WEIGHTED POINCARÉ INEQUALITY AND THE POISSON EQUATION

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Abstract. We develop Green's function estimates for manifolds satisfying a weighted Poincaré inequality together with a compatible lower bound on the Ricci curvature. This estimate is then applied to establish existence and sharp estimates of solutions to the Poisson equation on such manifolds. As an application, a Liouville property for finite energy holomorphic functions is proven on a class of complete Kähler manifolds. Consequently, such Kähler manifolds must be connected at infinity.

1. Introduction

Recently, in [25], we studied the existence and estimates of the solution u to the Poisson equation

$$\Delta u = -\varphi$$

on a complete Riemannian manifold (M^n, g), where φ is a given smooth function on M. Among other things, we obtained the following result.

Theorem 1.1. *Let (M^n, g) be a complete Riemannian manifold with bottom spectrum $\lambda_1(\Delta) > 0$ and Ricci curvature $\mathrm{Ric} \geq -(n-1)K$ for some constant $K \geq 0$. Let φ be a smooth function such that*

$$|\varphi|(x) \leq c\,(1 + r(x))^{-k}$$

for some $k > 1$, where $r(x)$ is the distance function from x to a fixed point $p \in M$. Then the Poisson equation $\Delta u = -\varphi$ admits a bounded solution u on M.

If, in addition, the volume of the unit ball $B(x,1)$ satisfies $\mathrm{V}(x,1) \geq v_0 > 0$ for all $x \in M$, then the solution u decays and

$$|u|(x) \leq C\,(1 + r(x))^{-k+1}.$$

Recall that the bottom spectrum $\lambda_1(\Delta)$ of the Laplacian can be characterized as the best constant of the Poincaré inequality

$$\lambda_1(\Delta) \int_M \phi^2 dx \leq \int_M |\nabla \phi|^2 dx.$$

It is known that $\lambda_1(\Delta) > 0$ implies that M is non-parabolic, that is, there exists a positive symmetric Green's function $G(x,y)$ for the Laplacian. The preceding theorem relies on the following sharp estimate of the minimal positive Green's function.

Theorem 1.2. *Let (M^n, g) be an n-dimensional complete manifold with $\lambda_1(\Delta) > 0$ and $\mathrm{Ric} \geq -(n-1)K$. Then for any $p, x \in M$ and $r > 0$ we have*

$$\int_{B(p,r)} G(x,y)\,dy \leq C\,(1+r)$$

for some constant C depending only on n, K and $\lambda_1(\Delta)$.

In the current paper, we continue to address similar issues for complete manifolds satisfying more generally a so-called weighted Poincaré inequality.

Definition 1.3. *A complete noncompact Riemannian manifold (M, g) satisfies a weighted Poincaré inequality if there exists a smooth function $\rho > 0$ on M such that*

$$\int_M \rho\phi^2 \leq \int_M |\nabla\phi|^2 \tag{1.1}$$

for any compactly supported function $\phi \in C_0^\infty(M)$.

Other than being a natural generalization of $\lambda_1(\Delta) > 0$, there are various motivations for considering weighted Poincaré inequality. First, it is well-known (see [18]) that M being nonparabolic is equivalent to the validity of a weighted Poincaré inequality for some $\rho > 0$. Second, according to a result of Cheng [5], when the Ricci curvature of manifold M is asymptotically nonnegative at infinity, its bottom spectrum $\lambda_1(\Delta) = 0$, and one is forced to work with weighted Poincaré inequalities. Third, by considering weighted Poincaré inequalities, it enables one to study manifolds with Ricci curvature bounded below by a function. Typically, in geometric analysis, curvature is assumed to be bounded by a constant so that various comparison theorems become available. As demonstrated in [18, 21], the weighted Poincaré inequality allows one to go beyond this realm. Indeed, they were able to prove some structure theorems for manifolds with Ricci curvature satisfying the inequality

$$\mathrm{Ric}(x) \geq -C\,\rho(x)$$

for a suitable constant C for all $x \in M$. Finally, weighted Poincaré inequalities occur naturally under various geometric settings. Indeed, a result of Minerbe [23] (see [13] for further development) implies that a complete manifold M with nonnegative Ricci curvature satisfies a weighted Poincaré inequality with $\rho(x) = c\,r^{-2}(x)$, where $r(x)$ is the distance from x to a fixed point p in M, provided that the following reverse volume comparison holds for some constant C and $\nu > 2$

$$\frac{\mathrm{V}(B(p,t))}{\mathrm{V}(B(p,s))} \geq C\left(\frac{t}{s}\right)^{\nu}$$

for all $0 < s < t < \infty$. Also, for a minimal submanifold M^n of the Euclidean space $\mathbb{R}^N$, a weighted Poincaré inequality is valid on M with $\rho(x) = \frac{(n-2)^2}{4}\bar{r}^{-2}(x)$, where $\bar{r}(x)$ denotes the extrinsic distance function from x to a fixed point (see [3, 18]). On the other hand, for a stable minimal hypersurface in a manifold with nonnegative Ricci curvature, by the second variation formula, a weighted Poincaré inequality holds with $\rho(x)$ being the length square of the second fundamental form.

We remark that weighted Poincaré inequalities in various forms have appeared in many important issues of analysis and mathematical physics. Agmon [1] has used them in his study of eigenfunctions for Schrödinger operators. In [8, 9], Fefferman and Phong considered the more general weighted Sobolev type inequalities for pseudodifferential operators. There are many interesting results concerning the sharp form of the weight ρ. The classical Hardy inequality for the Euclidean space $\mathbb{R}^n$ implies that $\rho(x) = \frac{(n-2)^2}{4}\, r^{-2}(x)$ and it is optimal. In [2], it is shown that a sharp ρ on the hyperbolic space $\mathbb{H}^n$ is given by $\rho(x) = \frac{(n-1)^2}{4} + \frac{1}{4r^2(x)} + \frac{(n-1)(n-3)}{4\sinh^2 r(x)}$. We also refer to [7] for a more systematic approach to finding an optimal ρ for more general second order elliptic operators.

Throughout the paper, we will assume the weight $\rho(x)$ additionally satisfies both (1.2) and (1.3), that is, the ρ-metric defined by

$$ds_\rho^2 = \rho\, ds^2 \tag{1.2}$$

is complete; and for some constants $A > 0$ and $\delta > 0$,

$$\sup_{B\left(x, \frac{\delta}{\sqrt{\rho(x)}}\right)} \rho \le A \inf_{B\left(x, \frac{\delta}{\sqrt{\rho(x)}}\right)} \rho \tag{1.3}$$

for all $x \in M$.

We point out that these two conditions obviously hold true for a weight of the form $\rho(x) = c\,(1 + r(x))^\alpha$ with $\alpha \ge -2$. The metric ds_ρ^2 was first used by Agmon [1] to obtain decay estimates for eigenfunctions. It was later employed to establish L^2 decay estimates for the minimal positive Green's function in [18].

Our first result is an integral estimate for the minimal positive Green's function $G(x,y)$ on M. In the following, we denote geodesic balls centered at point x of radius r with respect to the background metric ds^2 and the metric ds_ρ^2 by $B(x,r)$ and $B_\rho(x,r)$, respectively.

Theorem 1.4. *Let (M^n, g) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight ρ having properties (1.2) and (1.3). Assume that* $\mathrm{Ric} \ge -K\rho$ *on M for some $K \ge 0$. Then*

$$\int_{B_\rho(p,r)} \rho(y)\, G(x,y)\, dy \le C\,(r+1)$$

for all p and x in M, and all $r > 0$, where C depends only on n, K, δ and A.

While it is possible to prove Theorem 1.4 by following a similar strategy as in [25], we instead adopt a complete different approach here. The important observation is that the problem of estimating the Green's function for Δ may be transformed into one for the weighted Laplacian on a suitable smooth metric measure space with positive bottom spectrum. Recall a smooth metric measure space $(N,\, g,\, e^{-f}\, dv)$ is nothing but a smooth Riemannian manifold (N, g) equipped with a weighted measure $e^{-f}\, dv$, where f is a smooth function on N and dv the Riemannian measure induced by the metric g. The weighted Laplacian Δ_f is defined by $\Delta_f u = \Delta\, u - \langle \nabla f, \nabla u\rangle$. Indeed, the Laplacian Δ of (M, g) may be realized as $\Delta = \rho\, \tilde{\Delta}_f$, where $\tilde{\Delta}$ is the Laplacian of (M, ds_ρ^2) and $\tilde{\Delta}_f$ the weighted Laplacian of the smooth metric measure space $(M, ds_\rho^2, e^{-f}\, dv_\rho)$ with a suitably chosen f and dv_ρ the Riemannian measure with respect to the metric ds_ρ^2. In particular, both Δ and $\tilde{\Delta}_f$ have the same Green's function. Now the weighted Poincaré inequality (1.1) is translated into the fact that the bottom spectrum of the weighted Laplacian $\tilde{\Delta}_f$ satisfies $\lambda_1(\tilde{\Delta}_f) \ge 1$. This immediately leads to an exponential decay estimate for the heat kernel of the weighted Laplacian for large time t. For small time t, an estimate of the heat kernel follows from the Ricci curvature assumption. From these estimates and the fact that the Green's function is the time integral of the heat kernel, Theorem 1.4 follows. Incidentally, this new approach applies to Theorem 1.2 as well and seems to be simpler than the original argument in [25].

As an application of Theorem 1.4, we obtain the following solvability result for the Poisson equation.

Theorem 1.5. *Let (M^n, g) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight ρ having properties (1.2) and (1.3). Assume that* $\mathrm{Ric} \geq -K\rho$ *on M for some $K \geq 0$. Then for smooth function φ such that*

$$\left|\varphi\right|(x) \leq c\,\left(1 + r_\rho(x)\right)^{-k}$$

for some $k > 1$, where $r_\rho(x)$ is the ρ-distance function from x to a fixed point $p \in M$, the Poisson equation $\Delta u = -\rho\,\varphi$ admits a bounded solution u on M.

If, in addition, there exists $v_0 > 0$ such that

$$\mathcal{V}_\rho\left(x, 1\right) = \int_{B_\rho(x,1)} \rho\left(y\right) dy \geq v_0$$

for all $x \in M$, then the solution u decays and

$$\left|u\right|(x) \leq C\,\left(1 + r_\rho(x)\right)^{-k+1}.$$

Obviously, these results are faithful generalization of the ones from $\lambda_1\left(\Delta\right) > 0$. We also point out that Theorem 1.4 is sharp, see Section 3. In passing, we mention that recently Catino, Monticelli and Punzo [4] have studied the solvability of the Poisson equation by only assuming the essential spectrum of M is positive. In view of this, one may speculate that some of the preceding results generalize with the weighted Poincaré inequality holds only for smooth functions ϕ with support avoiding a fixed geodesic ball.

As an application of the solvability of the Poisson equation, we prove the following result concerning the connectivity at infinity.

Theorem 1.6. *Let (M, g) be a complete Kähler manifold satisfying (1.1) with weight ρ having properties (1.2), (1.3) and $\rho \leq C$. Assume that there exists $v_0 > 0$ so that for all $x \in M$*

$$\mathcal{V}_\rho\left(x, 1\right) = \int_{B_\rho(x,1)} \rho\left(y\right) dy \geq v_0 > 0$$

and that the Ricci curvature lower bound $\mathrm{Ric} \geq -\zeta\rho$ *holds for some function $\zeta\left(x\right) > 0$ converging to zero at infinity. Then M has only one end.*

The novelty of the result is that the assumption on the Ricci curvature is essentially imposed only at infinity, yet we are able to conclude that the manifold is connected at infinity. This is of course not true in the Riemannian setting. Indeed, the connected sum of copies of $\mathbb{R}^n$ for $n \geq 3$ has non-negative Ricci curvature outside a compact set and satisfies a weighted Poincaré inequality of the form $\rho(x) = c\,r^{-2}(x)$. Obviously, it can have as many ends as one wishes.

