\phi\right| S^{\epsilon} + cG + c\\ &\leq \frac{c}{\sqrt{R}} G^{\frac{3}{2}} + cG + c.\end{aligned} \tag{3.9}$$

We now choose $\epsilon := (\ln R)^{-1}$. It is easy to see that (3.9) implies

$$\sup_M G \leq \frac{c}{\epsilon} = c\ln R.$$

This proves that

$$\sup_{D(R)} \left(S^{2\epsilon} \left|\nabla \ln S\right|^2 \right) \leq c \ln R.$$

Using the bound in [9] that $S \geq \frac{c}{R}$ on $D\left(R\right)$, one easily concludes that $S^{2\epsilon} \geq c > 0$. Thus,

$$\sup_{\Sigma(R)} \left|\nabla \ln S\right|^2 \leq c \ln(f+2)$$

and the result follows. □

To prove Theorem 1.1 in the introduction, we need to improve the Ricci curvature estimate from Proposition 3.1. Let us first establish a parallel version of Lemma 2.2.

Lemma 3.5. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature $S \leq A$. Then the function $u := \left|\mathrm{Ric}\right|^2 S^{-2}$ satisfies the differential inequality*

$$\Delta_F u \geq 3\, u^2\, S - c\, u\, S.$$

Proof. Note that by Proposition 2.1 and Proposition 3.3 we have

$$\begin{aligned} \left|\mathrm{Rm}\right| &\leq c \left(\frac{\left|\nabla \mathrm{Ric}\right|}{\sqrt{f}} + \frac{\left|\mathrm{Ric}\right|^2 + 1}{f} + \left|\mathrm{Ric}\right| \right) \\ &\leq c \left(\frac{\sqrt{S}}{\sqrt{f}} + \frac{1}{f} + \left|\mathrm{Ric}\right| \right) \\ &\leq c\left(S + \left|\mathrm{Ric}\right|\right). \end{aligned}$$

In the last line we have used the fact that $S \geq \frac{c}{f}$ on M. Since $S \leq 2\left|\mathrm{Ric}\right|$, we conclude that

$$\left|\mathrm{Rm}\right| \leq c \left|\mathrm{Ric}\right|. \tag{3.10}$$

By (2.4) we have

$$\begin{aligned} \Delta_f \left|\mathrm{Ric}\right|^2 &\geq 2\left|\nabla \mathrm{Ric}\right|^2 + 2\left|\mathrm{Ric}\right|^2 - c\left|\mathrm{Rm}\right|\left|\mathrm{Ric}\right|^2 \\ &\geq 2\left|\nabla \mathrm{Ric}\right|^2 + 2\left|\mathrm{Ric}\right|^2 - c\left|\mathrm{Ric}\right|^3. \end{aligned}$$

Hence,

$$\begin{aligned} (3.11) \qquad & \Delta_f \left(\left|\mathrm{Ric}\right|^2 S^{-2} \right) \\ = \;& S^{-2} \Delta_f \left(\left|\mathrm{Ric}\right|^2 \right) + \left|\mathrm{Ric}\right|^2 \Delta_f \left(S^{-2}\right) + 2\left\langle \nabla S^{-2}, \nabla \left|\mathrm{Ric}\right|^2 \right\rangle \\ \geq \;& 2\left|\nabla \mathrm{Ric}\right|^2 S^{-2} + 2\left|\mathrm{Ric}\right|^2 S^{-2} - c\left|\mathrm{Ric}\right|^3 S^{-2} + 2\left\langle \nabla S^{-2}, \nabla \left|\mathrm{Ric}\right|^2 \right\rangle \\ + \;& \left|\mathrm{Ric}\right|^2 \left(-2S^{-2} + 4\left|\mathrm{Ric}\right|^2 S^{-3} + 6\left|\nabla S\right|^2 S^{-4} \right). \end{aligned}$$

We can estimate

$$
\begin{aligned}
2\left\langle \nabla S^{-2}, \nabla |\mathrm{Ric}|^2 \right\rangle &= S^2 \left\langle \nabla S^{-2}, \nabla \left(|\mathrm{Ric}|^2 S^{-2} \right) \right\rangle \\
&\quad - \left\langle \nabla S^{-2}, \nabla S^{-2} \right\rangle |\mathrm{Ric}|^2 S^2 + \left\langle \nabla S^{-2}, \nabla |\mathrm{Ric}|^2 \right\rangle \\
&\geq -2 \left\langle \nabla \ln S, \nabla \left(|\mathrm{Ric}|^2 S^{-2} \right) \right\rangle - 4 |\nabla S|^2 S^{-4} |\mathrm{Ric}|^2 \\
&\quad -4 |\nabla \mathrm{Ric}| \, |\nabla S| \, S^{-3} |\mathrm{Ric}| \\
&\geq -2 \left\langle \nabla \ln S, \nabla \left(|\mathrm{Ric}|^2 S^{-2} \right) \right\rangle - 6 |\nabla S|^2 S^{-4} |\mathrm{Ric}|^2 \\
&\quad -2 |\nabla \mathrm{Ric}|^2 S^{-2}.
\end{aligned}
$$

Plugging this in (3.11) we get

$$
\begin{aligned}
\Delta_F \left(|\mathrm{Ric}|^2 S^{-2} \right) &\geq 4 |\mathrm{Ric}|^4 S^{-3} - c |\mathrm{Ric}|^3 S^{-2} \\
&\geq 3 |\mathrm{Ric}|^4 S^{-3} - c |\mathrm{Ric}|^2 S^{-1},
\end{aligned}
\tag{3.12}
$$

where we have used the Cauchy-Schwarz inequality

$$
c |\mathrm{Ric}|^3 S^{-2} \leq |\mathrm{Ric}|^4 S^{-3} + c |\mathrm{Ric}|^2 S^{-1}
$$

in the last line. This proves the result. □

We are ready to prove the following result which was stated as Theorem 1.1 in the introduction.

