POISSON EQUATION ON COMPLETE MANIFOLDS

OVIDIU MUNTEANU, CHIUNG-JUE ANNA SUNG, AND JIAPING WANG

Abstract. We develop heat kernel and Green's function estimates for manifolds with positive bottom spectrum. The results are then used to establish existence and sharp estimates of the solution to the Poisson equation on such manifolds with Ricci curvature bounded below. As an application, we show that the curvature of a steady gradient Ricci soliton must decay exponentially if it decays faster than linear and the potential function is bounded above.

Mathematics Subject Classification (2010) 53C44

1. Introduction

For a complete noncompact manifold (M^n, g) without boundary, consider the Poisson equation

$$\Delta u = -\varphi,$$

where φ is a given smooth function on M. In this paper, we establish existence and sharp estimates of the solution u and provide applications to steady gradient Ricci solitons.

As well-known, the solvability of the Poisson equation is closely related to the existence of the so-called Green's function. In [20], Malgrange showed that M always admits a Green's function $G(x, y)$, namely, $G(x, y) = G(y, x)$ and $\Delta_y G(x, y) = -\delta_x(y)$. In particular, if $\varphi \in C_0^\infty(M)$, then a solution u to the Poisson equation exists and is given by

$$u(x) = \int_M G(x, y)\, \varphi(y)\, dy.$$

Malgrange's proof is rather abstract. Later, Li and Tam [13] provided a more constructive proof. The argument was further extended by Ganguly and Pinchover [11] recently to a more general class of operators. Among other things, the constructed Green's function satisfies

$$\sup_{y \in M \setminus B(p,2R)} \sup_{x \in B(p,R)} |G(p, y) - G(x, y)| < \infty,$$

for any $p \in M$ and any radius $R > 0$. This turns out to be very useful in applications. For example, it was used to prove an extension theorem for harmonic functions in [30].

The first author was partially supported by NSF grant DMS-1506220. The second author was partially supported by MOST. The third author was partially supported by NSF grant DMS-1606820.

Theorem 1.1. *(Sung-Tam-Wang) For any harmonic function u defined on $M \setminus \Omega$, where Ω is a bounded subset of M, there exists a harmonic function v on M such that $u - v$ is bounded on $M \setminus \Omega$.*

Obviously, a good control of the Green's function $G(x, y)$ will enable one to establish existence and estimates of the solution u to the Poisson equation for more general φ. Recall that M is nonparabolic if M admits a positive Green's function and parabolic otherwise. It is well-known that M is nonparabolic if and only if it admits a nonconstant bounded superharmonic function, see e.g. Chapter 20 in [18]. Therefore, when M is parabolic, the solution u to the Poisson equation must be unbounded if $\varphi \geq 0$ but not identically 0. Since we are primarily concerned on the existence of bounded solutions in this paper, we will restrict our attention to nonparabolic manifold M. In particular, there exists a unique minimal positive Green's function on M. In the following, it is understood that this is the Green's function we refer to.

For manifolds with nonnegative Ricci curvature, a sharp pointwise estimate for the Green's function is available.

Theorem 1.2. *(Li-Yau) Let M^n be a complete manifold with nonnegative Ricci curvature. If $\int_1^\infty \mathrm{V}^{-1}(p, \sqrt{t})\, dt < \infty$ for some point $p \in M$, then M is nonparabolic and its minimal positive Green's function $G(x, y)$ satisfies the estimate*

$$C_1 \int_{r^2(x,y)}^\infty \mathrm{V}^{-1}(x, \sqrt{t}) dt \leq G(x, y) \leq C_2 \int_{r^2(x,y)}^\infty \mathrm{V}^{-1}(x, \sqrt{t}) dt$$

for some constants C_1 and C_2 depending only on the dimension n.

Here and in the following, $\mathrm{V}(p, r)$ denotes the volume of the geodesic ball $B(p, r)$ centered at point p with radius r, and $r(x, y)$ the distance between points x and y in M. The estimate follows from their famous upper and lower bounds of the heat kernel [17] together with the fact that

$$G(x, y) = \int_0^\infty H(x, y, t) dt.$$

Based on the estimates of the Green's function, Ni, Shi and Tam [27] obtained existence result concerning the Poisson equation under the assumption that φ decays in the average sense over the geodesic balls centered at a fixed point. While the solution u in general may not be bounded, its growth is well controlled. Furthermore, they applied their result to study, among other things, the following uniformization conjecture of Yau.

Conjecture 1.3. *(Yau) A complete noncompact Kähler manifold with positive bisectional curvature is biholomorphic to the complex Euclidean space.*

Indeed, by first solving the Poisson equation $\Delta u = S$ on such manifold M, where S is the scalar curvature, they demonstrated that under suitable assumptions u is in fact a solution to the Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar{\partial} u = \rho,$$

where ρ is the Ricci form of M.

This line of ideas was initiated by Mok, Siu and Yau in [21]. While the conjecture in its most general form is still open, there are various partial results. We refer to the recent spectacular work of Liu [19] and the references therein for further information.

Our focus will be on manifolds with positive spectrum. Denote by $\lambda_1(\Delta)$ the smallest spectrum of the Laplacian or the bottom spectrum of M. It is well-known that M is nonparabolic if $\lambda_1(\Delta) > 0$. Recall that $\lambda_1(\Delta)$ can be characterized as the best constant of the Poincaré inequality.

$$\lambda_1(\Delta) = \inf_{\phi \in C_0^\infty(M)} \frac{\int_M |\nabla \phi|^2 dx}{\int_M \phi^2 dx}.$$

As observed by Strichartz [29], if $\lambda_1(\Delta) > 0$, then Δ^{-1} is in fact a bounded operator on $L^p(M)$ for $1 < p < \infty$. In particular, there exists a solution $u \in L^p(M)$ to the Poisson equation for $\varphi \in L^p(M)$.

Our achievement here is to establish an existence result with sharp control of the solution u by only assuming a modest decay on the function φ.

Theorem 1.4. *Let M be a complete Riemannian manifold with bottom spectrum $\lambda_1(\Delta) > 0$ and Ricci curvature $\mathrm{Ric} \geq -(n-1)K$ for some constant K. Let φ be a smooth function such that*

$$|\varphi|(x) \leq c\,(1 + r(x))^{-k}$$

for some $k > 1$, where $r(x)$ is the distance function from x to a fixed point $p \in M$. Then the Poisson equation $\Delta u = -\varphi$ admits a bounded solution u on M. If, in addition, the volume of the ball $B(x,1)$ satisfies $\mathrm{V}(x,1) \geq v_0 > 0$ for all $x \in M$, then the solution u decays and

$$|u|(x) \leq C\,(1 + r(x))^{-k+1}.$$

We point out that the existence of a solution u was previously proved by the first author and Sesum [22]. However, their estimate on the solution u takes the form

$$|u|(x) \leq C\, e^{c\, r(x)}.$$

It should also be emphasized that the assumption on the volume that $\mathrm{V}(x,1) \geq v_0$ is necessary to guarantee the solution u decays at infinity. Indeed, since u is a bounded super-harmonic function when φ is positive, u can not possibly decay to zero along a parabolic end of M.

The theorem is sharp as one can see from the following example. On the hyperbolic space $\mathbb{H}^n$, the Green's function is given by

$$G(x,y) = \int_{r(x,y)}^{\infty} \frac{dt}{A(t)},$$

where $A(t)$ is the area of geodesic sphere of radius t in $\mathbb{H}^n$. For $\varphi(x) = (1 + r(x))^{-k}$ with $k > 1$, a direct calculation gives

$$\begin{aligned} u(x) &= \int_{\mathbb{H}^n} G(x,y)\,\varphi(y)\,dy \\ &\geq c\,(1 + r(x))^{-k+1}. \end{aligned}$$

Our proof again relies on some sharp estimates of the Green's function. Recall the following result of the third author with Li [14, 16], which is a sharp version of Agmon's work [1].

Theorem 1.5. *(Li-Wang) Let M be a complete Riemannian manifold with $\lambda_1(\Delta) > 0$. Let u be a nonnegative subharmonic function defined on $M \backslash B(p,1)$. If u satisfies the growth condition*

$$\int_{B(p,R)\backslash B(p,1)} u^2\, e^{-2\sqrt{\lambda_1(\Delta)}\, r} = o(R)$$

as $R \to \infty$, then it must satisfy the decay estimate

$$\int_{B(p,R+1)\backslash B(p,R)} u^2 \le C\, e^{-2\sqrt{\lambda_1(\Delta)}\, R} \int_{B(p,2)\backslash B(p,1)} u^2,$$

for some constant $C > 0$ depending only on $\lambda_1(\Delta)$.

In particular, the theorem implies that the minimal positive Green's function satisfies

$$\int_{B(p,R+1)\backslash B(p,R)} G^2(p,y)\, dy \le C\, e^{-2\sqrt{\lambda_1(\Delta)}\, R} \int_{B(p,2)\backslash B(p,1)} G^2(p,y)\, dy.$$

While this result provides a version of sharp estimate on the Green's function, to prove our theorem, however, we also need the following double integral estimate

$$\int_A \int_B G(x,y)\, dy\, dx \le \frac{e^{\sqrt{\lambda_1(\Delta)}}}{\lambda_1(\Delta)} \sqrt{\mathrm{V}(A)}\sqrt{\mathrm{V}(B)}\, (1 + r(A,B))\, e^{-\sqrt{\lambda_1(\Delta)}\, r(A,B)},$$

for any bounded domains A and B of M, where $r(A,B)$ denotes the distance between A and B, and $\mathrm{V}(A), \mathrm{V}(B)$ their volumes.

For this purpose, we develop a parabolic version of the aforementioned result of Li and the third author.

Theorem 1.6. *Let M be a complete Riemannian manifold with $\lambda_1(\Delta) > 0$. Suppose that $(\Delta - \frac{\partial}{\partial t})u(x,t) \ge 0$ with $u(x,t) \ge 0$,*

$$\int_M u^2(x,0) e^{2\sqrt{\lambda_1(\Delta)}\, r(x,A)} dx < \infty$$

and

$$\int_0^T \int_{B(A,2R)\backslash B(A,R)} u^2(x,t)\, e^{-2\sqrt{\lambda_1(\Delta)}\, r(x,A)} dx dt = o(R)$$

for all $T > 0$ as $R \to \infty$. Then, for all $R > 0$,

$$\int_0^\infty \int_{B(A,R+2)\backslash B(A,R)} u^2(x,t)\, dx dt \le C\, e^{-2\sqrt{\lambda_1(\Delta)}\, R} \int_M u^2(x,0) e^{2\sqrt{\lambda_1(\Delta)}\, r(x,A)} dx,$$

for a constant $C > 0$ depending only on $\lambda_1(\Delta)$.

Here, A is a bounded subset in M, $r(x,A)$ the distance from x to A and $B(A,R) = \{x \in M \,|\, r(x,A) < R\}$.

By applying the theorem to the function $u(x,t) = \int_A H(x,y,t)dy$, one obtains a sharp integral estimate of the heat kernel. This may be of independent interest. The desired estimate of the Green's function follows from the fact that

$$G(x,y)=\int_0^\infty H(x,y,t)\,dt.$$

The following result is crucial to our proof of Theorem 1.4. It provides a sharp integral control of the Green's function.

Theorem 1.7. *Let M^n be an n-dimensional complete manifold with $\lambda_1(\Delta)>0$ and $\mathrm{Ric}\geq -(n-1)K$. Then for any $x\in M$ and $r>0$ we have*

$$\int_{B(p,r)} G(x,y)\,dy\leq C\,(1+r),$$

for some constant C depending only on n, K and $\lambda_1(\Delta)$.

On top of the double integral estimate, the proof of the theorem utilizes an idea originated in [15] and further illuminated in [22], where they used the co-area formula together with suitably chosen cut-off functions to justify that for any $x\in M$ and $0<\alpha<\beta$,

$$\int_{L_x(\alpha,\beta)} G(x,y)\,dy\leq c\left(1+\ln\frac{\beta}{\alpha}\right),$$

for c depending only on n, K and $\lambda_1(\Delta)$. Here

$$L_x(\alpha,\beta):=\{y\in M:\alpha<G(x,y)<\beta\}.$$

Partly motivated by applications to gradient Ricci solitons, we in fact consider more generally the weighted Poisson equation on smooth metric measure space $(M,g,e^{-f}\,dx)$, that is, Riemannian manifold (M,g) together with a weighted measure $e^{-f}\,dx$, where f is a smooth function on M. The weighted Poisson equation is given by

$$\Delta_f u=-\varphi,$$

where $\Delta_f\, u=\Delta u-\langle\nabla f,\nabla u\rangle$ is the weighted Laplacian. A natural curvature notion corresponding to the Ricci curvature in the Riemannian setting is the Bakry-Emery Ricci curvature, defined by

$$\mathrm{Ric}_f=\mathrm{Ric}+\mathrm{Hess}\,(f).$$

It is known (see [24, 25]) that results such as volume comparison, gradient estimates and mean value inequality are available on the smooth metric measure spaces with the Bakry-Emery Ricci curvature bounded below together with suitable assumptions on the weight function f. With this in mind, we have parallel versions of Theorems 1.6 and 1.7, see Theorems 2.2 and 3.5, respectively. Consequently, we obtain the following result in the smooth metric measure space setting.

Theorem 1.8. *Let $\left(M^n,\,g,\,e^{-f}dx\right)$ be an n-dimensional smooth metric measure space with $\mathrm{Ric}_f\geq -(n-1)K$ and the oscillation of f on any unit ball $B(x,1)$ bounded above by a fixed constant a. Assume that the bottom spectrum of the weighted Laplacian $\lambda_1(\Delta_f)$ is positive. Let φ be a smooth function such that*

$$|\varphi|(x)\leq c\,(1+r(x))^{-k}$$

for some $k>1$. Then $\Delta_f u=-\varphi$ admits a bounded solution u on M. If, in addition, the weighted volume of the ball $B(x,1)$ satisfies $\mathrm{V}_f(x,1)\geq v_0>0$ for all $x\in M$, then the solution u decays and satisfies

$$|u|(x)\leq C\,(1+r(x))^{-k+1}.$$

In fact, we establish a slightly more general result (see Theorem 4.2). As an immediate application, we obtain the following decay estimate concerning the sub-solutions to semi-linear equations. It would be interesting to see if the estimate can be improved to exponential decay.