We remark that our assumption is vacuous when $\rho = \lambda_1\left(\Delta\right)$ is constant according to the aforementioned result of Cheng [5]. However, in the case $\lambda_1\left(\Delta\right) > 0$, there are various results concerning the number of ends for both Riemannian and Kähler manifolds. We refer to the papers [17, 20, 19, 24] for more information and further references. It should also be noted, although not explicitly stated there, that the argument in [18] already implies that M necessarily has finitely many ends.

To prove Theorem 1.6, we first observe the assumption that

$$\mathcal{V}_\rho(x,1)=\int_{B_\rho(x,1)}\rho(y)\,dy\geq v_0>0$$

ensures all ends of M must be nonparabolic. Therefore, by the result of Li and Tam [16], M admits a nonconstant bounded harmonic function u with finite energy if it is not connected at infinity. According to [14], such u must be pluriharmonic as M is Kähler. One may view u as a holomorphic map from M into the hyperbolic disk. The proof is then completed by establishing a Liouville type result for such maps. It is well-known from Yau's Schwarz lemma [30] that such map u must be constant if the Ricci curvature of the domain manifold M is nonnegative. The result was generalized by Li and Yau [22] to the case that the negative part of the Ricci curvature of M is integrable. They concluded that u is necessarily a constant map if M is in addition nonparabolic. Our next result may be viewed as further development along this line.

Theorem 1.7. *Let (M,g) be a complete Kähler manifold satisfying the assumptions of Theorem 1.6. Assume that $F:M\to N$ is a finite energy holomorphic map into a complex Hermitian manifold N of non-positive bisectional curvature. Then F must be a constant map.*

The paper is organized as follows. In Section 2, we study Green's function estimates and the Poisson equation for the weighted Laplacian on smooth metric measure spaces with positive bottom spectrum. In Section 3, after making some preliminary observations relating ρ-balls to the background metric balls, we prove Theorem 1.4 and Theorem 1.5 by applying the results from Section 2. In Section 4, we discuss applications of the Poisson equation and prove Theorem 1.7 concerning the Liouville property of finite energy holomorphic maps.

2. Poisson equation for weighted Laplacian

In this section we study the Poisson equation for the weighted Laplacian on smooth metric measure spaces, strengthening our previous results in [25] by a new approach involving the heat kernel.

Throughout this section, (M,g) is assumed to be a complete noncompact Riemannian manifold. To a fixed smooth function $f\in C^\infty(M)$ we associate the weighted volume $dv_f=e^{-f}dv$ and call $\left(M,\,g,\,e^{-f}\,dv\right)$ a smooth metric measure space. The weighted Laplacian Δ_f acting on functions is defined by

$$\Delta_f u=\Delta u-\langle\nabla u,\nabla f\rangle\,.$$

It is self-adjoint with respect to the weighted volume dv_f. Its bottom spectrum is defined by

$$\lambda_1(\Delta_f)=\inf_{\phi\in C_0^\infty(M)}\frac{\int_M|\nabla\phi|^2\,e^{-f}}{\int_M\phi^2e^{-f}}.$$

Function $G_f(x,y)$ is called a Green's function of Δ_f if

$$\Delta_f G_f(x,y)=-\delta(x,y)\,.$$

To ease the notation, throughout this section we use $G(x,y)$ instead of $G_f(x,y)$. Recall that $\lambda_1(\Delta_f) > 0$ guarantees the existence of the minimal positive Green's function, obtained as the limit of the Dirichlet Green's function G_i of a compact exhaustion Ω_i of the manifold M.

The Bakry-Emery Ricci curvature Ric_f of $\left(M, g, e^{-f}dv\right)$ is given by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\,(f)\,.$$

In [25], by assuming a lower bound on the Bakry-Emery curvature of the form

$$\mathrm{Ric}_f \geq -Kg \tag{2.1}$$

and that the weight f satisfies

$$\sup_{y\in B(x,1)} |f(y) - f(x)| \leq a \tag{2.2}$$

for some fixed constants K and $a > 0$, we have proved a sharp integral estimate for the Green's function of Δ_f.

Theorem 2.1. *Let $\left(M, g, e^{-f}dv\right)$ be an n-dimensional smooth metric measure space satisfying (2.1) and (2.2). If $\lambda_1(\Delta_f) > 0$, then the minimal positive Green's function $G(x,y)$ of Δ_f satisfies*

$$\int_{B(p,r)} G(x,y)\, e^{-f(y)} dy \leq C(r+1)$$

for any $p, x \in M$ and any $r > 0$. The constant $C > 0$ depends only on n, K, a and $\lambda_1(\Delta_f)$.

The proof of Theorem 2.1 is based on integral estimate

$$\int_{L_x(\alpha,\beta)} G(x,y)\, e^{-f(y)} dy \leq C\left(1 + \ln\frac{\beta}{\alpha}\right)$$

for any $0 < \alpha < \beta$, where

$$L_x(\alpha,\beta) = \{y \in M : \alpha < G_f(x,y) < \beta\}$$

and C is a constant.

Our goal in this section is to improve Theorem 2.1 by relaxing the assumptions (2.1) and (2.2) to a local Sobolev inequality.

Definition 2.2. *Smooth metric measure space $\left(M, g, e^{-f}dv\right)$ is said to satisfy the Sobolev inequality $(\mathcal{S})$ if there exist positive constants $\mu > 1$, $r_0 > 0$ and $C_S > 0$ such that*

$$C_S\left(\fint_{B(x,r)} \phi^{2\mu} e^{-f}\right)^{\frac{1}{\mu}} \leq r^2 \fint_{B(x,r)} |\nabla\phi|^2 e^{-f} + \fint_{B(x,r)} \phi^2 e^{-f} \tag{2.3}$$

for all $x \in M$, $0 < r \leq r_0$ and any $\phi \in C_0^\infty(B(x,r))$.

Without loss of generality we may assume that $r_0 \leq 1$. Here and in the following, we use $\fint_{B(x,r)} ue^{-f}$ to denote the weighted average of a function u over the ball $B(x,r)$, namely,

$$\fint_{B(x,r)} ue^{-f} = \frac{1}{\mathrm{V}_f(x,r)} \int_{B(x,r)} ue^{-f},$$

where $\mathrm{V}_f(x,r) = \int_{B(x,r)} e^{-f}$ is the weighted volume of $B(x,r)$.

According to [26], assumptions (2.1) and (2.2) imply a Sobolev inequality (2.3) with $\mu = \mu(n)$, $r_0 = 1$ and the constant C_S depending only on dimension n, K in (2.1) and a in (2.2).

Let $H(x,y,t)$ denote the minimal positive heat kernel of Δ_f. That is, H satisfies

$$\begin{aligned} \partial_t H &= \Delta_f H \\ \lim_{t\to 0} H(x,y,t) &= \delta(x,y), \end{aligned}$$

where the second identity is understood in $L^2\left(e^{-f}dv\right)$ sense, and H is obtained as the limit of the Dirichlet heat kernel of Δ_f on compact exhaustion Ω_i of M.

Lemma 2.3. *Let $\left(M, g, e^{-f}dv\right)$ be a smooth metric measure space satisfying the Sobolev inequality $(\mathcal{S})$. Then there exists constant $C > 0$, depending only on μ, r_0 and C_S in (2.3), such that the following holds.*

- *For $x \in M$ and $t_0 = \sqrt{r_0}$, the weighted heat kernel satisfies*

$$H(x,x,t_0) \le \frac{C}{\mathrm{V}_f(x,r_0)}. \tag{2.4}$$

- *For $x \in M$ and $r \ge r_0$, the weighted volume satisfies*

$$\frac{\mathrm{V}_f(x,r)}{\mathrm{V}_f(x,r_0)} \le e^{Cr}. \tag{2.5}$$

Proof. Since the results are standard and can be found in [11, 12], we only sketch the ideas of proof. For (2.4), it is well known that the Sobolev inequality (2.3) implies a mean value inequality for positive subsolutions of the heat equation (see [29] or Chapter 19 of [15]).

$$u(x,t_0) \le \frac{C}{\mathrm{V}_f(x,r_0)} \int_0^{t_0} \int_{B(x,r_0)} u(y,s)\, e^{-f(y)} dy ds,$$

As the heat kernel satisfies

$$\int_M H(x,y,t)\, e^{-f(y)} dy \le 1,$$

(2.4) follows immediately.

Concerning (2.5), as pointed out in Section 2 of [12], the Sobolev inequality (2.3) implies the volume comparison property

$$\mathrm{V}_f(y,r_0) \le C\mathrm{V}_f\left(y, \frac{1}{4}r_0\right),$$

for any $y \in M$. By a covering argument (see [12]), this implies the weighted volume comparison estimate claimed in Theorem 2.3. ☐

We can now extend Theorem 2.1 to the more general setting with (2.1) and (2.2) replaced by (2.3).

Theorem 2.4. *Let $\left(M,g,e^{-f}dv\right)$ be a smooth metric measure space satisfying the Sobolev inequality $(\mathcal{S})$. If $\lambda_1\left(\Delta_f\right)>0$, then the minimal positive Green's function G of Δ_f satisfies the estimate*

$$\int_{B(p,r)} G\left(x,y\right)e^{-f(y)}dy\leq C\left(r+1\right)$$

for any p, $x\in M$ and $r>0$ with the constant $C>0$ depending only on μ, r_0, C_S in (2.3) and $\lambda_1\left(\Delta_f\right)$.

Proof. We first remark that it suffices to prove the result for $x\in B\left(p,r\right)$. Indeed, consider the function

$$\Phi\left(x\right)=\int_{B(p,r)} G\left(x,y\right)e^{-f(y)}dy.$$

We claim that the maximum value of Φ on $M\backslash B\left(p,r\right)$ must occur on $\partial B\left(p,r\right)$. In fact, for

$$\Phi_i\left(x\right)=\int_{B(p,r)} G_i\left(x,y\right)e^{-f(y)}dy,$$

where $G_i\left(x,y\right)$ is the Dirichlet Green's function of Δ_f on a compact exhaustion Ω_i of M, we have $\Phi_i\left(x\right)\to\Phi\left(x\right)$ as $i\to\infty$. This is because $G\left(x,y\right)$ is the limit of $G_i(x,y)$. Since $\Delta_f\Phi_i=0$ on $\Omega_i\setminus B\left(p,r\right)$, by the maximum principle, the maximum value of $\Phi_i\left(x\right)$ on $\Omega_i\setminus B\left(p,r\right)$ is achieved on $\partial B\left(p,r\right)$. Therefore, the same must be true for $\Phi\left(x\right)$.

From now on, we assume that $x\in B\left(p,r\right)$. It is well known (see Chapter 10 in [10]) that for the heat kernel $H\left(x,y,t\right)$ of Δ_f,

$$e^{\lambda_1(\Delta_f)t}H\left(x,x,t\right)\quad\text{is nonincreasing in } t>0. \tag{2.6}$$

In fact, by the semi-group property,

$$\begin{aligned}
\frac{d}{dt}H\left(x,x,t\right) &= \frac{d}{dt}\int_M H\left(x,y,\frac{t}{2}\right)^2 e^{-f(y)}dy\\
&= \int_M H\left(x,y,\frac{t}{2}\right)\Delta_f H\left(x,y,\frac{t}{2}\right)e^{-f(y)}dy\\
&= -\int_M \left|\nabla H\left(x,y,\frac{t}{2}\right)\right|^2 e^{-f(y)}dy\\
&\leq -\lambda_1\left(\Delta_f\right)\int_M H\left(x,y,\frac{t}{2}\right)^2 e^{-f(y)}dy\\
&= -\lambda_1\left(\Delta_f\right)H\left(x,x,t\right).
\end{aligned}$$

Therefore,

$$H\left(x,x,t\right)\leq e^{-\lambda_1(\Delta_f)(t-t_0)}H\left(x,x,t_0\right) \tag{2.7}$$

for all $t\geq t_0>0$.