Theorem 3.6. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature $S \leq A$. There exists a constant $c > 0$ so that*

$$
\sup_M \frac{|\mathrm{Rm}|}{S} \leq c. \tag{3.13}
$$

Proof. By (3.10) it suffices to show that

$$
\sup_M \frac{|\mathrm{Ric}|}{S} \leq c. \tag{3.14}
$$

By Lemma 3.5 the function $u := |\mathrm{Ric}|^2 S^{-2}$ satisfies the following differential inequality

$$
\Delta_F u \geq 3 u^2 S - c\, u\, S. \tag{3.15}
$$

Let $\psi(t) = \frac{R-t}{R}$ on $[0, R]$ and $\psi = 0$ for $t \geq R$. Using $\psi(f)$ as a cutoff function on M, we have

$$
\begin{aligned}
|\nabla \psi| &= \frac{1}{R} |\nabla f| \\
\Delta_f \psi &= \frac{1}{R} (f - 2).
\end{aligned}
\tag{3.16}
$$

By Lemma 3.4,

$$
\begin{aligned}
\Delta_F \psi &= \Delta_f \psi + 2 \langle \nabla \ln S, \nabla \psi \rangle \\
&\geq \frac{1}{R} (f - 2) - \frac{2}{R} |\nabla \ln S| \, |\nabla f| \\
&\geq \frac{1}{R} (f - 2) - \frac{c}{R} \sqrt{f} \ln (f + 2).
\end{aligned}
$$

The constant $c>0$ in the above estimate depends on A and $D(r_0)$. Therefore, there exists $t_0>0$, depending on A and $D(r_0)$, so that

$$\Delta_F \psi \geq 0, \quad \text{on } D(R)\backslash D(t_0). \tag{3.17}$$

Using (3.15) and (3.17), for the function $G:=\psi^2 u$ we have that on $M\backslash D(t_0)$,

$$\begin{aligned} \psi^2\Delta_F G &\geq 3G^2S - cGS + G\Delta_F\psi^2 + 2\left\langle \nabla u, \nabla\psi^2\right\rangle \psi^2 \\ &\geq 3G^2S - cGS + 2\left\langle \nabla\left(G\psi^{-2}\right), \nabla\psi^2\right\rangle\psi^2 \\ &= 3G^2S - cGS + 2\left\langle \nabla G, \nabla\psi^2\right\rangle - 8\left|\nabla\psi\right|^2 G. \end{aligned} \tag{3.18}$$

By (3.16) and the estimate $Sf\geq c>0$ we have

$$\begin{aligned} \left|\nabla\psi\right|^2 G &\leq \frac{1}{R}G \\ &\leq \frac{1}{c}SG. \end{aligned}$$

Therefore, (3.18) becomes

$$\psi^2\Delta_F G \geq \left(3G^2 - cG\right)S + 2\left\langle\nabla G, \nabla\psi^2\right\rangle.$$

Now the maximum principle implies that G must be bounded on $M\backslash D(t_0)$. Moreover, by Theorem 2.4, there exists $c>0$ so that $|\mathrm{Rm}|\leq c$ on $D(t_0)\backslash D(r_0)$. Since $S\geq \frac{c}{f}\geq\frac{c}{t_0}$ on $D(t_0)\backslash D(r_0)$, this proves (3.14) and hence the theorem. □

We can now improve the covariant derivative estimate in Proposition 3.3 as well.

Theorem 3.7. *Let (M,g,f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature $S\leq A$. There exists $c>0$ so that*

$$|\nabla\mathrm{Rm}|\leq cS \quad \text{on } M.$$

In particular,

$$\sup_M |\nabla \ln S|\leq c.$$

Proof. Using (3.2) we get

$$\begin{aligned} \Delta_f\left(\left|\nabla\mathrm{Rm}\right|^2 S^{-2}\right) &\geq S^{-2}\left(2\left|\nabla^2\mathrm{Rm}\right|^2 + 3\left|\nabla\mathrm{Rm}\right|^2 - c\left|\mathrm{Rm}\right|\left|\nabla\mathrm{Rm}\right|^2\right) \\ &\quad + \left|\nabla\mathrm{Rm}\right|^2\left(-2S^{-2} + 4\left|\mathrm{Ric}\right|^2 S^{-3} + 6\left|\nabla S\right|^2 S^{-4}\right) \\ &\quad + 2\left\langle\nabla\left|\nabla\mathrm{Rm}\right|^2, \nabla S^{-2}\right\rangle. \end{aligned} \tag{3.19}$$

Observe that

$$\begin{aligned} 2\left\langle\nabla\left|\nabla\mathrm{Rm}\right|^2, \nabla S^{-2}\right\rangle &= \left\langle\nabla\left(\left|\nabla\mathrm{Rm}\right|^2 S^{-2}\right)S^2, \nabla S^{-2}\right\rangle + \left\langle\nabla\left|\nabla\mathrm{Rm}\right|^2, \nabla S^{-2}\right\rangle \\ &\geq \left\langle\nabla\left(\left|\nabla\mathrm{Rm}\right|^2 S^{-2}\right), \nabla S^{-2}\right\rangle S^2 + \left|\nabla\mathrm{Rm}\right|^2 S^{-2}\left\langle\nabla S^2, \nabla S^{-2}\right\rangle \\ &\quad -4\left|\nabla^2\mathrm{Rm}\right|\left|\nabla S\right|\left|\nabla\mathrm{Rm}\right| S^{-3} \\ &\geq -2\left\langle\nabla\left(\left|\nabla\mathrm{Rm}\right|^2 S^{-2}\right), \nabla\ln S\right\rangle - 6\left|\nabla\mathrm{Rm}\right|^2\left|\nabla S\right|^2 S^{-4} \\ &\quad -2\left|\nabla^2\mathrm{Rm}\right|^2 S^{-2}. \end{aligned}$$

It now follows from (3.19) and Theorem 3.6 that the function $w := |\nabla \mathrm{Rm}|^2 S^{-2}$ satisfies the inequality

$$\begin{aligned} \Delta_F w &\geq w - c\,|\mathrm{Rm}|\,w \\ &\geq w\,(1 - cS)\,, \end{aligned} \tag{3.20}$$

where $F := f - 2\ln S$. We now show that a function $w \geq 0$ satisfying (3.20) must be bounded. Let $\psi(t) = \frac{R-t}{R}$ on $[0, R]$ and $\psi = 0$ for $t \geq R$. We view $\psi(f)$ as a cut-off function on $D(R)$.

