STRUCTURE AT INFINITY FOR SHRINKING RICCI SOLITONS

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ABSTRACT. This paper concerns the structure at infinity for complete gradient shrinking Ricci solitons. It is shown that for such a soliton with bounded curvature, if the round cylinder $\mathbb{R}\times\mathbb{S}^{n-1}/\Gamma$ occurs as a limit for a sequence of points going to infinity along an end, then the end is asymptotic to the same round cylinder at infinity. This result is applied to obtain structural results at infinity for four dimensional gradient shrinking Ricci solitons. It was previously known that such solitons with scalar curvature approaching zero at infinity must be smoothly asymptotic to a cone. For the case that the scalar curvature is bounded from below by a positive constant, we conclude that along each end the soliton is asymptotic to a quotient of $\mathbb{R}\times\mathbb{S}^{3}$ or converges to a quotient of $\mathbb{R}^{2}\times\mathbb{S}^{2}$ along each integral curve of the gradient vector field of the potential function.

Keywords: Ricci solitons, Ricci flow, asymptotic structure.

Mathematics Subject Classification (2010) 53C44, 53C21.

1. INTRODUCTION

The goal of this paper is to continue our study of complete four dimensional gradient shrinking Ricci solitons initiated in [29] and to obtain further information concerning the structure at infinity of such manifolds. Recall that a Riemannian manifold (M,g) is a gradient shrinking Ricci soliton if there exists a smooth function $f\in C^{\infty}(M)$ such that the Ricci curvature Ric of M and the hessian $\mathrm{Hess}(f)$ of f satisfy the following equation

$$\mathrm{Ric}+\mathrm{Hess}\,(f)=\frac{1}{2}g.$$

By defining ϕ_t to be the one-parameter family of diffeomorphisms generated by the vector field $\frac{\nabla f}{-t}$ for $-\infty<t<0$, one checks that $g(t)=(-t)\ \phi_t^{*}\,g$ is a solution to the Ricci flow

$$\frac{\partial g(t)}{\partial t}=-2\,\mathrm{Ric}(t)$$

on time interval $(-\infty,0)$. Since the Ricci flow equation is invariant under the action of the diffeomorphism group, such solution $g(t)$ is evidently a shrinking self-similar solution to the Ricci flow. Gradient shrinking Ricci solitons have played a crucial role in the singularity analysis of Ricci flows. A conjecture, generally attributed to Hamilton, asserts that the blow-ups around a type-I singularity point of a Ricci flow always converge to (nontrivial) gradient shrinking Ricci solitons. More precisely, a Ricci flow solution $(M,g(t))$ on a finite-time interval $[0,T)$, $T<\infty$, is said to

The first author was partially supported by NSF grant DMS-1506220. The second author was partially supported by NSF grant DMS-1606820 .

develop a Type-I singularity (and T is called a Type-I singular time) if there exists a constant $C > 0$ such that for all $t \in [0, T)$

$$\sup_M |\mathrm{Rm}_{g(t)}|_{g(t)} \leq \frac{C}{T-t}$$

and

$$\limsup_{t \to T} \sup_M |\mathrm{Rm}_{g(t)}|_{g(t)} = \infty.$$

Here $\mathrm{Rm}_{g(t)}$ denotes the Riemannian curvature tensor of the metric $g(t)$. A point $p \in M$ is a singular point if there exists no neighborhood of p on which $|\mathrm{Rm}_{g(t)}|_{g(t)}$ stays bounded as $t \to T$. Then the conjecture claims that for every sequence $\lambda_j \to \infty$, the rescaled Ricci flows $(M, g_j(t), p)$ defined on $[-\lambda_j T, 0)$ by $g_j(t) := \lambda_j \, g(T + \lambda_j^{-1} t)$ subconverge to a nontrivial gradient shrinking Ricci soliton.

While the conjecture was first confirmed by Perelman [34] for the dimension three case, in the most general form it has also been satisfactorily resolved. In the case where the blow-up limit is compact, it was confirmed by Sesum [37]. In the general case, blow-up to a gradient shrinking soliton was proved by Naber [32]. The nontriviality issue of the soliton was later taken up by Enders, Müller and Topping [16], see also Cao and Zhang [8].

In view of their importance, it is then natural to seek a classification of the gradient shrinking Ricci solitons. It is relatively simple to classify two dimensional ones, [18].

Theorem 1.1. *A two dimensional gradient shrinking Ricci soliton is isometric to the plane $\mathbb{R}^2$ or to a quotient of the sphere $\mathbb{S}^2$.*

For the three dimensional case, there is a parallel classification result as well.

Theorem 1.2. *A three dimensional gradient shrinking Ricci soliton is isometric to the Euclidean space $\mathbb{R}^3$ or to a quotient of the sphere $\mathbb{S}^3$ or of the cylinder $\mathbb{R} \times \mathbb{S}^2$.*

This theorem has a long history. Ivey [24] first showed that a three dimensional compact gradient shrinking Ricci soliton must be a quotient of the sphere $\mathbb{S}^3$. Later, it was realized from the Hamilton-Ivey estimate [18] that the curvature of a three dimensional gradient shrinking Ricci soliton must be nonnegative. Moreover, by the strong maximum principle of Hamilton [20], the manifold must split off a line, hence is a quotient of $\mathbb{R} \times \mathbb{S}^2$ or $\mathbb{R}^3$, if its sectional curvature is not strictly positive. When the sectional curvature is strictly positive, Perelman [35] showed that the soliton must be compact, hence a quotient of the sphere, provided that the soliton is noncollapsing with bounded curvature. Obviously, the classification result follows by combining all these together, at least for the ones which are noncollapsing with bounded curvature. The result in particular implies that a type I singularity of the Ricci flow on a compact three dimensional manifold is necessarily of spherical or neck-like, a fact crucial for Perelman [35] to define the Ricci flows with surgery and for the eventual resolution of the Poincaré or the more general Thurston's geometrization conjecture. The noncollapsing assumption was later removed by Naber [32]. By adopting a different argument, Ni and Wallach [33], and Cao, Chen and Zhu [3] showed the full classification result Theorem 1.2. Some relevant contributions were also made in [32, 36]. In passing, we mention that it is now known that a complete shrinking Ricci soliton of any dimension with positive sectional curvature is compact by [31].

The logical next step is to search for a classification of four dimensional gradient shrinking Ricci solitons. Such a result should be very much relevant in understanding the formation of singularities of the Ricci flows on four dimensional manifolds, just like the three dimensional case. However, in contrast to the dimension three case, for dimension four or higher, the curvature of a gradient shrinking Ricci soliton may change sign as demonstrated by the examples constructed in [17]. The existence of such examples, which are obviously not of the form of a sphere, or the Euclidean space, or their product, certainly complicates the classification outlook.

Note that in the case of dimension three, the curvature operator, being nonnegative, is bounded by the scalar curvature. In the case of dimension four, we showed that such a conclusion still holds even though the curvature operator no longer has a fixed sign, [29]. In particular, this implies that the curvature operator must be bounded if the scalar curvature is.

Theorem 1.3. *Let (M, g, f) be a four dimensional complete gradient shrinking Ricci soliton with bounded scalar curvature S. Then there exists a constant $c > 0$ so that*

$$|\mathrm{Rm}| \leq c\, S \quad \text{on } M.$$

In the theorem, the constant $c > 0$ depends only on the upper bound of the scalar curvature A and the geometry of the geodesic ball $B_p(r_0)$, where p is a minimum point of potential function f and r_0 is determined by A. We stress that the potential function f of the soliton is exploited in an essential way in our proof by working on the level sets of f.

As an application, we obtained the following structural result. Recall that a Riemannian cone is a manifold $[0, \infty) \times \Sigma$ endowed with Riemannian metric $g_c = dr^2 + r^2\, g_\Sigma$, where (Σ, g_Σ) is a closed $(n-1)$-dimensional Riemannian manifold. Denote $E_R = (R, \infty) \times \Sigma$ for $R \geq 0$ and define the dilation by λ to be the map $\rho_\lambda : E_0 \to E_0$ given by $\rho_\lambda(r, \sigma) = (\lambda\, r, \sigma)$. Then Riemannian manifold (M, g) is said to be C^k asymptotic to the cone (E_0, g_c) if, for some $R > 0$, there is a diffeomorphism $\Phi : E_R \to M \setminus \Omega$ such that $\lambda^{-2}\, \rho_\lambda^*\, \Phi^*\, g \to g_c$ as $\lambda \to \infty$ in $C^k_{loc}(E_0, g_c)$, where Ω is a compact subset of M. The following result was established in [29].

Theorem 1.4. *Let (M, g, f) be a complete four dimensional gradient shrinking Ricci soliton with scalar curvature converging to zero at infinity. Then there exists a cone E_0 such that (M, g) is C^k asymptotic to E_0 for all k.*

A recent result due to Kotschwar and L. Wang [26] states that two gradient shrinking Ricci solitons (of arbitrary dimensions) must be isometric if they are C^2 asymptotic to the same cone. Together with our result, this implies that the classification problem for four dimensional gradient shrinking Ricci solitons with scalar curvature going to zero at infinity is reduced to the one for the limiting cones.

In this paper, we take up the case that the scalar curvature is bounded from below by a positive constant and show the following structural result. Here, a Riemannian manifold (M, g) is said to be C^k asymptotic to the cylinder $L = (\mathbb{R} \times N, g_c)$, where g_c is the product metric, if there is a diffeomorphism $\Phi : L_0 = (0, \infty) \times N \to M \setminus \Omega$ such that $\rho_\lambda^*\, \Phi^*\, g \to g_c$ as $\lambda \to \infty$ in $C^k_{loc}(L_0, g_c)$, where Ω is a compact subset of M and $\rho_\lambda : L \to L$ is the translation given by $\rho_\lambda(r, \sigma) = (\lambda + r, \sigma)$ for $r \in \mathbb{R}$ and $\sigma \in N$.

Theorem 1.5. *Let (M, g, f) be a complete, four dimensional gradient shrinking Ricci soliton with bounded scalar curvature S. If S is bounded from below by a positive constant on end E of M, then E is smoothly asymptotic to the round cylinder $\mathbb{R} \times \mathbb{S}^3/\Gamma$, or for any sequence $x_i \in E$ going to infinity along an integral curve of ∇f, (M, g, x_i) converges smoothly to $\mathbb{R}^2 \times \mathbb{S}^2$ or its $\mathbb{Z}_2$ quotient. Moreover, the limit is uniquely determined by the integral curve and is independent of the sequence x_i.*

Here and throughout the paper, $\mathbb{S}^n$ denotes the n-dimensional standard sphere with metric normalized such that $\mathrm{Ric} = \frac{1}{2}\, g$. As pointed out in [32], in the second case, the limit in general may depend on the integral curve as demonstrated by the example $M = \mathbb{R} \times \left(\mathbb{R} \times \mathbb{S}^2\right)/\mathbb{Z}_2$. We remark that under the additional assumption that the Ricci curvature is non-negative, a similar version of Theorem 1.5 was proved in [32] by a different argument. Obviously, Theorem 1.5 together with Theorem 1.4 would provide a description of the geometry at infinity for all four dimensional gradient shrinking Ricci solitons with bounded scalar curvature if one could establish a dichotomy that the scalar curvature S either goes to 0 at infinity or is bounded from below by a positive constant. This question remains open presently.

Let us now briefly describe how Theorem 1.5 is proven. According to [32], for any n-dimensional shrinking gradient Ricci soliton (M^n, g, f) with bounded curvature and a sequence of points $x_i \in M$ going to infinity along an integral curve of ∇f, by choosing a subsequence if necessary, (M^n, g, x_i) converges smoothly to a product manifold $\mathbb{R} \times N^{n-1}$, where N is a gradient shrinking Ricci soliton. By the classification result of three dimensional gradient shrinking Ricci solitons Theorem 1.2 and the fact that the scalar curvature is assumed to be bounded from below by a positive constant, Theorem 1.5 will then follow from the following structural result for gradient shrinking Ricci solitons of arbitrary dimension.

Theorem 1.6. *Let (M, g, f) be an n-dimensional, complete, gradient shrinking Ricci soliton with bounded curvature. Assume that along an end E of M there exists a sequence of points $x_i \to \infty$ with (M, g, x_i) converging to the round cylinder $\mathbb{R} \times \mathbb{S}^{n-1}/\Gamma$. Then E is smoothly asymptotic to the same round cylinder.*

The proof of this theorem constitutes the major part of the paper. Conceptually speaking, we will view the level sets of f endowed with the induced metric as an approximate Ricci flow and adopt the argument due to Huisken [23] who proved that the Ricci flow starting from a manifold with sufficiently pinched sectional curvature must converge to a quotient of the round sphere. However, to actually carry out the argument, we have to overcome some serious technical hurdles, one of them being the control of the scalar curvature. Along the way, we have managed to obtain some localized estimates for the derivatives of the curvature tensor, which may be of independent interest. In particular, these estimates enabled us to derive a Harnack type estimate for the scalar curvature.

There are quite a few related works concerning the geometry and classification of high dimensional gradient shrinking Ricci solitons. The survey paper [1] contains a wealth of information and then current results. The paper by Naber [32] has strong influence on the present work. For some of the more recent progress, we refer to [9], [10], [4], [5], [7], [28]. In the other direction, Catino, Deruelle and Mazzieri [11] have attempted to address the rigidity issue for the complete gradient shrinking Ricci solitons which are asymptotic to the round cylinder at infinity, that is, whether the soliton M in Theorem 1.6 is in fact itself a round cylinder. Apparently, this issue remains unresolved, as it was stated there that the proof given is yet incomplete.

The paper is organized as follows. After recalling a few preliminary facts in section 2, we prove some useful localized curvature estimates in section 3. Theorem 1.6 is then proved in section 4. The applications to four dimensional gradient shrinking Ricci solitons are discussed in section 5.

We would like to thank Ben Chow, Brett Kotschwar, Aaron Naber and Lei Ni for their interest and helpful discussions. We also thank Yongjia Zhang for useful comments on a previous draft of this paper.