Furthermore, by the semi-group property and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} H(x,y,t) &= \int_M H\left(x,z,\frac{t}{2}\right) H\left(y,z,\frac{t}{2}\right) e^{-f(z)}dz \\ &\leq \left(\int_M H\left(x,z,\frac{t}{2}\right)^2 e^{-f(z)}dz\right)^{\frac{1}{2}} \left(\int_M H\left(y,z,\frac{t}{2}\right)^2 e^{-f(z)}dz\right)^{\frac{1}{2}} \\ &= H(x,x,t)^{\frac{1}{2}} H(y,y,t)^{\frac{1}{2}}. \end{aligned}$$

Together with (2.7), this proves that

$$H(x,y,t) \leq e^{-\lambda_1(\Delta_f)(t-t_0)} H(x,x,t_0)^{\frac{1}{2}} H(y,y,t_0)^{\frac{1}{2}} \tag{2.8}$$

for all x, $y \in M$ and $t \geq t_0 > 0$.

By Lemma 2.3, for all $x \in M$ we have

$$H(x,x,t_0) \leq \frac{C}{\mathrm{V}_f(x,r_0)} \tag{2.9}$$

and

$$\frac{\mathrm{V}_f(x,r)}{\mathrm{V}_f(x,r_0)} \leq e^{Cr}$$

for all $r \geq r_0$. Here $t_0 = \sqrt{r_0}$ and C depends only on μ, r_0 and C_S in (2.3). So for $x \in B(p,r)$ and any $r > 0$, the triangle inequality implies that

$$\begin{aligned} \frac{\mathrm{V}_f(p,r)}{\mathrm{V}_f(x,r_0)} &\leq \frac{\mathrm{V}_f(x,2r)}{\mathrm{V}_f(x,r_0)} \\ &\leq e^{Cr}. \end{aligned} \tag{2.10}$$

Hence, for $x, y \in B(p,r)$, we get from (2.9) and (2.10) that

$$H(x,x,t_0)^{\frac{1}{2}} H(y,y,t_0)^{\frac{1}{2}} \leq \frac{Ce^{Cr}}{\mathrm{V}_f(p,r)}.$$

Plugging this into (2.8) we conclude that

$$H(x,y,t) \leq \frac{C\, e^{-\lambda_1(\Delta_f)t + Cr}}{\mathrm{V}_f(p,r)}$$

for $x, y \in B(p,r)$ and $t \geq t_0$. This immediately implies there exists $C_1 > 0$ such that

$$\int_{B(p,r)} H(x,y,t)\, e^{-f(y)}dy \leq C_1 e^{-\lambda_1(\Delta_f)t + C_1 r} \tag{2.11}$$

for $x \in B(p,r)$ and $t \geq t_0$. In particular, for $t \geq \Lambda$ with

$$\Lambda = \max\left\{t_0, \frac{2C_1 r}{\lambda_1(\Delta_f)}\right\}, \tag{2.12}$$

one has

$$\int_{B(p,r)} H(x,y,t)\, e^{-f(y)}dy \leq C\, e^{-\frac{1}{2}\lambda_1(\Delta)t}$$

for $x \in B(p,r)$. We integrate this inequality from $t = \Lambda$ to $t = \infty$ and use Fubini's theorem to conclude that

$$\int_{B(p,r)} \left(\int_{\Lambda}^{\infty} H(x,y,t)\, dt \right) e^{-f(y)} dy \leq C \tag{2.13}$$

for $x \in B(p,r)$.

On the other hand, it is well know that the minimal heat kernel satisfies

$$\int_M H(x,y,t)\, e^{-f(y)} dy \leq 1$$

for all $x \in M$. Therefore,

$$\begin{aligned} \int_{B(p,r)} \left(\int_0^{\Lambda} H(x,y,t)\, e^{-f(y)} dt \right) dy &= \int_0^{\Lambda} \left(\int_{B(p,r)} H(x,y,t)\, e^{-f(y)} dy \right) dt \\ &\leq \Lambda. \end{aligned}$$

In view of the choice of Λ from (2.12) we conclude that

$$\int_{B(p,r)} \left(\int_0^{\Lambda} H(x,y,t)\, dt \right) e^{-f(y)} dy \leq C\ (r+1) \tag{2.14}$$

for $x \in B(p,r)$.

Combining (2.13) and (2.14), we obtain that

$$\int_{B(p,r)} \left(\int_0^{\infty} H(x,y,t)\, dt \right) e^{-f(y)} dy \leq C\,(r+1)$$

for $x \in B(p,r)$. Since

$$G(x,y) = \int_0^{\infty} H(x,y,t)\, dt,$$

this shows that

$$\int_{B(p,r)} G(x,y)\, e^{-f(y)} dy \leq C\,(r+1)$$

for all $x \in B(p,r)$. The theorem is proved. □

We now record several applications to the solvability of the Poisson equation. The methods are similar to those in [25].

We adopt the same convention that c and C denote positive constants depending on $\lambda_1(\Delta_f)$, and μ, r_0, C_S in (2.3). Fix $p \in M$ and let

$$r(x) = r(p,x)$$

be the distance function to p.

Theorem 2.5. *Let $\left(M, g, e^{-f} dv\right)$ be a smooth metric measure space with positive spectrum $\lambda_1(\Delta_f) > 0$ and the Sobolev inequality $(\mathcal{S})$. Then for any smooth function φ satisfying*

$$|\varphi|(x) \leq \omega(r(x)),$$

where $\omega(t)$ is a non-increasing function such that $\int_0^\infty \omega(t)\,dt < \infty$, the Poisson equation $\Delta_f u = -\varphi$ admits a bounded solution u on M with

$$\sup_M |u| \le c\left(\omega(0) + \int_0^\infty \omega(t)\,dt\right).$$

Proof. We first prove that

$$\int_M G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \le c\left(\omega(0) + \int_0^\infty \omega(t)\,dt\right) \tag{2.15}$$

for all $x \in M$. Note that by Theorem 2.4 we have

$$\begin{aligned}\int_{B(p,1)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy &\le c \sup_{B(p,1)} |\varphi| \\ &\le c\,\omega(0)\end{aligned}$$

as ω is non-increasing. Therefore,

$$\begin{aligned}&\int_M G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\ = \;& \sum_{j=0}^\infty \int_{B(p,2^{j+1})\setminus B(p,2^j)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\ &+ \int_{B(p,1)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\ \le \;& \sum_{j=0}^\infty \left(\int_{B(p,2^{j+1})\setminus B(p,2^j)} G(x,y)\,e^{-f(y)}dy\right) \sup_{B(p,2^{j+1})\setminus B(p,2^j)} |\varphi| \\ &+ c\omega(0).\end{aligned} \tag{2.16}$$

The hypothesis on φ implies

$$\sup_{B(p,2^{j+1})\setminus B(p,2^j)} |\varphi| \le \omega\left(2^j\right)$$

and Theorem 2.4 says that

$$\int_{B(p,2^{j+1})\setminus B(p,2^j)} G(x,y)\,e^{-f(y)}dy \le c\,2^{j-1}.$$

Using these estimates in (2.16) we obtain

$$\begin{aligned}\int_M G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy &\le c\omega(0) + c\sum_{j=0}^\infty 2^{j-1}\,\omega\left(2^j\right) \\ &\le c\omega(0) + c\sum_{j=0}^\infty \int_{2^{j-1}}^{2^j} \omega(t)\,dt \\ &\le c\left(\omega(0) + \int_0^\infty \omega(t)\,dt\right).\end{aligned}$$

This proves (2.15). As $\int_0^\infty \omega(t)\,dt < \infty$, it follows that the function

$$u(x) := \int_M G(x,y)\,\varphi(y)\,e^{-f(y)}dy$$

is well defined, bounded on M, and verifies

$$\Delta_f u = -\varphi.$$

Furthermore, we have the estimate

$$\sup_M |u| \leq c\left(\omega(0) + \int_0^\infty \omega(t)\,dt\right).$$

This proves the theorem. □

Our next step is to prove that the solution u in Theorem 2.5 decays to zero at infinity by assuming a uniform lower bound on $\mathrm{V}_f(x,1)$, that is,

$$\mathrm{V}_f(x,1) = \int_{B(x,1)} e^{-f(y)}dy \geq v_0 > 0 \tag{2.17}$$

for all $x \in M$.

We first establish a pointwise decay estimate for the Green's function. For the rest of the section, constants c and C depend only on $\lambda_1(\Delta_f)$, v_0 in (2.17) and μ, r_0, C_S in (2.3).

Let us note a general fact that if $w \geq 0$ satisfies

$$\Delta_f w \geq -Cw \ \text{ on } B(x,r_0),$$

then by (2.3) and the DeGiorgi-Nash-Moser iteration it follows that

$$\begin{aligned} w(x) &\leq \frac{C}{\mathrm{V}_f(x,r_0)} \int_{B(x,r_0)} w(y)\,e^{-f(y)}dy \\ &\leq C\int_{B(x,r_0)} w(y)\,e^{-f(y)}dy. \end{aligned} \tag{2.18}$$

The second line follows from (2.17) and (2.5), as

$$\begin{aligned} v_0 &\leq \mathrm{V}_f(x,1) \\ &\leq C\mathrm{V}_f(x,r_0). \end{aligned}$$

Theorem 2.6. *Let $(M,g,e^{-f}dv)$ be a smooth metric measure space with positive spectrum $\lambda_1(\Delta_f) > 0$ and the Sobolev inequality $(\mathcal{S})$. Assume (2.17) holds on M. Then*

$$G(x,z) \leq C\,e^{-\sqrt{\lambda_1(\Delta_f)}r(x,z)}$$

for $z \in M$ with $r(x,z) \geq 1$.

Proof. By Corollary 2.2 in [18] (cf. Theorem 2.5 in [25]) we have that

$$\begin{aligned} &\int_{B(x,r+1)\backslash B(x,r-1)} G^2(x,y)\,e^{-f(y)}dy \\ \leq\ & Ce^{-2\sqrt{\lambda_1(\Delta_f)}r} \int_{B(x,3)\backslash B(x,1)} G^2(x,y)\,e^{-f(y)}dy \end{aligned} \tag{2.19}$$

for any $r \geq 4$.

We first estimate the right hand side of (2.19). For fixed $y \in B(x,3)\backslash B(x,1)$, the function $w(z) = G(x,z)$ satisfies $\Delta_f w = 0$ on $B(y,1)$. Since $r_0 \leq 1$, (2.18) implies that

$$w(y) \leq C\int_{B(y,r_0)} w(z)\,e^{-f(z)}dz. \tag{2.20}$$

Hence, using (2.20) and Theorem 2.4, we get

$$\begin{aligned} G(x,y) &\leq C\int_{B(y,r_0)} G(x,z)\,e^{-f(z)}dz \\ &\leq C \end{aligned} \tag{2.21}$$

for $y \in B(x,3)\setminus B(x,1)$.

By (2.19) and Theorem 2.4 we conclude

$$\begin{aligned} &\int_{B(x,r+1)\setminus B(x,r-1)} G^2(x,y)\,e^{-f(y)}dy \\ \leq\; & Ce^{-2\sqrt{\lambda_1(\Delta_f)}r}\int_{B(x,3)\setminus B(x,1)} G(x,y)\,e^{-f(y)}dy \\ \leq\; & C\,e^{-2\sqrt{\lambda_1(\Delta_f)}r} \end{aligned} \tag{2.22}$$

for any $r \geq 4$.