For $G := \psi^2 w$ we have that

$$\Delta_F G \geq G\,(1 - cS) + 2\psi^{-1}(\Delta_F \psi)\,G - 6\psi^{-2}|\nabla\psi|^2 G + 2\psi^{-2}\left\langle \nabla G, \nabla \psi^2 \right\rangle. \tag{3.21}$$

Let $q \in D(R)$ be the maximum point of G. If $S(q) > \frac{1}{c}$, where $c > 0$ is the constant in (3.21), then from the definition of w and Theorem 2.4 one sees that $G \leq G(q) \leq C$ on $D(R)$. This proves that w is bounded on $D\left(\frac{R}{2}\right)$. So we may assume in (3.21) that $1 - cS(q) \geq 0$. Now the maximum principle implies that at q we have

$$0 \geq \psi^{-1}(\Delta_F \psi) - 3\psi^{-2}|\nabla\psi|^2. \tag{3.22}$$

We estimate

$$\begin{aligned} \Delta_F \psi &= -\frac{1}{R}\Delta_f(f) + \frac{2}{R}\langle \nabla \ln S, \nabla f\rangle \\ &\geq \frac{f-2}{R} - \frac{2}{R}|\nabla \ln S|\sqrt{f} \\ &\geq \frac{f - c\sqrt{f}\,\ln(f+2) - 2}{R}, \end{aligned}$$

By Lemma 3.4, there exists t_0 depending on A and $D(r_0)$ so that $\Delta_F \psi \geq \frac{f}{2R}$ on $M \backslash D(t_0)$. If $q \in D(t_0)$, then it follows as in the proof of Theorem 3.6 that $w \leq c$ on $D\left(\frac{R}{2}\right)$. So without loss of generality we may assume $q \in D(R) \backslash D(t_0)$, hence

$$\Delta_F \psi \geq \frac{f}{2R} \quad \text{at } q.$$

Therefore, (3.22) implies that at q,

$$\frac{f}{R}\psi \leq 6|\nabla\psi|^2 \leq 6\frac{1}{R^2}f.$$

This means $f(q) \geq R - 6$ and

$$\begin{aligned} \frac{1}{4}\sup_{D\left(\frac{R}{2}\right)}\left(|\nabla \mathrm{Rm}|^2 S^{-2}\right) &\leq \sup_{D\left(\frac{R}{2}\right)} G \\ &\leq G(q) \\ &\leq \frac{36}{R^2}\sup_{D(R)}\left(|\nabla \mathrm{Rm}|^2 S^{-2}\right) \\ &\leq \frac{c}{R}, \end{aligned}$$

where in the last line we have used Proposition 3.3 and $Sf \geq c > 0$. This again proves that $|\nabla \mathrm{Rm}|^2 S^{-2}$ is bounded. In conclusion, we have proved that

$$\sup_{D\left(\frac{R}{2}\right)}\left(|\nabla \mathrm{Rm}|\,S^{-1}\right) \leq c.$$

Since R is arbitrary, this proves the theorem. □

4. Curvature lower bound

In this section we prove Theorem 1.2. The argument uses the estimates from the previous sections and ideas of Hamilton-Ivey pinching estimate for three dimensional Ricci flows. We first establish the following result, which improves Proposition 2.1.

Lemma 4.1. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature. Then*

$$\begin{aligned} R_{abcd} &= (R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad}) \\ &\quad -\frac{S}{2}(g_{ac}g_{bd} - g_{ad}g_{bc}) + O\left(Sf^{-\frac{1}{2}}\right). \end{aligned}$$

Proof. Following the proof of Proposition 2.1 we have that

$$\begin{aligned} R_{abcd} &= R^{\Sigma}_{abcd} - h_{ac}h_{bd} + h_{ad}h_{bc} \\ &= \left(R^{\Sigma}_{ac}g_{bd} - R^{\Sigma}_{ad}g_{bc} + R^{\Sigma}_{bd}g_{ac} - R^{\Sigma}_{bc}g_{ad}\right) \\ &\quad -\frac{1}{2}S^{\Sigma}(g_{ac}g_{bd} - g_{ad}g_{bc}) \\ &\quad -h_{ac}h_{bd} + h_{ad}h_{bc} \end{aligned} \tag{4.1}$$

Recall also that

$$\begin{aligned} R^{\Sigma}_{ac} &= R_{ac} - R_{a4c4} + H\, h_{ac} - h_{ab}\, h_{bc} \\ S^{\Sigma} &= S - 2R_{44} + H^2 - |h|^2. \end{aligned}$$

Using Theorem 3.7 we can estimate

$$|R_{ijk4}| \le c\frac{|\nabla \mathrm{Ric}|}{\sqrt{f}} \le cSf^{-\frac{1}{2}}. \tag{4.2}$$

Hence, (4.1) implies

$$\begin{aligned} R_{abcd} &= (R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad}) \\ &\quad -\frac{S}{2}(g_{ac}g_{bd} - g_{ad}g_{bc}) + \mathcal{E} + O\left(Sf^{-\frac{1}{2}}\right), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \mathcal{E} &= (H\, h_{ac} - h_{ae}\, h_{ec})\, g_{bd} - (H\, h_{ad} - h_{ae}\, h_{ed})\, g_{bc} \\ &\quad + (H\, h_{bd} - h_{be}\, h_{ed})\, g_{ac} - (H\, h_{bc} - h_{be}\, h_{ec})\, g_{ad} \\ &\quad -\frac{1}{2}\left(H^2 - |h|^2\right)(g_{ac}g_{bd} - g_{ad}g_{bc}) \\ &\quad -h_{ac}h_{bd} + h_{ad}h_{bc}. \end{aligned} \tag{4.4}$$

Note also that by Theorem 3.6,

$$h_{ab} = \frac{f_{ab}}{|\nabla f|} = \frac{1}{2}g_{ab}f^{-\frac{1}{2}} + O\left(Sf^{-\frac{1}{2}}\right). \tag{4.5}$$

Plugging (4.5) into (4.4) and simplifying, we immediately obtain that

$$|\mathcal{E}| \le O\left(Sf^{-1}\right).$$

By (4.3), this proves the lemma. □

We are now in position to prove Theorem 1.2. For convenience, we restate it here.