2. Preliminaries

In this section, we recall some preliminary facts concerning gradient shrinking Ricci solitons. We will use the same notation as in [29]. Throughout this paper, (M, g) denotes an n-dimensional, complete noncompact gradient shrinking Ricci soliton. A result of Chen ([12, 2]) implies that the scalar curvature $S > 0$ on M, unless M is flat. This result was later refined in [14] to that

$$S \geq \frac{C_0}{f} \tag{2.1}$$

for some positive constant C_0 depending on the soliton. Furthermore, by adding a constant to the potential function f if necessary, one has the following important identity due to Hamilton [18].

$$S + |\nabla f|^2 = f.$$

For such a normalized potential function f, it is well known [6] that there exist positive constants c_1 and c_2 such that

$$\frac{1}{4}r^2(x) - c_1\, r(x) - c_2 \leq f(x) \leq \frac{1}{4}r^2(x) + c_1\, r(x) + c_2, \tag{2.2}$$

where $r(x)$ is the distance to a fixed point $p \in M$. Moreover, c_1 and c_2 can be chosen to depend only on n if p is a minimum point of f, see [22].

Consequently, if the scalar curvature of (M, g) is bounded, then there exists $t_0 > 0$ such that the level set

$$\Sigma(t) = \{x \in M : f(x) = t\}$$

of f is a compact Riemannian manifold for $t \geq t_0$. Also, the domain

$$D(t) := \{x \in M : f(x) \leq t\}$$

is a compact manifold with smooth boundary $\Sigma(t)$.

Since the volume of M grows polynomially of order at most n by [6], one sees that the weighted volume of M given by

$$V_f(M) = \int_M e^{-f}\, dv$$

must be finite.

We recall the following equations for various curvature quantities of M, see e.g. [29].

$$
\begin{aligned}
\nabla S &= 2\mathrm{Ric}(\nabla f) \\
\nabla_l R_{ijkl} &= R_{ijkl} f_l = \nabla_j R_{ik} - \nabla_i R_{jk} \\
\Delta_f S &= S - 2\left|\mathrm{Ric}\right|^2 \\
\Delta_f \mathrm{Ric} &= \mathrm{Ric} - 2\mathrm{Rm}\mathrm{Ric} \\
\Delta_f \mathrm{Rm} &= \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm} \\
\Delta_f \left(\nabla^k \mathrm{Rm}\right) &= \left(\frac{k}{2}+1\right)\nabla^k \mathrm{Rm} + \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j}\mathrm{Rm}.
\end{aligned}
\tag{2.3}
$$

Here, Δ_f is the weighted Laplacian defined by $\Delta_f T = \Delta T - \langle \nabla f, \nabla T\rangle$ for a tensor field T. The notation $\mathrm{Rm} * \mathrm{Rm}$ denotes a quadratic expression in the Riemann curvature tensor and $\nabla^j \mathrm{Rm}$ denotes the j-th covariant derivative of the curvature tensor Rm.

As mentioned in the introduction, M may be viewed as a self-similar solution to the Ricci flow. Therefore, if the curvature of (M, g) is bounded, that is, there exists a constant $C > 0$ such that $|\mathrm{Rm}| \leq C$ on M, then by Shi's derivative estimates [38], for each $k \geq 1$, there exists a constant $A_k > 0$ such that

$$
\left|\nabla^k \mathrm{Rm}\right| \leq A_k \quad \text{on } M \tag{2.4}
$$

with A_k depending only on n, k and C.

Using (2.3) we get, for any $k \geq 0$ and $\sigma > 0$, that

$$
\begin{aligned}
\Delta_f \left(\left|\nabla^k \mathrm{Rm}\right|^2 S^{-\sigma}\right) &\geq S^{-\sigma}\left(2\left|\nabla^{k+1}\mathrm{Rm}\right|^2 + (k+2)\left|\nabla^k \mathrm{Rm}\right|^2\right) \\
&- cS^{-\sigma}\Sigma_{j=0}^{k}\left|\nabla^j \mathrm{Rm}\right|\left|\nabla^{k-j}\mathrm{Rm}\right|\left|\nabla^k \mathrm{Rm}\right| \\
&+ \left|\nabla^k \mathrm{Rm}\right|^2\left(-\sigma S^{-\sigma} + 2\sigma\left|\mathrm{Ric}\right|^2 S^{-\sigma-1} + \sigma(\sigma+1)\left|\nabla S\right|^2 S^{-\sigma-2}\right) \\
&+ 2\left\langle \nabla\left|\nabla^k \mathrm{Rm}\right|^2, \nabla S^{-\sigma}\right\rangle.
\end{aligned}
\tag{2.5}
$$

Observe that

$$
2\left\langle \nabla\left|\nabla^k \mathrm{Rm}\right|^2, \nabla S^{-\sigma}\right\rangle \geq -2\left|\nabla^{k+1}\mathrm{Rm}\right|^2 S^{-\sigma} - 2\sigma^2\left|\nabla S\right|^2 S^{-\sigma-2}\left|\nabla^k \mathrm{Rm}\right|^2.
$$

This implies the function $w := \left|\nabla^k \mathrm{Rm}\right|^2 S^{-\sigma}$ satisfies

$$
\begin{aligned}
\Delta_f w \geq\; & \left(k+2-\sigma+\left(\sigma-\sigma^2\right)\left|\nabla \ln S\right|^2\right) w \\
& - c\Sigma_{j=0}^{k}\left|\nabla^j \mathrm{Rm}\right|\left|\nabla^{k-j}\mathrm{Rm}\right|\left|\nabla^k \mathrm{Rm}\right| S^{-\sigma}.
\end{aligned}
\tag{2.6}
$$

If instead in (2.5) we use

$$\begin{aligned}
& 2\left\langle \nabla\left|\nabla^{k}\mathrm{Rm}\right|^{2}, \nabla S^{-\sigma}\right\rangle \\
= & \left\langle \nabla\left(\left|\nabla^{k}\mathrm{Rm}\right|^{2} S^{-\sigma}S^{\sigma}\right), \nabla S^{-\sigma}\right\rangle + \left\langle \nabla\left|\nabla^{k}\mathrm{Rm}\right|^{2}, \nabla S^{-\sigma}\right\rangle \\
\geq & \left\langle \nabla\left(\left|\nabla^{k}\mathrm{Rm}\right|^{2} S^{-\sigma}\right), \nabla S^{-\sigma}\right\rangle S^{\sigma} + \left|\nabla^{k}\mathrm{Rm}\right|^{2} S^{-\sigma}\left\langle \nabla S^{\sigma}, \nabla S^{-\sigma}\right\rangle \\
& -2\sigma\left|\nabla^{k+1}\mathrm{Rm}\right|\left|\nabla S\right|\left|\nabla^{k}\mathrm{Rm}\right| S^{-\sigma-1} \\
\geq & -\sigma\left\langle \nabla\left(\left|\nabla^{k}\mathrm{Rm}\right|^{2} S^{-\sigma}\right), \nabla \ln S\right\rangle - \frac{3}{2}\sigma^{2}\left|\nabla^{k}\mathrm{Rm}\right|^{2}\left|\nabla S\right|^{2} S^{-\sigma-2} \\
& -2\left|\nabla^{k+1}\mathrm{Rm}\right|^{2} S^{-\sigma},
\end{aligned}$$

then the function $w := \left|\nabla^{k}\mathrm{Rm}\right|^{2} S^{-\sigma}$ satisfies

$$\begin{aligned}
\Delta_F w \geq & \left(k+2-\sigma+\left(\sigma-\frac{\sigma^2}{2}\right)\left|\nabla \ln S\right|^2\right) w \\
& -c\Sigma_{j=0}^{k}\left|\nabla^{j}\mathrm{Rm}\right|\left|\nabla^{k-j}\mathrm{Rm}\right|\left|\nabla^{k}\mathrm{Rm}\right| S^{-\sigma},
\end{aligned} \tag{2.7}$$

where $F := f - \sigma \ln S$.

3. Curvature estimates for shrinkers

In this section, we establish some localized derivative estimates for the curvature tensor of a gradient shrinking Ricci soliton. The estimates will be applied in next section to prove Theorem 1.6.

Throughout this section, (M, g) denotes an n-dimensional gradient shrinking Ricci soliton with bounded curvature. Hence, we may assume that (2.4) holds everywhere on M.

Everywhere in this paper, we will denote by $\{e_1, e_2, \cdots, e_n\}$ a local orthonormal frame of M with

$$e_n := \frac{\nabla f}{|\nabla f|}.$$

Clearly, e_n is a unit normal vector to $\Sigma(t)$ and $\{e_1, e_2, \cdots, e_{n-1}\}$ a local orthonormal frame of $\Sigma(t)$. Throughout this paper, the indices $a, b, c, d = 1, 2, \cdots, n-1$ and $i, j, k, l = 1, 2, \cdots, n$. In this notation, the second fundamental form of $\Sigma(t)$ is given by

$$h_{ab} = \frac{f_{ab}}{|\nabla f|}, \tag{3.1}$$

for any $a, b = 1, 2, \cdots, n-1$.

By (2.3) we have that

$$\begin{aligned}
|R_{ijkn}| &= \frac{|R_{ijkl} f_l|}{|\nabla f|} \\
&= \frac{1}{|\nabla f|}\left|\nabla_j R_{ik} - \nabla_i R_{jk}\right| \\
&\leq \frac{2\left|\nabla \mathrm{Ric}\right|}{|\nabla f|}.
\end{aligned} \tag{3.2}$$

Denote with

$$
\begin{aligned}
\overset{\circ}{R}_{ab} &:= R_{ab} - \frac{1}{n-1} S\, g_{ab}, \\
U_{abcd} &:= \frac{1}{(n-1)(n-2)} S\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right), \\
\overset{\circ}{R}_{abcd} &:= R_{abcd} - U_{abcd}, \\
V_{abcd} &:= \frac{1}{n-3}\left(\overset{\circ}{R}_{ac}g_{bd} + \overset{\circ}{R}_{bd}g_{ac} - \overset{\circ}{R}_{ad}g_{bc} - \overset{\circ}{R}_{bc}g_{ad}\right), \\
W_{abcd} &:= R_{abcd} - U_{abcd} - V_{abcd},
\end{aligned} \tag{3.3}
$$

where $a, b, c, d = 1, 2, \cdots, n-1$. It should be pointed out that W is not the Weyl curvature tensor of the manifold (M, g), restricted to the level set $\Sigma(t)$, rather it is an approximation of the one of $\Sigma(t)$.

Denote

$$
\begin{aligned}
\left|\overset{\circ}{\mathrm{Ric}}_{\Sigma}\right|^2 &:= \left|\overset{\circ}{R}_{ab}\right|^2, \\
\left|\overset{\circ}{\mathrm{Rm}}_{\Sigma}\right|^2 &:= \left|\overset{\circ}{R}_{abcd}\right|^2.
\end{aligned}
$$

We now state the main result of this section. Fix $t_0 > 0$ large enough, depending only on dimension n and the constant A_0 in (2.4). Since $S \leq nA_0$, using Hamilton's identity $S + |\nabla f|^2 = f$ we get that the level sets $\Sigma(t)$ of f are all smooth for $t \geq t_0$. Also fix some $T > t_0$. We have the following.

Theorem 3.1. *Let (M, g, f) be an n-dimensional, complete, gradient shrinking Ricci soliton with bounded curvature such that*

$$
\begin{aligned}
\left|\overset{\circ}{\mathrm{Rm}}_{\Sigma}\right|^2 &\leq \eta_1 S^2 \quad \text{on } D(T) \setminus D(t_0), \\
S &\geq \eta_2 \quad \text{on } \Sigma(t_0)
\end{aligned} \tag{3.4}
$$

for some $\eta_1, \eta_2 > 0$. Then for each $k \geq 0$, there exists constant $c_k > 0$ such that

$$
\left|\nabla^k \mathrm{Rm}\right|^2 \leq c_k\, S^{k+2} \quad \text{on } D(T) \setminus D(t_0).
$$

For given C_0 from (2.1) and A_k from (2.4), the constant c_k in Theorem 3.1 only depends on n, η_1, η_2, C_0 and A_0,....,A_{Kk}, where K is an absolute constant ($K = 100$ suffices). We stress that all c_k are independent of t_0 and T.

As a useful corollary of this theorem, if

$$
\begin{aligned}
|\mathrm{Rm}|^2 &\leq \eta_1 S^2 \quad \text{on } D(T) \setminus D(t_0), \\
S &\geq \eta_2 \quad \text{on } \Sigma(t_0),
\end{aligned} \tag{3.5}
$$

then

$$
\left|\nabla^k \mathrm{Rm}\right|^2 \leq c_k S^{k+2} \quad \text{on } D(T) \setminus D(t_0).
$$

In fact, in the proof of Theorem 3.1, we will show that (3.4) implies (3.5). The converse is obviously true.

According to Theorem 1.3, (3.5) is true for $T = \infty$ on a four dimensional gradient shrinking Ricci soliton with bounded scalar curvature. Hence, we obtain the following.

Corollary 3.2. *Let (M, g, f) be a complete, four dimensional, gradient shrinking Ricci soliton with bounded scalar curvature. Then for each $k \geq 0$, there exists constant $c_k > 0$ so that*

$$\left|\nabla^k \mathrm{Rm}\right|^2 \leq c_k\, S^{k+2} \quad \text{on } M.$$

The rest of the section is devoted to proving Theorem 3.1. First, we observe that T may be assumed to be large compared to t_0. Indeed, define ϕ_t by

$$\begin{aligned} \frac{d\phi_t}{dt} &= \frac{\nabla f}{|\nabla f|^2} \\ \phi_{t_0} &= \mathrm{Id} \ \text{ on } \Sigma(t_0). \end{aligned}$$

For a fixed $x \in \Sigma(t_0)$, denote $S(t) := S(\phi_t(x))$, where $t \geq t_0$. Then, as $\langle \nabla S, \nabla f \rangle = \Delta S - S + 2|\mathrm{Ric}|^2$, it follows from (2.4) that

$$\left|\frac{dS}{dt}\right| = \frac{|\langle \nabla S, \nabla f\rangle|}{|\nabla f|^2} \leq \frac{c_0}{t}. \tag{3.6}$$

Integrating this in t we get

$$\begin{aligned} S(t) &\geq S(t_0) - c_0 \ln \frac{t}{t_0} \\ &\geq \eta_2 - c_0 \ln \frac{t}{t_0}. \end{aligned} \tag{3.7}$$

Hence, if $T \leq e^{\frac{\eta_2}{2c_0}} t_0$, then (3.7) implies $S \geq \frac{1}{2}\eta_2$ on $D(T) \backslash D(t_0)$. In this case, Theorem 3.1 follows directly from (2.4). So we may assume from now on that there exists $\nu > 1$, depending only on η_2, A_0 and A_2, satisfying

$$T \geq \nu t_0. \tag{3.8}$$

Using (3.2) and (2.1), we see from (3.4) that

$$|\mathrm{Rm}| \leq c\left(S + \frac{|\nabla \mathrm{Ric}|}{\sqrt{f}}\right) \leq c\sqrt{S} \ \text{ on } D(T) \backslash D(t_0). \tag{3.9}$$

The proof of Theorem 3.1 is divided in two parts.