For $z \in \partial B(x,r)$ with $r \geq 4$, since

$$B(z,r_0) \subset B(x,r+1)\setminus B(x,r-1),$$

it follows that

$$\int_{B(z,r_0)} G^2(x,y)\,e^{-f(y)}dy \leq Ce^{-2\sqrt{\lambda_1(\Delta_f)}r}. \tag{2.23}$$

As the function $w(y) = G^2(x,y)$ satisfies $\Delta_f w \geq 0$ on $B(z,r_0)$, by (2.18) we conclude

$$G(x,z) \leq Ce^{-\sqrt{\lambda_1(\Delta_f)}r(x,z)}$$

for $z \in M$ with $r(x,z) \geq 4$. Together with (2.21), this proves the result. □

We now establish a decay estimate of the solution u to the Poisson equation.

Theorem 2.7. *Let $\left(M, g, e^{-f}dv\right)$ be a smooth metric measure space with positive spectrum $\lambda_1(\Delta_f) > 0$ and the Sobolev inequality $(\mathcal{S})$. Assume (2.17) holds on M. Then for any smooth function φ satisfying*

$$|\varphi|(x) \leq \omega(r_\rho(x)),$$

where $\omega(t)$ is a non-increasing function such that $\int_0^\infty \omega(t)\,dt < \infty$, the Poisson equation $\Delta_f u = -\varphi$ admits a bounded solution u on M such that

$$|u|(x) \leq C\left(\int_{\alpha r(x)}^\infty \omega(t)\,dt + \mathrm{V}_f(p,1)\,\omega(0)\,e^{-\frac{1}{2}\sqrt{\lambda_1(\Delta_f)}r(x)}\right)$$

for all $x \in M$, where $0 < \alpha < \frac{1}{4}$ is a constant depending only on μ, r_0 and C_S in (2.3).

Proof. According to (2.5), there exists a constant $c_1 > 0$ so that

$$\mathrm{V}_f(p,t) \leq e^{c_1\sqrt{\lambda_1(\Delta_f)}\,t}\,\mathrm{V}_f(p,1) \tag{2.24}$$

for all $t > 0$. For c_1 specified in (2.24), set

$$\alpha = \frac{1}{4(c_1+1)}. \tag{2.25}$$

For $x \in M$ fixed, let

$$R = r(x) = r(p,x). \tag{2.26}$$

We may assume $R \geq 2$ as the theorem is obviously true for $R \leq 2$ by adjusting the constant C.

Similar to Theorem 2.5 we have

$$\begin{aligned}
&\int_{M\backslash B(p,2\alpha R)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\
&= \sum_{j=1}^{\infty} \int_{B(p,2^{j+1}\alpha R)\backslash B(p,2^{j}\alpha R)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\
&\leq \sum_{j=1}^{\infty} \left(\int_{B(p,2^{j+1}\alpha R)\backslash B(p,2^{j}\alpha R)} G(x,y)\,e^{-f(y)}dy \right) \sup_{B(p,2^{j+1}\alpha R)\backslash B(p,2^{j}\alpha R)} |\varphi| \\
&\leq C \sum_{j=1}^{\infty} \left(2^{j-1}\alpha R\right)\,\omega\left(2^{j}\alpha R\right),
\end{aligned}$$

where in the last line we have used the decay hypothesis on φ and Theorem 2.4.

Since $\omega(t)$ is nonincreasing, it is easy to see that

$$\begin{aligned}
\sum_{j=1}^{\infty} \left(2^{j-1}\alpha R\right)\omega\left(2^{j}\alpha R\right) &\leq \sum_{j=1}^{\infty} \int_{2^{j-1}\alpha R}^{2^{j}\alpha R} \omega(t)\,dt \\
&\leq \int_{\alpha R}^{\infty} \omega(t)\,dt.
\end{aligned}$$

It follows that

$$\int_{M\backslash B(p,2\alpha R)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \leq c\int_{\alpha R}^{\infty} \omega(t)\,dt. \tag{2.27}$$

We now proceed to obtain an estimate on $B(p,2\alpha R)$. For $y \in B(p,2\alpha R)$, we get by the triangle inequality and (2.26) that

$$\begin{aligned}
r(x,y) &\geq r(p,x) - r(p,y) \\
&\geq (1-2\alpha)\,R.
\end{aligned}$$

Hence, by Theorem 2.6,

$$G(x,y) \leq ce^{-\sqrt{\lambda_1(\Delta_f)}(1-2\alpha)R}$$

for $y \in B(p,2\alpha R)$.

Furthermore, by (2.24),

$$\mathrm{V}_f(p,2\alpha R) \leq e^{2\,\alpha\,c_1\,\sqrt{\lambda_1(\Delta_f)}\,R}\,\mathrm{V}_f(p,1).$$

Combining these estimates, we get

$$\begin{aligned}
\int_{B(p,2\alpha R)} G(x,y)e^{-f(y)}dy &\leq c\,e^{-\sqrt{\lambda_1(\Delta_f)}\,(1-2\alpha(c_1+1))\,R}\,\mathrm{V}_f(p,1) \\
&= c\,e^{-\frac{1}{2}\sqrt{\lambda_1(\Delta_f)}\,R}\,\mathrm{V}_f(p,1),
\end{aligned} \tag{2.28}$$

where the second line follows from (2.25). Together with (2.27) we arrive at

$$\int_M G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \le c\int_{\alpha R}^{\infty}\omega(t)\,dt + c\mathrm{V}_f(p,1)\,\omega(0)\,e^{-\frac{1}{2}\sqrt{\lambda_1(\Delta_f)}\,R},$$

where $R = r(x)$. This proves the theorem. □

3. Weighted Poincaré inequality

In this section, we prove Theorem 1.4 and Theorem 1.5 by applying the results of Section 2. In order to do so, we first relate both the geometry and the analysis of the ρ-balls to the background metric balls.

Consider the ρ-distance function, defined to be

$$r_\rho(x,y) = \inf_\gamma l_\rho(\gamma),$$

the infimum of the length with respect to metric ds_ρ^2 of all smooth curves joining x and y. For a fixed point $x \in M$, one checks readily that $|\nabla r_\rho|^2(x,y) = \rho(y)$. When there is no confusion, the ρ-distance from x to a fixed point p is simply denoted by $r_\rho(x)$. More generally, for any function $v \in C^\infty(M)$, denote by $\nabla_\rho v$ the gradient of v with respect to ds_ρ^2. Then its length with respect to ds_ρ^2 is given by

$$|\nabla_\rho v|_\rho^2 = \frac{1}{\rho}|\nabla v|^2.$$

We denote geodesic balls with center x and radius r with respect to ds^2 by $B(x,r)$ and those with respect to ds_ρ^2 by $B_\rho(x,r)$. Our first result shows that $B\left(x, \frac{r}{\sqrt{\rho(x)}}\right)$ and $B_\rho(x,r)$ are comparable when $r \le 1$. Without loss of generality, we may assume the constants A and δ specified in (1.3) satisfy $A > 16$ and $\delta < 1$.

Throughout this section, we use c and C to denote constants depending only on dimension n, the constant K from the Ricci curvature lower bound, and the constants A and δ in (1.3). Any other dependencies will be explicitly stated.

Proposition 3.1. *Let M be a complete Riemannian manifold satisfying weighted Poincaré inequality (1.1) with weight ρ having properties (1.2) and (1.3). Then there exists $C > 0$ depending only on A and δ such that for any $x \in M$,*

$$\sup_{B_\rho(x,1)} \rho \le C \inf_{B_\rho(x,1)} \rho.$$

Furthermore, there exist $c_0 > 0$ and $C_0 > 0$ depending only on A and δ such that

$$B\left(x, \frac{c_0}{\sqrt{\rho(x)}}r\right) \subset B_\rho(x,r) \subset B\left(x, \frac{C_0}{\sqrt{\rho(x)}}r\right)$$

for all $x \in M$ and $0 < r \le 1$.

Proof. Let $x \in M$ and $0 < r \le 1$. Let $\tau(t)$, $0 \le t \le T$, be a minimizing ρ-geodesic starting from x. We claim that either

$$\tau([0,T]) \subset B\left(x, \frac{\delta}{\sqrt{\rho(x)}}r\right) \quad \text{or } l_\rho(\tau) > \frac{\delta}{A}r. \tag{3.1}$$

Indeed, if τ is not entirely contained in $B\left(x, \frac{\delta}{\sqrt{\rho(x)}}r\right)$, then there exists $0 < t_1 < T$ so that $\tau(t) \in B\left(x, \frac{\delta}{\sqrt{\rho(x)}}r\right)$ for all $0 \le t \le t_1$ and $\tau(t_1) \in \partial B\left(x, \frac{\delta}{\sqrt{\rho(x)}}r\right)$. Let $\bar{\tau}$ be the restriction of τ to $[0, t_1]$. Then

$$\begin{aligned} l_\rho(\bar{\tau}) &= \int_{\bar{\tau}} |\bar{\tau}'|_\rho (t)\, dt \\ &= \int_{\bar{\tau}} \sqrt{\rho(\bar{\tau}(t))}\, |\bar{\tau}'|(t)\, dt \\ &\ge \frac{1}{\sqrt{A}} \sqrt{\rho(x)} \int_{\bar{\tau}} |\bar{\tau}'|(t)\, dt \\ &= \frac{1}{\sqrt{A}} \sqrt{\rho(x)}\, l(\bar{\tau}), \end{aligned}$$

where in the third line we have used (1.3) and that $\bar{\tau}(t) \in B\left(x, \frac{\delta}{\sqrt{\rho(x)}}r\right)$ for all $t \le t_1$. Since $\tau(t_1) \in \partial B\left(x, \frac{\delta}{\sqrt{\rho(x)}}r\right)$, we have $l(\bar{\tau}) \ge \frac{\delta}{\sqrt{\rho(x)}}r$. Consequently,

$$l_\rho(\bar{\tau}) \ge \frac{\delta}{A} r.$$

This proves (3.1).

We infer from the claim that $r(x,y) < \frac{\delta}{\sqrt{\rho(x)}} r$ when $r_\rho(x,y) < \frac{\delta}{A} r$. In other words,

$$B_\rho\left(x, \frac{\delta}{A} r\right) \subset B\left(x, \frac{\delta}{\sqrt{\rho(x)}} r\right) \tag{3.2}$$

for all $x \in M$ and all $0 < r \le 1$. By (1.3), this implies

$$\sup_{B_\rho\left(x, \frac{\delta}{A}\right)} \rho \le A \inf_{B_\rho\left(x, \frac{\delta}{A}\right)} \rho. \tag{3.3}$$

Now for $x, y \in M$ with $r_\rho(x,y) \le 1$, let τ be a minimizing ρ-geodesic from x to y. Applying (3.3) successively on each interval of ρ-length $\frac{\delta}{A}$ along τ, we conclude that

$$\frac{1}{C}\rho(x) \le \rho(y) \le C\rho(x),$$

where $C = A^{\frac{2A}{\delta}}$. Therefore,

$$\sup_{B_\rho(x,1)} \rho \le C \inf_{B_\rho(x,1)} \rho \tag{3.4}$$

for all $x \in M$. This proves the first part of the proposition.