Theorem 1.9. *Let $\left(M^n,\, g,\, e^{-f}dx\right)$ be an n-dimensional smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)\,K$ and the oscillation of f on any unit ball $B(x,1)$ bounded above by a fixed constant a. Assume that the bottom spectrum $\lambda_1(\Delta_f)$ of the weighted Laplacian is positive and the weighted volume has lower bound $\mathrm{V}_f(x,1) \geq v_0 > 0$ for all $x \in M$. Suppose $\psi \geq 0$ satisfies*

$$\Delta_f \psi \geq -c\psi^q$$

for some $q > 1$, and

$$\lim_{x\to\infty} \psi(x)\, r^{\frac{1}{q-1}}(x) = 0.$$

Then there exist $\delta > 0$ and $C > 0$ such that

$$\psi(x) \leq Ce^{-r^{\delta}(x)}.$$

This result motivated us to study the curvature behavior of steady gradient Ricci solitons.

Definition 1.10. *A steady gradient Ricci soliton is a complete manifold (M, g) on which there exists a smooth potential function f such that*

$$\mathrm{Ric} + \mathrm{Hess}(f) = 0.$$

Steady gradient Ricci solitons are self-similar solutions to the Ricci flow. Indeed, if we let $g(t) = \psi(t)^*\, g$, where $\psi(t)$ is the diffeomorphism generated by the vector field ∇f with $\psi(0) = id_M$, then $g(t)$ is a solution to the Ricci flow

$$\frac{\partial}{\partial t}\, g(t) = -2\mathrm{Ric}(g(t)).$$

As such, they play important role in the study of the Ricci flows. Some prominent examples of steady gradient Ricci solitons include the Euclidean space $\mathbb{R}^n$ with f being a linear function, Hamilton's cigar soliton $(\Sigma,\, g_\Sigma)$, where $\Sigma = \mathbb{R}^2$ and

$$g_\Sigma = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

with the potential function $f(x, y) = -\ln(1 + x^2 + y^2)$, and Bryant soliton $(\mathbb{R}^n, g)$, $n \geq 3$, where both g and f are rotationally symmetric. The scalar curvature of the cigar satisfies $S = e^f$ and decays exponentially $S \simeq ce^{-r(x)}$ in the distance function. However, the curvature of the Bryant soliton decays linearly in distance. Both examples show that Theorem 1.8 is sharp, see Section 5.

For a steady gradient Ricci soliton, its Riemann curvature Rm satisfies

$$\Delta_f |\mathrm{Rm}| \geq -c\, |\mathrm{Rm}|^2$$

for some constant $c > 0$. Moreover, $|\nabla f|$ is bounded and $\lambda_1(\Delta_f) > 0$ by [25]. So Theorem 1.9 becomes applicable once the weighted volume assumption is verified. This more or less follows from potential f being bounded above by a constant. These considerations motivate the following theorem.

Theorem 1.11. *Let (M^n, g, f) be a complete steady gradient Ricci soliton with potential f bounded above by a constant. If its Riemann curvature satisfies*

$$|\mathrm{Rm}|(x)\, r(x) = o(1)$$

as $x \to \infty$, then

$$|\mathrm{Rm}|(x) \le c\,(1 + r(x))^{3(n+1)}\, e^{-r(x)}.$$

It is unclear at this point whether the assumption on f is necessary. It is known that the assumption automatically holds true when $\mathrm{Ric} > 0$. We also note that the exponential decay rate in the theorem is sharp as seen from $M = N \times \Sigma$, where Σ is the cigar soliton and N a compact Ricci flat manifold.

It view of our result, one may wonder whether there is a dichotomy for the curvature decay rate of steady gradient Ricci solitons, namely, either exactly linear or exponential. This dichotomy, if confirmed, should be very useful for the classification of steady gradient Ricci solitons. In the three dimensional case, very recently, Deng and Zhu [8] have shown that such a soliton must be the Bryant soliton if its curvature decays exactly linearly. On the other hand, if the curvature decays faster than linear, then it must be the product of the cigar soliton and a circle (see Corollary 5.5). We should also mention that Brendle [2] has confirmed Perelman's assertion in [28] that a noncollapsed three dimensional steady gradient Ricci soliton must be the Bryant soliton.

Structure of the paper. In Section 2 we study heat kernel estimates and prove Theorem 1.6. Section 3 is devoted to the study of Green's function and the proof of Theorem 1.7. In both sections, and in the subsequent ones, we work in the general setting of smooth metric measure spaces. In Section 4 we prove Theorem 1.8 and 1.9. The applications concerning steady solitons are presented in Section 5.

Acknowledgement: We would like to dedicate this paper to our teacher Professor Peter Li. It can not be overstated how much we have benefited from his teaching, encouragement and support over the years.

2. Heat kernel estimates

In this section we extend the decay estimate for subharmonic functions developed in [14] and [15] to the subsolutions of the heat equation. As a consequence, we obtain heat kernel estimate on complete manifolds with positive bottom spectrum. The estimate will be applied in next section to derive integral estimates for the minimal Green's function.

We will cast our result in a more general setting of smooth metric measure space $\left(M, g, e^{-f}dx\right)$, where the following weighted Poincaré inequality holds true for a positive function ρ.

$$\int_M \rho(x)\, \phi^2(x) e^{-f}\, dx \le \int_M |\nabla \phi|^2(x) e^{-f}\, dx, \tag{2.1}$$

for any compactly supported function $\phi \in C_0^\infty(M)$.

Let us define the ρ-metric by

$$ds_\rho^2 = \rho\, ds^2.$$

Using this metric, we consider the ρ-distance function defined to be

$$r_\rho(x,y) = \inf_\gamma \ell_\rho(\gamma),$$

the infimum of the length of all smooth curves joining x and y with respect to ds_ρ^2. For a fixed point $p \in M$, one checks readily that $|\nabla r_\rho|^2(p,x) = \rho(x)$.

Definition 2.1. *We say that manifold M has property (P_ρ) if the ρ-metric is complete.*

Similarly, for a compact domain $A \subset M$, we denote

$$r_\rho(x,A) = \inf_{y\in A} r_\rho(y,x)$$

to be the ρ-distance to A and

$$B_\rho(A,R) = \{x \in M \,|\, r_\rho(x,A) < R\}$$

to be the set of points in M that have ρ-distance less than R from set A.

Consider $u\,(x,t)$ a nonnegative subsolution to the weighted heat equation

$$\left(\Delta_f - \frac{\partial}{\partial t}\right) u \geq 0. \tag{2.2}$$

We assume that $u\,(x,t)$ satisfies the growth condition that

$$\int_M u^2\,(x,0)\, e^{2r_\rho(x,A)} e^{-f(x)} dx < \infty \tag{2.3}$$

and that for all $T > 0$,

$$\int_0^T \int_{B_\rho(A,2R)\setminus B_\rho(A,R)} \rho\,(x)\, u^2\,(x,t)\, e^{-2r_\rho(x,A)} e^{-f(x)} dxdt = o(R) \tag{2.4}$$

as $R \to \infty$.

Theorem 2.2. *Let $\left(M, g, e^{-f}dx\right)$ be a complete smooth metric measure space with property (P_ρ). Let $u(x,t)$ satisfy (2.2), (2.3) and (2.4). Then for all $R > 0$,*

$$\begin{aligned} &\int_0^\infty \int_{B_\rho(A,R+2)\setminus B_\rho(A,R)} \rho\,(x)\; u^2\,(x,t)\; e^{-f(x)} dx\, dt \\ \leq\;& C\, e^{-2R} \int_M u^2\,(x,0)\, e^{2r_\rho(x,A)} e^{-f(x)}\, dx \end{aligned}$$

for some absolute constant $C > 0$.

Proof. Throughout the proof, we will denote by C an absolute constant which may change from line to line. We also suppress the dependency of A and write $B_\rho(R) = B_\rho(A,R)$ and $r_\rho(x) = r_\rho(x,A)$. The first step is to prove that for any $0 < \delta < 1$, there exists a constant $0 < C < \infty$ such that

$$\int_0^\infty \int_M \rho(x) e^{2\delta r_\rho(x)} u^2(x,t) e^{-f(x)}\, dx\, dt \leq \frac{C}{1-\delta} \int_M u^2\,(x,0)\, e^{2r_\rho(x)} e^{-f(x)}\, dx.$$

Indeed, let $\phi(x)$ be a non-negative cut-off function on M. Then for any function $h(x)$ integration by parts yields

$$
\begin{aligned}
(2.5)\int_M |\nabla(\phi\, u\, e^h)|^2 e^{-f} &= \int_M |\nabla(\phi e^h)|^2\, u^2 e^{-f} + \int_M (\phi\, e^h)^2\, |\nabla u|^2 e^{-f} \\
&\quad +2\int_M \left(\phi\, e^h\right) u\langle \nabla(\phi\, e^h), \nabla u\rangle e^{-f} \\
&= \int_M |\nabla(\phi\, e^h)|^2 u^2 e^{-f} + \int_M \phi^2 |\nabla u|^2 e^{2h} e^{-f} \\
&\quad +\frac{1}{2}\int_M \langle \nabla(\phi^2\, e^{2h}), \nabla u^2\rangle e^{-f} \\
&= \int_M |\nabla(\phi e^h)|^2 u^2 e^{-f} + \int_M \phi^2 |\nabla u|^2 e^{2h} e^{-f} \\
&\quad -\frac{1}{2}\int_M \phi^2 \Delta_f(u^2) e^{2h} e^{-f} \\
&= \int_M |\nabla(\phi\, e^h)|^2\, u^2 e^{-f} - \int_M \phi^2\, u\, (\Delta_f u)\, e^{2h} e^{-f} \\
&\leq \int_M |\nabla\phi|^2\, u^2 e^{2h} e^{-f} + 2\int_M \phi\, \langle \nabla\phi, \nabla h\rangle u^2 e^{2h} e^{-f} \\
&\quad + \int_M \phi^2\, |\nabla h|^2\, u^2\, e^{2h} e^{-f} - \int_M \phi^2 u u_t e^{2h} e^{-f},
\end{aligned}
$$

where in the last line we have used (2.2). On the other hand, using the weighted Poincaré inequality (2.1), we have

$$\int_M \rho\, \phi^2\, u^2 e^{2h} e^{-f} \leq \int_M |\nabla(\phi\, u e^h)|^2 e^{-f}.$$

Hence (2.5) becomes

$$
\begin{aligned}
(2.6)\ \int_M \rho\, \phi^2\, u^2\, e^{2h} e^{-f} &\leq \int_M |\nabla\phi|^2\, u^2\, e^{2h} e^{-f} + 2\int_M \phi\, \langle \nabla\phi, \nabla h\rangle u^2 e^{2h} e^{-f} \\
&\quad + \int_M \phi^2\, |\nabla h|^2\, u^2 e^{2h} e^{-f} - \frac{1}{2}\frac{d}{dt}\int_M \phi^2 u^2 e^{2h} e^{-f}.
\end{aligned}
$$

Integrating with respect to t, we conclude

$$
\begin{aligned}
(2.7)\qquad & \int_0^T \int_M \rho\, \phi^2\, u^2\, e^{2h}\, e^{-f} dx\, dt + \frac{1}{2}\int_M \phi^2\, u^2(x,T) e^{2h}\, e^{-f} dx \\
\leq\ & \int_0^T \int_M |\nabla\phi|^2\, u^2\, e^{2h} e^{-f} dx\, dt + 2\int_0^T \int_M \phi\, u^2 \langle \nabla\phi, \nabla h\rangle e^{2h}\, e^{-f}\, dx\, dt \\
& + \int_0^T \int_M \phi^2\, |\nabla h|^2 u^2\, e^{2h} e^{-f} dx\, dt + \frac{1}{2}\int_M \phi^2 u^2\, (x, 0)\, e^{2h} e^{-f}\, dx.
\end{aligned}
$$

Let us first choose

$$
(2.8)\qquad \phi\left(r_\rho\left(x\right)\right) = \begin{cases} 1 & \text{on} \quad B_\rho\left(R\right) \\ R^{-1}(2R - r_\rho(x)) & \text{on} \quad B_\rho(2R)\backslash B_\rho(R) \\ 0 & \text{on} \quad M\backslash B_\rho(2R) \end{cases}
$$

and

$$h\left(r_{\rho}\left(x\right)\right)=\left\{\begin{array}{cl} \delta r_{\rho}\left(x\right) & \text{on } B_{\rho}\left(\left(1+\delta\right)^{-1}K\right) \\ K-r_{\rho}\left(x\right) & \text{on } M\backslash B_{\rho}\left(\left(1+\delta\right)^{-1}K\right) \end{array}\right.$$

for some fixed $K>1$. Note that when $R\geq\left(1+\delta\right)^{-1}K$,

$$\left|\nabla\phi\right|^{2}\left(x\right)=\left\{\begin{array}{ccc} R^{-2}\rho\left(x\right) & \text{on} & B_{\rho}(2R)\backslash B_{\rho}(R) \\ 0 & \text{on} & \left(M\backslash B_{\rho}(2R)\right)\cup B_{\rho}\left(R\right) \end{array}\right.$$

and

$$\left\langle\nabla\phi,\nabla h\right\rangle\left(x\right)=\left\{\begin{array}{ccc} R^{-1}\rho\left(x\right) & \text{on} & B_{\rho}(2R)\backslash B_{\rho}(R) \\ 0 & \text{on} & \left(M\backslash B_{\rho}(2R)\right)\cup B_{\rho}\left(R\right), \end{array}\right.$$

whereas

$$\left|\nabla h\right|^{2}\left(x\right)=\left\{\begin{array}{cl} \delta^{2}\rho & \text{on } B_{\rho}\left(\left(1+\delta\right)^{-1}K\right) \\ \rho & \text{on } M\backslash B_{\rho}\left(\left(1+\delta\right)^{-1}K\right). \end{array}\right.$$