Theorem 4.2. *Let (M, g, f) be a four dimensional shrinking gradient Ricci soliton with bounded scalar curvature $S \leq A$. Then the curvature operator is bounded below by*

$$\mathrm{Rm} \geq -\left(\frac{c}{\ln(r+1)}\right)^{\frac{1}{4}}, \tag{4.6}$$

for a constant $c > 0$ depending only on A and $D(r_0)$, where r_0 depends only on A.

Proof. By (4.2), to establish (4.6) for the curvature operator of M, it is enough to do so for its restriction to the subspace $\wedge^2(T\Sigma)$. Restrict the Ricci curvature of M to Σ and let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the eigenvalues of the resulting operator. Our goal is to show that

$$\nu \geq -\left(\frac{c}{\ln f}\right)^{\frac{1}{4}}, \tag{4.7}$$

where

$$\nu := \lambda_1 + \lambda_2 - \lambda_3.$$

In view of Lemma 4.1, it is clear that (4.7) implies (4.6).

We will also denote

$$\begin{aligned} \lambda &:= \lambda_1 + \lambda_3 - \lambda_2 \\ \mu &:= \lambda_2 + \lambda_3 - \lambda_1. \end{aligned} \tag{4.8}$$

Note that $\nu \leq \lambda \leq \mu$.

We now prove (4.7). Using Lemma 4.1 it follows that

$$\begin{aligned} \Delta_f R_{ac} &= R_{ac} - 2R_{aicj}R_{ij} \\ &= R_{ac} - 2R_{abcd}R_{bd} + O\left(Sf^{-1}\right) \\ &= R_{ac} - 3\bar{S}R_{ac} + \bar{S}^2 g_{ac} + 4R_{ad}R_{dc} - 2\left|R_{cd}\right|^2 g_{ac} + O\left(Sf^{-\frac{1}{2}}\right), \end{aligned} \tag{4.9}$$

where

$$\bar{S} = g^{ac}R_{ac} = \lambda_1 + \lambda_2 + \lambda_3 = \nu + \lambda + \mu.$$

Observe that $\left|S - \bar{S}\right| \leq cSf^{-\frac{1}{2}}$, by (4.2). Note that (4.9) implies

$$\Delta_f \bar{S} = \bar{S} - 2\left|R_{ac}\right|^2 + O\left(Sf^{-\frac{1}{2}}\right). \tag{4.10}$$

We use (4.9) to obtain, in the sense of barrier,

$$\Delta_f \lambda_3 \geq \lambda_3 - 3\bar{S}\lambda_3 + \bar{S}^2 + 4\lambda_3^2 - 2\left|R_{ab}\right|^2 - cSf^{-\frac{1}{2}}.$$

Since $\nu = \bar{S} - 2\lambda_3$, it follows that

$$\Delta_f \nu \leq \nu + 6\bar{S}\lambda_3 - 2\bar{S}^2 - 8\lambda_3^2 + 2\left|R_{ab}\right|^2 + cSf^{-\frac{1}{2}}. \tag{4.11}$$

Using (4.8), we can express the right side of (4.11) as

$$\begin{aligned} 6\bar{S}\lambda_3 - 2\bar{S}^2 - 8\lambda_3^2 + 2\left|R_{ab}\right|^2 &= 3(\lambda + \mu + \nu)(\lambda + \mu) \\ &\quad - 2(\lambda + \mu + \nu)^2 - 2(\lambda + \mu)^2 \\ &\quad + \left(\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \mu\nu + \nu\lambda\right) \\ &= -\nu^2 - \lambda\mu. \end{aligned}$$

Hence, using this in (4.11), implies the following

$$\Delta_f \nu \leq \nu - \nu^2 - \lambda\mu + cSf^{-\frac{1}{2}}. \tag{4.12}$$

Modulo the error term $Sf^{-\frac{1}{2}}$, this is the inequality one gets for three dimensional shrinking gradient Ricci solitons.

Let

$$F := f - 2\ln S.$$

We have

$$\begin{aligned}
& \Delta_F\left(\nu S^{-1}\right) \\
= & \left(\Delta_f \nu\right) S^{-1} + \nu\left(\Delta_f S^{-1}\right) + 2\left\langle \nabla\nu, \nabla S^{-1}\right\rangle + 2\left\langle \nabla \ln S, \nabla\left(\nu S^{-1}\right)\right\rangle \\
\leq & \left(\nu - \nu^2 - \lambda\mu + cSf^{-\frac{1}{2}}\right) S^{-1} + 2\left\langle \nabla\ln S, \nabla\left(\nu S^{-1}\right)\right\rangle \\
& +\nu\left(-S^{-1} + 2\left|\mathrm{Ric}\right|^2 S^{-2} + 2\left|\nabla S\right|^2 S^{-3}\right) \\
& +2\left\langle \nabla\left(\nu S^{-1}\right), \nabla S^{-1}\right\rangle S + 2\left\langle \nabla S, \nabla S^{-1}\right\rangle\left(\nu S^{-1}\right) \\
= & -S^{-2}\left(\left(\nu^2 + \lambda\mu\right) S - 2\left|\mathrm{Ric}\right|^2 \nu\right) + cf^{-\frac{1}{2}}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\left(\nu^2 + \lambda\mu\right) S - 2\left|\mathrm{Ric}\right|^2 \nu &= \left(\nu^2 + \lambda\mu\right)\left(\nu + \lambda + \mu\right) \\
& \quad - \left(\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \mu\nu + \nu\lambda\right)\nu + O\left(S^2 f^{-\frac{1}{2}}\right) \\
&= \lambda^2\left(\mu - \nu\right) + \mu^2\left(\lambda - \nu\right) + O\left(S^2 f^{-\frac{1}{2}}\right).
\end{aligned}$$

Hence, the function

$$u := \frac{\nu}{S}$$

satisfies, in the sense of barrier, the following inequality

$$\Delta_F u \leq -S^{-2}\left(\lambda^2\left(\mu - \nu\right) + \mu^2\left(\lambda - \nu\right)\right) + cf^{-\frac{1}{2}}. \tag{4.13}$$

We remark that a function similar to u was used to classify locally conformally flat shrinking Ricci solitons of arbitrary dimension in [10]. This function also appears in Hamilton-Ivey pinching estimate for three dimensional ancient solutions [8].