Note that by (3.2), for any $z_1, z_2 \in M$ and $0 < r \le 1$,

$$r(z_1, z_2) < \frac{\delta}{\sqrt{\rho(z_1)}} r \text{ whenever } r_\rho(z_1, z_2) < \frac{\delta}{A} r. \tag{3.5}$$

So for $x, y \in M$ with $r_\rho(x,y) \le r$, applying (3.5) successively on intervals of ρ-length $\frac{\delta}{A} r$ along a minimizing ρ-geodesic τ from x to y and using (3.4), one concludes that

$$r(x,y) \le \frac{C_0}{\sqrt{\rho(x)}} r$$

for some $C_0 > 0$ depending on A and δ. Hence,

$$B_\rho(x,r) \subset B\left(x, \frac{C_0}{\sqrt{\rho(x)}} r\right) \tag{3.6}$$

for all $x \in M$ and $r \leq 1$.

We now show that

$$B\left(x, \frac{c_0}{\sqrt{\rho(x)}} r\right) \subset B_\rho(x,r) \tag{3.7}$$

for all $x \in M$ and $r \leq 1$ with $c_0 = \frac{\delta}{A}$.

Indeed, for $y \in B\left(x, \frac{c_0}{\sqrt{\rho(x)}} r\right)$ and $\gamma(t)$, $0 \leq t \leq T < \frac{c_0}{\sqrt{\rho(x)}} r$, a minimizing geodesic joining x and y, we have

$$\begin{aligned} l_\rho(\gamma) &= \int_\gamma |\gamma'|_\rho (t)\, dt \\ &= \int_\gamma \sqrt{\rho(\gamma(t))}\, |\gamma'|(t)\, dt \\ &\leq \sqrt{A}\sqrt{\rho(x)} \int_\gamma |\gamma'|(t)\, dt \\ &= \sqrt{A}\sqrt{\rho(x)}\, l(\gamma) \\ &\leq c_0 \sqrt{A}\, r \\ &< r, \end{aligned}$$

where in the third line we have used (1.3) together with $\gamma(t) \in B\left(x, \frac{\delta}{\sqrt{\rho(x)}} r\right)$ for all $0 \leq t \leq T$. This proves (3.7).

From (3.7) and (3.6) we conclude that

$$B\left(x, \frac{c_0}{\sqrt{\rho(x)}} r\right) \subset B_\rho(x,r) \subset B\left(x, \frac{C_0}{\sqrt{\rho(x)}} r\right)$$

for all $x \in M$ and $r \leq 1$. This proves the proposition. □

The previous result enables us to translate some properties on geodesic balls of metric ds^2 to those of ds_ρ^2.

Denote by $C_S(B_\rho(x,r))$ the optimal constant for the following Dirichlet Sobolev inequality on $B_\rho(x,r)$.

$$\begin{aligned} & C_S(B_\rho(x,r)) \left(\frac{1}{\mathrm{V}(B_\rho(x,r))} \int_{B_\rho(x,r)} \phi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ \leq\ & \frac{1}{\mathrm{V}(B_\rho(x,r))} \int_{B_\rho(x,r)} |\nabla\phi|^2 + \frac{\rho(x)}{r^2} \frac{1}{\mathrm{V}(B_\rho(x,r))} \int_{B_\rho(x,r)} \phi^2 \end{aligned} \tag{3.8}$$

for any $\phi \in C_0^\infty(B_\rho(x,r))$, where $\mathrm{V}(B_\rho(x,r))$ is the volume of $B_\rho(x,r)$ with respect to the metric ds^2.

Evidently, we have implicitly assumed above that the dimension $n \geq 3$. When $n = 2$, then (3.9) is understood to hold with n replaced by any fixed $n' > 2$.

We refer to $C_S(B_\rho(x,r))$ as the Dirichlet Sobolev constant for $B_\rho(x,r)$.

Proposition 3.2. *Let (M^n,g) be a complete manifold satisfying (1.1), (1.2) and (1.3). Assume that* $\mathrm{Ric} \geq -K\rho$ *on M for some $K \geq 0$. Then for some $C>0$,*

$$C_S(B_\rho(x,r)) \geq \frac{1}{Cr^2}\rho(x)$$

for any $x \in M$ and $0 < r \leq r_0 = \frac{\delta}{2C_0}$. Here C_0 is the constant specified in Proposition 3.1.

Proof. According to Saloff-Coste [29], the following Sobolev inequality holds on $B(x,R)$ if $\mathrm{Ric} \geq -H$ on $B(x,2R)$ and the dimension $n \geq 3$.

$$\begin{aligned} &\frac{1}{R^2} e^{-C\left(1+\sqrt{H}R\right)} \mathrm{V}(B(x,R))^{\frac{2}{n}} \left(\int_{B(x,R)} \phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq \int_{B(x,R)} |\nabla\phi|^2 + \frac{1}{R^2}\int_{B(x,R)} \phi^2 \end{aligned} \tag{3.9}$$

for any $\phi \in C_0^\infty(B(x,R))$. When $n=2$, the inequality (3.9) holds with n replaced by any fixed $n'>2$. The constant C in (3.9) depends only on n (or n', respectively).

For $r \leq \frac{\delta}{2C_0}$ we have

$$B\left(x, \frac{2C_0}{\sqrt{\rho(x)}}r\right) \subset B\left(x, \frac{\delta}{\sqrt{\rho(x)}}\right).$$

The Ricci curvature lower bound assumption together with (1.3) implies that

$$\mathrm{Ric} \geq -c\rho(x) \ \text{ on } B\left(x, \frac{2C_0}{\sqrt{\rho(x)}}r\right). \tag{3.10}$$

Now for $R = \frac{C_0}{\sqrt{\rho(x)}}r$, in view of (3.10) and (3.9), we get

$$\frac{1}{C}\frac{\rho(x)}{r^2}\mathrm{V}(B(x,R))^{\frac{2}{n}}\left(\int_{B(x,R)}\phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \int_{B(x,R)}|\nabla\phi|^2 + \frac{\rho(x)}{r^2}\int_{B(x,R)}\phi^2$$

for any $\phi \in C_0^\infty(B(x,R))$.

However, according to Proposition 3.1 we have

$$B_\rho(x,r) \subset B(x,R).$$

It follows for $\phi \in C_0^\infty(B_\rho(x,r))$ that

$$\frac{1}{C}\frac{\rho(x)}{r^2}\mathrm{V}(B_\rho(x,r))^{\frac{2}{n}}\left(\int_{B_\rho(x,r)}\phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \int_{B_\rho(x,r)}|\nabla\phi|^2 + \frac{\rho(x)}{r^2}\int_{B_\rho(x,r)}\phi^2.$$

This completes the proof of the proposition. □

Another important ingredient for us is the Cheng-Yau [6] gradient estimate for positive harmonic functions. This result says that for $u>0$ satisfying $\Delta u = 0$ on $B(x,R)$,

$$|\nabla \ln u|\,(x) \leq c\left(\sqrt{H}+\frac{1}{R}\right) \tag{3.11}$$

for some constant $c>0$ depending only on dimension n, provided that the Ricci curvature $\mathrm{Ric} \geq -H$ on $B\,(x,R)$ for some constant $H \geq 0$. We now use Proposition 3.1 to translate this estimate to ρ-balls.

Lemma 3.3. *Let (M^n, g) be a complete manifold satisfying (1.1), (1.2) and (1.3). Assume that* $\mathrm{Ric} \geq -K\rho$ *on M for some $K \geq 0$. Then there exists $c>0$ such that for $u>0$ a harmonic function on $B_\rho\,(x,r)$ with* $0<r\leq 1$,

$$|\nabla_\rho \ln u|_\rho\,(x) \leq \frac{c}{r}.$$

Proof. By Proposition 3.1 and (1.3) we have

$$\mathrm{Ric} \geq -c\rho\,(x) \quad \text{on } B_\rho\,(x,r)\,,$$

and

$$B\left(x, \frac{c_0}{\sqrt{\rho\,(x)}}r\right) \subset B_\rho\,(x,r)\,.$$

Hence, applying (3.11) for the harmonic function u on $B\left(x, \frac{c_0}{\sqrt{\rho(x)}}r\right)$, where $\mathrm{Ric} \geq -c\rho\,(x)$ on $B\left(x, \frac{c_0}{\sqrt{\rho(x)}}r\right)$, we obtain that

$$|\nabla \ln u|\,(x) \leq \frac{c}{r}\sqrt{\rho\,(x)}. \tag{3.12}$$

This can be rewritten into

$$|\nabla_\rho \ln u|_\rho\,(x) \leq \frac{c}{r},$$

which proves the lemma. □

We can now relate the geometry of (M,g) with that of a smooth metric measure space. Consider the smooth metric measure space $\left(M,\, g_\rho,\, e^{-f}dv_\rho\right)$, where

$$g_\rho = \rho g \tag{3.13}$$

and

$$f = \left(\frac{n}{2}-1\right)\ln \rho. \tag{3.14}$$

By a well known formula, the Laplacian $\tilde{\Delta}$ with respect to the conformal metric $g_\rho = \rho g$ and Δ, the Laplacian with respect to g, are related by

$$\begin{aligned}\tilde{\Delta} u &= \frac{1}{\rho}\Delta u + \left(\frac{n}{2}-1\right)\frac{1}{\rho^2}\langle \nabla u, \nabla \rho\rangle \\ &= \frac{1}{\rho}\Delta u + \left(\frac{n}{2}-1\right)\langle \nabla_\rho u, \nabla_\rho \ln \rho\rangle_\rho\,.\end{aligned}$$

We have denoted with ∇_ρ the Levi-Civita connection of g_ρ. Hence, we see from above that the weighted Laplacian $\tilde{\Delta}_f$ associated to $\left(M,\, g_\rho,\, e^{-f}dv_\rho\right)$ satisfies

$$\begin{aligned}\tilde{\Delta}_f\, u &= \tilde{\Delta}\, u - \langle \nabla_\rho u, \nabla_\rho f\rangle_\rho \\ &= \frac{1}{\rho}\Delta u,\end{aligned} \tag{3.15}$$

where the last line follows from (3.14).

Note moreover that

$$e^{-f}\,dv_\rho = \rho\,dv. \tag{3.16}$$

Hence, using $|\nabla\phi|^2 = \rho\,|\nabla_\rho\phi|_\rho^2$, we see that the weighted Poincaré inequality (1.1) is equivalent to

$$\int_M \phi^2 e^{-f} dv_\rho \le \int_M |\nabla_\rho\phi|_\rho^2 e^{-f} dv_\rho$$

for any $\phi \in C_0^\infty(M)$. In conclusion,

$$\lambda_1\left(\tilde{\Delta}_f\right) \ge 1. \tag{3.17}$$

Furthermore, by Proposition 3.2 there exists $C > 0$, depending only on n, K, A and δ, such that for any $x \in M$ and $0 < r \le r_0$,

$$\begin{aligned} & C\left(\frac{1}{\mathrm{V}(B_\rho(x,r))}\int_{B_\rho(x,r)} \phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ \le\ & \frac{r^2}{\rho(x)}\frac{1}{\mathrm{V}(B_\rho(x,r))}\int_{B_\rho(x,r)} |\nabla\phi|^2 + \frac{1}{\mathrm{V}(B_\rho(x,r))}\int_{B_\rho(x,r)} \phi^2 \end{aligned} \tag{3.18}$$

for all $\phi \in C_0^\infty(B_\rho(x,r))$, where $r_0 = \frac{\delta}{2C_0} < 1$ depends only on A and δ. In dimension $n = 2$, we replace n in (3.19) by any fixed $n' > 2$.

By Proposition 3.1, for any function $u > 0$ we have

$$\frac{1}{C}\fint_{B_\rho(x,r)} ue^{-f} dv_\rho \le \frac{1}{\mathrm{V}(B_\rho(x,r))}\int_{B_\rho(x,r)} u \le C\fint_{B_\rho(x,r)} ue^{-f} dv_\rho, \tag{3.19}$$

where

$$\fint_{B_\rho(x,r)} ue^{-f} dv_\rho = \frac{1}{\int_{B_\rho(x,r)} e^{-f} dv_\rho}\int_{B_\rho(x,r)} ue^{-f} dv_\rho$$

is the average weighted integral of u over $B_\rho(x,r)$.