Substituting all these into (2.7) implies

$$\begin{aligned}
&\int_{0}^{T}\int_{M}\rho\,\phi^{2}\,u^{2}e^{2h}e^{-f}dx\,dt+\frac{1}{2}\int_{M}\phi^{2}u^{2}(x,T)e^{2h}\,e^{-f}dx\\
\leq\;& R^{-2}\int_{0}^{T}\int_{B_{\rho}(2R)\backslash B_{\rho}(R)}\rho\,u^{2}e^{2h}e^{-f}\,dx\,dt\\
&+2R^{-1}\int_{0}^{T}\int_{B_{\rho}(2R)\backslash B_{\rho}(R)}\rho\,u^{2}e^{2h}e^{-f}\,dx\,dt\\
&+\delta^{2}\int_{0}^{T}\int_{B_{\rho}((1+\delta)^{-1}K)}\rho\,\phi^{2}\,u^{2}\,e^{2h}e^{-f}\,dx\,dt\\
&+\int_{0}^{T}\int_{B_{\rho}(2R)\backslash B_{\rho}((1+\delta)^{-1}K)}\rho\,\phi^{2}\,u^{2}\,e^{2h}e^{-f}\,dx\,dt\\
&+\frac{1}{2}\int_{M}u^{2}\left(x,0\right)e^{2h}e^{-f}\,dx.
\end{aligned}$$

This proves that

$$\begin{aligned}
&(1-\delta^{2})\int_{0}^{T}\int_{B_{\rho}((1+\delta)^{-1}K)}\rho\,u^{2}e^{2h}e^{-f}\,dx\,dt\\
&+\frac{1}{2}\int_{B_{\rho}((1+\delta)^{-1}K)}u^{2}(x,T)e^{2h}e^{-f}\,dx\\
\leq\;& R^{-2}\int_{0}^{T}\int_{B_{\rho}(2R)\backslash B_{\rho}(R)}\rho\,u^{2}e^{2h}\,e^{-f}dx\,dt\\
&+2R^{-1}\int_{0}^{T}\int_{B_{\rho}(2R)\backslash B_{\rho}(R)}\rho\,u^{2}e^{2h}\,e^{-f}dx\,dt\\
&+\frac{1}{2}\int_{M}u^{2}\left(x,0\right)e^{2r_{\rho}}e^{-f}\,dx.
\end{aligned}$$

In view of the definition of h and (2.4), the first two terms on the right hand side of this inequality tend to 0 as $R\rightarrow\infty$. Therefore, we obtain the estimate

$$
\begin{aligned}
&(1-\delta^2)\int_0^T\int_{B_\rho((1+\delta)^{-1}K)}\rho\, u^2 e^{2\delta r_\rho}e^{-f}\,dx\,dt\\
&+\frac{1}{2}\int_{B_\rho((1+\delta)^{-1}K)}u^2(x,T)e^{2\delta r_\rho}e^{-f}\,dx\\
\le\ &\frac{1}{2}\int_M u^2\,(x,0)\,e^{2r_\rho}e^{-f}\,dx.
\end{aligned}
$$

Since the right hand side is independent of K, by letting $K\to\infty$ we conclude that

$$
\int_0^T\int_M \rho\, u^2 e^{2\delta r_\rho}e^{-f}\,dx\,dt\le\frac{1}{2(1-\delta^2)}\int_M u^2\,(x,0)\,e^{2r_\rho}e^{-f}\,dx \tag{2.9}
$$

and

$$
\int_M u^2(x,T)\,e^{2\delta r_\rho}e^{-f}dx\le\int_M u^2\,(x,0)\,e^{2r_\rho}e^{-f}\,dx
$$

for all $T>0$ and $0<\delta<1$.

Our next step is to improve this estimate by setting $h=r_\rho$ in the preceding argument. Note that with this choice of h, (2.6) asserts that

$$
\begin{aligned}
-2\int_M\phi\langle\nabla\phi,\nabla r_\rho\rangle\,u^2e^{2r_\rho}e^{-f}\ \le\ &\int_M|\nabla\phi|^2u^2\,e^{2r_\rho}e^{-f}\\
&-\frac{1}{2}\frac{d}{dt}\int_M\phi^2u^2e^{2r_\rho}e^{-f}.
\end{aligned}
$$

For $0<R_1<R$, let us choose ϕ to be

$$
\phi\,(x)=\begin{cases} R_1^{-1}r_\rho(x) & \text{on}\quad B_\rho(R_1)\\ (R-R_1)^{-1}\,(R-r_\rho(x)) & \text{on}\quad B_\rho(R)\backslash B_\rho(R_1)\\ 0 & \text{on}\quad M\backslash B_\rho\,(R)\,. \end{cases}
$$

We conclude that

$$
\begin{aligned}
&\frac{2}{R-R_1}\int_{B_\rho(R)\backslash B_\rho(R_1)}\frac{R-r_\rho}{R-R_1}\,\rho u^2\,e^{2r_\rho}e^{-f}\\
\le\ &\frac{2}{R_1^2}\int_{B_\rho(R_1)}r_\rho\,\rho\,u^2\,e^{2r_\rho}e^{-f}+\frac{1}{(R-R_1)^2}\int_{B_\rho(R)\backslash B_\rho(R_1)}\rho\,u^2\,e^{2r_\rho}e^{-f}\\
&+\frac{1}{R_1^2}\int_{B_\rho(R_1)}\rho\,u^2\,e^{2r_\rho}e^{-f}-\frac{1}{2}\frac{d}{dt}\int_M\phi^2u^2e^{2r_\rho}e^{-f}.
\end{aligned}
$$

Integrating with respect to t, we obtain

$$\begin{aligned}
&\frac{2}{R-R_1}\int_0^T\int_{B_\rho(R)\setminus B_\rho(R_1)}\frac{R-r_\rho}{R-R_1}\,\rho\,u^2\,e^{2r_\rho}e^{-f}\,dx\,dt\\
\leq\ &\frac{2}{R_1^2}\int_0^T\int_{B_\rho(R_1)} r_\rho\,\rho\,u^2\,e^{2r_\rho}e^{-f}dx\,dt\\
&+\frac{1}{(R-R_1)^2}\int_0^T\int_{B_\rho(R)\setminus B_\rho(R_1)}\rho\,u^2\,e^{2r_\rho}e^{-f}\,dx\,dt\\
&+\frac{1}{R_1^2}\int_0^T\int_{B_\rho(R_1)}\rho\,u^2e^{2r_\rho}e^{-f}\,dx\,dt+\frac{1}{2}\int_M u^2\,(x,0)\,e^{2r_\rho}e^{-f}\,dx.
\end{aligned}$$

Consequently, for any $0<\tau<R-R_1$, since

$$\begin{aligned}
&\frac{2\tau}{(R-R_1)^2}\int_{B_\rho(R-\tau)\setminus B_\rho(R_1)}\rho\,u^2\,e^{2r_\rho}e^{-f}\\
\leq\ &\frac{2}{(R-R_1)^2}\int_{B_\rho(R)\setminus B_\rho(R_1)}(R-r_\rho)\,\rho\,u^2e^{2r_\rho}e^{-f},
\end{aligned}$$

we deduce that

$$\begin{aligned}
&\frac{2\tau}{(R-R_1)^2}\int_0^T\int_{B_\rho(R-\tau)\setminus B_\rho(R_1)}\rho\,u^2e^{2r_\rho}e^{-f}\,dx\,dt\\
\leq\ &\left(\frac{2}{R_1}+\frac{1}{R_1^2}\right)\int_0^T\int_{B_\rho(R_1)}\rho\,u^2e^{2r_\rho}e^{-f}\,dx\,dt\\
&+\frac{1}{(R-R_1)^2}\int_0^T\int_{B_\rho(R)\setminus B_\rho(R_1)}\rho\,u^2e^{2r_\rho}\,e^{-f}dx\,dt\\
&+\frac{1}{2}\int_M u^2\,(x,0)\,e^{2r_\rho}e^{-f}\,dx.
\end{aligned}\tag{2.10}$$

Take $R_1=1$, $\tau=1$, and set

$$g(R)=\int_0^T\int_{B_\rho(R)\setminus B_\rho(1)}\rho\,u^2e^{2r_\rho}e^{-f}\,dx\,dt.$$

Using (2.9) for $\delta=0$ we may rewrite the inequality (2.10) as

$$g(R-1)\leq C_1\,R^2\int_M u^2\,(x,0)\,e^{2r_\rho}e^{-f}\,dx+\frac{1}{2}\,g(R),$$

where

$$C_1=\frac{3}{4}\,\left(e^2+1\right)$$

is an absolute constant. Iterating this inequality, we obtain that for any positive integer k and $R\geq 1$,

$$\begin{aligned}
g(R)\ &\leq\ C_1\left(\sum_{i=1}^k\frac{(R+i)^2}{2^{i-1}}\right)\int_M u^2\,(x,0)\,e^{2r_\rho}e^{-f}\,dx+2^{-k}\,g(R+k)\\
&\leq\ C\,R^2\int_M u^2\,(x,0)\,e^{2r_\rho}e^{-f}\,dx+2^{-k}\,g(R+k)
\end{aligned}$$

for some absolute constant C. However, our previous estimate (2.9) asserts that

$$\int_0^T \int_M \rho\, u^2 e^{2\delta r_\rho} e^{-f}\, dx\, dt \le \frac{1}{2\left(1-\delta^2\right)} \int_M u^2\left(x,0\right) e^{2r_\rho} e^{-f}\, dx$$

for any $\delta < 1$. This implies that

$$\begin{aligned} g(R+k) &= \int_0^T \int_{B_\rho(R+k)\setminus B_\rho(1)} \rho\, u^2 e^{2r_\rho} e^{-f} dx\, dt \\ &\le e^{2(R+k)(1-\delta)} \int_0^T \int_{B_\rho(R+k)\setminus B_\rho(1)} \rho\, u^2 e^{2\delta r_\rho} e^{-f}\, dx\, dt \\ &\le \frac{1}{2\left(1-\delta^2\right)} e^{2(R+k)(1-\delta)} \int_M u^2\left(x,0\right) e^{2r_\rho} e^{-f}\, dx. \end{aligned}$$

Hence,

$$2^{-k}\, g(R+k) \to 0$$

as $k \to \infty$ by choosing $2(1-\delta) < \ln 2$. This proves the estimate

$$g(R) \le C\; R^2 \int_M u^2\left(x,0\right) e^{2r_\rho} e^{-f}\, dx.$$

By adjusting the constant, we have

$$\int_0^T \int_{B_\rho(R)} \rho\, u^2\, e^{2r_\rho} e^{-f}\, dx\, dt \le C\, R^2 \int_M u^2\left(x,0\right) e^{2r_\rho} e^{-f}\, dx \tag{2.11}$$

for all $R > 1$.

Using inequality (2.10) again and choosing $R_1 = 1$ and $\tau = \frac{R}{2}$ this time, we conclude that

$$\begin{aligned} R \int_0^T \int_{B_\rho(\frac{R}{2})\setminus B_\rho(1)} \rho\, u^2 e^{2r_\rho} e^{-f}\, dx\, dt \quad &\le \quad CR^2 \int_M u^2\left(x,0\right) e^{2r_\rho} e^{-f}\, dx \\ &\quad + \int_0^T \int_{B_\rho(R)\setminus B_\rho(1)} \rho\, u^2 e^{2r_\rho} e^{-f} dx\, dt. \end{aligned}$$

However, applying the estimate (2.11) to the second term on the right hand side, we have

$$\int_0^T \int_{B_\rho(\frac{R}{2})\setminus B_\rho(1)} \rho\, u^2 e^{2r_\rho} e^{-f} dx\, dt \le CR \int_M u^2\left(x,0\right) e^{2r_\rho} e^{-f}\, dx.$$

Therefore, for $R > 1$,

$$\int_0^T \int_{B_\rho(R)} \rho\, u^2 e^{2r_\rho} e^{-f}\, dx\, dt \le C\, R \int_M u^2\left(x,0\right) e^{2r_\rho} e^{-f}\, dx. \tag{2.12}$$

We are now ready to prove the theorem by using (2.12). Setting $\tau = 2$ and $R_1 = R - 4$ in (2.10), we obtain

$$\begin{aligned}
&\int_0^T \int_{B_\rho(R-2)\setminus B_\rho(R-4)} \rho\, u^2 e^{2r_\rho} e^{-f} dx\, dt \\
\leq\ & \left(\frac{8}{R-4} + \frac{4}{(R-4)^2}\right) \int_0^T \int_{B_\rho(R-4)} \rho\, u^2 e^{2r_\rho} e^{-f}\, dx\, dt \\
&+\frac{1}{4}\int_0^T \int_{B_\rho(R)\setminus B_\rho(R-4)} \rho\, u^2 e^{2r_\rho} e^{-f} dx\, dt \\
&+2\int_M u^2(x,0)\, e^{2r_\rho} e^{-f}\, dx.
\end{aligned}$$

According to (2.12), the first term of the right hand side is bounded by

$$\begin{aligned}
&\left(\frac{8}{R-4} + \frac{4}{(R-4)^2}\right) \int_0^T \int_{B_\rho(R-4)} \rho\, u^2 e^{2r_\rho} e^{-f}\, dx\, dt \\
\leq\ & C\int_A u^2(x,0)\, e^{2r_\rho} e^{-f}\, dx.
\end{aligned}$$