We want to prove a lower bound for the function u based on (4.13). For this, let $R > r_0$ be large enough so that

$$R_1 := \ln R \geq r_0.$$

According to Theorem 3.6, there exists a constant $c_0 > 0$ so that

$$u > -c_0 \text{ on } M. \tag{4.14}$$

Consider the function

$$w := u + kf^{-\varepsilon} + \varepsilon S^{-1},$$

where

$$\begin{aligned}
\varepsilon &= \frac{1}{\sqrt{R_1}} = \frac{1}{\sqrt{\ln R}} \\
k &= c_0\left(R_1\right)^{\varepsilon}.
\end{aligned} \tag{4.15}$$

The constant $c_0 > 0$ in (4.15) is the same as that in (4.14). The choice of k guarantees that

$$w > 0 \text{ on } \partial D(R_1). \tag{4.16}$$

On $M \backslash D(R_1)$,

$$\begin{aligned}
\Delta_F f^{-\varepsilon} &= -\varepsilon(\Delta_f(f)) f^{-\varepsilon-1} + \varepsilon(\varepsilon+1)|\nabla f|^2 f^{-\varepsilon-2} - 2\varepsilon \langle \nabla \ln S, \nabla f\rangle f^{-\varepsilon-1} \\
&\le \varepsilon(f-2) f^{-\varepsilon-1} + \varepsilon(\varepsilon+1) f^{-\varepsilon-1} + 2\varepsilon|\nabla \ln S| f^{-\varepsilon-\frac{1}{2}} \\
&\le 2\varepsilon f^{-\varepsilon},
\end{aligned}$$

where in the last line we have used Theorem 3.7.

Next, we have that

$$\begin{aligned}
\Delta_F S^{-1} &= \Delta_f S^{-1} + 2\left\langle \nabla \ln S, \nabla S^{-1}\right\rangle \\
&= -(\Delta_f S) S^{-2} + 2|\nabla S|^2 S^{-3} - 2|\nabla S|^2 S^{-3} \\
&= -S^{-1} + 2|\mathrm{Ric}|^2 S^{-2} \\
&\le -S^{-1} + c,
\end{aligned}$$

where in the last line we have used Theorem 3.6.

Hence, on $M \backslash D(R_1)$,

$$\Delta_F w \le -S^{-2}\left(\lambda^2(\mu-\nu) + \mu^2(\lambda-\nu)\right) + 2\varepsilon k f^{-\varepsilon} - \varepsilon S^{-1} + c\varepsilon, \tag{4.17}$$

where we have used the fact that $cf^{-\frac{1}{2}} \le c(R_1)^{-\frac{1}{2}} = c\varepsilon$ on $M \backslash D(R_1)$.

Let $\phi(t) = \frac{R-t}{R}$ on $[0, R]$ and consider the cutoff function $\phi(f)$ on $D(R)$. On $D(R) \backslash D(R_1)$ we have

$$\begin{aligned}
|\nabla \phi| &= \frac{|\nabla f|}{R} \le \frac{1}{\sqrt{R}} \\
\Delta_F \phi &= \Delta_f \phi + 2\langle \nabla \ln S, \nabla \phi\rangle \\
&\ge \frac{1}{R}(f-2) - \frac{c}{R}\sqrt{f} \\
&\ge \frac{1}{2R} f.
\end{aligned} \tag{4.18}$$

In the second line above we have used Theorem 3.7.

Now define the function $G := \phi^2 w$ on $M \backslash D(R_1)$, which is positive on $\partial D(R_1)$ by (4.16) and zero on $M \backslash D(R)$. Let us first assume that G is negative somewhere in $D(R) \backslash D(R_1)$. Then there exists an interior point q of $D(R) \backslash D(R_1)$ at which G achieves its minimum. In particular, $G(q) < 0$ and $\nu(q) < 0$.

Using (4.17) and (4.18) it follows that at q,

$$\begin{aligned}
0 &\le \phi^2 \Delta_F G \\
&= \phi^4 \Delta_F w + G \Delta_F \phi^2 + 2\left\langle \nabla w, \nabla \phi^2\right\rangle \phi^2 \\
&\le -S^{-2}\left(\lambda^2(\mu-\nu) + \mu^2(\lambda-\nu)\right)\phi^4 + \left(2k\varepsilon f^{-\varepsilon} - \varepsilon S^{-1} + c\varepsilon\right)\phi^4 \\
&\quad + \left(\phi \frac{f}{R} - 6|\nabla \phi|^2\right) G,
\end{aligned} \tag{4.19}$$

where we have used

$$2\phi(\Delta_F \phi) G \le \frac{1}{R} \phi f G \text{ at } q.$$

We now discuss two cases.