Therefore, by (3.19) and (3.19) we obtain that

$$\begin{aligned} C\left(\fint_{B_\rho(x,r)} \phi^{2\mu} e^{-f} dv_\rho\right)^{\frac{1}{\mu}} \le\ & r^2 \fint_{B_\rho(x,r)} |\nabla_\rho\phi|_\rho^2 e^{-f} dv_\rho \\ & + \fint_{B_\rho(x,r)} \phi^2 e^{-f} dv_\rho \end{aligned}$$

for any $0 < r \le r_0$, where $\mu = \frac{n}{n-2}$ if $n \ge 3$ and $\mu > 1$ is any fixed number if $n = 2$.

In conclusion, we have established the following result.

Proposition 3.4. *Let (M^n, g) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight ρ having properties (1.2) and (1.3). Assume that* $\mathrm{Ric} \ge -K\rho$ *on M for some* $K \ge 0$. *Then for* $f = \left(\frac{n}{2} - 1\right)\ln\rho$, *the smooth metric measure space* $\left(M, g_\rho, e^{-f} dv_\rho\right)$ *has positive bottom spectrum* $\lambda_1\left(\tilde{\Delta}_f\right) \ge 1$ *and satisfies the Sobolev inequality ($\mathcal{S}$) with* $\mu = \mu(n)$, $r_0 = r_0(\delta, A)$ *and Sobolev constant* $C_S = C_S(n, K, \delta, A)$.

Proposition 3.4 enables us to apply the results of Section 2 to our context of weighted Poincaré inequality (1.1). First, let us note that the minimal positive Green's function G of Δ is the same as that of the weighted Laplacian $\tilde{\Delta}_f$ as $\Delta = \rho\,\tilde{\Delta}_f$. By Theorem 2.4 we obtain the following.

Theorem 3.5. *Let (M^n, g) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight ρ having properties (1.2) and (1.3). Assume that* $\mathrm{Ric} \geq -K\,\rho$ *on M for some $K \geq 0$. Then*

$$\int_{B_\rho(p,r)} \rho(y)\, G(x,y)\, dy \leq C\,(r+1)$$

for all p and x in M, and all $r > 0$, where C depends only on n, K, δ and A.

Let us point out that Theorem 3.5 is sharp. Indeed, for any $\varepsilon > 0$ small enough so that $B(x,\varepsilon) \subset B_\rho(x,t)$, we have

$$\begin{aligned} 0 &= \int_{B_\rho(x,t)\setminus B(x,\varepsilon)} \Delta_y G(x,y)\, dy \\ &= \int_{\partial B_\rho(x,t)} \frac{\partial G}{\partial \nu}(x,\xi)\, dA(\xi) \\ &\quad - \int_{\partial B(x,\varepsilon)} \frac{\partial G}{\partial r}(x,\xi)\, dA(\xi), \end{aligned}$$

where ν is the outward unit normal vector of $\partial B_\rho(x,t)$ with respect to ds^2. Using the asymptotics of G near its pole, we obtain

$$\int_{\partial B(x,\varepsilon)} \frac{\partial G}{\partial r}(x,\xi)\, dA(\xi) = -1$$

for any $\varepsilon > 0$. So

$$\begin{aligned} 1 &= -\int_{\partial B_\rho(x,t)} \frac{\partial G}{\partial \nu}(x,\xi)\, dA(\xi) \\ &\leq \int_{\partial B_\rho(x,t)} |\nabla G|(x,\xi)\, dA(\xi) \end{aligned} \tag{3.20}$$

for any $t > 0$. Combining with the gradient estimate in Lemma 3.3 that

$$|\nabla G|(x,y) \leq C\sqrt{\rho(y)}\, G(x,y)$$

for $y \in M\setminus B_\rho(x,1)$, where the gradient is taken in variable y, we conclude

$$\int_{\partial B_\rho(x,t)} \sqrt{\rho(\xi)}\, G(x,\xi)\, dA(\xi) \geq \frac{1}{C}$$

for all $t \geq 1$. Now the co-area formula yields

$$\begin{aligned} (3.21) \qquad & \int_{B_\rho(x,r)\setminus B_\rho(x,1)} \rho\left(y\right) G\left(x,y\right) dy \\ = & \int_1^r \int_{\partial B_\rho(x,t)} \frac{1}{\left|\nabla r_\rho\right|\left(x,\xi\right)} \rho\left(\xi\right) G\left(x,\xi\right) dA\left(\xi\right) dt \\ = & \int_1^r \int_{\partial B_\rho(x,t)} \sqrt{\rho\left(\xi\right)} G\left(x,\xi\right) dA\left(\xi\right) dt \\ \geq & \frac{1}{C}\left(r-1\right). \end{aligned}$$

This shows that

$$\int_{B_\rho(x,r)} \rho\left(y\right) G\left(x,y\right) dy \geq \frac{1}{C}\left(r-1\right)$$

for all $r > 1$, confirming the sharpness of Theorem 3.5.

Combining Theorem 2.5 and Theorem 2.7 we have the following result. Define

$$\mathcal{V}_\rho\left(x,r\right) = \int_{B_\rho(x,r)} \rho\left(y\right) dy,$$

which corresponds to the weighted volume in $\left(M, g_\rho, e^{-f} dv_\rho\right)$.

Theorem 3.6. *Let (M^n, g) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight ρ having properties (1.2) and (1.3). Assume that* $\mathrm{Ric} \geq -K\rho$ *on M for some $K \geq 0$. Then for any smooth function φ satisfying*

$$\left|\varphi\right|\left(x\right) \leq \omega\left(r_\rho\left(x\right)\right),$$

where $\omega\left(t\right)$ is a non-increasing function such that $\int_0^\infty \omega\left(t\right) dt < \infty$, the Poisson equation $\Delta u = -\rho\varphi$ admits a bounded solution u on M with

$$\sup_M \left|u\right| \leq c\left(\omega\left(0\right) + \int_0^\infty \omega\left(t\right) dt\right),$$

for a constant c depending only on n, K A and δ.

If, in addition, there exists $v_0 > 0$ such that for all $x \in M$,

$$\mathcal{V}_\rho\left(x,1\right) = \int_{B_\rho(x,1)} \rho\left(y\right) dy \geq v_0,$$

then u decays to zero at infinity. Moreover

$$(3.22) \qquad \left|u\right|\left(x\right) \leq C\left(\int_{\alpha r_\rho(x)}^\infty \omega\left(t\right) dt + \mathcal{V}_\rho\left(p,1\right) \omega\left(0\right) e^{-\frac{1}{2} r_\rho(x)}\right)$$

for all $x \in M$, where $0 < \alpha < \frac{1}{5}$ is a constant depending only on n, K, δ, A, and $C > 0$ may additionally depend on v_0.

Finally, let us note that Theorem 1.5 follows from Theorem 3.6. Indeed, in the case that the function φ decays as

$$\left|\varphi\right|\left(x\right) \leq c\left(1 + r_\rho\left(x\right)\right)^{-k}$$

for some $k > 1$ and

$$\mathcal{V}_\rho\left(x,1\right) \geq v_0 > 0$$

holds for all $x \in M$, Theorem 3.6 readily implies that the solution u satisfies

$$|u|(x) \le C(k)(1+r(x))^{-k+1}$$

as claimed in Theorem 1.5.

We also note the following property. Assume that $\eta \ge 0$ is a C^1 function satisfying

$$\Delta\eta \ge -c\,\rho\,\eta \text{ on } B_\rho(x,r_0). \tag{3.23}$$

Then we have

$$\eta(x) \le \frac{C}{\mathcal{V}_\rho(x,r_0)} \int_{B_\rho(x,r_0)} \rho(y)\,\eta(y)\,dy. \tag{3.24}$$

Here c, C are constants depending only on n, K, A and δ.

Indeed, by (3.15) and (3.23) we have $\tilde{\Delta}_f \eta \ge -c\,\eta$ on $B_\rho(x,r_0)$. Then (3.24) follows from (2.18) and (3.16).

We now present a two-sided volume estimate for geodesic ρ-balls. Let $0 < r_0 < 1$ be the constant specified in Proposition 3.2. We have the following result.

Theorem 3.7. *Let (M^n, g) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight ρ having properties (1.2) and (1.3). Assume that* Ric $\ge -K\rho$ *on M for some $K \ge 0$. Then for all $x \in M$,*

$$\frac{1}{C}e^{2R}\mathcal{V}_\rho(x,r_0) \le \mathcal{V}_\rho(x,R) \le e^{CR}\mathcal{V}_\rho(x,r_0)$$

for all $R \ge r_0$.

Proof. We apply the volume growth estimate of (2.5) and note that $e^{-f}dv_\rho = \rho dv$ by (3.16). It follows that

$$\mathcal{V}_\rho(x,R) \le e^{CR}\mathcal{V}_\rho(x,r_0), \tag{3.25}$$

for all $R > 0$, which proves the upper bound estimate.

We now turn to the lower bound. The same argument as in (3.21) implies that

$$\frac{1}{C} \le \int_{B_\rho(x,R)\setminus B_\rho(x,R-1)} \rho(y)\,G(x,y)\,dy$$

for $R > 2$. By the Cauchy-Schwarz inequality it follows that

$$\frac{1}{C} \le \mathcal{V}_\rho(x,R)\int_{B_\rho(x,R)\setminus B_\rho(x,R-1)} \rho(y)\,G^2(x,y)\,dy. \tag{3.26}$$

By Corollary 2.2 in [18] (cf. Theorem 2.5 in [25]) we have that

$$\begin{aligned} &\int_{B_\rho(x,R)\setminus B_\rho(x,R-1)} \rho(y)\,G^2(x,y)\,dy \\ \le\ & Ce^{-2R}\int_{B_\rho(x,2)\setminus B_\rho(x,1)} \rho(y)\,G^2(x,y)\,dy. \end{aligned}$$

Therefore, combining with (3.26), we obtain

$$\frac{1}{C}e^{2R} \le \mathcal{V}_\rho(x,R)\int_{B_\rho(x,2)\setminus B_\rho(x,1)} \rho(y)\,G^2(x,y)\,dy. \tag{3.27}$$

Let us set

$$\sigma(x) = \sup_{y\in B_\rho(x,2)\setminus B_\rho(x,1)} G(x,y). \tag{3.28}$$

Then by Theorem 3.5 we have

$$\begin{aligned}\int_{B_\rho(x,2)\setminus B_\rho(x,1)} \rho(y)\, G^2(x,y)\, dy &\le \sigma(x) \int_{B_\rho(x,2)\setminus B_\rho(x,1)} \rho(y)\, G(x,y)\, dy \\ &\le C\sigma(x).\end{aligned}$$

Hence, (3.27) implies that

$$\frac{1}{C} e^{2R} \le \mathcal{V}_\rho(x,R)\,\sigma(x). \tag{3.29}$$

Now let $z \in B_\rho(x,2) \setminus B_\rho(x,1)$ such that $\sigma(x) = G(x,z)$. Since the function $w(y) = G(x,y)$ satisfies $\Delta w = 0$ on $B_\rho(z, r_0)$, (3.24) implies that

$$\begin{aligned}\sigma(x) &= G(x,z) \\ &\le \frac{C}{\mathcal{V}_\rho(z,r_0)} \int_{B_\rho(z,r_0)} \rho(y)\, G(x,y)\, dy \\ &\le \frac{C}{\mathcal{V}_\rho(z,r_0)},\end{aligned} \tag{3.30}$$

where the last line follows from Theorem 3.5. As $z \in B_\rho(x,2) \setminus B_\rho(x,1)$, using (3.25), we have that

$$\begin{aligned}\frac{\mathcal{V}_\rho(x,1)}{\mathcal{V}_\rho(z,r_0)} &\le \frac{\mathcal{V}_\rho(z,3)}{\mathcal{V}_\rho(z,r_0)} \\ &\le C.\end{aligned}$$

Hence, (3.30) implies

$$\sigma(x) \le \frac{C}{\mathcal{V}_\rho(x,1)}.$$

In conclusion, by (3.29) we get that

$$\frac{1}{C} e^{2R} \le \frac{\mathcal{V}_\rho(x,R)}{\mathcal{V}_\rho(x,1)},$$

which proves the lower bound. □

We end this section with a remark concerning the regularity of ρ. The smoothness assumption on ρ is mostly for convenience. It suffices to assume ρ is continuous for our purposes.