Proposition 3.3. *Let (M, g, f) be an n-dimensional, complete, gradient shrinking Ricci soliton such that (3.4) holds. Then, for any $k \geq 0$, there exists constant c_k such that*

$$\left|\nabla^k \mathrm{Rm}\right| \leq c_k S \quad \text{on } D(T) \backslash D(t_0). \tag{3.10}$$

Proof. We first prove by induction on $k \geq 0$ that

$$\left|\nabla^k \mathrm{Rm}\right|^2 \leq c_k S \ \text{ on } D(a_k T) \backslash D(t_0), \tag{3.11}$$

where

$$a_k := 1 - \left(1 - \frac{1}{2^k}\right)\frac{1}{\sqrt{T}}.$$

For $k = 0$ we get (3.11) from (3.9). Let us assume (3.11) is true for $k = 0, 1, .., l-1$ and prove it for $k = l$. By (2.6), on $D(T) \backslash D(t_0)$ the function $w := \left|\nabla^l \mathrm{Rm}\right|^2 S^{-1}$ satisfies

$$\begin{aligned} \Delta_f w &\geq 2w - c\Sigma_{j=0}^{l} \left|\nabla^j \mathrm{Rm}\right| \left|\nabla^{l-j} \mathrm{Rm}\right| \left|\nabla^l \mathrm{Rm}\right| S^{-1} \\ &\geq w - c\Sigma_{j=0}^{l} \left|\nabla^j \mathrm{Rm}\right|^2 \left|\nabla^{l-j} \mathrm{Rm}\right|^2 S^{-1}. \end{aligned}$$

By the induction hypothesis,

$$\left|\nabla^j \mathrm{Rm}\right|^2 \left|\nabla^{l-j}\mathrm{Rm}\right|^2 S^{-1} \leq c \quad \text{on } D\left(a_{l-1}T\right)\backslash D\left(t_0\right).$$

Hence, it follows from above that

$$\Delta_f w \geq w - c_l \quad \text{on } D\left(a_{l-1}T\right)\backslash D\left(t_0\right). \tag{3.12}$$

Define the cut-off function

$$\psi\left(f\left(x\right)\right) = \frac{e^{\sqrt{a_{l-1}T}} - e^{\frac{f}{\sqrt{a_{l-1}T}}}}{e^{\sqrt{a_{l-1}T}}} \tag{3.13}$$

with support in $D\left(a_{l-1}T\right)\backslash D\left(t_0\right)$ and let $G := \psi^2 w$. By (3.12) we get

$$\begin{aligned} \Delta_f G \quad \geq \quad & G - c_l + 2\psi^{-1}\left(\Delta_f \psi\right) G - 6\psi^{-2}\left|\nabla \psi\right|^2 G \\ & + 2\psi^{-2}\left\langle \nabla G, \nabla \psi^2 \right\rangle. \end{aligned} \tag{3.14}$$

Let x_0 be the maximum point of G on $D\left(a_{l-1}T\right)\backslash D\left(t_0\right).$ If $x_0 \in \Sigma\left(t_0\right),$ then $G\left(x_0\right) \leq c_l$ by (3.4) and (2.4). So, without loss of generality, we may assume that x_0 is an interior point. If at x_0 we have $\psi^{-1}\left(\Delta_f\psi\right) - 3\psi^{-2}\left|\nabla\psi\right|^2 \geq 0$, then applying the maximum principle to (3.14) we get $G\left(x_0\right) \leq c_l$. Now suppose that

$$\psi^{-1}\left(\Delta_f \psi\right) - 3\psi^{-2}\left|\nabla\psi\right|^2 < 0 \quad \text{at } x_0. \tag{3.15}$$

Since

$$\begin{aligned} \Delta_f \psi \quad &= \quad \psi' \Delta_f\left(f\right) + \psi''\left|\nabla f\right|^2 \\ &= \quad \frac{e^{\frac{f}{\sqrt{a_{l-1}T}}}}{\sqrt{a_{l-1}T}\, e^{\sqrt{a_{l-1}T}}}\left(f - \frac{n}{2} - \frac{\left|\nabla f\right|^2}{\sqrt{a_{l-1}T}}\right) \\ &\geq \quad \frac{1}{2}\frac{e^{\frac{f}{\sqrt{a_{l-1}T}}}}{e^{\sqrt{a_{l-1}T}}}\frac{f}{\sqrt{a_{l-1}T}}, \end{aligned} \tag{3.16}$$

by (3.15) it follows that

$$\begin{aligned} \frac{1}{2}\frac{e^{\frac{f}{\sqrt{a_{l-1}T}}}}{e^{\sqrt{a_{l-1}T}}}\frac{f}{\sqrt{a_{l-1}T}}\psi\left(x_0\right) \quad &\leq \quad \psi \Delta_f \psi \\ &< \quad 3\left|\nabla\psi\right|^2 \\ &\leq \quad 3\frac{e^{\frac{2f}{\sqrt{a_{l-1}T}}}}{e^{2\sqrt{a_{l-1}T}}}\frac{f}{a_{l-1}T}. \end{aligned}$$

This immediately implies

$$\psi\left(x_0\right) \leq \frac{c}{\sqrt{T}}.$$

Hence, we obtain from (2.1) that

$$G\left(x_0\right) \leq \frac{c}{T}\left|\nabla^l \mathrm{Rm}\right|^2 S^{-1} \leq c_l.$$

In conclusion, this proves

$$G \leq c_l \quad \text{on} \quad D\left(a_{l-1}T\right)\backslash D\left(t_0\right). \tag{3.17}$$

Since $(a_{l-1}-a_l)\sqrt{T}=\frac{1}{2^l}$, on $D(a_lT)\backslash D(t_0)$ by (3.13) we have

$$\begin{aligned}\psi &\geq 1-e^{\frac{1}{\sqrt{a_{l-1}}}(a_l-a_{l-1})\sqrt{T}}\\ &\geq 1-e^{-2^{-l}}.\end{aligned}$$

By (3.17) we get that $\left|\nabla^l \mathrm{Rm}\right|^2 S^{-1}\leq c_l$ on $D(a_lT)\backslash D(t_0)$. This completes the induction step and proves (3.11). In particular, we have

$$\left|\nabla^k \mathrm{Rm}\right|^2 \leq c_k S \ \text{ on } D\left(T-\sqrt{T}\right)\backslash D(t_0). \tag{3.18}$$

We now prove that for all $k\geq 0$

$$\left|\nabla^k \mathrm{Rm}\right|^2 \leq c_k S^2 \ \text{ on } D\left(T-2\sqrt{T}\right)\backslash D(t_0). \tag{3.19}$$

For $k=0$, (3.19) follows from (3.9) and (3.18). Indeed, by (3.18), one has

$$|\nabla \mathrm{Ric}|\leq c\sqrt{S}$$

on $D(b_0T)=D\left(T-\sqrt{T}\right)$. Plugging this into (3.9) and using (2.1), one sees that

$$|\mathrm{Rm}|\leq c\,S$$

on $D(b_0T)$. We now prove (3.19) for $k\geq 1$.

By (2.7), on $D\left(T-\sqrt{T}\right)\backslash D(t_0)$, the function $w:=\left|\nabla^k \mathrm{Rm}\right|^2 S^{-2}$ satisfies

$$\begin{aligned}\Delta_F w &\geq w-c\Sigma_{j=0}^k\left|\nabla^j \mathrm{Rm}\right|\left|\nabla^{k-j}\mathrm{Rm}\right|\left|\nabla^k \mathrm{Rm}\right|S^{-2}\\ &\geq \frac{1}{2}w-c\Sigma_{j=0}^k\left|\nabla^j \mathrm{Rm}\right|^2\left|\nabla^{k-j}\mathrm{Rm}\right|^2 S^{-2}\\ &\geq \frac{1}{2}w-c_k,\end{aligned} \tag{3.20}$$

where $F:=f-2\ln S$ and in the last line we have used (3.18). Let $G:=\psi^2 w$ with

$$\psi:=\frac{e^{\sqrt{T-\sqrt{T}}}-e^{\frac{f}{\sqrt{T-\sqrt{T}}}}}{e^{\sqrt{T-\sqrt{T}}}}. \tag{3.21}$$

By (3.20) we obtain

$$\begin{aligned}\Delta_F G &\geq \frac{1}{2}G-c_k+2\psi^{-1}(\Delta_F\psi)G-6\psi^{-2}|\nabla\psi|^2 G\\ &\quad +2\psi^{-2}\left\langle \nabla G,\nabla\psi^2\right\rangle.\end{aligned} \tag{3.22}$$

Let x_0 be the maximum point of G on $D\left(T-\sqrt{T}\right)\backslash D(t_0)$. As above, we may assume that x_0 is an interior point. If at x_0 we have $\psi^{-1}(\Delta_F\psi)-3\psi^{-2}|\nabla\psi|^2\geq 0$, then the maximum principle implies $G(x_0)\leq c_k$. So it remains to consider the case that

$$\psi^{-1}(\Delta_F\psi)-3\psi^{-2}|\nabla\psi|^2<0 \ \text{ at } x_0. \tag{3.23}$$

As in (3.16), we have

$$\Delta_f\psi\geq \frac{1}{2}\frac{e^{\frac{f}{\sqrt{T-\sqrt{T}}}}}{e^{\sqrt{T-\sqrt{T}}}}\frac{f}{\sqrt{T-\sqrt{T}}}. \tag{3.24}$$

Furthermore,

$$|\langle \nabla \ln S, \nabla \psi \rangle| = \frac{e^{\frac{f}{\sqrt{T-\sqrt{T}}}}}{\sqrt{T-\sqrt{T}}e^{\sqrt{T-\sqrt{T}}}} |\langle \nabla S, \nabla f \rangle| S^{-1}. \tag{3.25}$$

However, (3.18) implies that

$$\begin{aligned} |\langle \nabla S, \nabla f \rangle| &\leq |\Delta S| + S + 2|\mathrm{Ric}|^2 \\ &\leq c\sqrt{S}. \end{aligned} \tag{3.26}$$

For $c > 0$ specified in (3.26) and C_0 the constant in (2.1), we consider the following cases.

First, assume that

$$c\sqrt{S} < C_0^{\frac{3}{4}} S^{\frac{1}{4}} \text{ at } x_0.$$

Then (3.26) implies that

$$\begin{aligned} |\langle \nabla S, \nabla f \rangle| S^{-1}(x_0) &\leq \left(C_0 S^{-1}(x_0)\right)^{\frac{3}{4}} \\ &\leq f^{\frac{3}{4}}(x_0), \end{aligned}$$

where the last line follows from (2.1). By (3.25), this implies that

$$|\langle \nabla \ln S, \nabla \psi \rangle|(x_0) \leq \frac{e^{\frac{f(x_0)}{\sqrt{T-\sqrt{T}}}}}{\sqrt{T-\sqrt{T}}e^{\sqrt{T-\sqrt{T}}}} f^{\frac{3}{4}}(x_0).$$

From (3.24), we get that at x_0, the maximum point of G on $D\left(T-\sqrt{T}\right)\setminus D(t_0)$, we have

$$\Delta_F \psi \geq \frac{1}{3} \frac{e^{\frac{f}{\sqrt{T-\sqrt{T}}}}}{e^{\sqrt{T-\sqrt{T}}}} \frac{f}{\sqrt{T-\sqrt{T}}}.$$

Plugging this into (3.23), we get

$$\psi(x_0) \leq \frac{c}{\sqrt{T}}.$$

Therefore, by (2.1) and (3.18),

$$\begin{aligned} G(x_0) &= \psi^2(x_0) w(x_0) \\ &\leq \frac{c}{T} \left|\nabla^k \mathrm{Rm}\right|^2 S^{-2} \\ &\leq c_k. \end{aligned}$$

Finally, assume that

$$c\sqrt{S} \geq C_0^{\frac{3}{4}} S^{\frac{1}{4}} \text{ at } x_0,$$

where c is the constant in (3.26) and C_0 the constant in (2.1).

We then get $S(x_0) \geq C_0^3 c^{-4}$, which by (2.4) proves that $G(x_0) \leq c_k$. In conclusion, from these two cases we get that

$$G \leq c_k \text{ on } D\left(T-\sqrt{T}\right)\setminus D(t_0). \tag{3.27}$$

Note by (3.21) we have on $D\left(T-2\sqrt{T}\right)\setminus D(t_0)$

$$\begin{aligned} \psi &\geq 1 - e^{-\frac{\sqrt{T}}{\sqrt{T-\sqrt{T}}}} \\ &\geq 1 - e^{-1}. \end{aligned}$$

So (3.27) implies that $\left|\nabla^k \mathrm{Rm}\right|^2 S^{-2} \leq c_k$ on $D\left(T-2\sqrt{T}\right)\backslash D\left(t_0\right)$, which proves (3.19).