Hence, the above inequality can be rewritten as

$$\begin{aligned}
\int_0^T \int_{B_\rho(R-2)\setminus B_\rho(R-4)} \rho\, u^2 e^{2r_\rho} e^{-f} dx\, dt \quad \leq\ & \frac{1}{3}\int_0^T \int_{B_\rho(R)\setminus B_\rho(R-2)} \rho\, u^2 e^{2r_\rho} e^{-f}\, dx\, dt \\
&+C\int_M u^2(x,0)\, e^{2r_\rho} e^{-f}\, dx.
\end{aligned}$$

Iterating this inequality k times, we arrive at

$$\begin{aligned}
&\int_0^T \int_{B_\rho(R+2)\setminus B_\rho(R)} \rho\, u^2 e^{2r_\rho} e^{-f}\, dx\, dt \\
\leq\ & 3^{-k}\int_0^T \int_{B_\rho(R+2(k+1))\setminus B_\rho(R+2k)} \rho\, u^2 e^{2r_\rho} e^{-f}\, dx\, dt \\
&+C\left(\sum_{i=0}^{k-1} 3^{-i}\right)\int_M u^2(x,0)\, e^{2r_\rho} e^{-f}\, dx\,.
\end{aligned}$$

However, using (2.12) again, we conclude that the second term is bounded by

$$\begin{aligned}
&3^{-k}\int_0^T \int_{B_\rho(R+2(k+1))\setminus B_\rho(R+2k)} \rho\, u^2 e^{2r_\rho} dx\, dt \\
\leq\ & C\, 3^{-k}(R+2(k+1))\int_M u^2(x,0)\, e^{2r_\rho} e^{-f}\, dx
\end{aligned}$$

and tends to 0 as $k \to \infty$. Hence,

$$\int_0^T \int_{B_\rho(R+2)\setminus B_\rho(R)} \rho\, u^2 e^{2r_\rho} e^{-f}\, dx\, dt \leq C\int_M u^2(x,0)\, e^{2r_\rho} e^{-f}\, dx \tag{2.13}$$

for some absolute constant $C > 0$. The theorem now follows from (2.13) by letting $T \to \infty$. □

We now apply this theorem to establish integral estimates for the heat kernel. For a compact set $A \subset M$, let

$$u(x,t) = \int_A H(x,y,t)\, e^{-f(y)} dy, \tag{2.14}$$

where $H(x,y,t)$ is the minimal heat kernel of Δ_f on M. Clearly, $u(x,0) = \chi_A(x)$, therefore

$$\int_M u^2\,(x,0)\, e^{2r_\rho(x,A)} e^{-f(x)} dx = \mathrm{V}_f\,(A)\,. \tag{2.15}$$

Furthermore, as the heat semigroup is a contraction in $L^2\left(e^{-f}dx\right)$, see e.g. Theorem 4.9 in [12], we have

$$\begin{aligned} \int_M u^2(x,t) e^{-f(x)}\, dx &\leq \int_M u^2(x,0) e^{-f(x)}\, dx \\ &\leq \mathrm{V}_f(A), \end{aligned} \tag{2.16}$$

where $\mathrm{V}_f(A) := \int_A e^{-f} dv$ is the weighted volume of the set A. Theorem 2.2 then implies the following.

Corollary 2.3. *Let $\left(M, g, e^{-f}dx\right)$ be a complete smooth metric measure space with property (P_ρ). Then $u\,(x,t)$ defined in (2.14) satisfies*

$$\int_0^\infty \int_{B_\rho(A,R+1)\setminus B_\rho(A,R)} \rho\,(x)\, u^2\,(x,t)\, e^{-f(x)} dx dt \leq C\, e^{-2R}\, \mathrm{V}_f\,(A)$$

for all $R > 0$, where $C > 0$ is an absolute constant. Furthermore, for all $0 < \delta < 1$,

$$\int_0^\infty \int_M \rho\,(x)\; u^2\,(x,t)\, e^{2\delta r_\rho(x,A)} e^{-f(x)}\, dx\, dt \leq \frac{1}{2(1-\delta^2)} \mathrm{V}_f\,(A)\,.$$

Proof. To apply Theorem 2.2 we need to verify the assumptions (2.3) and (2.4). Choosing $h = 0$ and ϕ as in (2.8), we get from (2.7) and (2.15) that

$$\begin{aligned} \int_0^T \int_{B_\rho(R)} \rho u^2 e^{-f} dx dt &\leq \frac{1}{R^2} \int_0^T \int_{B_\rho(2R)\setminus B_\rho(R)} u^2 e^{-f} dx dt \\ &\quad + \frac{1}{2} \int_{B_\rho(2R)} u^2\,(x,0)\, e^{-f} dx \\ &\leq \left(\frac{T}{R^2} + \frac{1}{2}\right) \mathrm{V}_f(A), \end{aligned}$$

where we have used (2.16) in the last line. Letting $R \to \infty$, one sees that

$$\int_0^T \int_M \rho u^2 e^{-2r_\rho} e^{-f} dx dt < \infty$$

and (2.4) follows.

Now the first estimate of the corollary follows from (2.13) and (2.15), and the second from (2.9) and (2.15). $\square$

Finally, in the case $\lambda_1\,(\Delta_f) > 0$, obviously one may take $\rho = \lambda_1\,(\Delta_f)\,.$

Corollary 2.4. *Let $\left(M, g, e^{-f}dx\right)$ be a smooth metric measure space with $\lambda_1(\Delta_f) > 0$. Then the function $u(x,t)$ defined in (2.14) satisfies*

$$\int_0^\infty \int_{B(A,R+1)\setminus B(A,R)} u^2(x,t)\, e^{-f(x)} dx dt \le C\, e^{-2\sqrt{\lambda_1(\Delta_f)}R}\, \mathrm{V}_f(A),$$

where C depends only on $\lambda_1(\Delta_f)$. Furthermore, for all $0<\delta<1$,

$$\int_0^\infty \int_M u^2(x,t)\, e^{2\delta\sqrt{\lambda_1(\Delta_f)}r(x,A)} e^{-f(x)}\, dx\, dt \le \frac{1}{2(1-\delta^2)\lambda_1(\Delta_f)} \mathrm{V}_f(A).$$

Proof. Since $ds_\rho^2 = \lambda_1(\Delta_f)\, ds^2$, we get $r_\rho(x,y) = \sqrt{\lambda_1(\Delta_f)}\, r(x,y)$ and $B_\rho(A,R) = B\left(A, \frac{R}{\sqrt{\lambda_1(\Delta_f)}}\right)$. The result now follows from Corollary 2.3. The constant C can be taken as $C = c \cdot \max\left\{1, \lambda_1(\Delta_f)^{-1}\right\}$, where c is an absolute constant. □

For later use, we also record the elliptic version of Theorem 2.2. Fix $p \in M$ and denote $r_\rho(x) = r_\rho(x,p)$.

Theorem 2.5. *Let $\left(M, g, e^{-f}dx\right)$ be a complete smooth metric measure space with property (P_ρ). Let $u \ge 0$ be an f-subharmonic function defined on $M\setminus B_\rho(R_0)$, for some $R_0 > 0$. Assume that*

$$\int_{B_\rho(2R)\setminus B_\rho(R)} \rho u^2 e^{-2r_\rho} e^{-f} = o(R) \tag{2.17}$$

as $R\to\infty$. Then there exists an absolute constant $C>0$ so that

$$\int_{B_\rho(R+1)\setminus B_\rho(R)} \rho u^2 e^{-f} \le C\left(1+\frac{1}{a}\right)^2 e^{-2R} \int_{B_\rho(R_0+a)\setminus B_\rho(R_0)} \rho u^2 e^{2r_\rho} e^{-f},$$

for any $a>0$ and any $R \ge 2(R_0+a)$.

Proof. The proof follows as in [15, 16], the only modification being that we use the weighted measure instead of the Riemannian one. For completeness, let us sketch the argument. Following the proof of (2.6), we get that

$$\begin{aligned} \int_M \rho\,\phi^2\, u^2 e^{2h} e^{-f} &\le \int_M |\nabla(\phi\, u\, e^h)|^2 e^{-f} \\ &= \int_M |\nabla(\phi\, e^h)|^2\, u^2 e^{-f} - \int_M \phi^2\, u\,(\Delta_f u)\, e^{2h} e^{-f} \\ &\le \int_M |\nabla\phi|^2\, u^2 e^{2h} e^{-f} + 2\int_M \phi\,\langle\nabla\phi, \nabla h\rangle u^2 e^{2h} e^{-f} \\ &+ \int_M \phi^2\, |\nabla h|^2\, u^2\, e^{2h} e^{-f}, \end{aligned} \tag{2.18}$$

for any function $\phi \ge 0$ with compact support in $M\setminus B_\rho(R_0)$, and any function h. We first choose

$$\phi(r_\rho(x)) = \begin{cases} 1 & \text{on } B_\rho(R)\setminus B_\rho(R+a) \\ R^{-1}(2R - r_\rho(x)) & \text{on } B_\rho(2R)\setminus B_\rho(R) \\ a^{-1}(r_\rho(x) - R_0) & \text{on } B_\rho(R_0+a)\setminus B_\rho(R_0) \\ 0 & \text{on } (M\setminus B_\rho(2R)) \cup B_\rho(R_0) \end{cases} \tag{2.19}$$

and

$$h\left(r_\rho(x)\right)=\begin{cases}\delta r_\rho(x) & \text{on } B_\rho\left(\left(1+\delta\right)^{-1}K\right)\\ K-r_\rho(x) & \text{on } M\backslash B_\rho\left(\left(1+\delta\right)^{-1}K\right)\end{cases}$$

for some fixed $K>1$.

As in Theorem 2.2 we obtain from (2.18) that when $R\geq\left(1+\delta\right)^{-1}K\geq R_0+a$,

$$\begin{aligned}&\left(1-\delta^2\right)\int_{B_\rho((1+\delta)^{-1}K)\backslash B_\rho(R_0+a)}\rho\, u^2e^{2h}e^{-f}\\ \leq\ &\left(R^{-2}+2R^{-1}\right)\int_{B_\rho(2R)\backslash B_\rho(R)}\rho\, u^2e^{2h}e^{-f}\\ &+\left(a^{-2}+2\delta a^{-1}\right)\int_{B_\rho(R_0+a)\backslash B_\rho(R_0)}\rho\, u^2e^{2h}e^{-f}.\end{aligned}$$

Note that the first term on the right side converges to zero as $R\to\infty$ by (2.17). This implies after letting $K\to\infty$,

$$\left(1-\delta^2\right)\int_{M\backslash B_\rho(R_0+a)}\rho\, u^2e^{2h}e^{-f}\leq\left(a^{-2}+2\delta a^{-1}\right)\int_{B_\rho(R_0+a)\backslash B_\rho(R_0)}\rho\, u^2e^{2h}e^{-f}$$

Now in (2.18) we set $h=r_\rho$ and for $R_0<R_1<R$ take

$$\phi(x)=\begin{cases}\left(R_1-R_0\right)^{-1}r_\rho(x) & \text{on } B_\rho(R_1)\backslash B_\rho(R_0)\\ \left(R-R_1\right)^{-1}\left(R-r_\rho(x)\right) & \text{on } B_\rho(R)\backslash B_\rho(R_1)\\ 0 & \text{on } \left(M\backslash B_\rho(R)\right)\cup B_\rho(R_0)\end{cases}$$

to obtain as in (2.10)

$$\begin{aligned}&\frac{2\tau}{\left(R-R_1\right)^2}\int_{B_\rho(R-\tau)\backslash B_\rho(R_1)}\rho u^2\, e^{2r_\rho}e^{-f}\\ \leq\ &\left(\frac{2}{R_1-R_0}+\frac{1}{\left(R_1-R_0\right)^2}\right)\int_{B_\rho(R_1)\backslash B_\rho(R_0)}\rho\, u^2\, e^{2r_\rho}e^{-f}\\ +\ &\frac{1}{(R-R_1)^2}\int_{B_\rho(R)\backslash B_\rho(R_1)}\rho\, u^2\, e^{2r_\rho}e^{-f}.\end{aligned}$$

The result then follows by a similar iteration argument as Theorem 2.2 by choosing appropriate R_1 and τ. □

3. Green's function estimates

In this section, we develop estimates for the Green's function. The results, while of independent interest, will be applied to solve the Poisson equation in next section. We continue to assume that $\left(M,g,e^{-f}dx\right)$ is a smooth metric measure space with positive bottom spectrum $\lambda_1\left(\Delta_f\right)>0$. It is known (see e.g. Theorem 13.4 in [12]) that M must be f-nonparabolic, that is, there exists a positive Green's function for the weighted Laplacian Δ_f. Denote by $G(x,y)$ the minimal positive Green's function. As noted in [14], Theorem 2.5 implies the following estimate.

$$\begin{aligned}(3.1)\qquad &\int_{B(x,R+1)\backslash B(x,R)} G^2\,(x,y)\,e^{-f(y)}dy \\ \leq\;& Ce^{-2\sqrt{\lambda_1(\Delta_f)}R}\int_{B(x,2)\backslash B(x,1)} G^2\,(x,y)\,e^{-f(y)}dy,\end{aligned}$$

where $C > 0$ depends only on $\lambda_1(\Delta_f)$. Indeed, by Theorem 2.5, the inequality (3.1) obviously holds for the Dirichlet Green's function $G_i(x,y)$ of $B(p,R_i)$ as function $u(y) = G_i(x,y)$ is f-harmonic on $M\backslash B(x,1)$ and has compact support. Since $G(x,y)$ is the uniform limit of $G_i(x,y)$ as $R_i \to \infty$, after passing to the limit, the inequality (3.1) holds for $G(x,y)$ as well.

Now using Theorem 2.2 we obtain an estimate of different flavor.