4. Applications

In this section, we discuss some applications of the Poisson equation and prove Theorem 1.6. We continue to assume that (M, g) is a complete manifold satisfying the weighted Poincaré inequality (1.1), together with (1.2) and (1.3). Furthermore, we assume that there exists $v_0 > 0$ such that the weighted volume

$$\mathcal{V}_\rho(x,1) = \int_{B_\rho(x,1)} \rho(y)\, dy \ge v_0 > 0 \tag{4.1}$$

for all $x \in M$. In the following, unless otherwise specified, the constants c and C depend only on n, K, δ, A and v_0.

We begin with a Liouville type result.

Theorem 4.1. *Let (M^n, g) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight ρ having properties (1.2), (1.3), and (4.1), and* $\mathrm{Ric} \geq -K\rho$ *for some constant $K \geq 0$. Let $\eta \geq 0$ be a C^1 function satisfying*

$$\eta\Delta\eta \geq -\zeta\rho\eta^2 + |\nabla\eta|^2 \tag{4.2}$$

for some positive continuous function $\zeta(x)$ which converges to zero at infinity. If there exist $\varepsilon > 0$ and $\Lambda > 0$ such that

$$\eta(x) \leq \Lambda e^{-\varepsilon r_\rho(x)} \tag{4.3}$$

on M, then $\eta = 0$ on M.

Proof. We assume by contradiction that η is not identically zero. We first normalize η by defining

$$h = \frac{1}{\Lambda e}\eta. \tag{4.4}$$

Then

$$h \leq e^{-\varepsilon r_\rho - 1} \quad \text{on } M.$$

As h satisfies

$$\Delta h \geq -\zeta\rho h + \frac{|\nabla h|^2}{h}$$

at all points where $h > 0$, it is easy to see that

$$\Delta \ln h \geq -\zeta\rho. \tag{4.5}$$

In addition, we have

$$-\ln h \geq 1 + \varepsilon r_\rho \quad \text{on } M. \tag{4.6}$$

Denote by

$$v = \frac{1}{(-\ln h)}, \tag{4.7}$$

where we set $v = 0$ whenever $h = 0$. Hence, $v \in C^0(M)$.

Computing directly, we have

$$\Delta v = (\Delta \ln h)\, v^2 + 2\,|\nabla \ln h|^2\, v^3.$$

Hence, by (4.5) v satisfies

$$\begin{aligned} \Delta v &\geq -\zeta\rho v^2 + \frac{|\nabla v|^2}{v} \\ &\geq -\zeta\rho v^2 \end{aligned} \tag{4.8}$$

whenever $v > 0$. Also, by (4.6),

$$0 \leq v \leq \frac{1}{1 + \varepsilon r_\rho} \quad \text{on } M. \tag{4.9}$$

Define the continuous function

$$\varphi = \zeta v^2 \text{ on } M \tag{4.10}$$

and let

$$\omega(t) = \frac{1}{(1+\varepsilon t)^2} \sup_{M \setminus B_\rho(p,t)} \zeta. \tag{4.11}$$

Clearly, ω is non-increasing and $\int_0^\infty \omega(t)\,dt < \infty$. Furthermore, (4.9) implies that

$$|\varphi|(x) \le \omega(r_\rho(x)) \quad \text{on } M.$$

By Theorem 3.6, the Poisson equation

$$\Delta u = -\rho\varphi \tag{4.12}$$

admits a bounded positive solution $u > 0$ such that

$$0 < u(x) \le C\left(\int_{\alpha r_\rho(x)}^\infty \omega(t)\,dt + \mathcal{V}_\rho(p,1)\,\omega(0)\,e^{-\frac{1}{2}r_\rho(x)}\right) \quad \text{on } M$$

for some $0 < \alpha < \frac{1}{4}$. Since $\rho\varphi$ is continuous, we have $u \in W^{2,p}_{\text{loc}}(M)$ for any p.

By (4.11) we have that

$$\begin{aligned} 0 \quad < \quad u(x) &\le \frac{C}{1+\alpha\varepsilon r_\rho(x)} \sup_{M\setminus B_\rho(p,\alpha r_\rho(x))} \zeta \\ &+ C\mathcal{V}_\rho(p,1)\,e^{-\frac{1}{2}r_\rho(x)} \sup_M \zeta. \end{aligned}$$

As $\zeta \to 0$ at infinity we conclude that for any $\sigma > 0$ there exists $R_0 > 0$ such that

$$u(x) \le \frac{1}{\sigma r_\rho(x)} \tag{4.13}$$

for all $x \in M\setminus B_\rho(p,R_0)$.

We claim that

$$v \le u \quad \text{on } M. \tag{4.14}$$

Suppose by contradiction that (4.14) is not true. Since by (4.9) and (4.13) both u and v approach 0 at infinity, the function $v-u$ must achieve its maximum at some point $x_0 \in M$, where in particular $v(x_0) > 0$. Observe that by (4.10) and (4.12) we have $\Delta u = -\zeta\rho v^2$, whereas by (4.8) we have $\Delta v \ge -\zeta\rho v^2$ at any point where $v > 0$. Then $v - u \in W^{1,2}_{\text{loc}}(\{v>0\})$ is subharmonic in a neighborhood of x_0 and achieves its maximum at x_0. The strong maximum principle implies that $v-u$ is in fact constant on M. Obviously, the constant must be 0, which contradicts (4.8). This contradiction implies that (4.14) is true.

In view of (4.13) and (4.14) we have proved that for any large $\sigma > 0$, there exists $R_0 > 0$ sufficiently large such that

$$v(x) \le \frac{1}{\sigma r_\rho(x)} \quad \text{for all } x \in M\setminus B_\rho(p,R_0). \tag{4.15}$$

We now follow the proof of Theorem 4.4 in [25] and show that v decays faster than any polynomial order in the ρ-distance. This will be done by iterating the previous argument.

First, let us note the following fact. Define

$$|\zeta|_\infty := \sup_M \zeta.$$

Then (4.8) implies that

$$\Delta v \ge -|\zeta|_\infty\,\rho v^2 \tag{4.16}$$

whenever $v > 0$. Assume that

$$v(x) \le \theta(r_\rho(x))$$

for some decreasing function $\theta(t)$ such that $\int_0^\infty \theta^2(t)\,dt < \infty$. Then there exists $0<\alpha<\frac{1}{4}$ and $\Upsilon>0$, independent of v or θ, such that

$$v(x) \leq \Upsilon \left(\int_{\alpha r_\rho(x)}^\infty \theta^2(t)\,dt + e^{-\frac{1}{2} r_\rho(x)} \theta^2(0) \right) \tag{4.17}$$

for all $x \in M$.

Indeed, (4.17) follows in the same manner as (4.15). Define the continuous function

$$\varphi(x) = |\zeta|_\infty v^2$$

and note that

$$0 \leq \varphi(x) \leq \omega(r_\rho(x)),$$

where

$$\omega(t) = |\zeta|_\infty \theta^2(t).$$

By Theorem 3.6, there exists a bounded solution $u \in W^{2,p}_{\text{loc}}(M)$ of

$$\begin{aligned} \Delta u &= -\rho\varphi \\ &= -|\zeta|_\infty \rho v^2 \end{aligned} \tag{4.18}$$

such that

$$0 < u(x) \leq C \left(\int_{\alpha r_\rho(x)}^\infty \omega(t)\,dt + \mathcal{V}_\rho(p,1)\,\omega(0)\, e^{-\frac{1}{2} r_\rho(x)} \right) \quad \text{on } M$$

for some $0<\alpha<\frac{1}{4}$. Using that $\omega(t) = |\zeta|_\infty \theta^2(t)$ and taking

$$\Upsilon := C\,|\zeta|_\infty \max\{1, \mathcal{V}_\rho(p,1)\},$$

we have

$$0 < u(x) \leq \Upsilon \left(\int_{\alpha r_\rho(x)}^\infty \theta^2(t)\,dt + e^{-\frac{1}{2} r_\rho(x)} \theta^2(0) \right) \text{ on } M.$$

By (4.16) and (4.18) the function $v - u \in W^{1,2}_{\text{loc}}(\{v>0\})$ is subharmonic and converges to zero at infinity. Using the maximum principle we obtain $v \leq u$ on M, thus proving (4.17).

Fix $b>0$ small enough, depending only on α and Υ in (4.17), to be specified later. Note that by (4.15), there exists $B_0 > 0$ so that

$$v(x) \leq \frac{b^6}{\alpha^2 r_\rho(x) + 1} + B_0^2 e^{-\alpha^2 r_\rho(x)} \text{ on } M. \tag{4.19}$$

We prove by induction on $m \geq 2$ that

$$v(x) \leq \frac{b^{2^m+m}}{\alpha^m r_\rho(x) + 1} + B^{2^m - m} e^{-\alpha^m r_\rho(x)} \text{ on } M, \tag{4.20}$$

where B is a large enough constant depending only on α, Υ and B_0.

Clearly, (4.20) holds for $m=2$ from (4.19). We now assume (4.20) holds for $m \geq 2$ and prove

$$v(x) \leq \frac{b^{2^{m+1}+(m+1)}}{\alpha^{m+1} r_\rho(x) + 1} + B^{2^{m+1}-(m+1)} e^{-\alpha^{m+1} r_\rho(x)} \text{ on } M. \tag{4.21}$$

By the induction hypothesis we have $v(x) \leq \theta(r_\rho(x))$, where

$$\theta(t) := \frac{b^{2^m+m}}{\alpha^m t+1} + B^{2^m-m} e^{-\alpha^m t}.$$

By (4.17) we obtain that

$$v(x) \leq \Upsilon \left(\int_{\alpha r_\rho(x)}^{\infty} \theta^2(t)\, dt + e^{-\frac{1}{2} r_\rho(x)} \theta^2(0) \right). \tag{4.22}$$

Obviously,

$$\theta^2(t) \leq \frac{2b^{2^{m+1}+2m}}{(\alpha^m t+1)^2} + 2B^{2^{m+1}-2m} e^{-2\alpha^m t}. \tag{4.23}$$

It follows that

$$\begin{aligned} \int_{\alpha r_\rho(x)}^{\infty} \theta^2(t)\, dt &\leq \frac{2}{\alpha^m} \frac{b^{2^{m+1}+2m}}{\alpha^{m+1} r_\rho(x)+1} \\ &\quad + \frac{1}{\alpha^m} B^{2^{m+1}-2m} e^{-\alpha^{m+1} r_\rho(x)}. \end{aligned} \tag{4.24}$$

Furthermore, since $b<1<B$, we have by (4.23) that

$$\begin{aligned} e^{-\frac{1}{2} r_\rho(x)} \theta^2(0) &\leq 2\left(b^{2^{m+1}+2m} + B^{2^{m+1}-2m}\right) e^{-\frac{1}{2} r_\rho(x)} \\ &\leq 4B^{2^{m+1}-2m} e^{-\frac{1}{2} r_\rho(x)} \\ &\leq \frac{1}{\alpha^m} B^{2^{m+1}-2m} e^{-\alpha^{m+1} r_\rho(x)}. \end{aligned} \tag{4.25}$$

The last line follows from

$$4e^{-\frac{1}{2} r_\rho(x)} \leq \frac{1}{\alpha^m} e^{-\alpha^{m+1} r_\rho(x)},$$

using that $0<\alpha<\frac{1}{4}$.