We now complete the proof of the proposition. Define ϕ_t by

$$\begin{aligned} \frac{d\phi_t}{dt} &= \frac{\nabla f}{\left|\nabla f\right|^2} \\ \phi_{T-2\sqrt{T}} &= \mathrm{Id} \text{ on } \Sigma\left(T-2\sqrt{T}\right). \end{aligned} \tag{3.28}$$

For $q \in \Sigma\left(t_1\right)$ with $T-2\sqrt{T} \leq t_1 \leq T$, let $q_0 \in \Sigma\left(T-2\sqrt{T}\right)$ be such that $\phi_{t_1}\left(q_0\right) = q$. We obtain, as in (3.6), that

$$\left|\frac{d}{dt} S\left(\phi_t\left(q_0\right)\right)\right| \leq \frac{c}{t}.$$

Integrating this from $t = T-2\sqrt{T}$ to $t = t_1$ implies that

$$\begin{aligned} S\left(q_0\right) &\leq S\left(q\right) + c\ln\frac{t_1}{T-2\sqrt{T}} \\ &\leq S\left(q\right) + \frac{c}{\sqrt{T}}. \end{aligned} \tag{3.29}$$

By (2.1) we know that $S\left(q\right) \geq \frac{c}{T}$. Hence, (3.29) implies that

$$S\left(q_0\right) \leq c\sqrt{S}\left(q\right).$$

Since $q_0 \in D\left(T-2\sqrt{T}\right)\backslash D\left(t_0\right)$, by (3.19) we get that

$$\left|\nabla^k \mathrm{Rm}\right|\left(q_0\right) \leq c_k\sqrt{S}\left(q\right). \tag{3.30}$$

Using (2.3) we compute

$$\begin{aligned} \frac{d}{dt}\left|\nabla^k \mathrm{Rm}\right|^2\left(\phi_t\left(q_0\right)\right) &= \frac{\left\langle \nabla\left|\nabla^k \mathrm{Rm}\right|^2, \nabla f\right\rangle}{\left|\nabla f\right|^2} \\ &= \frac{1}{\left|\nabla f\right|^2}\left(\nabla^k \mathrm{Rm} * \nabla^{k+2}\mathrm{Rm} + \nabla^k \mathrm{Rm} * \nabla^k \mathrm{Rm}\right) \\ &+ \frac{1}{\left|\nabla f\right|^2}\left(\sum_{j=0}^{k} \nabla^k \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^{k-j}\mathrm{Rm}\right). \end{aligned} \tag{3.31}$$

Therefore, by (2.4)

$$\left|\frac{d}{dt}\left|\nabla^k \mathrm{Rm}\right|\left(\phi_t\left(q_0\right)\right)\right| \leq \frac{c_k}{t}.$$

Integrating this from $t = T-2\sqrt{T}$ to $t = t_1$ and using (3.30) we conclude that

$$\begin{aligned} \left|\nabla^k \mathrm{Rm}\right|\left(q\right) &\leq \left|\nabla^k \mathrm{Rm}\right|\left(q_0\right) + c_k\ln\frac{t_1}{T-2\sqrt{T}} \\ &\leq c_k\sqrt{S}\left(q\right) + \frac{c_k}{\sqrt{T}}. \end{aligned}$$

Using (2.1) again that $S\left(q\right) \geq \frac{c}{T}$, we get

$$\left|\nabla^k \mathrm{Rm}\right|\left(q\right) \leq c_k\sqrt{S}\left(q\right),$$

for any $q\in D(T)\backslash D\left(T-2\sqrt{T}\right)$. Together with (3.19), this proves that

$$\left|\nabla^k \mathrm{Rm}\right| \leq c_k\sqrt{S} \text{ on } D(T)\backslash D(t_0). \tag{3.32}$$

For any $q\in\Sigma(t_1)$ with $T-2\sqrt{T}\leq t_1\leq T$, let $q_0\in\Sigma\left(T-2\sqrt{T}\right)$ be such that $\phi_{t_1}(q_0)=q$, where ϕ_t is defined by (3.28). By (3.6) and (2.3) we have that

$$\begin{aligned}\left|\frac{d}{dt}S(\phi_t(q_0))\right| &\leq \frac{c}{t}\left(S+2|\mathrm{Ric}|^2+|\Delta S|\right)(\phi_t(q_0))\\ &\leq \frac{c}{t}\sqrt{S}(\phi_t(q_0)),\end{aligned}$$

where for the last line we used (3.32). We rewrite this as $\left|\frac{d}{dt}\sqrt{S}(\phi_t(q_0))\right|\leq\frac{c}{t}$, and integrate in $t\in\left[T-2\sqrt{T},t_1\right]$. It follows, as for (3.29), that

$$\begin{aligned}\sqrt{S}(\phi_t(q_0)) &\leq \sqrt{S}(q)+\frac{c}{\sqrt{T}}\\ &\leq c\sqrt{S}(q).\end{aligned}\tag{3.33}$$

In particular, $S(q_0)\leq cS(q)$. Since $q_0\in D\left(T-2\sqrt{T}\right)\backslash D(t_0)$, by (3.19) we get that

$$\left|\nabla^k\mathrm{Rm}\right|(q_0)\leq c_k S(q). \tag{3.34}$$

By (3.31), (3.32) and (3.33) we now get

$$\begin{aligned}\left|\frac{d}{dt}\left|\nabla^k\mathrm{Rm}\right|(\phi_t(q_0))\right| &\leq \frac{c}{t}\sqrt{S}(\phi_t(q_0))\\ &\leq \frac{c}{t}\sqrt{S}(q).\end{aligned}$$

Integrating from $t=T-2\sqrt{T}$ to $t=t_1$ it follows that

$$\begin{aligned}\left|\nabla^k\mathrm{Rm}\right|(q) &\leq \left|\nabla^k\mathrm{Rm}\right|(q_0)+\frac{c_k}{\sqrt{T}}\sqrt{S}(q)\\ &\leq c_k S(q),\end{aligned}$$

where in last line we used (3.34) and (2.1). This inequality is true for any $q\in D(T)\backslash D\left(T-2\sqrt{T}\right)$. Together with (3.19), it follows that $\left|\nabla^k\mathrm{Rm}\right|\leq c_k S$ on $D(T)\backslash D(t_0)$. This proves the proposition. □

To improve Proposition 3.3 we use a different strategy. Let us first record some useful consequences. Note that (2.3) and (3.10) imply

$$\begin{aligned}\left|\left\langle\nabla f,\nabla^k\mathrm{Rm}\right\rangle\right| &\leq \left|\Delta_f\left(\nabla^k\mathrm{Rm}\right)\right|+\left|\Delta\left(\nabla^k\mathrm{Rm}\right)\right|\\ &\leq c_k S.\end{aligned}\tag{3.35}$$

In particular,

$$|\langle\nabla f,\nabla S\rangle|\leq cS. \tag{3.36}$$

We can easily see from (2.3) that

$$\begin{aligned} S_{ij} f_i f_j &= \langle \nabla (S_i f_i), \nabla f \rangle - f_{ij} S_i f_j \\ &= 2 \left\langle \nabla |\mathrm{Ric}|^2, \nabla f \right\rangle + \langle \nabla (\Delta S), \nabla f \rangle \\ &\quad - \frac{3}{2} \langle \nabla S, \nabla f \rangle + \frac{1}{2} |\nabla S|^2 . \end{aligned}$$

By (3.10) and (3.35) it follows that

$$|S_{ij} f_i f_j| \le c\, S. \tag{3.37}$$

We now complete the proof of Theorem 3.1 by proving the following.

Proposition 3.4. *Let (M, g, f) be an n-dimensional, complete, gradient shrinking Ricci soliton such that (3.4) holds. Then for any $k \ge 0$ there exists $c_k > 0$ such that*

$$\left|\nabla^k \mathrm{Rm}\right|^2 \le c_k S^{k+2} \quad \text{on } D(T) \backslash D(t_0).$$

Proof. For $k = 0$ this follows from (3.10). For the case $k = 1$, we first prove a weaker statement that

$$|\nabla \mathrm{Rm}|^2 \le c S^{\frac{11}{4}} \quad \text{on } D(T) \backslash D(t_0). \tag{3.38}$$

For any $2 \le \sigma \le 3$, by (2.7) we have

$$\begin{aligned} \Delta_F \left(|\nabla \mathrm{Rm}|^2 S^{-\sigma} \right) &\ge \left(3 - \sigma - \sigma \left(\frac{\sigma}{2} - 1 \right) |\nabla \ln S|^2 \right) |\nabla \mathrm{Rm}|^2 S^{-\sigma} \\ &- c\, |\mathrm{Rm}|\, |\nabla \mathrm{Rm}|^2 S^{-\sigma}. \end{aligned}$$

Hence, it follows from (3.10) that

$$w := |\nabla \mathrm{Rm}|^2 S^{-\sigma}$$

satisfies the inequality

$$\Delta_F w \ge \left((3 - \sigma) - cS - \sigma \left(\frac{\sigma}{2} - 1 \right) |\nabla S|^2 S^{-2} \right) w \tag{3.39}$$

on $D(T) \backslash D(t_0)$, where $F := f - \sigma \ln S$. We will rewrite this as an inequality on $\Sigma(t)$. Note that

$$\Delta w = \Delta_\Sigma w + w_{nn} + H w_n, \tag{3.40}$$

where Δ_Σ is the Laplacian on $\Sigma(t)$ and H is the mean curvature of $\Sigma(t)$. We first estimate $w_{nn} = \mathrm{Hess}(w)(e_n, e_n)$ from above.

Denote by

$$u := |\nabla \mathrm{Rm}|^2$$

and write $w = u S^{-\sigma}$. Then

$$\begin{aligned} w_{nn} &= \frac{1}{|\nabla f|^2} (u_{ij} f_i f_j) S^{-\sigma} - \frac{2\sigma}{|\nabla f|^2} \langle \nabla u, \nabla f \rangle \langle \nabla S, \nabla f \rangle S^{-\sigma-1} \\ &\quad + \frac{\sigma(\sigma+1)}{|\nabla f|^2} \langle \nabla S, \nabla f \rangle^2 S^{-\sigma-2} u - \frac{\sigma}{|\nabla f|^2} (S_{ij} f_i f_j) S^{-\sigma-1} u. \end{aligned} \tag{3.41}$$

We argue that all these terms can be bounded. Note that by (2.3)

$$\begin{aligned} 2 \left\langle \nabla f, \nabla \left(\nabla^k \mathrm{Rm} \right) \right\rangle &= -(k+2) \nabla^k \mathrm{Rm} \\ &+ \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j} \mathrm{Rm} + 2 \Delta \nabla^k \mathrm{Rm}. \end{aligned} \tag{3.42}$$

Consequently,

$$\langle \nabla u, \nabla f \rangle = \nabla \mathrm{Rm} * \nabla \mathrm{Rm} + \mathrm{Rm} * \nabla \mathrm{Rm} * \nabla \mathrm{Rm} + \nabla^3 \mathrm{Rm} * \nabla \mathrm{Rm}. \tag{3.43}$$

It can be similarly checked by using (3.42) and (3.43) that

$$\begin{aligned} \langle \nabla \langle \nabla u, \nabla f \rangle, \nabla f \rangle &= \nabla \mathrm{Rm} * \nabla \mathrm{Rm} + \mathrm{Rm} * \nabla \mathrm{Rm} * \nabla \mathrm{Rm} \\ &+ \mathrm{Rm} * \mathrm{Rm} * \nabla \mathrm{Rm} * \nabla \mathrm{Rm} + \nabla^2 \mathrm{Rm} * \nabla \mathrm{Rm} * \nabla \mathrm{Rm} \\ &+ \nabla^3 \mathrm{Rm} * \nabla \mathrm{Rm} + \nabla^3 \mathrm{Rm} * \nabla \mathrm{Rm} * \mathrm{Rm} + \nabla^3 \mathrm{Rm} * \nabla^3 \mathrm{Rm} * \mathrm{Rm} \\ &+ \nabla^3 \mathrm{Rm} * \nabla^3 \mathrm{Rm} + \nabla^5 \mathrm{Rm} * \nabla \mathrm{Rm}. \end{aligned} \tag{3.44}$$

Hence, by (3.10) we get $|\langle \nabla \langle \nabla u, \nabla f \rangle, \nabla f \rangle| \leq c\, S^2$. Moreover, (3.43) and (3.10) imply

$$\begin{aligned} |f_{ij} u_i f_j| &= \left| \frac{1}{2} \langle \nabla u, \nabla f \rangle - R_{ij} f_j u_i \right| \\ &\leq \frac{1}{2} |\langle \nabla u, \nabla f \rangle| + \frac{1}{2} |\langle \nabla S, \nabla u \rangle| \\ &\leq c\, S^2. \end{aligned}$$

This shows

$$\begin{aligned} |u_{ij} f_i f_j| &\leq |\langle \nabla (u_i f_i), \nabla f \rangle| + |f_{ij} u_i f_j| \\ &\leq cS^2. \end{aligned} \tag{3.45}$$

Since $2 \leq \sigma \leq 3$, using (2.1) we get

$$\frac{1}{|\nabla f|^2} |u_{ij} f_i f_j| \, S^{-\sigma} \leq c.$$

Also, note that by (3.36) and (3.10)

$$\frac{1}{|\nabla f|^2} \langle \nabla S, \nabla f \rangle^2 S^{-\sigma-2} u \leq cS^{3-\sigma} \leq c.$$

Furthermore, using (3.43), one finds that

$$\frac{1}{|\nabla f|^2} |\langle \nabla u, \nabla f \rangle| \, |\langle \nabla S, \nabla f \rangle| \, S^{-\sigma-1} \leq cS^{3-\sigma} \leq c.$$

According to (3.37), we have

$$\frac{1}{|\nabla f|^2} |S_{ij} f_i f_j| \, S^{-\sigma-1} u \leq cS^{3-\sigma} \leq c.$$

From these estimates we get that for any $2 \leq \sigma \leq 3$,

$$w_{nn} \leq c. \tag{3.46}$$

Also, note that

$$\langle \nabla w, \nabla \ln S \rangle = \langle \nabla w, \nabla \ln S \rangle_\Sigma + \frac{1}{|\nabla f|^2} \langle \nabla w, \nabla f \rangle \langle \nabla \ln S, \nabla f \rangle.$$

The last term above can be bounded by using (3.36) together with

$$\begin{aligned} |\langle \nabla w, \nabla f \rangle| &\leq |\langle \nabla u, \nabla f \rangle| \, S^{-\sigma} + c \, |\langle \nabla \ln S, \nabla f \rangle| \, S^{-\sigma} u \\ &\leq cS^{-\sigma+2}. \end{aligned}$$

It follows that

$$\sigma \langle \nabla w, \nabla \ln S \rangle \leq \sigma \langle \nabla w, \nabla \ln S \rangle_\Sigma + c.$$

Similarly, using (3.1) we have

$$|H\, w_n| \leq \frac{c}{|\nabla f|^2} \, |\langle \nabla f, \nabla w \rangle| \leq c.$$

Plugging this and (3.46) into (3.39) we obtain

$$\begin{aligned} \Delta_\Sigma w \quad &\geq \quad \langle \nabla f, \nabla w \rangle - \sigma \langle \nabla w, \nabla \ln S \rangle_\Sigma \\ & \quad + \left((3-\sigma) - cS - \sigma \left(\frac{\sigma}{2} - 1 \right) |\nabla S|^2 S^{-2} \right) w - c, \end{aligned} \tag{3.47}$$

where $w = |\nabla \mathrm{Rm}|^2 S^{-\sigma}$.