Theorem 3.1. *Let $\left(M, g, e^{-f}dx\right)$ be a complete smooth metric measure space with positive bottom spectrum $\lambda_1(\Delta_f) > 0$. Then the minimal positive Green's function $G(x,y)$ of Δ_f satisfies*

$$\begin{aligned}&\int_A\int_B G(x,y)\,e^{-f(x)}e^{-f(y)}dy\,dx \\ \leq\;& \frac{e^{\sqrt{\lambda_1(\Delta_f)}}}{\lambda_1(\Delta_f)}\sqrt{\mathrm{V}_f(A)}\sqrt{\mathrm{V}_f(B)}\,(1+r(A,B))\,e^{-\sqrt{\lambda_1(\Delta_f)}\,r(A,B)}\end{aligned}$$

for any bounded domains A and B of M.

Proof. By Corollary 2.4, for any $0 < \delta < 1$,

$$\begin{aligned}&\int_0^\infty\int_M\left(\int_A H(x,y,t)\,e^{-f(y)}dy\right)^2 e^{2\delta\sqrt{\lambda_1(\Delta_f)}\,r(x,A)}e^{-f(x)}\,dx\,dt \\ \leq\;& \frac{1}{2(1-\delta^2)\lambda_1(\Delta_f)}\mathrm{V}_f(A).\end{aligned}$$

Of course, the same inequality holds for domain B as well. Therefore, noting that $r(A,B) \leq r(x,A) + r(x,B)$, we get

$$\begin{aligned}&\int_0^\infty\int_M\left(\int_A H(x,y,t)e^{-f(y)}\,dy\right) \\ &\times\left(\int_B H(x,z,t)\,e^{-f(z)}dz\right)e^{\delta\sqrt{\lambda_1(\Delta_f)}\,r(A,B)}e^{-f(x)}\,dx\,dt \\ \leq\;& \left(\int_0^\infty\int_M\left(\int_A H(x,y,t)e^{-f(y)}\,dy\right)^2 e^{2\delta\sqrt{\lambda_1(\Delta_f)}r(x,A)}e^{-f(x)}\,dx\,dt\right)^{1/2} \\ &\times\left(\int_0^\infty\int_M\left(\int_B H(x,z,t)\,e^{-f(z)}dz\right)^2 e^{2\delta\sqrt{\lambda_1(\Delta_f)}\,r(x,B)}e^{-f(x)}\,dx\,dt\right)^{1/2} \\ \leq\;& \frac{1}{2(1-\delta^2)\lambda_1(\Delta_f)}\sqrt{\mathrm{V}_f(A)}\sqrt{\mathrm{V}_f(B)}.\end{aligned}$$

Recall that by the semi-group property (e.g. Theorem 7.13 in [12]) we have

$$H(y,z,2t) = \int_M H(x,y,t)\,H(x,z,t)e^{-f(x)}dx.$$

Therefore, since

$$G(x,y)=\int_0^\infty H(x,y,t)\,dt,$$

we have that

$$\begin{aligned}
&\frac{1}{2}\int_A\int_B G(y,z)\,e^{-f(y)}e^{-f(z)}e^{\delta\sqrt{\lambda_1(\Delta_f)}\,r(A,B)}dz\,dy\\
=\;&\int_A\int_B\int_0^\infty H(y,z,2t)\,e^{-f(y)}e^{-f(z)}e^{\delta\sqrt{\lambda_1(\Delta_f)}\,r(A,B)}dt\,dz\,dy\\
=\;&\int_A\int_B\int_0^\infty\int_M H(x,y,t)\,H(x,z,t)e^{-f(x)}e^{-f(y)}e^{-f(z)}\,e^{\delta\sqrt{\lambda_1(\Delta_f)}\,r(A,B)}dx\,dt\,dz\,dy\\
=\;&\int_0^\infty\int_M\left(\int_A H(x,y,t)e^{-f(y)}dy\right)\\
&\times\left(\int_B H(x,z,t)\,e^{-f(z)}dz\right)\,e^{\delta\sqrt{\lambda_1(\Delta_f)}\,r(A,B)}e^{-f(x)}\,dx\,dt.
\end{aligned}$$

Combining with the previous inequality, we conclude

$$\begin{aligned}
&\int_A\int_B G(y,z)e^{-f(y)}e^{-f(z)}\,dz\,dy\\
\le\;&\frac{1}{(1-\delta^2)\,\lambda_1(\Delta_f)}\sqrt{\mathrm{V}_f(A)}\sqrt{\mathrm{V}_f(B)}e^{-\delta\sqrt{\lambda_1(\Delta_f)}\,r(A,B)}.
\end{aligned}\tag{3.2}$$

Clearly, this proves the theorem if $r(A,B)\le 1$. When $r(A,B)>1$, the theorem follows by setting

$$\delta:=1-\frac{1}{r(A,B)}$$

in (3.2). □

Before continuing, let us first recall some results from [25] concerning smooth metric measure space $\left(M,g,e^{-f}dx\right)$. Suppose that the associated Bakry-Emery Ricci curvature is bounded below by

$$\mathrm{Ric}_f\ge-(n-1)\,K\tag{3.3}$$

and weight f satisfies

$$\sup_{y\in B(x,1)}|f(x)-f(y)|\le a.\tag{3.4}$$

Then Sobolev inequality of the following form holds.

$$\left(\int_{B(x,1)}\phi^{\frac{2n}{n-2}}e^{-f}\right)^{\frac{n-2}{n}}\le\frac{c(n,K,a)}{\mathrm{V}_f(x,1)^{\frac{2}{n}}}\left(\int_{B(x,1)}|\nabla\phi|^2\,e^{-f}+\int_{B(x,1)}\phi^2e^{-f}\right)\tag{3.5}$$

for any ϕ with support in $B(x,1)$. Also, volume comparison of the form

$$\frac{\mathrm{V}_f\left(x,r_2\right)}{\mathrm{V}_f\left(x,r_1\right)}\leq c\left(a\right)\left(\frac{\int_0^{r_2}\sinh^{n-1+2a}\left(\sqrt{K}t\right)dt}{\int_0^{r_1}\sinh^{n-1+2a}\left(\sqrt{K}t\right)dt}\right) \tag{3.6}$$

is valid for any $0<r_1<r_2<1$. In view of (3.4), both inequalities hold with respect to the Riemannian volume as well. One also has the gradient estimate of the form

$$\left|\nabla \ln u\right|\leq C\left(n,K,a\right)\ \text{ on } B\left(x,\frac{1}{20}\right) \tag{3.7}$$

for any $u>0$ with $\Delta_f u=0$ in $B\left(x,\frac{1}{10}\right)$.

As a consequence of Theorem 3.1, we have the following.

Proposition 3.2. *Let $\left(M^n,\, g,\, e^{-f}dx\right)$ be a smooth metric measure space satisfying (3.3) and (3.4). If $\lambda_1\left(\Delta_f\right)>0$, then the minimal positive Green's function $G(x,y)$ of Δ_f satisfies*

$$\int_{B(x,1)} G(x,y)\,e^{-f(y)}dy\leq C,$$

for any $x\in M$, with C depending only on n, K, a and $\lambda_1\left(\Delta_f\right)$.

Proof. Applying Theorem 3.1 to $A=B=B\left(x,1\right)$ we get

$$\int_{B(x,1)}\int_{B(x,1)} G(y,z)\,e^{-f(y)}e^{-f(z)}dy\,dz\leq C\,\mathrm{V}_f\left(x,1\right).$$

Note that the function

$$u\left(z\right):=\int_{B(x,1)} G(y,z)\,e^{-f(y)}dy$$

satisfies $\Delta_f u=-1$ on $B\left(x,1\right)$ and

$$\int_{B(x,1)} u\left(z\right)e^{-f(z)}dz\leq C\,\mathrm{V}_f\left(x,1\right).$$

Since $u>0$, we have

$$\Delta_f\left(u+1\right)\geq -\left(u+1\right)\ \text{ on } B\left(x,1\right).$$

Using (3.5) and (3.6), the standard DeGiorgi-Nash-Moser iteration implies that

$$\begin{aligned} u\left(x\right)+1 &\leq C\mathrm{V}_f^{-1}\left(x,1\right)\int_{B(x,1)}\left(u\left(y\right)+1\right)e^{-f(y)}dy \\ &\leq C. \end{aligned}$$

This proves the proposition. ∎

For the following results, we will work with the level sets of the Green's function. For $x\in M$ fixed, any $0<\alpha<\beta$ and any $t>0$ denote with

$$\begin{aligned} L_x\left(\alpha,\beta\right) &:= \left\{y\in M:\alpha<G\left(x,y\right)<\beta\right\} \\ l_x\left(t\right) &:= \left\{y\in M:G\left(x,y\right)=t\right\}. \end{aligned} \tag{3.8}$$

Lemma 3.3. *Let $\left(M,\, g,\, e^{-f}dx\right)$ be a smooth metric measure space satisfying (3.3) and (3.4). If $\lambda_1\left(\Delta_f\right)>0$, then for any $t>0$ we have*

$$\int_{l_x(t)} |\nabla G|\left(x,\xi\right)e^{-f(\xi)}dA\left(\xi\right)=1, \tag{3.9}$$

where dA is the Riemannian area form of $l_x\left(t\right)$.

Proof. We first note that this integral is finite and independent of $t>0$ by following the argument in [15]. Indeed, for any ϕ with compact support in $M\backslash B\left(x,1\right)$ we have by the co-area formula that

$$\int_M \phi\left|\nabla G\right|^2 e^{-f}=\int_0^\infty\int_{l_x(t)}\phi\left|\nabla G\right|e^{-f}dAdt. \tag{3.10}$$

Since $G\left(x,\cdot\right)$ is f-harmonic on $M\backslash B\left(x,1\right)$, by (3.1) and the gradient estimate (3.7) we get

$$\int_{M\backslash B(x,1)}\left|\nabla G\right|^2\left(x,y\right)e^{-f(y)}dy<\infty.$$

Hence, (3.10) implies that

$$\int_{l_x(t)}\left|\nabla G\right|\left(x,\xi\right)e^{-f(\xi)}dA\left(\xi\right)<\infty$$

for almost all $t>0$. Now for $r\left(y\right)=r\left(x,y\right)$ let

$$\phi\left(y\right)=\begin{cases}1 & \text{on}\quad B(x,R)\\ R^{-1}\left(2R-r(y)\right) & \text{on}\quad B(x,2R)\backslash B(x,R)\\ 0 & \text{on}\quad M\backslash B\left(x,2R\right).\end{cases}$$

Since $G\left(x,\cdot\right)$ is f-harmonic on $L_x\left(\alpha,\beta\right)$, integrating by parts yields that

$$\begin{aligned}0 &= \int_{L_x(\alpha,\beta)}\phi\left(y\right)\Delta_f G\left(x,y\right)e^{-f(y)}dy\\ &= -\int_{L_x(\alpha,\beta)}\left\langle\nabla\phi\left(y\right),\nabla G\left(x,y\right)\right\rangle e^{-f(y)}dy\\ &\quad+\int_{l_x(\beta)}\phi\left(\xi\right)\frac{\partial G}{\partial\nu}\left(x,\xi\right)e^{-f(\xi)}dA\left(\xi\right)\\ &\quad-\int_{l_x(\alpha)}\phi\left(\xi\right)\frac{\partial G}{\partial\nu}\left(x,\xi\right)e^{-f(\xi)}dA\left(\xi\right).\end{aligned}$$

Note that the normal vector of $l_x\left(t\right)$ is given by $\nu=\left|\nabla G\right|^{-1}\left(x,\cdot\right)\nabla G\left(x,\cdot\right)$. Therefore,

$$\left|\int_{l_x(\beta)}\phi\left(\xi\right)\left|\nabla G\right|\left(x,\xi\right)e^{-f(\xi)}dA\left(\xi\right)-\int_{l_x(\alpha)}\phi\left(\xi\right)\left|\nabla G\right|\left(x,\xi\right)e^{-f(\xi)}dA\left(\xi\right)\right| \tag{3.11}$$

$$\begin{aligned}&\leq \int_{L_x(\alpha,\beta)}\left|\nabla\phi\right|\left(y\right)\left|\nabla G\right|\left(x,y\right)e^{-f(y)}dy\\ &= \frac{1}{R}\int_{(B(x,2R)\backslash B(x,R))\cap L_x(\alpha,\beta)}\left|\nabla G\right|\left(x,y\right)e^{-f(y)}dy.\end{aligned}$$

On the other hand, by the gradient estimate (3.7) and the integral estimate (3.1), we get

$$\begin{aligned}
&\int_{(B(x,2R)\setminus B(x,R))\cap L_x(\alpha,\beta)} |\nabla G|(x,y)\, e^{-f(y)}dy \\
\leq\ & c\int_{(B(x,2R)\setminus B(x,R))\cap L_x(\alpha,\beta)} G(x,y)\, e^{-f(y)}dy \\
\leq\ & \frac{c}{\alpha}\int_{(B(x,2R)\setminus B(x,R))\cap L_x(\alpha,\beta)} G^2(x,y)\, e^{-f(y)}dy \\
\leq\ & \frac{c}{\alpha}e^{-2\sqrt{\lambda_1(\Delta_f)}R}\int_{B(x,2)\setminus B(x,1)} G^2(x,y)\, e^{-f(y)}dy.
\end{aligned}$$

Clearly, taking $R\to\infty$ in (3.11), one concludes that

$$\int_{l_x(\beta)} |\nabla G|(x,\xi)\, e^{-f(\xi)}dA(\xi) = \int_{l_x(\alpha)} |\nabla G|(x,\xi)\, e^{-f(\xi)}dA(\xi)$$

for any $0<\alpha<\beta$.