Plugging (4.24) and (4.25) into (4.22) yields

$$\begin{aligned} v(x) &\leq \frac{2\Upsilon}{\alpha^m} \frac{b^{2^{m+1}+2m}}{\alpha^{m+1} r_\rho(x)+1} + \frac{2\Upsilon}{\alpha^m} B^{2^{m+1}-2m} e^{-\alpha^{m+1} r_\rho(x)} \\ &= \left(\frac{2\Upsilon}{\alpha^2} b\right) \left(\frac{b}{\alpha}\right)^{m-2} \frac{b^{2^{m+1}+(m+1)}}{\alpha^{m+1} r_\rho(x)+1} \\ &\quad + \left(\frac{2\Upsilon}{\alpha^2 B}\right) \left(\frac{1}{\alpha B}\right)^{m-2} B^{2^{m+1}-(m+1)} e^{-\alpha^{m+1} r_\rho(x)}. \end{aligned} \tag{4.26}$$

Now take b sufficiently small so that $\frac{b}{\alpha} \leq 1$ and $\frac{2\Upsilon}{\alpha^2} b \leq 1$, and B sufficiently large so that $\frac{1}{\alpha B} \leq 1$ and $\frac{2\Upsilon}{\alpha^2 B} \leq 1$. Since $m \geq 2$, it follows by (4.26) that

$$v(x) \leq \frac{b^{2^{m+1}+(m+1)}}{\alpha^{m+1} r_\rho(x)+1} + B^{2^{m+1}-(m+1)} e^{-\alpha^{m+1} r_\rho(x)}.$$

This proves (4.21). Hence,

$$v(x) \leq \frac{b^{2^m+m}}{\alpha^m r_\rho(x)+1} + B^{2^m-m} e^{-\alpha^m r_\rho(x)} \text{ on } M, \tag{4.27}$$

for all $m \geq 2$.

For $x \in M$ with $r_\rho(x)$ large, apply (4.27) by setting

$$m := \left[\frac{\ln r_\rho(x)}{4 \ln(\alpha^{-1})} \right],$$

where $[\cdot]$ denotes the greatest integer function. Let us note that

$$r_\rho^a(x) \le 2^m \le r_\rho^{2a}(x),$$

where

$$a = \frac{1}{8 \ln(\alpha^{-1})} \in \left(0, \frac{1}{8}\right).$$

Hence, as in particular $b < \frac{1}{e}$, we obtain

$$\begin{aligned} \frac{b^{2^m+m}}{\alpha^m r_\rho(x) + 1} &\le b^{2^m} \\ &\le e^{-r_\rho^a(x)}. \end{aligned}$$

Moreover, from $\alpha^m \ge r_\rho^{-\frac{1}{4}}(x)$ we have

$$\begin{aligned} B^{2^m-m} e^{-\alpha^m r_\rho(x)} &\le e^{-\left(\sqrt{r_\rho(x)} - (\ln B) r_\rho^{2a}(x)\right)} \\ &\le e^{-r_\rho^a(x)}. \end{aligned}$$

In conclusion, there exists constant $a > 0$ such that

$$v(x) \le C e^{-r_\rho^a(x)} \quad \text{on } M. \tag{4.28}$$

We now complete the proof of the theorem. By (4.7) we have that

$$-\ln h \ge \frac{1}{C} e^{r_\rho^a} \quad \text{on } M \tag{4.29}$$

and satisfies

$$\Delta(-\ln h) \le \zeta \rho.$$

Consider the function

$$f = \ln(-\ln h).$$

Note that f satisfies

$$\begin{aligned} \Delta f &\le \frac{\zeta \rho}{(-\ln h)} - \frac{|\nabla h|^2}{h^2 (\ln h)^2} \\ &\le \frac{\zeta \rho}{(-\ln h)} \end{aligned} \tag{4.30}$$

whenever $h > 0$. Moreover, from (4.29), f is bounded below by

$$f(x) \ge r_\rho^a(x) - C \quad \text{on } M. \tag{4.31}$$

Define

$$\varphi = \frac{\zeta}{(-\ln h)},$$

where φ is continuously extended as $\varphi = 0$ at points where $h = 0$. By Theorem 3.6 and (4.29) we can solve the Poisson equation

$$\begin{aligned} \Delta u &= -\rho \varphi \\ &= -\frac{\zeta \rho}{(-\ln h)} \end{aligned} \tag{4.32}$$

and obtain a solution $u \in W^{2,p}_{\text{loc}}(M)$ that decays to zero at infinity.

According to (4.31), the function $f+u$ achieves its minimum at some point $x_0 \in M$. Then $h(x_0) > 0$. So by (4.30) and (4.32), $f+u \in W^{1,2}_{\text{loc}}(\{h>0\})$ satisfies

$$\Delta(f+u) \leq 0$$

in a neighborhood of x_0. By the maximum principle, this implies that $f+u$ is constant, which is a contradiction to (4.30).

Hence h, as well as η, must be identically zero on M. □

Let us point out that the hypothesis (4.3) on η is necessary and optimal. Indeed, consider

$$\eta(x) = e^{-\ln^a\left(|x|^2+e\right)} \text{ on } \mathbb{R}^n,$$

where $0 < a < 1$ is fixed. It can be checked directly that

$$\begin{aligned} \Delta\eta - \frac{|\nabla\eta|^2}{\eta} &= \left(-\Delta \ln^a\left(|x|^2+e\right)\right)\eta \\ &\geq -a\frac{\Delta|x|^2}{\left(|x|^2+e\right)\ln^{1-a}\left(|x|^2+e\right)}\eta \\ &= -\frac{2na}{\left(|x|^2+e\right)\ln^{1-a}\left(|x|^2+e\right)}\eta. \end{aligned}$$

Now $\mathbb{R}^n$ satisfies weighted Poincaré inequality with weight $\frac{(n-2)^2}{4}\frac{1}{|x|^2}$, so we may take $\rho(x) = \frac{(n-2)^2}{4}\frac{1}{|x|^2+1}$, which is continuous. Moreover, there exists $C(n) > 0$ so that

$$\frac{1}{C}\ln(|x|+1) \leq r_\rho(x) \leq C\ln(|x|+1)$$

for all $|x| \geq 1$. Hence, η satisfies

$$\Delta\eta \geq -\zeta\rho\eta + \frac{|\nabla\eta|^2}{\eta}$$

with

$$\zeta(x) = \frac{c(n,a)}{(r_\rho(x)+1)^{1-a}}.$$

However, η violates the hypothesis (4.3) as

$$e^{-2c(n)(r_\rho(x)+1)^a} \leq \eta(x) \leq e^{-c(n)((r_\rho(x))+1)^a}.$$

We also point out that various vanishing results for differential inequalities of the form (4.2) have appeared in the literature due to its connection with the Bochner technique. Comparing to the existing results, we note that Theorem 4.1 does not require integrability of $\zeta\rho$ on M as in [22], nor the smallness of $\sup_M \zeta$ as in [17, 18, 19, 20] and [28].

Theorem 4.1 leads to the following vanishing result for holomorphic maps.

Theorem 4.2. *Let (M^n, g) be a complete Kähler manifold satisfying the weighted Poincaré inequality (1.1) with weight ρ having properties (1.2), (1.3), (4.1) and $\rho \leq C$. Assume that the Ricci curvature has lower bound* $\text{Ric} \geq -\zeta\rho$ *for some function $\zeta(x) > 0$ that converges to zero at infinity. Then any finite energy holomorphic map $F: M \to N$, where N is a complex Hermitian manifold of non-positive bisectional curvature, is identically constant.*

Proof. It is well known (see e.g. Theorem 1.24 in [27]) that the differential $\eta = |dF|$ satisfies

$$\eta\Delta\eta \geq -\zeta\rho\eta^2 + |\nabla\eta|^2 . \tag{4.33}$$

To be in the context of Theorem 4.1, we first show that η decays exponentially fast in the ρ-distance based on the assumption that $\int_M \eta^2 < \infty$. Since ζ converges to zero at infinity, by (4.33) there exists $R_0 > 0$ so that

$$\Delta\eta \geq -\frac{1}{2}\rho\eta \quad \text{on } M\backslash B_\rho(p, R_0) .$$

Note that since $\rho \leq C$, we have

$$\int_M \rho\eta^2 < \infty. \tag{4.34}$$

Hence, applying Theorem 2.1 in [18] we conclude that

$$\int_{M\backslash B_\rho(p,r)} \rho\eta^2 \leq Ce^{-r} \int_{B_\rho(p,R_0)} \rho\eta^2$$

for $r \geq 2R_0$.

Consequently, there exists $\Lambda > 0$ so that

$$\int_{B_\rho(x,1)} \rho\eta^2 \leq \Lambda e^{-r_\rho(x)} \tag{4.35}$$

for all $x \in M$. In fact, we may take $\Lambda = C \int_M \rho\eta^2$.

Since $\Delta\eta^2 \geq -C\eta^2$ on M , by (3.24) and (4.35) we obtain

$$\eta^2(x) \leq \frac{C}{\mathcal{V}_\rho(x, r_0)} \int_{B_\rho(x,r_0)} \rho\eta^2. \tag{4.36}$$

According to Theorem 3.7 and (4.1) we have

$$\mathcal{V}_\rho(x, r_0) \geq \frac{1}{C} v_0 > 0$$

for all $x \in M$. Then (4.36) and (4.35) imply that

$$\eta(x) \leq \Lambda e^{-\frac{1}{2} r_\rho(x)} \tag{4.37}$$

for all $x \in M$, where Λ is a constant depending on the total energy of η on M.

Applying Theorem 4.1, we conclude $\eta = 0$ and F is a constant map. □

We point out that in [22] Li and Yau proved a vanishing theorem for holomorphic maps $F : M \to N$, where M is assumed to be non-parabolic and its Ricci curvature is bounded from below by $\mathrm{Ric} \geq -\bar{\rho}$ with $\bar{\rho}$ being an integrable function. An alternative proof of this result using the Poisson equation is given as Theorem 8.6 in [27].

As a consequence of Theorem 4.2 we obtain the following structural result.

Corollary 4.3. *Let (M^n, g) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight ρ having properties (1.2), (1.3), (4.1) and $\rho \leq C$. Assume that the Ricci curvature is bounded by $\mathrm{Ric} \geq -\zeta\rho$ for some function $\zeta(x) > 0$ that converges to zero at infinity. Then M has only one end.*

Proof. Let us assume by contradiction that M has at least two ends. We denote by E a nonparabolic end and let $F = M\backslash E$. Note that E exists because M is nonparabolic. We claim that F is nonparabolic as well. Indeed, if F were parabolic, then by [18],

$$\int_{(M\backslash B_\rho(p,R))\cap F} \rho(y)\,dy \le C\,e^{-2R}$$

for all R. This obviously contradicts with (4.1). Hence, both E and F are nonparabolic ends. By Li-Tam [16], there exists a harmonic function w on M with the following properties.

$$\begin{aligned} \int_M |\nabla w|^2 &< \infty \\ \limsup_F w &= 1 \\ \liminf_E w &= 0. \end{aligned} \tag{4.38}$$

Such w is necessarily pluriharmonic according to [14]. Therefore, Theorem 4.2 is applicable to w and w must be constant. This shows that M must be connected at infinity. □

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