Let $\sigma = 2 + \alpha$, where $\alpha > 0$ is to be determined later. Note that by (3.10), $|\nabla S|^2 S^{-2} \leq c$. It follows from (3.47) that

$$\begin{aligned} \Delta_\Sigma w \quad &\geq \quad \langle \nabla f, \nabla w \rangle - \sigma \langle \nabla w, \nabla \ln S \rangle_\Sigma \\ & \quad + \left(\frac{1}{2} - cS - c\alpha \right) w - c. \end{aligned} \tag{3.48}$$

Now we take α small so that $c\,\alpha < \frac{1}{4}$. Then (3.48) becomes

$$\Delta_\Sigma w \geq \langle \nabla f, \nabla w \rangle - \sigma \langle \nabla w, \nabla \ln S \rangle_\Sigma + \left(\frac{1}{4} - cS \right) w - c, \tag{3.49}$$

where $w = |\nabla \mathrm{Rm}|^2 S^{-2-\alpha}$.

Let x_0 be the maximum point of w in $D(T) \setminus D(t_0)$. If $x_0 \in \Sigma(t_0)$, then $w(x_0) \leq c$ by (2.4). So we may assume that $x_0 \notin \Sigma(t_0)$. By maximum principle, we have $\Delta_\Sigma w \leq 0$, $\langle \nabla w, \nabla \ln S \rangle_\Sigma = 0$ and $\langle \nabla f, \nabla w \rangle \geq 0$ at x_0. So (3.49) implies that $\left(\frac{1}{4} - cS(x_0) \right) w(x_0) \leq c$. If $S(x_0) < \frac{1}{8c}$, it follows that $w(x_0) \leq c$. On the other hand, if $S(x_0) \geq \frac{1}{8c}$, then (2.4) implies $w(x_0) \leq c$. In conclusion, we have proved that

$$|\nabla \mathrm{Rm}|^2 S^{-2-\alpha} \leq c \quad \text{on } D(T) \setminus D(t_0). \tag{3.50}$$

Using this estimate, we get from (3.47) that for any $2 \leq \sigma \leq 3$, the function

$$w := |\nabla \mathrm{Rm}|^2 S^{-\sigma}$$

satisfies

$$\Delta_\Sigma w \geq \langle \nabla f, \nabla w \rangle - \sigma \langle \nabla w, \nabla \ln S \rangle_\Sigma + ((3-\sigma) - cS^\alpha) w - c. \tag{3.51}$$

Let $\sigma = \frac{11}{4}$. Then (3.51) becomes

$$\Delta_\Sigma w \geq \langle \nabla f, \nabla w \rangle - \sigma \langle \nabla w, \nabla \ln S \rangle_\Sigma + \left(\frac{1}{4} - cS^\alpha \right) w - c.$$

Applying the maximum principle as in the proof of (3.50), one concludes that w is bounded on $D(T) \setminus D(t_0)$. This shows that

$$|\nabla \mathrm{Rm}|^2 \leq cS^{\frac{11}{4}} \quad \text{on } D(T) \setminus D(t_0) \tag{3.52}$$

and (3.38) is established.

We now prove that for any $k \geq p \geq 1$ there exists a constant $c_{k,p}$, depending on k and p, such that

$$\left| \nabla^k \mathrm{Rm} \right|^2 \leq c_{k,p}\, S^{p+1} \quad \text{on } D(T) \setminus D(t_0). \tag{3.53}$$

The proof is by induction on p. By (3.10), clearly (3.53) is true for $p=1$. Now assume that (3.53) holds for $p=1,2,\cdots,l$. We will prove it for $p=l+1$. That is, if

$$\left|\nabla^j \mathrm{Rm}\right|^2 \leq c_j\, S^{j+1} \quad \text{for all} \quad j \leq l \tag{3.54}$$

and

$$\left|\nabla^j \mathrm{Rm}\right|^2 \leq c_{j,l}\, S^{l+1} \quad \text{for all} \quad j > l, \tag{3.55}$$

then

$$\left|\nabla^k \mathrm{Rm}\right|^2 \leq c_{k,l+1}\, S^{l+2} \text{ for all } k \geq l+1. \tag{3.56}$$

For any $\sigma>0$, we have by (2.6) that

$$\begin{aligned}\Delta_f\left(\left|\nabla^k \mathrm{Rm}\right|^2 S^{-\sigma}\right) \geq & \left((k+2)-\sigma-c\sigma^2\left|\nabla \ln S\right|^2\right)\left|\nabla^k \mathrm{Rm}\right|^2 S^{-\sigma} \\ & -c\Sigma_{j=0}^k\left|\nabla^j \mathrm{Rm}\right|\left|\nabla^{k-j} \mathrm{Rm}\right|\left|\nabla^k \mathrm{Rm}\right| S^{-\sigma}.\end{aligned} \tag{3.57}$$

Note that (3.54) and (3.55) imply that for $k \geq l+1$,

$$\left|\nabla^j \mathrm{Rm}\right|\left|\nabla^{k-j} \mathrm{Rm}\right|\left|\nabla^k \mathrm{Rm}\right| S^{-l-2} \leq c_{k,l+1},$$

where $c_{k,l+1}$ is a constant depending on c_j from (3.54) for $j \leq l$, and on $c_{h,l}$ from (3.55) for $l<h \leq k$. It now follows from (3.52) and (3.57) that for any $k \geq l+1$, by letting $\sigma=l+2$, the function

$$w:=\left|\nabla^k \mathrm{Rm}\right|^2 S^{-l-2}$$

satisfies the inequality

$$\Delta_f w \geq\left(1-c_k \sqrt{S}\right) w-c_{k,l+1}. \tag{3.58}$$

Now we follow a similar strategy as in the proof of (3.52) to show that $w_{nn} \leq c_{k,l+1}$. Denote by

$$u:=\left|\nabla^k \mathrm{Rm}\right|^2,$$

so that $w=uS^{-l-2}$. Note that by (3.36) and (3.55),

$$\frac{1}{|\nabla f|^2}\langle\nabla S, \nabla f\rangle^2 S^{-l-4} u \leq c S^{-l-1} u \leq c_{k,l+1}$$

and by (3.37) and (3.55),

$$\frac{1}{|\nabla f|^2}\left|S_{ij} f_i f_j\right| S^{-l-3}\, u \leq c\, S^{-l-1} u \leq c_{k,l+1}.$$

Furthermore, according to (2.3) we have

$$\begin{aligned}\langle\nabla u, \nabla f\rangle &= \nabla^k \mathrm{Rm} * \nabla^k \mathrm{Rm}+\nabla^k \mathrm{Rm} * \nabla^{k+2} \mathrm{Rm} \\ & \quad +\Sigma_{j=0}^k \nabla^j \mathrm{Rm} * \nabla^{k-j} \mathrm{Rm} * \nabla^k \mathrm{Rm}.\end{aligned} \tag{3.59}$$

It follows immediately from (3.54) and (3.55) that $|\langle\nabla u, \nabla f\rangle| \leq c_{k,l+1}\, S^{l+1}$, where $c_{k,l+1}$ depends on c_j from (3.54) for $j \leq l$ and on $c_{h,l}$ from (3.55) for $l<h \leq k+2$. Hence, this proves that

$$\frac{1}{|\nabla f|^2}|\langle\nabla u, \nabla f\rangle||\langle\nabla S, \nabla f\rangle| S^{-l-3} \leq c_{k,l+1}.$$

Similarly, we have $|\langle\nabla\langle\nabla u, \nabla f\rangle, \nabla f\rangle| \leq c_{k,l+1}\, S^{l+1}$. As in (3.45) we get

$$\left|u_{ij} f_i f_j\right| S^{-l-1} \leq c_{k,l+1}.$$

The above estimates, together with (3.41), imply

$$w_{nn} \leq c_{k,l+1}. \tag{3.60}$$

Finally, we also get from above and from (3.1) that

$$\begin{aligned} Hw_n &\leq \frac{c}{|\nabla f|^2} \left| \langle \nabla w, \nabla f \rangle \right| \\ &\leq \frac{c}{|\nabla f|^2} \left(\langle \nabla u, \nabla f \rangle S^{-l-2} + (l+2) \left| \langle \nabla \ln S, \nabla f \rangle \right| S^{-l-2} u \right) \\ &\leq c_{k,l+1}. \end{aligned}$$

It is easy to see from (3.58) and (3.40) that

$$w := \left| \nabla^k \mathrm{Rm} \right|^2 S^{-l-2}$$

satisfies

$$\Delta_\Sigma w \geq \langle \nabla w, \nabla f \rangle + \left(1 - c_k \sqrt{S} \right) w - c_{k,l+1} \tag{3.61}$$

for any $k \geq l+1$. By the maximum principle, (3.61) implies that $w \leq c_{k,l+1}$ on $D(T) \setminus D(t_0)$ for all $k \geq l+1$. This proves (3.56) and completes the induction step.

In conclusion, we have established (3.53). In particular, for all $p \geq 1$, there exists $c_p > 0$ such that

$$\left| \nabla^p \mathrm{Rm} \right|^2 \leq c_p \, S^{p+1} \quad \text{on } D(T) \setminus D(t_0). \tag{3.62}$$

We are now ready to prove an estimate like (3.38) for all $k \geq 1$, that is,

$$\left| \nabla^k \mathrm{Rm} \right|^2 \leq c_k \, S^{k+\frac{7}{4}} \quad \text{on } D(T) \setminus D(t_0). \tag{3.63}$$

Note that (3.62) implies

$$\Sigma_{j=0}^k \left| \nabla^j \mathrm{Rm} \right| \left| \nabla^{k-j} \mathrm{Rm} \right| \leq c_k \, S^{\frac{k}{2}+1}.$$

So from (3.57) we get

$$\begin{aligned} \Delta_f \left(\left| \nabla^k \mathrm{Rm} \right|^2 S^{-k-\frac{7}{4}} \right) &\geq \left(\frac{1}{4} - c_k \sqrt{S} \right) \left| \nabla^k \mathrm{Rm} \right|^2 S^{-k-\frac{7}{4}} - c_k \left| \nabla^k \mathrm{Rm} \right| S^{-\frac{k}{2}-\frac{3}{4}} \\ &\geq \left(\frac{1}{6} - c_k \sqrt{S} \right) \left| \nabla^k \mathrm{Rm} \right|^2 S^{-k-\frac{7}{4}} - c_k. \end{aligned}$$

Hence the function

$$w := \left| \nabla^k \mathrm{Rm} \right|^2 S^{-k-\frac{7}{4}}$$

satisfies

$$w_{nn} + Hw_n + \Delta_\Sigma w \geq \langle \nabla f, \nabla w \rangle + \left(\frac{1}{6} - c_k \sqrt{S} \right) w - c_k. \tag{3.64}$$

Following the proof of (3.60) it can be seen that $w_{nn} + Hw_n \leq c_k$. Therefore, by applying the maximum principle to (3.64), we have $w \leq c_k$ on $D(T) \setminus D(t_0)$. This shows that (3.63) is indeed true.

We now finish the proof of the proposition by showing

$$\left| \nabla^k \mathrm{Rm} \right|^2 \leq c_k \, S^{k+2} \quad \text{on } D(T) \setminus D(t_0) \tag{3.65}$$

for each $k \geq 1$.