This proves that indeed $\int_{l_x(t)} |\nabla G|(x,\xi)\, e^{-f(\xi)}dA(\xi)$ is independent of $t>0$. It remains to compute its value. Here we use the fact that $G(x,y)$ is the Green's function. In particular,

$$\int_M \langle\nabla\phi(y), \nabla G(x,y)\rangle\, e^{-f(y)}dy = \phi(x), \tag{3.12}$$

for any compactly supported function ϕ. Observe that by the construction, $G(x,\cdot)$ is bounded on $M\setminus B(x,1)$ and has a pole at x. Consequently, $l_x(T)$ is a compact hypersurface in M for $T>0$ sufficiently large. For fixed $s>0$ we take $T>0$ large enough so that $L_x(T,\infty)\subset B(x,s)$. By the Stokes theorem, we have that

$$\begin{aligned}
0 &= \int_{B(x,s)\setminus L_x(T,\infty)} \Delta_f G(x,y)\, e^{-f(y)}dy \\
&= \int_{\partial B(x,s)} \frac{\partial G}{\partial r}(x,\xi)\, e^{-f(\xi)}dA(\xi) \\
&\quad + \int_{l_x(T)} |\nabla G|(x,\xi)\, e^{-f(\xi)}dA(\xi).
\end{aligned}$$

In other words,

$$\int_{l_x(t)} |\nabla G|(x,\xi)\, e^{-f(\xi)}dA(\xi) = -\int_{\partial B(x,s)} \frac{\partial G}{\partial r}(x,\xi)\, e^{-f(\xi)}dA(\xi) \tag{3.13}$$

for any $s,t>0$.

We now set

$$\phi(y) = \begin{cases} 1-r(y) & \text{on} \quad B(x,1) \\ 0 & \text{on} \quad M\setminus B(x,1) \end{cases}$$

in (3.12) and get that

$$
\begin{aligned}
1 &= \phi(x) \\
&= \int_M \langle \nabla\phi(y), \nabla G(x,y)\rangle e^{-f(y)} dy \\
&= -\int_0^1 \int_{\partial B(x,s)} \frac{\partial G}{\partial r}(x,\xi) e^{-f(\xi)} dA(\xi) \\
&= \int_{l_x(t)} |\nabla G|(x,\xi) e^{-f(\xi)} dA(\xi),
\end{aligned}
$$

where in the last line we have used (3.13). This proves the lemma. □

In the following, we will use c and C to denote constants depending only on n, K, a and $\lambda_1(\Delta_f)$. These constants may change from line to line.

Proposition 3.4. *Let $\left(M, g, e^{-f}dx\right)$ be a smooth metric measure space satisfying (3.3) and (3.4). If $\lambda_1(\Delta_f) > 0$, then*
(i) for any $r > 0$,

$$
\sup_{y\in B(p,r)\setminus B(x,1)} G(x,y) \le e^{c\,r} \inf_{y\in B(p,r)\setminus B(x,1)} G(x,y);
$$

(ii) for $x \in M$ and $0 < \alpha < \beta$,

$$
\int_{L_x(\alpha,\beta)} G(x,y) e^{-f(y)} dy \le c\left(1 + \ln\frac{\beta}{\alpha}\right).
$$

Proof. For (i), we first show that

$$
G(x,y) \le c\,G(x,z) \tag{3.14}
$$

for $y, z \in \partial B(x,1)$.

For $y \in \partial B(x,1)$, since the function $G(x,\cdot)$ is f-harmonic on $B(y,1)$, by (3.7),

$$
G(x,y) \le c\,G(x,z)
$$

for $z \in B\left(y, \frac{1}{5}\right)$. Hence, it suffices to prove (3.14) for y and z satisfying $r(y,z) \ge \frac{1}{5}$. Let $\gamma(t)$ and $\sigma(t)$ be minimizing geodesics from x to y and from x to z respectively, $t \in [0,1]$. We have that $r(y,\sigma) \ge \frac{1}{10}$ and $r(z,\gamma) \ge \frac{1}{10}$. Indeed, suppose that there exists $t_0 \in (0,1)$ such that $r(y,\sigma(t_0)) < \frac{1}{10}$. Since $r(x,y) = r(x,z) = 1$ and $r(y,z) \ge \frac{1}{5}$, the triangle inequality implies

$$
\begin{aligned}
r(z,\sigma(t_0)) &\ge r(y,z) - r(y,\sigma(t_0)) \\
&> \frac{1}{10}
\end{aligned}
$$

and

$$
\begin{aligned}
r(x,\sigma(t_0)) &\ge r(x,y) - r(y,\sigma(t_0)) \\
&> \frac{9}{10}.
\end{aligned}
$$

Adding up these two inequalities we get

$$
\begin{aligned}
r(x,z) &= r(x,\sigma(t_0)) + r(\sigma(t_0),z) \\
&> 1.
\end{aligned}
$$

This contradiction shows that $r(y,\sigma) \geq \frac{1}{10}$ as claimed. The proof of $r(z,\gamma) \geq \frac{1}{10}$ is similar.

Consequently, $G(y,\cdot)$ is f-harmonic on $B\left(\sigma(t), \frac{1}{10}\right)$ for all $t \in [0,1]$. It follows from (3.7) that

$$G(y,x) \leq c\, G(y,z). \tag{3.15}$$

Similarly, as $r(z,\gamma) \geq \frac{1}{10}$, $G(z,\cdot)$ is f-harmonic on $B\left(\gamma(t), \frac{1}{10}\right)$ for $t \in [0,1]$. By (3.7) we get

$$G(z,y) \leq c\, G(z,x). \tag{3.16}$$

Combining (3.15) with (3.16) we conclude that $G(x,y) \leq c\, G(x,z)$ as claimed in (3.14).

Now for given $r > 0$, suppose first that $r > \frac{1}{2} r(p,x)$. For $y, z \in B(p,r) \setminus B(x,1)$, let $\tau(t)$ and $\eta(t)$ be minimizing normal geodesics from x to y and from x to z respectively. We denote $y_1 := \tau(1)$ and $z_1 := \eta(1)$. Since $y_1, z_1 \in \partial B(x,1)$, by (3.14) we have

$$G(x,y_1) \leq c\, G(x,z_1). \tag{3.17}$$

On the other hand, the function $G(x,\cdot)$ is f-harmonic on $B\left(\tau(t), \frac{1}{10}\right)$ for all $t \geq 1$. Integrating (3.7) along $\tau(t)$ we obtain

$$\begin{aligned} G(x,y) &\leq e^{c\, r(y,y_1)}\, G(x,y_1) \\ &\leq e^{c\, r(x,y)}\, G(x,y_1). \end{aligned}$$

Similarly, we have

$$G(x,z_1) \leq e^{c\, r(x,z)}\, G(x,z).$$

In view of (3.17) we conclude that

$$G(x,y) \leq e^{c(r(x,y)+r(x,z))}\, G(x,z). \tag{3.18}$$

Since $r(p,x) < 2r$ and $y, z \in B(p,r)$, by the triangle inequality, $r(x,y) < 3r$ and $r(x,z) < 3r$. Hence, (3.18) implies that

$$G(x,y) \leq e^{cr}\, G(x,z)$$

for $y, z \in B(p,r) \setminus B(x,1)$. This proves (i) in the case that $r > \frac{1}{2} r(p,x)$.

Suppose now that $r \leq \frac{1}{2} r(p,x)$. We may assume that

$$B(p,r) \setminus B(x,1) \neq \varnothing.$$

For $q \in B(p,r) \setminus B(x,1)$, we have $r(q,x) \geq 1$ and

$$\begin{aligned} r(p,q) &< r \\ &\leq \frac{1}{2} r(p,x). \end{aligned}$$

The triangle inequality implies

$$\begin{aligned} r(p,x) &\geq r(q,x) - r(p,q) \\ &\geq 1 - \frac{1}{2} r(p,x). \end{aligned}$$

This shows that

$$r(p,x) \geq \frac{2}{3}. \tag{3.19}$$

We claim that

$$x \notin B\left(p, r+\frac{1}{10}\right). \tag{3.20}$$

Otherwise, $r(p,x) < r + \frac{1}{10}$. Since $r(p,x) \geq 2r$, it follows that $r < \frac{1}{10}$ and $r(p,x) < \frac{1}{5}$. This contradicts with (3.19). So (3.20) holds and $G(x,\cdot)$ is f-harmonic on $B\left(p, r+\frac{1}{10}\right)$. It is then easy to see from (3.7) that

$$\sup_{y\in B(p,r)} G(x,y) \leq e^{cr} \inf_{y\in B(p,r)} G(x,y).$$

This proves (i) in the remaining case that $r \leq \frac{1}{2} r(p,x)$.

To prove part (ii), let $\phi = \chi\psi$ be a cut-off function with compact support on M, where

$$\chi(y) := \begin{cases} \ln(e\beta) - \ln G(x,y) & \text{on } L_x(\beta, e\beta) \\ 1 & \text{on } L_x(\alpha,\beta) \\ \ln G(x,y) - \ln\left(e^{-1}\alpha\right) & \text{on } L_x\left(e^{-1}\alpha, \alpha\right) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi(y) = \begin{cases} 1 & \text{on } B(x,R) \\ R+1-r(x,y) & \text{on } B(x,R+1)\setminus B(x,R) \\ 0 & \text{on } M\setminus B(x,R+1) \end{cases}$$

Obviously,

$$\begin{aligned} & \lambda_1(\Delta_f) \int_M \phi^2(y)\, G(x,y)\, e^{-f(y)} dy \\ \leq\ & \int_M \left|\nabla\left(\phi G^{\frac{1}{2}}\right)\right|^2 (x,y)\, e^{-f(y)} dy \\ \leq\ & \frac{1}{2}\int_M \phi^2(y) \left|\nabla G\right|^2 (x,y)\, G^{-1}(x,y)\, e^{-f(y)} dy \\ & +2\int_M G(x,y) \left|\nabla \phi\right|^2 (y)\, e^{-f(y)} dy. \end{aligned} \tag{3.21}$$

To compute the integrals on the right side of (3.21) we use the co-area formula and Lemma 3.3.

$$\begin{aligned} & \int_{L_x(t_0,t_1)} \left|\nabla G\right|^2 (x,y)\, G^{-1}(x,y)\, e^{-f(y)} dy \\ =\ & \int_{t_0}^{t_1} t^{-1} \left(\int_{l_x(t)} \left|\nabla G\right| (x,\xi)\, e^{-f(\xi)} dA(\xi)\right) dt \\ =\ & \ln\frac{t_1}{t_0}. \end{aligned} \tag{3.22}$$

This implies that

$$\begin{aligned}(3.23) \qquad & \int_M \phi^2(y)\,|\nabla G|^2(x,y)\,G^{-1}(x,y)\,e^{-f(y)}dy \\ \leq & \int_{L_x(e^{-1}\alpha,e\beta)} |\nabla G|^2(x,y)\,G^{-1}(x,y)\,e^{-f(y)}dy \\ \leq & \ 2+\ln\frac{\beta}{\alpha}.\end{aligned}$$

To estimate the second term of the right hand side of (3.21), note that

$$\begin{aligned}(3.24) \qquad & \int_M G(x,y)\,|\nabla\phi|^2(y)\,e^{-f(y)}dy \\ \leq & \ 2\int_M G(x,y)\,|\nabla\psi|^2(y)\,\chi^2(y)\,e^{-f(y)}dy \\ & +2\int_M G(x,y)\,|\nabla\chi|^2(y)\,\psi^2(y)\,e^{-f(y)}dy.\end{aligned}$$

Since $G>e^{-1}\alpha$ on the support of χ, it follows that

$$\begin{aligned}(3.25) \qquad & \int_M G(x,y)\,|\nabla\psi|^2(y)\,\chi^2(y)\,e^{-f(y)}dy \\ \leq & \ \frac{1}{\alpha}\int_{B(x,R+1)\setminus B(x,R)} G^2(x,y)\,e^{-f(y)}dy \\ \leq & \ \frac{c}{\alpha}e^{-2\sqrt{\lambda_1(\Delta_f)}R}\int_{B(x,2)\setminus B(x,1)} G^2(x,y)\,e^{-f(y)}dy,\end{aligned}$$

where in the last line we have used (3.1). Furthermore, (3.22) yields

$$\begin{aligned}& \int_M G(x,y)\,|\nabla\chi|^2(y)\,\psi^2(y)\,e^{-f(y)}dy \\ \leq & \int_{L_x(\beta,e\beta)} |\nabla G|^2(x,y)\,G^{-1}(x,y)\,e^{-f(y)}dy \\ & +\int_{L_x(e^{-1}\alpha,\alpha)} |\nabla G|^2(x,y)\,G^{-1}(x,y)\,e^{-f(y)}dy \\ = & \ 2.\end{aligned}$$

Combining this with (3.25) and (3.24) we obtain

$$\begin{aligned}& \int_M G(x,y)\,|\nabla\phi|^2(y)\,e^{-f(y)}dy \\ \leq & \ \frac{c}{\alpha}e^{-2\sqrt{\lambda_1(\Delta_f)}R}\int_{B(x,2)\setminus B(x,1)} G^2(x,y)\,e^{-f(y)}dy+4.\end{aligned}$$

Together with (3.23) and (3.21), this implies that

$$\begin{aligned} & \lambda_1(\Delta_f)\int_{L_x(\alpha,\beta)\cap B(x,R)} G(x,y)\, e^{-f(y)}dy \\ \leq\ & \lambda_1(\Delta_f)\int_M \phi^2(y)\, G(x,y)\, e^{-f(y)}dy \\ \leq\ & \frac{c}{\alpha} e^{-2\sqrt{\lambda_1(\Delta_f)}R}\int_{B(x,2)\setminus B(x,1)} G^2(x,y)\, e^{-f(y)}dy \\ & +\frac{1}{2}\ln\frac{\beta}{\alpha}+c. \end{aligned}$$

Letting $R\to\infty$ we conclude that

$$\int_{L_x(\alpha,\beta)} G(x,y)\, e^{-f(y)}dy \leq c\left(1+\ln\frac{\beta}{\alpha}\right).$$

So the proposition is proved. □

We now come to the following crucial estimate.