Case 1. Suppose first that at q we have

$$\left(\phi\frac{f}{R}-6\left|\nabla\phi\right|^{2}\right)G\leq 0.$$

Then, we see from (4.19) that

$$S^{-2}\left(\lambda^{2}\left(\mu-\nu\right)+\mu^{2}\left(\lambda-\nu\right)\right)\leq 2k\varepsilon f^{-\varepsilon}-\varepsilon S^{-1}+c\varepsilon\ . \tag{4.20}$$

In particular, (4.20) implies that

$$\begin{aligned}\varepsilon S^{-1} &\leq 2k\varepsilon f^{-\varepsilon}+c\varepsilon\\ &\leq 2k\varepsilon\left(R_{1}\right)^{-\varepsilon}+c\varepsilon\\ &= \left(2c_{0}+c\right)\varepsilon.\end{aligned}$$

Here we have used the definition of k in (4.15). This shows that there exists a constant $c_1 := 2c_0 + c > 0$ so that $S\left(q\right)\geq\frac{1}{c_{1}}>0$. Now (4.20) implies that

$$\begin{aligned}\lambda^{2}\left(\mu-\nu\right)+\mu^{2}\left(\lambda-\nu\right) &\leq c\left(2k\varepsilon f^{-\varepsilon}+c\varepsilon\right)\\ &\leq c\varepsilon.\end{aligned}$$

Hence,

$$\begin{aligned}\lambda^{2}\left(\mu-\nu\right) &\leq c\varepsilon\\ \mu^{2}\left(\lambda-\nu\right) &\leq c\varepsilon.\end{aligned} \tag{4.21}$$

In addition, we know that $\bar{S}(q)=\mu+\lambda+\nu\geq\frac{1}{2c_{1}}>0$, which implies that $\mu\geq\frac{1}{4c_{1}}$. Therefore, one concludes from the first inequality in (4.21) (recall $\nu\left(q\right)<0$) that $\left|\lambda\right|\leq c_{2}\sqrt{\varepsilon}$. Using this in the second inequality of (4.21), we obtain

$$\begin{aligned}-\nu-c_{2}\sqrt{\varepsilon} &\leq \lambda-\nu\\ &\leq \frac{c\varepsilon}{\mu^{2}}\\ &\leq c\varepsilon.\end{aligned}$$

This proves that $-\nu\left(q\right)\leq c\sqrt{\varepsilon}$ and

$$G\left(q\right)\geq -c\sqrt{\varepsilon}$$

as $S\left(q\right)\geq\frac{1}{c_{1}}>0$. In conclusion,

$$\inf_{D\left(\frac{R}{2}\right)\setminus D\left(R_{1}\right)} w\geq 4G\left(q\right)\geq -c\sqrt{\varepsilon}. \tag{4.22}$$

Case 2. Suppose now that at q we have

$$\left(\phi\frac{f}{R}-6\left|\nabla\phi\right|^{2}\right)G>0.$$

Since $G\left(q\right)<0$, we conclude that

$$\begin{aligned}\frac{R-f}{R}\frac{f}{R} &\leq 6\left|\nabla\phi\right|^{2}\\ &\leq \frac{6}{R^{2}}f.\end{aligned}$$

Hence, $f(q) \geq R-6$ and $\phi(q) \leq \frac{6}{R}$. We now conclude that

$$\begin{aligned}\inf_{D\left(\frac{R}{2}\right)\backslash D(R_1)} G &\geq G(q)\\ &= w(q)\,\phi^2(q)\\ &\geq -\frac{c}{R^2},\end{aligned}$$

where the last line follows from $w \geq u > -c_0$. In particular,

$$\inf_{D\left(\frac{R}{2}\right)\backslash D(R_1)} w \geq -\frac{c}{R^2}. \tag{4.23}$$

By (4.22) and (4.23) we conclude that if G is negative somewhere in $D(R)\backslash D(R_1)$, then on $D\left(\frac{R}{2}\right)\backslash D(R_1)$

$$\frac{\nu}{S} \geq -\frac{k}{f^{\varepsilon}} - \frac{\varepsilon}{S} - c\sqrt{\varepsilon}. \tag{4.24}$$

Certainly, the same conclusion holds true if G is non-negative on $D(R)\backslash D(R_1)$. Therefore, from (4.15) and (4.24) we see that on $D\left(\frac{R}{2}\right)\backslash D(R_1)$,

$$\nu \geq -c\left(\frac{R_1}{f}\right)^{\varepsilon} - c\sqrt{\varepsilon}, \tag{4.25}$$

Recall that $\varepsilon = \frac{1}{\sqrt{\ln R}}$. So on $\Sigma\left(\frac{R}{2}\right) = \partial D\left(\frac{R}{2}\right)$ we get from (4.25) that

$$\nu \geq -\left(\frac{c}{\ln R}\right)^{\frac{1}{4}}.$$

The constant c depends only on A and $\sup_{D(r_0)}|\mathrm{Rm}|$. Since R is arbitrary, this proves the result. $\square$

5. Conical structure

Our goal in this section is to prove the following theorem.

Theorem 5.1. *Let (M, g, f) be a complete four dimensional shrinking gradient Ricci soliton with scalar curvature converging to zero at infinity. Then there exists a cone E_0 such that (M, g) is C^k asymptotic to E_0 for all k.*

Proof. We prove that there exist constants $c, C > 0$ so that

$$c \leq S\,f \leq C \quad \text{on } M. \tag{5.1}$$

The lower bound was established in [9]. Here we use the above estimates to prove the upper bound.

Using Theorem 3.6 we see that there exists a constant $c_0 > 0$ for which

$$\begin{aligned}\Delta_f S &= S - 2|\mathrm{Ric}|^2\\ &\geq S - c_0 S^2.\end{aligned} \tag{5.2}$$

Using that $\Delta_f(f) = 2 - f$, we obtain

$$\begin{aligned}\Delta_f\left(f^{-1}\right) &= -\Delta_f(f)\,f^{-2} + 2|\nabla f|^2 f^{-3}\\ &\leq (f-2)\,f^{-2} + 2f^{-2}\\ &= f^{-1}.\end{aligned}$$

Choose $r_0 \geq 1$ large enough so that on $M \backslash D(r_0)$

$$S < \frac{1}{4c_0} \tag{5.3}$$

for $c_0 > 0$ the constant in (5.2) and also so that $6|\nabla f|^2 \geq 4f$. Then,

$$\begin{aligned} \Delta_f\left(f^{-2}\right) &= 2(f-2)f^{-3} + 6|\nabla f|^2 f^{-4} \\ &\geq 2f^{-2}. \end{aligned}$$