Let

$$w := \left| \nabla^k \mathrm{Rm} \right|^2 S^{-k-2}.$$

Using (3.57) and (3.63), we get

$$\begin{aligned}\Delta_f w &\geq -c_k \left|\nabla \ln S\right|^2 w - c\Sigma_{j=0}^k \left|\nabla^j \mathrm{Rm}\right| \left|\nabla^{k-j}\mathrm{Rm}\right| \left|\nabla^k \mathrm{Rm}\right| S^{-k-2} \\ &\geq -c_k S^{\frac{1}{2}}.\end{aligned} \tag{3.66}$$

On the other hand, it is easy to check that

$$\begin{aligned}\Delta_f S^{\frac{1}{2}} &= \frac{1}{2}\left(\Delta_f S\right) S^{-\frac{1}{2}} - \frac{1}{4}\left|\nabla S\right|^2 S^{-\frac{3}{2}} \\ &\geq \frac{1}{2} S^{\frac{1}{2}} \left(1 - cS^{\frac{3}{4}}\right).\end{aligned} \tag{3.67}$$

From (3.66) and (3.67) we see that there exists a constant $C_k > 0$ such that $\upsilon := w + C_k S^{\frac{1}{2}}$ satisfies

$$\Delta_f \upsilon \geq S^{\frac{1}{2}} \left(1 - c_k S^{\frac{3}{4}}\right). \tag{3.68}$$

We now bound υ_{nn} from above. By (3.37), we have

$$\left(S^{\frac{1}{2}}\right)_{nn} \leq \frac{1}{2} \frac{S_{ij} f_i f_j}{\left|\nabla f\right|^2} S^{-\frac{1}{2}} \leq cS^{\frac{3}{2}}. \tag{3.69}$$

To bound w_{nn}, we use (3.41). By (3.63), we get

$$\begin{aligned}\frac{1}{\left|\nabla f\right|^2} \left\langle \nabla S, \nabla f\right\rangle^2 S^{-k-4} u &\leq cS^{-k-1} u \\ &\leq c_k S^{\frac{3}{4}}.\end{aligned}$$

Also, by (3.37) and (3.63),

$$\begin{aligned}\frac{1}{\left|\nabla f\right|^2} \left|S_{ij} f_i f_j\right| S^{-k-3} u &\leq cS^{-k-1} u \\ &\leq c_k S^{\frac{3}{4}}.\end{aligned}$$

Furthermore, using (3.59) we see that $\left|\left\langle \nabla u, \nabla f\right\rangle\right| \leq c_k S^{k+\frac{7}{4}}$, hence

$$\frac{1}{\left|\nabla f\right|^2} \left|\left\langle \nabla u, \nabla f\right\rangle\right| \left|\left\langle \nabla S, \nabla f\right\rangle\right| S^{-k-3} \leq c_k S^{\frac{3}{4}}.$$

In a similar way it can be shown that

$$\frac{1}{\left|\nabla f\right|^2} \left|u_{ij} f_i f_j\right| S^{-k-2} \leq c_k S^{\frac{3}{4}}.$$

Combining all these estimates implies that

$$w_{nn} \leq c_k S^{\frac{3}{4}}. \tag{3.70}$$

From (3.68), (3.69) and (3.70) it can be easily seen that

$$\Delta_\Sigma \upsilon \geq \left\langle \nabla \upsilon, \nabla f\right\rangle + S^{\frac{1}{2}} \left(1 - c_k S^{\frac{1}{4}}\right).$$

Therefore, if the maximum of υ does not occur on $\Sigma\left(t_0\right)$, then $S^{\frac{1}{4}} \geq \frac{1}{c_k}$ at the maximum point. By (2.4), we have $\upsilon \leq c_k$ on $D\left(T\right) \backslash D\left(t_0\right)$ and

$$\left|\nabla^k \mathrm{Rm}\right|^2 \leq c_k S^{k+2} \quad \text{on } D\left(T\right) \backslash D\left(t_0\right).$$

This proves the proposition. □

Proposition 3.4 allows us to establish the following Harnack estimate for the scalar curvature. Assume that (3.4) holds. For $t \geq t_0$, define ϕ_t as follows.

$$\begin{aligned} \frac{d\phi_t}{dt} &= \frac{\nabla f}{|\nabla f|^2} \\ \phi_{t_0} &= \mathrm{Id} \ \text{ on } \Sigma(t_0). \end{aligned} \tag{3.71}$$

For $x \in \Sigma(t_0)$, let $S(t) := S(\phi_t(x))$, where $t_0 \leq t \leq T$. Then

$$\begin{aligned} \frac{dS}{dt} &= \frac{\langle \nabla S, \nabla f\rangle}{|\nabla f|^2} \\ &= \frac{\Delta S - S + 2|\mathrm{Ric}|^2}{t - S}. \end{aligned}$$

Using the estimate $|\Delta S| \leq cS^2$ from Proposition 3.4, we get

$$\left|\frac{dS}{dt} + \frac{S}{t}\right| \leq C_1 \frac{S^2}{t}$$

for some constant $C_1 > 0$. This can be rewritten into

$$\left|\frac{(tS)'}{(tS)^2}\right| \leq \frac{C_1}{t^2}.$$

Integrating in t gives

$$\left|\frac{1}{t_2 S(t_2)} - \frac{1}{t_1 S(t_1)}\right| \leq C_1 \left(\frac{1}{t_1} - \frac{1}{t_2}\right) \tag{3.72}$$

for any t_1 and t_2 with $t_0 < t_1 < t_2 < T$. Hence, if there exists $t_0 < t_1 < T$ with $S(t_1) \leq \frac{1}{2C_1}$, then

$$S(t) \leq \frac{1}{C_1}\frac{t_1}{t} \ \text{ for all } t_1 \leq t \leq T. \tag{3.73}$$

4. Ricci shrinkers asymptotic to round cylinder

In this section, we use the estimates from section 3 to prove Theorem 1.6. We continue to denote by M an n-dimensional, complete, gradient shrinking Ricci soliton with bounded curvature, and by $\{e_1, e_2, \cdots, e_n\}$ a local orthonormal frame with

$$e_n := \frac{\nabla f}{|\nabla f|}.$$

As before, the indices $a, b, c, d = 1, 2, \cdots, n-1$ and $i, j, k, l = 1, 2, \cdots, n$.

In the following, let us assume that (3.4) hold on $D(T)\backslash D(t_0)$. By Theorem 3.1 and (3.2) we have

$$\begin{aligned} |R_{ijkn}| &\leq \frac{2|\nabla \mathrm{Ric}|}{|\nabla f|} \\ &\leq cS^{\frac{3}{2}} f^{-\frac{1}{2}}. \end{aligned} \tag{4.1}$$

Using (2.3) we get

$$
\begin{aligned}
|R_{inkn}| &= \frac{|R_{ijkl}f_j f_l|}{|\nabla f|^2} \\
&= \frac{|f_j \nabla_j R_{ik} - f_j \nabla_i R_{jk}|}{|\nabla f|^2} \\
&\leq \frac{|\langle \nabla f, \nabla R_{ik}\rangle| + |R_{jk} f_{ij} - \nabla_i (R_{jk} f_j)|}{|\nabla f|^2}.
\end{aligned}
$$

Since

$$\langle \nabla f, \nabla R_{ik}\rangle = \Delta R_{ik} - R_{ik} + 2R_{ijkl}R_{jl},$$

we obtain from Theorem 3.1 that

$$|\langle \nabla f, \nabla R_{ik}\rangle| \leq cS.$$

Similarly, we have

$$
\begin{aligned}
|R_{jk} f_{ij} - \nabla_i (R_{jk} f_j)| &= \left| \frac{1}{2} R_{ik} - R_{jk} R_{ij} - \frac{1}{2} \nabla_i \nabla_k S \right| \\
&\leq cS.
\end{aligned}
$$

In conclusion, we get from above that

$$|R_{inkn}| \leq cSf^{-1}. \tag{4.2}$$

Consequently, for U, V, W defined in (3.3), we have on $D(T)\backslash D(t_0)$,

$$\langle U, V\rangle = O\left(S^2 f^{-1}\right),\ \langle U, W\rangle = O\left(S^2 f^{-1}\right),\ \langle V, W\rangle = O\left(S^2 f^{-1}\right) \tag{4.3}$$

and

$$
\begin{aligned}
|U|^2 &= \frac{2}{(n-1)(n-2)} S^2, \\
|V|^2 &= \frac{4}{n-3} \left| \overset{\circ}{\mathrm{Ric}}_{\Sigma} \right|^2 + O(S^2 f^{-1}), \\
|\mathrm{Rm}|^2 &= \left| \overset{\circ}{\mathrm{Rm}}_{\Sigma} \right|^2 + \frac{2}{(n-1)(n-2)} S^2 + O\left(S^2 f^{-1}\right), \\
|\mathrm{Ric}|^2 &= \left| \overset{\circ}{\mathrm{Ric}}_{\Sigma} \right|^2 + \frac{1}{n-1} S^2 + O\left(S^2 f^{-1}\right), \\
|\mathrm{Rm}|^2 &= |U|^2 + |V|^2 + |W|^2 + O\left(S^2 f^{-1}\right).
\end{aligned} \tag{4.4}
$$

Here and below, the constants implicit in the big O notation depend only on n, η_1, η_2, C_0 and $A_0, ..., A_{Kk}$, as specified in Theorem 3.1, so they are independent of t_0 and T.

We restate Theorem 1.6 here. Without loss of generality, we assume M has only one end.

Theorem 4.1. *Let (M, g, f) be an n-dimensional, complete, gradient shrinking Ricci soliton with $|\mathrm{Rm}| \leq C$. Assume that there exists a sequence of points $x_k \to \infty$ such that (M, g, x_k) converges to a round cylinder $\mathbb{R} \times \mathbb{S}^{n-1}/\Gamma$. Then M is smoothly asymptotic to the same round cylinder.*

Proof. Without loss of generality, we may assume that x_k converges to a point in $\{0\} \times \mathbb{S}^{n-1}/\Gamma$. We first claim that $\Sigma(t_k)$, the level set of f containing x_k, must converge to $\{0\} \times \mathbb{S}^{n-1}/\Gamma$. Indeed, consider the vector field defined on $M \backslash D(t_0)$,

$$X := \frac{\nabla f}{|\nabla f|}.$$

Since $|X| = 1$, we get that X converges smoothly to a vector field X_∞ on $\mathbb{R}\times\mathbb{S}^{n-1}/\Gamma$. It is easy to see that X_∞ is in fact parallel, because

$$\begin{aligned} |\nabla X| &\leq 2\frac{|\mathrm{Hess}\,(f)|}{|\nabla f|} \\ &\leq cf^{-\frac{1}{2}}. \end{aligned}$$

This proves that X_∞ is the radial vector on $\mathbb{R} \times \mathbb{S}^{n-1}/\Gamma$, and hence the level set corresponding to x_k converges to $\{0\} \times \mathbb{S}^{n-1}/\Gamma$. In particular, it follows that for any $\varepsilon > 0$ there exists sufficiently large t_0 such that

$$\begin{aligned} \sup_{\Sigma(t_0)} \left|\mathring{\mathrm{Rm}}_\Sigma\right|^2 S^{-2} &< \frac{\varepsilon}{2}, \\ \sup_{\Sigma(t_0)} \left|S - \frac{n-1}{2}\right| &< \varepsilon. \end{aligned} \tag{4.5}$$

Claim 4.2. *For $\varepsilon > 0$ and $t_0 > 0$ such that (4.5) holds we have*

$$\sup_{\Sigma(t)} \left|\mathring{\mathrm{Rm}}_\Sigma\right|^2 S^{-2} < \varepsilon \quad \textit{for all } t \geq t_0. \tag{4.6}$$

To prove Claim 4.2, let

$$T := \sup\left\{t : \sup_{\Sigma(r)} \left|\mathring{\mathrm{Rm}}_\Sigma\right|^2 S^{-2} < \varepsilon \;\text{ for all } t_0 \leq r \leq t\right\}. \tag{4.7}$$

If $T < \infty$, then

$$\sup_{\Sigma(T)} \left|\mathring{\mathrm{Rm}}_\Sigma\right|^2 S^{-2} = \varepsilon. \tag{4.8}$$

Note that (4.7) and (4.5) imply that (3.4) holds on $D(T)\backslash D(t_0)$, for η_1 and η_2 depending only on dimension. Hence, (4.3) and (4.4) hold on $D(T)\backslash D(t_0)$ as well.

We have the following formula (see Ch. 2.7 in [13] or [36])

$$\Delta_f |\mathrm{Rm}|^2 = 2|\nabla \mathrm{Rm}|^2 + 2|\mathrm{Rm}|^2 - 8R_{ijkl}R_{piqk}R_{pjql} - 2R_{ijkl}R_{ijpq}R_{pqkl}. \tag{4.9}$$

For the function G given by

$$G := |\mathrm{Rm}|^2 S^{-2} - \frac{2}{(n-1)(n-2)},$$

using (4.9) and arguing as in (2.7) we obtain the following inequality (cf. Lemma 3.2 in [23]).

$$\Delta_f G \geq -2\langle \nabla G, \nabla \ln S\rangle + 4S^{-3}P, \tag{4.10}$$

where

$$P := -2SR_{ijkl}R_{piqk}R_{pjql} - \frac{1}{2}SR_{ijkl}R_{ijpq}R_{pqkl} + |\mathrm{Rm}|^2 |\mathrm{Ric}|^2. \tag{4.11}$$

Note that by (4.4),

$$G=\left|\overset{\circ}{\mathrm{Rm}}_{\Sigma}\right|^{2}S^{-2}+O\left(f^{-1}\right). \tag{4.12}$$

By (4.3) and (4.4) we get that

$$\begin{aligned}
&\left|\mathrm{Rm}\right|^{2}\left|\mathrm{Ric}\right|^{2} \\
&=\left(|U|^{2}+|V|^{2}+|W|^{2}\right)\left(\left|\overset{\circ}{\mathrm{Ric}}_{\Sigma}\right|^{2}+\frac{1}{n-1}S^{2}\right)+O\left(S^{4}f^{-1}\right) \\
&=|W|^{2}\left|\overset{\circ}{\mathrm{Ric}}_{\Sigma}\right|^{2}+\frac{1}{n-1}S^{2}|W|^{2}+\frac{4}{n-3}\left|\overset{\circ}{\mathrm{Ric}}_{\Sigma}\right|^{4} \\
&+\frac{2\left(3n-7\right)}{\left(n-1\right)\left(n-2\right)\left(n-3\right)}S^{2}\left|\overset{\circ}{\mathrm{Ric}}_{\Sigma}\right|^{2} \\
&+\frac{2}{\left(n-1\right)^{2}\left(n-2\right)}S^{4}+O\left(S^{4}f^{-1}\right).
\end{aligned} \tag{4.13}$$

A much longer computation of similar nature implies (see Theorem 3.3 in [23])

$$\begin{aligned}
&-2SR_{ijkl}R_{piqk}R_{pjql}-\frac{1}{2}SR_{ijkl}R_{ijpq}R_{pqkl} \\
=\;&-2SR_{abcd}R_{eagc}R_{ebgd}-\frac{1}{2}SR_{abcd}R_{abeg}R_{egcd}+O\left(S^{4}f^{-1}\right) \\
=\;&-2SW_{abcd}W_{eagc}W_{ebgd}-\frac{1}{2}SW_{abcd}W_{abeg}W_{egcd} \\
&-\frac{6}{n-3}SW_{abcd}\overset{\circ}{R}_{ac}\overset{\circ}{R}_{bd}-\frac{6}{\left(n-1\right)\left(n-2\right)}S^{2}\left|\overset{\circ}{\mathrm{Ric}}_{\Sigma}\right|^{2} \\
&+\frac{8}{\left(n-3\right)^{2}}S\overset{\circ}{R}_{ab}\overset{\circ}{R}_{bc}\overset{\circ}{R}_{ac}-\frac{2}{\left(n-1\right)^{2}\left(n-2\right)}S^{4}+O\left(S^{4}f^{-1}\right).
\end{aligned} \tag{4.14}$$