Theorem 3.5. *Let $\left(M, g, e^{-f}dx\right)$ be a smooth metric measure space satisfying (3.3) and (3.4). If $\lambda_1(\Delta_f)>0$, then for any $p, x\in M$ and $r>0$,*

$$\int_{B(p,r)} G(x,y)\, e^{-f(y)}dy \leq C\ (1+r)$$

for some constant C depending only on n, K, a and $\lambda_1(\Delta_f)$.

Proof. We first prove that

$$\int_{B(p,r)\setminus B(x,1)} G(x,y)\, e^{-f(y)}dy \leq C\,(1+r)\,. \tag{3.26}$$

Let

$$\alpha := \inf_{y\in B(p,r)\setminus B(x,1)} G(x,y) \ \text{ and } \beta := \sup_{y\in B(p,r)\setminus B(x,1)} G(x,y)\,.$$

It follows from part (ii) of Proposition 3.4 that

$$\begin{aligned} \int_{B(p,r)\setminus B(x,1)} G(x,y)\, e^{-f(y)}dy &\leq \int_{L_x(\alpha,\beta)} G(x,y)\, e^{-f(y)}dy \\ &\leq c\left(\ln\frac{\beta}{\alpha}+1\right). \end{aligned}$$

However, part (i) of Proposition 3.4 implies that

$$\beta \leq e^{c\,r}\alpha.$$

Therefore, (3.26) follows. In view of Proposition 3.2, one concludes that

$$\int_{B(p,r)} G(x,y)\, e^{-f(y)}dy \leq C\ (1+r)$$

for any $r>0$. This proves the theorem. □

Let us note that Theorem 3.5 is sharp for any manifold with positive spectrum. Indeed, according to Lemma 3.3,

$$\int_{\partial B(p,s)} \frac{\partial G}{\partial r}(p,\xi)\, e^{-f(\xi)} dA(\xi) = -1$$

for any $s > 0$. So

$$\begin{aligned} R-1 &= -\int_1^R \int_{\partial B(p,s)} \frac{\partial G}{\partial r}(p,\xi)\, e^{-f(\xi)} dA(\xi)\, ds \\ &= -\int_{B(p,R)\setminus B(p,1)} \frac{\partial G}{\partial r}(p,y)\, e^{-f(y)} dy \\ &\leq \int_{B(p,R)\setminus B(p,1)} |\nabla G|(p,y)\, e^{-f(y)} dy. \end{aligned}$$

Now the gradient estimate (3.7) implies that

$$\int_{B(p,R)} G(p,y)\, e^{-f(y)} dy \geq c^{-1} R$$

for all $R \geq 2$. In conclusion, the linear upper bound obtained in Theorem 3.5 is optimal.

Finally, we point out that

$$\int_{\partial B(p,t)} G(x,y)\, e^{-f(y)} dy \leq c \tag{3.27}$$

for all $x \in M$ and $0 < t \leq \frac{1}{2}$, where c is a constant depending on n, K, a, $\lambda_1(\Delta_f)$ and possibly the geometry of $B(p,1)$. Indeed, if $x \in B(p,1)$ then (3.27) is clearly true as c is allowed to depend on the geometry of $B(p,1)$. In the case of $r(p,x) \geq 1$, since $G(x,\cdot)$ is f-harmonic on $B(p,1)$, by (3.7) we have

$$\sup_{y\in B\left(p,\frac{1}{2}\right)} G(x,y) \leq c \inf_{y\in B\left(p,\frac{1}{2}\right)} G(x,y).$$

Note by Theorem 3.5,

$$\int_{B\left(p,\frac{1}{2}\right)} G(x,y)\, e^{-f(y)} dy \leq c.$$

It follows that $\inf_{y\in B\left(p,\frac{1}{2}\right)} G(x,y) \leq c$. Therefore, $\sup_{y\in B\left(p,\frac{1}{2}\right)} G(x,y) \leq c$ as well. It is then easy to see that (3.27) is indeed true.

Whether or not (3.27) is true for all $t > 0$ remains an open question.

4. Solving Poisson equation

In this section, we solve the Poisson equation. We continue to denote by $\left(M, g, e^{-f}dx\right)$ an n-dimensional smooth metric measure space satisfying the assumptions that

$$\lambda_1(\Delta_f) > 0, \tag{4.1}$$

the Bakry-Emery Ricci tensor is bounded below

$$\mathrm{Ric}_f \geq -(n-1)K, \tag{4.2}$$

for some $K \geq 0$, and the oscillation of f on any unit ball $B(x,1) \subset M$ is uniformly bounded above by some fixed constant

$$\sup_{y \in B(x,1)} |f(y) - f(x)| \leq a. \tag{4.3}$$

For simplicity, write $r(x) := r(p,x)$, where $p \in M$ is a fixed point. We use c, C and C_0 to denote positive constants depending on n, K, a, $\lambda_1(\Delta_f)$ and possibly the geometry of $B(p,1)$.

Lemma 4.1. *Let $\left(M, g, e^{-f}dx\right)$ be a smooth metric measure space satisfying (4.2) and (4.3). Then the weighted volume satisfies*

$$\mathrm{V}_f(p,R) \leq c\, R^{n+2a}\, e^{\left((n-1+2a)\sqrt{K}+a\right)R}$$

for all $R \geq 1$.

Proof. Let us denote the volume form in geodesic coordinates centered at p by

$$dV|_{\exp_p(r\xi)} = J(p,r,\xi)\, dr d\xi$$

for $r > 0$ and $\xi \in S_pM$, the unit tangent sphere at p. Let $\gamma(s)$ be a minimizing normal geodesic with $\gamma(0) = p$. Along γ, according to the Laplace comparison theorem [31] we have

$$\begin{aligned} m_f(r) \quad &\leq \quad (n-1)\sqrt{K}\coth\left(\sqrt{K}r\right) \\ &\quad + \frac{2K}{\sinh^2\left(\sqrt{K}r\right)} \int_0^r (f(s) - f(r))\cosh\left(2\sqrt{K}s\right) ds, \end{aligned} \tag{4.4}$$

where $m_f(r) := \frac{d}{dr}\ln J_f(p,r,\xi)$ and $f(t) := f(\gamma(t))$. Using (4.3) we have that $|f(s) - f(r)| \leq a(r - s + 1)$. It follows that

$$\begin{aligned} &\int_0^r (f(s) - f(r))\cosh\left(2\sqrt{K}s\right) ds \\ \leq \quad & \frac{a}{2\sqrt{K}} \int_0^r \sinh\left(2\sqrt{K}s\right) ds + \frac{a}{2\sqrt{K}} \sinh\left(2\sqrt{K}r\right) \\ = \quad & \frac{a}{2K}\sinh^2\left(\sqrt{K}r\right) + \frac{a}{2\sqrt{K}}\sinh\left(2\sqrt{K}r\right). \end{aligned}$$

Therefore, we get from (4.4) that

$$\begin{aligned} m_f(r) \quad &\leq \quad (n-1+2a)\sqrt{K}\coth\left(\sqrt{K}r\right) + a \\ &\leq \quad (n-1+2a)\left(\sqrt{K} + r^{-1}\right) + a. \end{aligned}$$

Thus, after integrating with respect to r,

$$J_f(p,r,\xi) \leq r^{n-1+2a} e^{\left((n-1+2a)\sqrt{K}+a\right)r} J_f(p,1,\xi).$$

Integrating in $\xi \in S_pM$ then shows that

$$\mathrm{A}_f(p,r) \leq c\, r^{n-1+2a}\, e^{\left((n-1+2a)\sqrt{K}+a\right)r}.$$

This proves the result by integrating from 0 to R once more. ☐

We note that the volume growth rate in Lemma 4.1 is achieved by the Euclidean and hyperbolic spaces, as well as by steady Ricci solitons [23].

As $B(x,1) \subset B(p, r(x)+1)$, an immediate consequence of Lemma 4.1 is that

$$\mathrm{V}_f(x,1) \leq c\,(1+r(x))^{n+2a}\; e^{\left((n-1+2a)\sqrt{K}+a\right)r(x)} \tag{4.5}$$

for all $x \in M$.

We are now ready to prove the main result of this paper. In the following, α_0 is an arbitrary but fixed constant with

$$0 < \alpha_0 < 1. \tag{4.6}$$

Theorem 4.2. *Let* $\left(M, g, e^{-f}dx\right)$ *be a smooth metric measure space satisfying (4.1), (4.2) and (4.3). Let* φ *be a smooth function satisfying*

$$|\varphi|(x) \leq \omega(r(x)),$$

where $\omega(t)$ *is a non-increasing function such that* $\int_0^\infty \omega(t)\,dt < \infty$. *Then the Poisson equation* $\Delta_f u = -\varphi$ *admits a bounded solution* u *on* M *with*

$$\sup_M |u| \leq c \int_0^\infty \omega(t)\,dt.$$

Furthermore, there exists $C>0$ *such that for all* $\alpha \in [\alpha_0, 1]$ *and* $x \in M$,

$$\begin{aligned} |u|(x) \leq\ & C\int_{\frac{3}{4}\alpha r(x)}^\infty \omega(t)\,dt \\ & + C(1+r(x))^{n+a}\; e^{-\sqrt{\lambda_1(\Delta_f)}r(x)}\,\mathrm{V}_f^{-\frac{1}{2}}(x,1)\int_0^{\alpha r(x)} \omega(t)\,e^{bt}dt, \end{aligned} \tag{4.7}$$

where

$$b := \sqrt{\lambda_1(\Delta_f)} + \frac{1}{2}\left(((n-1)+2a)\sqrt{K}+a\right).$$

The constant c *depends only on* $n, K, a, \lambda_1(\Delta_f)$ *and on* $B(p,1)$, *while* C *depends additionally on* α_0.

Proof. We first prove that

$$\int_M G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \leq c\int_0^\infty \omega(t)\,dt \tag{4.8}$$

for all $x \in M$. Note that

$$\begin{aligned} \int_{B(p,1)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy =& \int_{B\left(p,\frac{1}{2}\right)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\ &+ \int_{B(p,1)\setminus B\left(p,\frac{1}{2}\right)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy. \end{aligned}$$

By (3.27) and the co-area formula,

$$\begin{aligned}
&\int_{B\left(p,\frac{1}{2}\right)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\
\leq\;& \int_0^{\frac{1}{2}}\left(\int_{\partial B(p,t)} G(x,\xi)\,e^{-f(\xi)}dA(\xi)\right)\sup_{\partial B(p,t)}|\varphi|\,dt \\
\leq\;& c\int_0^{\frac{1}{2}}\omega(t)\,dt,
\end{aligned}$$

where we have used that $\sup_{\partial B(p,t)}|\varphi|\leq\omega(t)$. By Theorem 3.5 we get

$$\begin{aligned}
\int_{B(p,1)\setminus B\left(p,\frac{1}{2}\right)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy &\leq c\sup_{B(p,1)\setminus B\left(p,\frac{1}{2}\right)}|\varphi| \\
&\leq c\,\omega\left(\frac{1}{2}\right) \\
&\leq c\int_0^{\frac{1}{2}}\omega(t)\,dt
\end{aligned}$$

as ω is non-increasing. In conclusion,

$$\int_{B(p,1)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy\leq c\int_0^{\frac{1}{2}}\omega(t)\,dt. \tag{4.9}$$

Therefore,

$$\begin{aligned}
&\int_M G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\
=\;& \sum_{j=0}^{\infty}\int_{B(p,2^{j+1})\setminus B(p,2^j)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\
&+\int_{B(p,1)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy \\
\leq\;& \sum_{j=0}^{\infty}\left(\int_{B(p,2^{j+1})\setminus B(p,2^j)} G(x,y)\,e^{-f(y)}dy\right)\sup_{B(p,2^{j+1})\setminus B(p,2^j)}|\varphi| \\
&+c\int_0^{\frac{1}{2}}\omega(t)\,dt.
\end{aligned} \tag{4.10}$$

The hypothesis on φ implies

$$\sup_{B(p,2^{j+1})\setminus B(p,2^j)}|\varphi|\leq\omega\left(2^j\right)$$

and Theorem 3.5 says that

$$\int_{B(p,2^{j+1})\setminus B(p,2^j)} G(x,y)\,e^{-f(y)}dy\leq c\,2^{j-1}.$$

Using these estimates in (4.10) we obtain

$$\begin{aligned}\int_M G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy &\le c\int_0^{\frac{1}{2}}\omega(t)\,dt + c\sum_{j=0}^{\infty}2^{j-1}\,\omega\left(2^j\right)\\ &\le c\int_0^{\infty}\omega(t)\,dt + c\sum_{j=0}^{\infty}\int_{2^{j-1}}^{2^j}\omega(t)\,dt\\ &\le c\int_0^{\infty}\omega(t)\,dt.\end{aligned}$$

This proves (4.8). As $\int_0^\infty \omega(t)\,dt < \infty$, it follows that the function

$$u(x) := \int_M G(x,y)\,\varphi(y)\,e^{-f(y)}dy$$

is well defined, bounded on M, and verifies $\Delta_f u = -\varphi$. Furthermore, we have the estimate

$$\sup_M |u| \le c\int_0^{\infty}\omega(t)\,dt. \tag{4.11}$$

This proves the first part of the theorem.

We now prove the decay estimate (4.7). For $x \in M$, denote

$$R := r(x) = r(p,x).$$

Given $\alpha \in [\alpha_0, 1]$, let us first assume that $\alpha R < 4$. It follows that $r(x) \le C$. By (4.5), it is obvious that

$$\begin{aligned}\int_0^{\infty}\omega(t)\,dt &\le C\int_{\frac{3}{4}\alpha r(x)}^{\infty}\omega(t)\,dt\\ &\quad + C\,(1+r(x))^{n+a}\,e^{-\sqrt{\lambda_1(\Delta_f)}r(x)}\mathrm{V}_f^{-\frac{1}{2}}(x,1)\int_0^{\alpha r(x)}\omega(t)\,e^{bt}dt.\end{aligned}$$

In view of (4.11), this proves (4.7) when $\alpha R < 4$.