Define the function

$$u := S - af^{-1} + c_0 a^2 f^{-2}, \tag{5.4}$$

where

$$a := \frac{r_0}{2c_0}.$$

By the choice of a and (5.3) it follows that

$$u < 0 \quad \text{on} \quad \partial D(r_0). \tag{5.5}$$

Indeed, on $\partial D(r_0)$ we have that

$$S - af^{-1} + 2c_0 a^2 f^{-2} < \frac{1}{4c_0} - \frac{1}{2c_0} + \frac{1}{4c_0} = 0.$$

Now note that

$$\begin{aligned} \Delta_f u &\geq S - c_0 S^2 - af^{-1} + 2c_0 a^2 f^{-2} \\ &= u - c_0 S^2 + c_0 a^2 f^{-2} \\ &= u - c_0\left(S - af^{-1}\right)\left(S + af^{-1}\right) \\ &\geq u - c_0 u\left(S + af^{-1}\right). \end{aligned}$$

Therefore, on $M \backslash D(r_0)$,

$$\Delta_f u \geq u\left(1 - c_0 S - c_0 a f^{-1}\right). \tag{5.6}$$

We now claim that

$$u \leq c f^{-2} \quad \text{on } M \backslash D(r_0). \tag{5.7}$$

To prove this claim, let $\psi(t) = \frac{R-t}{R}$ on $[0, R]$ and $\psi = 0$ for $t \geq R$. Define $G := \psi^2 u$ and compute

$$\begin{aligned} \Delta_f G &= \psi^2 \Delta_f u + u \Delta_f \psi^2 + 2\left\langle \nabla u, \nabla \psi^2 \right\rangle \\ &\geq G\left(1 - c_0 S - c_0 a f^{-1}\right) \\ &\quad + 2\psi^{-1}(\Delta_f \psi) G - 6\psi^{-2}|\nabla \psi|^2 G + 2\psi^{-2}\left\langle \nabla G, \nabla \psi^2 \right\rangle. \end{aligned} \tag{5.8}$$

Let q be the maximum point of G on $D(R) \backslash D(r_0)$. If $G(q) \leq 0$, then $u \leq 0$ and the claim (5.7) is true. So we may assume $G(q) > 0$. In this case (5.5) implies that q is an interior point of $D(R) \backslash D(r_0)$. At q, by the maximum principle and (5.8), we have

$$\begin{aligned} 0 &\geq 1 - c_0 S - c_0 a f^{-1} + 2\psi^{-1}(\Delta_f \psi) - 6\psi^{-2}|\nabla \psi|^2 \\ &> 2\psi^{-1}(\Delta_f \psi) - 6\psi^{-2}|\nabla \psi|^2. \end{aligned} \tag{5.9}$$

Since

$$\Delta_f \psi = \frac{f-2}{R} \geq \frac{f}{2R},$$

it follows from (5.9) that

$$\frac{f}{R}\psi \leq 6\left|\nabla\psi\right|^2 \leq 6\frac{1}{R^2}f.$$

This means that $\psi(q) \leq \frac{6}{R}$. Hence,

$$\begin{aligned} G(q) &= u(q)\,\psi^2(q) \\ &\leq \frac{c}{R^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{4}\sup_{D\left(\frac{R}{2}\right)\setminus D(r_0)} u &\leq \sup_{D\left(\frac{R}{2}\right)\setminus D(r_0)} G \\ &\leq G(q) \\ &\leq \frac{c}{R^2}. \end{aligned}$$

Since R is arbitrary, this again proves the claim (5.7).

In conclusion, on $M\setminus D(r_0)$,

$$S - af^{-1} + c_0a^2f^{-2} \leq cf^{-2}$$

or $S\,f \leq a + c$. This proves (5.1).

The theorem now follows as in [20]. Indeed, using Shi's derivative estimates, one can get corresponding sharp decay estimates for all covariant derivatives of the curvature. Certainly, this can also be done directly, by working with the elliptic equations instead of parabolic ones. These curvature estimates then prove the convergence to a cone as required by the theorem. We omit these details and refer the reader to [20]. □

6. Diameter estimate

In this section, we work with compact shrinking gradient Ricci solitons of arbitrary dimension and establish a diameter estimate from above. We are motivated by the work in [12], which implies that a complete gradient shrinking Ricci soliton with non-negative Ricci curvature and injectivity radius bounded away from zero must have finite topological type. Indeed, with these assumptions, a second variation argument shows that all critical points of f are contained in a compact set of M, whose size depends on the injectivity radius bound.

An more careful examination of this technique allows us to establish the following.

Theorem 6.1. *Let (M, g, f) be a compact gradient shrinking Ricci soliton of dimension n. Assume the injectivity radius of (M, g) is bounded below by* $\operatorname{inj}(M) \geq \delta > 0$. *Then the diameter of (M, g) has an upper bound of the form*

$$\operatorname{diam}(M) \leq c(n)\left(1 + \frac{1}{\delta}\right).$$

Proof. Since M is compact, the potential f assumes a maximum and a minimum value. Let us fix

$$\begin{aligned} f(p) &= \min_M f \\ f(q) &= \max_M f. \end{aligned}$$

We continue to normalize f so that

$$S+|\nabla f|^2-f=0, \tag{6.1}$$

where S is the scalar curvature. Recall again that

$$\begin{aligned} f(x) &\leq \left(\frac{1}{2}d(p,x)+c(n)\right)^2 \text{ for all } x\in M, \\ f(x) &\geq \frac{1}{4}d^2(p,x)-c(n)\,d(p,x) \text{ for all } x\in M\backslash B_p(r_0(n)). \end{aligned} \tag{6.2}$$

Here both $c(n)$ and $r_0(n)$ depend only on dimension n. Since $S\geq 0$, we see that (6.2) provides a uniform upper bound estimate for $|\nabla f|$ as well. Indeed, $|\nabla f|^2\leq f$.