From (4.11), (4.13) and (4.14) we conclude that

$$\begin{aligned}
P\;=\;&\frac{1}{n-1}S^{2}|W|^{2}-2SW_{abcd}W_{eafc}W_{ebfd} \\
&-\frac{1}{2}SW_{abcd}W_{abef}W_{efcd} \\
&+\frac{4}{\left(n-1\right)\left(n-2\right)\left(n-3\right)}S^{2}\left|\overset{\circ}{\mathrm{Ric}}_{\Sigma}\right|^{2}+\frac{4}{n-3}\left|\overset{\circ}{\mathrm{Ric}}_{\Sigma}\right|^{4} \\
&+\frac{8}{\left(n-3\right)^{2}}S\overset{\circ}{R}_{ab}\overset{\circ}{R}_{bc}\overset{\circ}{R}_{ac}+|W|^{2}\left|\overset{\circ}{\mathrm{Ric}}_{\Sigma}\right|^{2} \\
&-\frac{6}{n-3}SW_{abcd}\overset{\circ}{R}_{ac}\overset{\circ}{R}_{bd}+O\left(S^{4}f^{-1}\right).
\end{aligned} \tag{4.15}$$

Therefore, we have

$$\begin{aligned} P &\geq \frac{1}{n-1}S^2\left|W\right|^2+\frac{4}{(n-1)(n-2)(n-3)}S^2\left|\overset{\circ}{\mathrm{Ric}}_\Sigma\right|^2 \\ &\quad -\frac{5}{2}S\left|W\right|^3-\frac{8}{(n-3)^2}S\left|\overset{\circ}{\mathrm{Ric}}_\Sigma\right|^3 \\ &\quad -\frac{6}{n-3}S\left|W\right|\left|\overset{\circ}{\mathrm{Ric}}_\Sigma\right|^2-cS^4f^{-1}. \end{aligned}$$

Since by (4.7) for all $t_0\leq t\leq T$,

$$\begin{aligned} \left|W\right|^2+\frac{4}{n-3}\left|\overset{\circ}{\mathrm{Ric}}_\Sigma\right|^2 &\leq \left|\overset{\circ}{\mathrm{Rm}}_\Sigma\right|^2+cS^2f^{-1} \\ &\leq \varepsilon S^2+cS^2f^{-1} \\ &\leq 2\varepsilon S^2, \end{aligned} \tag{4.16}$$

it follows that

$$\begin{aligned} P &\geq \left(\frac{1}{n-1}-c\sqrt{\varepsilon}\right)S^2\left|W\right|^2 \\ &\quad +\left(\frac{4}{(n-1)(n-2)(n-3)}-c\sqrt{\varepsilon}\right)S^2\left|\overset{\circ}{\mathrm{Ric}}_\Sigma\right|^2 \\ &\quad -c\,S^4\,f^{-1} \end{aligned}$$

for some constant $c>0$ depending only on n. In particular, there exists $\theta>0$ depending only on n such that

$$P\geq\theta S^2\left|\overset{\circ}{\mathrm{Rm}}_\Sigma\right|^2-c\,S^4\,f^{-1}.$$

As $\left|\overset{\circ}{\mathrm{Rm}}_\Sigma\right|^2\geq GS^2-cS^2f^{-1}$, it follows from (4.10) that on $D(T)\setminus D(t_0)$

$$\Delta G\geq\left\langle\nabla G,\nabla f\right\rangle-2\left\langle\nabla G,\nabla\ln S\right\rangle+\theta SG-cSf^{-1}. \tag{4.17}$$

Note that

$$\Delta G=\Delta_\Sigma G+G_{nn}+HG_n,$$

where Δ_Σ is the Laplacian on $\Sigma(t)$. We now bound G_{nn} by a similar argument as in the proof of Proposition 3.4. Let $u:=\left|\mathrm{Rm}\right|^2$ and $w:=\frac{u}{S^2}$. Then $G_{nn}=w_{nn}$. By (3.41) we have

$$w_{nn}=\frac{f_if_j}{\left|\nabla f\right|^2}\left(u_{ij}S^{-2}-4u_iS_jS^{-3}+6S_iS_jS^{-4}u-2S_{ij}S^{-3}u\right). \tag{4.18}$$

Now,

$$u_{ij}f_if_j=\left\langle\nabla\left(u_if_i\right),\nabla f\right\rangle-f_{ij}u_if_j.$$

Note that by (2.3) and Proposition 3.4,

$$\begin{aligned} \left\langle\nabla u,\nabla f\right\rangle &= -2u+\mathrm{Rm}*\mathrm{Rm}*\mathrm{Rm} \\ &\quad +\nabla^2\mathrm{Rm}*\mathrm{Rm}. \end{aligned} \tag{4.19}$$

Therefore,

$$\left\langle\nabla u,\nabla f\right\rangle=-2u+O\left(S^3\right).$$

Using (4.19) and (2.3) we similarly get

$$\begin{aligned} \langle \nabla \langle \nabla u, \nabla f \rangle, \nabla f \rangle &= -2 \langle \nabla u, \nabla f \rangle + O\left(S^3\right) \\ &= 4u + O\left(S^3\right). \end{aligned} \tag{4.20}$$

Finally, we have

$$\begin{aligned} f_{ij} u_i f_j &= \frac{1}{2} \langle \nabla u, \nabla f \rangle - \langle \nabla u, \nabla S \rangle \\ &= -u + O\left(S^3\right). \end{aligned}$$

Hence, by (4.19) and (4.20) we conclude that

$$u_{ij} f_i f_j = 5u + O\left(S^3\right) \tag{4.21}$$

and

$$\frac{1}{|\nabla f|^2} u_{ij} f_i f_j S^{-2} = \frac{5}{|\nabla f|^2} w + O\left(S f^{-1}\right). \tag{4.22}$$

Note that the second and the third term in (4.18) can be rewritten as

$$\begin{aligned} & \frac{f_i f_j}{|\nabla f|^2} \left(-4 u_i S_j S^{-3} + 6 u S_i S_j S^{-4}\right) \\ = \; & -\frac{4}{|\nabla f|^2} \left(w_i f_i\right) S_j f_j S^{-1} - \frac{2u}{|\nabla f|^2} \langle \nabla S, \nabla f \rangle^2 S^{-4}. \end{aligned} \tag{4.23}$$

The first term above is estimated by (3.36) as

$$\frac{4}{|\nabla f|^2} \left|\langle \nabla w, \nabla f \rangle \langle \nabla S, \nabla f \rangle\right| S^{-1} \leq c f^{-1} \left|\langle \nabla w, \nabla f \rangle\right|.$$

The second term can be computed as

$$\frac{2u}{|\nabla f|^2} \langle \nabla S, \nabla f \rangle^2 S^{-4} = \frac{2}{|\nabla f|^2} w + O\left(S f^{-1}\right)$$

by noting that

$$\langle \nabla S, \nabla f \rangle = -S + O(S^2),$$

where we have used (2.3) and Proposition 3.4. Thus, we have proved that

$$\begin{aligned} & \frac{f_i f_j}{|\nabla f|^2} \left(-4 u_i S_j S^{-3} + 6 u S_i S_j S^{-4}\right) \\ \leq \; & c\, t_0^{-1} \left|\langle \nabla w, \nabla f \rangle\right| - \frac{2}{|\nabla f|^2} w + O\left(S f^{-1}\right). \end{aligned} \tag{4.24}$$

To estimate the last term in (4.18), we write

$$\begin{aligned} S_{ij} f_i f_j &= \langle \nabla \langle \nabla S, \nabla f \rangle, \nabla f \rangle - f_{ij} S_i f_j \\ &= \langle \nabla \langle \nabla S, \nabla f \rangle, \nabla f \rangle - \frac{1}{2} \langle \nabla S, \nabla f \rangle + \frac{1}{2} |\nabla S|^2. \end{aligned}$$

Following a similar idea as in the proof of (4.21), one sees that $\langle \nabla S, \nabla f \rangle = -S + O(S^2)$ and $\langle \nabla \langle \nabla S, \nabla f \rangle, \nabla f \rangle = S + O\left(S^2\right)$. Therefore,

$$S_{ij} f_i f_j = \frac{3}{2} S + O\left(S^2\right).$$

This implies that

$$\frac{2}{|\nabla f|^2} u\left(S_{ij} f_i f_j\right) S^{-3} = \frac{3}{|\nabla f|^2} w + O\left(S f^{-1}\right). \tag{4.25}$$

By (4.18), (4.22), (4.24) and (4.25) we obtain

$$w_{nn} \leq c_0\, t_0^{-1}\ |\langle \nabla G, \nabla f\rangle| + O(S f^{-1}). \tag{4.26}$$

Note that by (3.36),

$$\begin{aligned} \langle \nabla G, \nabla \ln S\rangle &= \langle \nabla G, \nabla \ln S\rangle_\Sigma + \frac{1}{|\nabla f|^2} \langle \nabla G, \nabla f\rangle \langle \nabla \ln S, \nabla f\rangle \\ &\leq \langle \nabla G, \nabla \ln S\rangle_\Sigma + c_0\, t_0^{-1}\ |\langle \nabla G, \nabla f\rangle| \end{aligned}$$

and, using (3.1),

$$H\, G_n \leq c_0\, t_0^{-1}\ |\langle \nabla G, \nabla f\rangle|.$$

Combining this with (4.17) and (4.26), we conclude that

$$\begin{aligned} \Delta_\Sigma G &\geq \langle \nabla G, \nabla f\rangle - c_0\, t_0^{-1}\ |\langle \nabla G, \nabla f\rangle| \\ &\quad -2\langle \nabla G, \nabla \ln S\rangle_\Sigma + \theta S G - c\, S\, f^{-1}. \end{aligned} \tag{4.27}$$

Here $\theta > 0$ depends only on n, whereas the constants c_0 and c depend only on n, C_0 from (2.1) and A_0,....,A_K from (2.4), for some absolute constant K.

Now if the maximum of G is achieved on $\Sigma(t_0)$, then (4.5) and (4.12) imply that $G \leq \frac{2\varepsilon}{3}$. Otherwise, by the maximum principle we get $G \leq c\, t_0^{-1}$, for a constant c that is independent of t_0 and T. Hence, by assuming t_0 to be large enough, one concludes that $G \leq \frac{2\varepsilon}{3}$. In either case, it shows that $G \leq \frac{2\varepsilon}{3}$ on $D(T)\backslash D(t_0)$. Now (4.12) implies that

$$\sup_{\Sigma(T)} \left|\overset{\circ}{\mathrm{Rm}}_\Sigma\right|^2 S^{-2} < \varepsilon.$$

This contradicts with (4.8). So the assumption that $T < \infty$ is false. Therefore,

$$\sup_{\Sigma(t)} \left|\overset{\circ}{\mathrm{Rm}}_\Sigma\right|^2 S^{-2} < \varepsilon \tag{4.28}$$

for all $t \geq t_0$.

We now claim that $S \geq C > 0$ on M. Indeed, if there exists $x \in M\backslash D(t_0)$ with $S(x) < \frac{1}{2C_1}$, where C_1 is the constant in (3.73), then (3.73) implies that $S\, f \leq c$ along the integral curve of ∇f through x. But this contradicts with the fact that (M, g, x_k) converges to $\mathbb{R}\times\mathbb{S}^{n-1}/\Gamma$. In conclusion, S is bounded below by a positive constant.

Let us assume that $z_k \to \infty$ is a sequence so that (M, g, z_k) converges smoothly to $\mathbb{R}\times N$, where (N, h) is a shrinking Ricci soliton. By (4.28) and the fact that $S \geq C > 0$, it follows that (N, h) is isometric to a quotient of the round sphere $\mathbb{S}^{n-1}$. By hypothesis $\Sigma(t_k)$, the level set of f containing x_k, converges to $\mathbb{S}^{n-1}/\Gamma$. Since all level sets $\Sigma(t)$ are diffeomorphic for $t \geq t_0$ large enough, we conclude that (N, h) is isometric to $\mathbb{S}^{n-1}/\Gamma$, for any such sequence z_k.

If $y_k \to \infty$ is an arbitrary sequence, according to Proposition 5.2 in [32], there exist sequences y_k^+ and y_k^- so that $\left(M, g, y_k^\pm\right)$ converge smoothly to shrinking Ricci solitons. We have established that any such shrinking solitons are isometric to the same quotient of the round cylinder $\mathbb{R}\times\mathbb{S}^{n-1}/\Gamma$, so using Proposition 5.2 in [32] it follows that (M, g, y_k) converges itself to $\mathbb{R}\times\mathbb{S}^{n-1}/\Gamma$.

From here we get that

$$\lim_{x\to\infty}\left|\overset{\circ}{\mathrm{Rm}}_{\Sigma}\right|(x)=0 \quad\text{and}\quad \lim_{x\to\infty}\left|S(x)-\frac{n-1}{2}\right|=0. \tag{4.29}$$

We now strengthen the above conclusion and show that M is asymptotic to $\mathbb{R}\times\mathbb{S}^{n-1}/\Gamma$. For this, we first obtain an explicit convergence rate for (4.29). According to (4.17), there exists $\alpha>0$ depending only on n so that

$$\Delta_F G\geq \alpha G-cf^{-1} \text{ on } M\backslash D(t_0), \tag{4.30}$$

where $G:=|\mathrm{Rm}|^2 S^{-2}-\frac{2}{(n-1)(n-2)}$ and $F:=f-2\ln S$. We may assume that $\alpha\leq 1$. Define

$$H:=G-t_0^{\frac{\alpha}{2}}f^{-\frac{\alpha}{2}}.$$

Then, choosing t_0 large enough, (4.29) implies that $H<0$ on $\Sigma(t_0)$ and $H\to 0$ at infinity. Furthermore, it is easy to check that

$$\begin{aligned}\Delta_F f^{-\frac{\alpha}{2}} &= -\frac{\alpha}{2}(\Delta_F f)f^{-\frac{\alpha}{2}-1}+\frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)|\nabla f|^2 f^{-\frac{\alpha}{2}-2}\\ &\leq \frac{\alpha}{2}f^{-\frac{\alpha}{2}}+cf^{-\frac{\alpha}{2}-1}\\ &\leq \frac{3}{4}\alpha f^{-\frac{\alpha}{2}}.\end{aligned}$$

Hence, (4.30) implies that

$$\begin{aligned}\Delta_F H &\geq \alpha H+\frac{\alpha}{4}t_0^{\frac{\alpha}{2}}f^{-\frac{\alpha}{2}}-cf^{-1}\\ &\geq \alpha H\end{aligned}$$

on $M\backslash D(t_0)$. For the last inequality, we used that $\frac{\alpha}{2}<1$. Using the maximum principle, we now conclude that $H\leq 0$ on $M\backslash D(t_0)$. Hence, this proves that there exists b_0 depending only on n so that

$$\left|\overset{\circ}{\mathrm{Rm}}_{\Sigma}\right|\leq cf^{-b_0} \tag{4.31}$$

on M.