Consider now a minimizing normal geodesic σ joining p and q, parametrized so that $\sigma(0)=p$ and $\sigma(R)=q$. We apply the second variation formula of arc length to $\sigma(s)$, $0\leq s\leq R$, and obtain

$$\int_0^R \operatorname{Ric}(\sigma'(s),\sigma'(s))\,\phi^2(s)\,ds\leq (n-1)\int_0^R(\phi'(s))^2\,ds$$

for any Lipschitz function ϕ with compact support in $[0,R]$. Using the fact that

$$\operatorname{Ric}(\sigma'(s),\sigma'(s))+f''(s)=\frac{1}{2}$$

and integrating by parts, we obtain

$$\frac{1}{2}\int_0^R\phi^2(s)\,ds\leq (n-1)\int_0^R(\phi'(s))^2\,ds-2\int_0^R f'(s)\,\phi(s)\,\phi'(s)\,ds, \tag{6.3}$$

where

$$f(s):=f(\sigma(s)).$$

For any $R-\frac{1}{2}\delta\leq t\leq R-\frac{1}{4}\delta$, let us take

$$\phi(s):=\begin{cases} s & \text{for } 0\leq s\leq 1 \\ 1 & \text{for } 1\leq s\leq t \\ \frac{R-s}{R-t} & \text{for } t\leq s\leq R \end{cases}$$

Then we get from (6.3) that

$$\begin{aligned} \frac{1}{2}(t-1) &\leq \frac{1}{2}\int_0^R\phi^2(s)\,ds \\ &\leq (n-1)\int_0^R(\phi'(s))^2\,ds-2\int_0^R f'(s)\,\phi(s)\,\phi'(s)\,ds \\ &= (n-1)\left(1+\frac{1}{R-t}\right)-2\int_0^1 f'(s)\,s\,ds+\frac{2}{(R-t)^2}\int_t^R f'(s)\,(R-s)\,ds. \end{aligned}$$

By (6.2) and the subsequent comments, it is easy to see that

$$\sup_{B_p(1)} |\nabla f| \leq c\,(n)\,.$$

This implies that

$$\int_t^R f'(s)\,(R-s)\,ds \geq \frac{\delta^2}{64}\left(R - c\,(n)\left(1+\frac{1}{\delta}\right)\right).$$

Integration by parts then yields

$$-\,(R-t)\,f\,(t) + \int_t^R f\,(s)\,ds \geq \frac{\delta^2}{64}\left(R - c\,(n)\left(1+\frac{1}{\delta}\right)\right).$$

Since $f\,(s) \leq f\,(R) = \max f$ and $\frac{1}{4}\delta \leq R-t \leq \frac{1}{2}\delta$, we see that

$$\begin{aligned} -\,(R-t)\,f\,(t) + \int_t^R f\,(s)\,ds &\leq (R-t)\,(f\,(R) - f\,(t)) \\ &\leq \frac{1}{2}\delta\,(f\,(R) - f\,(t))\,. \end{aligned}$$

Thus,

$$f\,(R) - f\,(t) \geq \frac{\delta}{32}\left(R - c\,(n)\left(1+\frac{1}{\delta}\right)\right), \tag{6.4}$$

for all $R - \frac{1}{2}\delta \leq t \leq R - \frac{1}{4}\delta$.

Now the assumption that $\operatorname{inj}(M) \geq \delta$ implies that the geodesic $\sigma\,(s)\,,\ R-\frac{1}{2}\delta \leq s \leq R$, can be extended into a minimizing normal geodesic over $R-\frac{1}{2}\delta \leq s \leq R+\frac{1}{2}\delta$.

We consider the cutoff function ψ on $\left[R-\frac{1}{2}\delta, R+\frac{1}{2}\delta\right]$ defined by

$$\psi\,(t) := \begin{cases} t - \left(R - \frac{1}{2}\delta\right) & \text{for } R-\frac{1}{2}\delta \leq t \leq R \\ \left(R+\frac{1}{2}\delta\right) - t & \text{for } R \leq t \leq R+\frac{1}{2}\delta \end{cases}$$

Applying the second variation formula to $\sigma\,(t)$ for $R-\frac{1}{2}\delta \leq t \leq R+\frac{1}{2}\delta$, we have that (see (6.3))

$$\frac{1}{2}\int_{R-\frac{1}{2}\delta}^{R+\frac{1}{2}\delta} \psi^2\,(t)\,dt \leq (n-1)\int_{R-\frac{1}{2}\delta}^{R+\frac{1}{2}\delta} \left(\psi'\,(t)\right)^2 dt - 2\int_{R-\frac{1}{2}\delta}^{R+\frac{1}{2}\delta} f'\,(t)\,\psi\,(t)\,\psi'\,(t)\,dt.$$

This implies

$$\int_{R-\frac{1}{2}\delta}^{R} f'\,(t)\,\psi\,(t)\,dt - \int_{R}^{R+\frac{1}{2}\delta} f'\,(t)\,\psi\,(t)\,dt \leq c\,(n)\,\delta.$$

After integrating by parts, this can be rewritten as

$$\delta f\,(R) \leq \int_{R-\frac{1}{2}\delta}^{R+\frac{1}{2}\delta} f\,(t)\,dt + c\,(n)\,\delta. \tag{6.5}$$

Note that $f\,(R) = f\,(q) = \max f$. So

$$\int_{R-\frac{1}{2}\delta}^{R+\frac{1}{2}\delta} f\,(t)\,dt \leq \int_{R-\frac{1}{2}\delta}^{R-\frac{1}{4}\delta} f\,(t)\,dt + \frac{3}{4}\delta f\,(R)\,.$$

By (6.5), this implies

$$\frac{1}{4}\delta f\,(R) \leq \int_{R-\frac{1}{2}\delta}^{R-\frac{1}{4}\delta} f\,(t)\,dt + c\,(n)\,\delta. \tag{6.6}$$

Using (6.4), we conclude

$$d\,(p,q) = R \leq c\,(n)\left(1+\frac{1}{\delta}\right).$$

Therefore, by (6.2),

$$\max_M f \leq c\,(n)\left(1+\frac{1}{\delta}\right)^2.$$

Now the lower bound of f from (6.2) implies

$$d\,(p,x) \leq c\,(n)\left(1+\frac{1}{\delta}\right)$$

for all $x \in M$. By the triangle inequality, one immediately sees that

$$\operatorname{diam}(M) \leq c\,(n)\left(1+\frac{1}{\delta}\right).$$

This proves the theorem. □

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