Define the tensor Q on M by

$$\begin{aligned}Q_{ijkl} &= R_{ijkl}-\frac{S}{(n-1)(n-2)}(g_{ik}g_{jl}-g_{il}g_{jk})\\ &\quad +\frac{S}{(n-1)(n-2)}\left(g_{ik}\frac{f_jf_l}{|\nabla f|^2}-g_{jk}\frac{f_if_l}{|\nabla f|^2}+g_{jl}\frac{f_if_k}{|\nabla f|^2}-g_{il}\frac{f_jf_k}{|\nabla f|^2}\right).\end{aligned}$$

Observe that $Q_{ijkl}=R_{ijkl}$ if at least one of the indices i,j,k,l is equal to n, and

$$Q_{abcd}=\overset{\circ}{R}_{abcd}.$$

By (4.31) and (4.1), we obtain that

$$|\mathrm{Q}|\leq cf^{-b_0} \tag{4.32}$$

on M.

We claim that for all $k\geq 1$ there exists $b_k>0$, which depends only on n, and c_k so that

$$\left|\nabla^k\mathrm{Rm}\right|\leq c_k f^{-b_k}. \tag{4.33}$$

Indeed, for $x \in \Sigma(t)$ and $\theta := t^{-\frac{1}{2}b_0}$ let ϕ be a cut-off on $B_x(\theta)$ so that $\phi = 1$ on $B_x\left(\frac{\theta}{2}\right)$ and $|\nabla \phi| \leq c\theta^{-1}$. Integrating by parts and using (2.3) and (2.4), we have

$$\begin{aligned} \int_{B_x(\theta)} |\nabla \mathrm{Q}|^2 \phi^2 &= -\int_{B_x(\theta)} (\Delta Q_{ijkl}) Q_{ijkl} \phi^2 \\ &\quad - \int_{B_x(\theta)} \left\langle \nabla Q_{ijkl}, \nabla \phi^2 \right\rangle Q_{ijkl} \\ &\leq c\left(1 + \frac{1}{\theta}\right) \int_{B_x(\theta)} |\mathrm{Q}|. \end{aligned}$$

It follows from (4.32) that

$$\int_{B_x(\theta)} |\nabla \mathrm{Q}|^2 \phi^2 \leq c t^{-\frac{1}{2}b_0} \mathrm{Vol}(B_x(\theta)). \tag{4.34}$$

As (M, g) has bounded curvature, the volume comparison implies that $\mathrm{Vol}(B_x(\theta)) \leq c\mathrm{Vol}\left(B_x\left(\frac{\theta}{2}\right)\right)$. Together with (4.34), this proves that

$$\inf_{B_x\left(\frac{\theta}{2}\right)} |\nabla \mathrm{Q}|^2 \leq c t^{-\frac{1}{2}b_0}. \tag{4.35}$$

By (2.4) we have that $\left|\nabla |\nabla \mathrm{Q}|^2\right| \leq c$. This together with (4.35) immediately leads to

$$|\nabla \mathrm{Q}| \leq c f^{-\frac{1}{4}b_0}.$$

Therefore, as the hessian of f is bounded, we get that

$$\begin{aligned} &\nabla_p R_{ijkl} - \frac{\nabla_p S}{(n-1)(n-2)} (g_{ik} g_{jl} - g_{il} g_{jk}) \\ &+ \frac{\nabla_p S}{(n-1)(n-2)} \left(g_{ik} \frac{f_j f_l}{|\nabla f|^2} - g_{jk} \frac{f_i f_l}{|\nabla f|^2} + g_{jl} \frac{f_i f_k}{|\nabla f|^2} - g_{il} \frac{f_j f_k}{|\nabla f|^2} \right) \\ = \; & O\left(f^{-\frac{1}{4}b_0}\right). \end{aligned} \tag{4.36}$$

Tracing this formula, we obtain

$$\nabla_p R_{ik} - \frac{\nabla_p S}{n-1} g_{ik} - \frac{\nabla_p S}{n-1} \frac{f_i f_k}{|\nabla f|^2} = O\left(f^{-\frac{1}{4}b_0}\right).$$

Tracing this in $p = k$, and using that

$$|\langle \nabla S, \nabla f \rangle| \leq c,$$

we conclude from above that

$$|\nabla S| \leq c f^{-\frac{1}{4}b_0}. \tag{4.37}$$

By (4.36) and (4.37) it follows that

$$|\nabla \mathrm{Rm}| \leq c f^{-\frac{1}{4}b_0}. \tag{4.38}$$

By induction on k we get (4.33).

We now consider ϕ_t defined by

$$\begin{aligned} \frac{d\phi_t}{dt} &= \frac{\nabla f}{|\nabla f|^2} \\ \phi_{t_0} &= \mathrm{Id} \text{ on } \Sigma(t_0). \end{aligned} \tag{4.39}$$

For a fixed $x \in \Sigma(t_0)$ we denote $S(t) := S(\phi_t(x))$, where $t \geq t_0$. Then

$$\frac{dS}{dt} = \frac{\langle \nabla S, \nabla f \rangle}{|\nabla f|^2} = \frac{\Delta S - S + 2|\mathrm{Ric}|^2}{t - S}. \tag{4.40}$$

Tracing (4.31) we get that

$$\left| R_{ab} - \frac{S}{n-1} g_{ab} \right| \leq c f^{-\delta},$$

whereas by (4.33) we have

$$|\Delta S| \leq c f^{-\delta},$$

for some $\delta > 0$. Hence, (4.40) implies that

$$t \frac{dS}{dt} = \frac{2}{n-1} S^2 - S + O\left(f^{-\delta}\right).$$

Consequently, the function $\rho := S - \frac{n-1}{2}$ satisfies

$$t\rho' = \rho + \frac{2}{n-1}\rho^2 + O\left(f^{-\delta}\right)$$

and $\rho \to 0$ at infinity. Integrating this in t we find that there exists $\delta > 0$ so that $|\rho(t)| \leq ct^{-\delta}$. Hence, we have proved that

$$\left| S - \frac{n-1}{2} \right| \leq c f^{-\delta} \quad \text{on } M \backslash D(t_0). \tag{4.41}$$

This and (4.31) imply that

$$\left| R_{abcd} - \frac{1}{2(n-2)} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) \right| \leq c f^{-\delta}, \tag{4.42}$$

for some $\delta > 0$ depending only on n.

We can now prove that M is smoothly asymptotic to $\mathbb{R} \times \mathbb{S}^{n-1}/\Gamma$. Indeed, for ϕ_t defined in (4.39) consider $\widetilde{g}(t) = \phi_t^*(g)$ on $\Sigma(t_0)$, the pullback of the metric g on $\Sigma(t)$. Then

$$\frac{d}{dt}\widetilde{g}_{ab}(t) = 2\,\phi_t^*\left(\frac{f_{ab}}{|\nabla f|^2}\right) = \frac{1}{|\nabla f|^2}\phi_t^*\left(g_{ab} - 2R_{ab}\right).$$

Hence, by (4.42),

$$-ct^{-1-\delta}\widetilde{g}_{ab}(t) \leq \frac{d}{dt}\widetilde{g}_{ab}(t) \leq ct^{-1-\delta}\widetilde{g}_{ab}(t).$$

Integrating in t implies that

$$\left|\widetilde{g}_{ab}(t) - g_{ab}^{\infty}\right| \leq ct^{-\delta},$$

where g_{ab}^{∞} is the round metric on $\Sigma(t_0)$. Note that (4.38) implies decay estimates for $\left|\partial^k \widetilde{g}_{ab}\right|$ for all k. It is easy to see that this implies M is smoothly asymptotic to $\mathbb{R} \times \mathbb{S}^{n-1}/\Gamma$. The theorem is proved. □

5. Asymptotic geometry of four dimensional shrinkers

We are now in position to prove Theorem 1.5 in the introduction. For the convenience of the reader, we restate it here.

Theorem 5.1. *Let (M, g, f) be a complete, four dimensional gradient shrinking Ricci soliton with bounded scalar curvature S. If S is bounded from below by a positive constant on end E of M, then E is smoothly asymptotic to the round cylinder $\mathbb{R} \times \mathbb{S}^3/\Gamma$, or for any sequence $x_i \in E$ going to infinity along an integral curve of ∇f, (M, g, x_i) converges smoothly to $\mathbb{R}^2 \times \mathbb{S}^2$ or its $\mathbb{Z}_2$ quotient. Moreover, the limit is uniquely determined by the integral curve and is independent of the sequence x_i.*

Proof. Since S is bounded, by Theorem 1.3, M has bounded curvature. Recall that $(M, g(t))$ is an ancient solution to the Ricci flow defined on $(-\infty, 0)$, where

$$g(t) := (-t)\, \phi_t^* g$$

and ϕ_t is the family of diffeomorphisms defined by

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{\nabla f}{(-t)} \\ \phi_{-1} &= \mathrm{Id}. \end{aligned}$$

For any sequence $\tau_i \to 0$, consider the rescaled flow $(M, g_i(t))$ for $t < 0$, where

$$g_i(t) := \frac{1}{\tau_i} g(\tau_i t).$$

By Theorem 1.5 in [32], for any $x_0 \in \Sigma(t_0)$, a subsequence of $(M, g_i(t), (x_0, -1))$ converges smoothly to a gradient shrinking Ricci soliton $(M_\infty, g_\infty(t), (x_\infty, -1))$.

Now for any sequence $x_i \in E$ going to infinity along an integral curve of ∇f, obviously one may write $x_i := \phi_{-\tau_i}(x_0)$ for some point x_0 and $\tau_i \to 0$. However, as $g_i(-1) = \phi_{-\tau_i}^* g$, we see that a subsequence of (M, g, x_i) converges to $(M_\infty, g_\infty(-1), x_\infty)$. Since $x_i \to \infty$, invoking Proposition 5.1 in [32] we conclude that $(M_\infty, g_\infty(-1))$ splits as $(\mathbb{R}, ds^2) \times (N, h)$, where (N, h) is a normalized three dimensional gradient shrinking Ricci soliton. Theorem 1.2 implies that (N, h) is isometric to a quotient of either $\mathbb{S}^3$ or $\mathbb{R} \times \mathbb{S}^2$. If the quotient of $\mathbb{S}^3$ ever occurs, then Theorem 1.6 implies that E is smoothly asymptotic to $\mathbb{R} \times \mathbb{S}^3/\Gamma$. So we may assume that N is never isometric to a quotient of $\mathbb{S}^3$. In this case, for any sequence $x_i \in E$ going to infinity along an integral curve of ∇f, a subsequence of (M, g, x_i) converges smoothly to $\mathbb{R} \times N$, where N is either $\mathbb{R} \times \mathbb{S}^2$ or its $\mathbb{Z}_2$ quotient. However, by Remark 5.1 in [32], such N is uniquely determined by the integral curve. This proves the theorem. □

We conclude with a rigidity result for four dimensional gradient shrinking Kähler Ricci soliton.

Proposition 5.2. *Let (M, g, f) be a complete, non-flat, four dimensional, gradient shrinking Kähler Ricci soliton with bounded nonnegative Ricci curvature. Then (M, g) is isometric to a quotient of $\mathbb{R}^2 \times \mathbb{S}^2$.*

Proof. In view of (2.3), since (M, g) has nonnegative Ricci curvature, it follows that S increases along each integral curve of ∇f. Hence, S is bounded below by a positive constant. Since M is Kähler, it can never be asymptotic to a quotient of the round cylinder. In view of Theorem 5.1, we conclude that (M, g) converges along each integral curve to $\left(\mathbb{R}^2 \times \mathbb{S}^2\right)/\Gamma$ and S must converge to 1 at infinity. In particular, this means that there exists a compact set $K \subset M$ so that $S \leq 1$ on $M\backslash K$, where the compact set K contains all critical points of f.

We diagonalize the Ricci curvature and denote the eigenvalues by $\alpha \leq \beta$. Then it follows that

$$\begin{aligned} S^2 - 2\left|\mathrm{Ric}\right|^2 &= 4\left(\alpha+\beta\right)^2 - 4\left(\alpha^2+\beta^2\right) \\ &= 8\alpha\beta \geq 0. \end{aligned}$$

Hence, on $M\backslash K$ we have

$$\begin{aligned} \Delta_f S &= S - S^2 + S^2 - 2\left|\mathrm{Ric}\right|^2 \\ &\geq S - S^2 \\ &\geq 0. \end{aligned} \tag{5.1}$$

Without loss of generality, we may assume that $K = D\left(t_0\right)$ for some $t_0 > 0$. Then by the Stokes theorem,

$$0 \leq \int_{M\backslash D(t_0)} \left(\Delta_f S\right) e^{-f} = -\int_{\Sigma(t_0)} \frac{\langle \nabla S, \nabla f\rangle}{|\nabla f|} e^{-f} \leq 0, \tag{5.2}$$

where the last inequality is because $\langle \nabla S, \nabla f\rangle = 2\mathrm{Ric}\left(\nabla f, \nabla f\right) \geq 0$. It follows from (5.1) and (5.2) that $S = 1$ on $M\backslash K$ and the eigenvalue α of the Ricci curvature is zero. By Corollary 1.3 of [25], (M, g) is real analytic. Therefore, $S = 1$, $\alpha = 0$ and $\beta = \frac{1}{2}$ on M. The proposition follows from the de Rham splitting theorem. $\square$

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