From now on we assume that $\alpha R \ge 4$. Note that

$$\begin{aligned}&\int_{M\setminus B(p,\alpha R)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy\\ =\ &\sum_{j=0}^{\infty}\int_{B(p,2^{j+1}\alpha R)\setminus B(p,2^j\alpha R)} G(x,y)\,|\varphi|(y)\,e^{-f(y)}dy\\ \le\ &\sum_{j=0}^{\infty}\left(\int_{B(p,2^{j+1}\alpha R)\setminus B(p,2^j\alpha R)} G(x,y)\,e^{-f(y)}dy\right)\sup_{B(p,2^{j+1}\alpha R)\setminus B(p,2^j\alpha R)}|\varphi|.\end{aligned}$$

Using the decay hypothesis on φ we get

$$\sup_{B(p,2^{j+1}\alpha R)\setminus B(p,2^j\alpha R)}|\varphi| \le \omega\left(2^j\alpha R\right).$$

We also infer from Theorem 3.5 that

$$\int_{B(p,2^{j+1}\alpha R)\setminus B(p,2^j\alpha R)} G(x,y)\,e^{-f(y)}dy \le c\,2^{j-1}\alpha R.$$

In conclusion,

$$\int_{M\setminus B(p,\alpha R)} G\left(x,y\right)\left|\varphi\right|\left(y\right)e^{-f(y)}dy \leq c\sum_{j=0}^{\infty}\left(2^{j-1}\alpha R\right)\,\omega\left(2^{j}\alpha R\right). \tag{4.12}$$

However,

$$\begin{aligned}\sum_{j=0}^{\infty}2^{j-1}\alpha R\omega\left(2^{j}\alpha R\right) &= \frac{1}{2}\alpha\, R\omega\left(\alpha R\right)+\sum_{j=1}^{\infty}\left(2^{j-1}\alpha R\right)\omega\left(2^{j}\alpha R\right)\\ &\leq 2\int_{\frac{3}{4}\alpha R}^{\alpha R}\omega\left(t\right)dt+\sum_{j=1}^{\infty}\int_{2^{j-1}\alpha R}^{2^{j}\alpha R}\omega\left(t\right)dt\\ &\leq 2\int_{\frac{3}{4}\alpha R}^{\infty}\omega\left(t\right)dt.\end{aligned}$$

It follows that

$$\int_{M\setminus B(p,\alpha R)} G\left(x,y\right)\left|\varphi\right|\left(y\right)e^{-f(y)}dy \leq c\int_{\frac{3}{4}\alpha R}^{\infty}\omega\left(t\right)dt. \tag{4.13}$$

We now proceed to obtain an estimate on $B\left(p,\alpha R\right)$. By Theorem 3.1,

$$\begin{aligned}&\int_{B(x,1)}\int_{B(p,j+1)\setminus B(p,j)} G(z,y)\,e^{-f(z)}e^{-f(y)}dy\,dz\\ \leq\ & c\,R\sqrt{\mathrm{V}_f\left(x,1\right)}\sqrt{\mathrm{V}_f\left(p,j+1\right)}\,e^{-\sqrt{\lambda_1(\Delta_f)}(R-j)}\end{aligned}$$

for any $j\in\{0,1,...,[R]-3\}$, where $[R]$ denotes the greatest integer less than or equal to R. According to Lemma 4.1,

$$\sqrt{\mathrm{V}_f\left(p,j+1\right)}\leq c\,R^{\frac{n}{2}+a}\,e^{\frac{1}{2}\left((n-1+2a)\sqrt{K}+a\right)j}.$$

Hence, we conclude that

$$\begin{aligned}&\int_{B(x,1)}\int_{B(p,j+1)\setminus B(p,j)} G(z,y)\,e^{-f(z)}e^{-f(y)}dy\,dz\\ \leq\ & c\,R^{n+a}\,e^{-\sqrt{\lambda_1(\Delta_f)}R}\sqrt{\mathrm{V}_f\left(x,1\right)}\,e^{bj},\end{aligned} \tag{4.14}$$

where

$$b=\sqrt{\lambda_1\left(\Delta_f\right)}+\frac{1}{2}\left(\left((n-1)+2a\right)\sqrt{K}+a\right).$$

Note that for any $j\leq[R]-3$,

$$B\left(x,2\right)\cap\left(B\left(p,j+1\right)\setminus B\left(p,j\right)\right)=\varnothing.$$

Hence the function

$$H\left(z\right):=\int_{B(p,j+1)\setminus B(p,j)} G(z,y)\,e^{-f(y)}dy$$

is f-harmonic on $B\left(x,2\right)$. Applying (3.7) we get that

$$H\left(x\right)\leq c\,\mathrm{V}_f^{-1}\left(x,1\right)\int_{B(x,1)} H\left(z\right)e^{-f(z)}dz.$$

Together with (4.14), this gives

$$\int_{B(p,j+1)\setminus B(p,j)} G(x,y)\, e^{-f(y)} dy \leq c\, R^{n+a}\, e^{-\sqrt{\lambda_1(\Delta_f)}R} \mathrm{V}_f^{-\frac{1}{2}}(x,1)\, e^{bj} \tag{4.15}$$

for $0 \leq j \leq [R]-3$. We claim that (4.15) holds for $[R]-3 \leq j \leq [R]$ as well. Indeed, in this case (4.15) is equivalent to

$$\int_{B(p,j+1)\setminus B(p,j)} G(x,y)\, e^{-f(y)} dy \leq cR^{n+a} \mathrm{V}_f^{-\frac{1}{2}}(x,1)\, e^{\frac{1}{2}\left((n-1+2a)\sqrt{K}+a\right)R}.$$

This follows from Theorem 3.5 that

$$\int_{B(p,j+1)\setminus B(p,j)} G(x,y)\, e^{-f(y)} dy \leq c\, R$$

for $[R]-3 \leq j \leq [R]$ together with (4.5). In conclusion, (4.15) holds true for all $j \in \{0, 1, \cdots, [R]\}$.

Following a similar argument as in (3.27), we have

$$\int_{\partial B(p,t)} G(x,y)\, e^{-f(y)} dy \leq cR^{n+a} e^{-\sqrt{\lambda_1(\Delta_f)}R} \mathrm{V}_f^{-\frac{1}{2}}(x,1) \tag{4.16}$$

for all $t \in [0,1]$. Indeed, (4.15) implies that

$$\begin{aligned} \inf_{B(p,1)} G(x,y) &\leq c \int_{B(p,1)} G(x,y)\, e^{-f(y)} dy \\ &\leq c\, R^{n+a}\, e^{-\sqrt{\lambda_1(\Delta_f)}R} \mathrm{V}_f^{-\frac{1}{2}}(x,1). \end{aligned}$$

As the function $G(x,\cdot)$ is f-harmonic on $B(p,2)$, one sees that

$$\sup_{B(p,1)} G(x,y) \leq c\, R^{n+a}\, e^{-\sqrt{\lambda_1(\Delta_f)}R} \mathrm{V}_f^{-\frac{1}{2}}(x,1).$$

This immediately implies (4.16).

We now write

$$\begin{aligned} &\int_{B(p,\alpha R)} G(x,y)\, |\varphi|(y)\, e^{-f(y)} dy \\ \leq\; &\sum_{j=1}^{[\alpha R]} \int_{B(p,j+1)\setminus B(p,j)} G(x,y)\, |\varphi|(y)\, e^{-f(y)} dy \\ &+ \int_{B(p,1)} G(x,y)\, |\varphi|(y)\, e^{-f(y)} dy. \end{aligned} \tag{4.17}$$

Using (4.15) we get

$$\begin{aligned}
&\sum_{j=1}^{[\alpha R]}\int_{B(p,j+1)\backslash B(p,j)} G(x,y)\ |\varphi|\,(y)\,e^{-f(y)}dy \\
\leq\ & c\sum_{j=1}^{[\alpha R]}\left(R^{n+a}\,e^{-\sqrt{\lambda_1(\Delta_f)}R}\mathrm{V}_f^{-\frac{1}{2}}\,(x,1)\,e^{bj}\right)\sup_{B(p,j+1)\backslash B(p,j)}|\varphi| \\
\leq\ & c\,R^{n+a}\,e^{-\sqrt{\lambda_1(\Delta_f)}R}\mathrm{V}_f^{-\frac{1}{2}}\,(x,1)\sum_{j=1}^{[\alpha R]}\omega\,(j)\,e^{bj} \\
\leq\ & c\,R^{n+a}\,e^{-\sqrt{\lambda_1(\Delta_f)}R}\mathrm{V}_f^{-\frac{1}{2}}\,(x,1)\int_0^{\alpha R}\omega\,(t)\,e^{bt}dt,
\end{aligned}$$

where we have used that $\sup_{\partial B(p,t)}|\varphi|\leq\omega\,(t)$ and that ω is non-increasing in t. By (4.16),

$$\begin{aligned}
&\int_{B(p,1)} G(x,y)\ |\varphi|\,(y)\,e^{-f(y)}dy \\
\leq\ & \int_0^1\left(\int_{\partial B(p,t)} G\,(x,\xi)\,e^{-f(\xi)}dA(\xi)\right)\sup_{\partial B(p,t)}|\varphi|\,dt \\
\leq\ & c\,R^{n+a}\,e^{-\sqrt{\lambda_1(\Delta_f)}R}\mathrm{V}_f^{-\frac{1}{2}}\,(x,1)\int_0^1\omega\,(t)\,dt.
\end{aligned}$$

Plugging these estimates in (4.17) and using that $\alpha R\geq 4$, we conclude

$$\begin{aligned}
&\int_{B(p,\alpha R)} G(x,y)\ |\varphi|\,(y)\,e^{-f(y)}dy \\
\leq\ & c\,R^{n+a}\,e^{-\sqrt{\lambda_1(\Delta_f)}R}\mathrm{V}_f^{-\frac{1}{2}}\,(x,1)\int_0^{\alpha R}\omega\,(t)\,e^{bt}dt.
\end{aligned}\tag{4.18}$$

Finally, combining (4.18) and (4.13) we arrive at

$$\begin{aligned}
\int_M G(x,y)\ |\varphi|\,(y)\,e^{-f(y)}dy\ \leq\ & c\int_{\frac{3}{4}\alpha R}^{\infty}\omega\,(t)\,dt \\
& +c\,R^{n+a}\,e^{-\sqrt{\lambda_1(\Delta_f)}R}\mathrm{V}_f^{-\frac{1}{2}}\,(x,1)\int_0^{\alpha R}\omega\,(t)\,e^{bt}dt.
\end{aligned}$$

This proves the theorem. □

Let us note that the solution u provided by Theorem 4.2 is generally not unique (modulo constants) among bounded ones, because there may exist non-constant bounded f-harmonic functions. On the other hand, any bounded f-harmonic function on a manifold with $\mathrm{Ric}_f\geq 0$ is constant [3]. So, when $K=0$, the bounded solution u of the Poisson equation $\Delta_f u=-\varphi$ is unique, modulo constants, among all bounded solutions.

Corollary 4.3. *Let $\left(M,g,e^{-f}dx\right)$ be a smooth metric measure space satisfying (4.1), (4.2) and (4.3). Let φ be a smooth function satisfying*

$$|\varphi|\,(x)\leq\omega\,(r\,(x))\,,$$

where $\omega(t)$ is a non-increasing function such that $\int_0^\infty \omega(t)\,dt < \infty$. Assume that the weighted volume of the unit ball $B(x,1)$ satisfies $\mathrm{V}_f(x,1) \geq v_0 > 0$, for any $x \in M$. Then the Poisson equation $\Delta_f u = -\varphi$ admits a bounded solution u on M satisfying

$$|u|(x) \leq C\left(\int_{\beta r(x)}^{\infty} \omega(t)\,dt + e^{-\beta r(x)}\omega(0)\right),$$

for a constant $\beta > 0$ depending only on $n, K, a, \lambda_1(\Delta_f)$ and for $C > 0$ depending additionally on v_0 and $B(p,1)$.

Proof. According to Theorem 4.2 and the hypothesis $\mathrm{V}_f(x,1) \geq v_0 > 0$, we get

$$|u|(x) \leq c\left(\int_{\frac{3}{4}\alpha r(x)}^{\infty} \omega(t)\,dt + (1+r(x))^{n+a}\, e^{-\sqrt{\lambda_1(\Delta_f)}r(x)} \int_0^{\alpha r(x)} \omega(t)\,e^{bt}dt\right).$$

We let

$$\alpha := \frac{1}{4}\frac{\sqrt{\lambda_1(\Delta_f)}}{b}$$

and obtain

$$\begin{aligned}\int_0^{\alpha r(x)} \omega(t)\,e^{bt}dt &\leq \omega(0)\int_0^{\alpha r(x)} e^{bt}dt \\ &\leq \frac{1}{b}\omega(0)\,e^{\frac{1}{4}\sqrt{\lambda_1(\Delta_f)}r(x)}.\end{aligned}$$

It follows from above that

$$|u|(x) \leq c\left(\int_{\frac{3}{4}\alpha r(x)}^{\infty} \omega(t)\,dt + e^{-\frac{1}{4}\sqrt{\lambda_1(\Delta_f)}r(x)}\omega(0)\right).$$

The result now follows by setting

$$\beta := \min\left\{\frac{3\alpha}{4}, \frac{1}{4}\sqrt{\lambda_1(\Delta_f)}\right\}.$$

□

In the case that the function φ decays as

$$|\varphi|(x) \leq c\,(1+r(x))^{-k}$$

for some $k > 1$ and the weighted volume of unit balls is uniformly bounded from below

$$\mathrm{V}_f(x,1) \geq v_0 > 0$$

for all $x \in M$, Corollary 4.3 implies that the solution u satisfies

$$|u|(x) \leq C(k)\,(1+r(x))^{-k+1}$$

as claimed by Theorem 1.8.

As an application of Theorem 4.2 we prove the following.

Theorem 4.4. *Let $\left(M, g, e^{-f}dx\right)$ be a smooth metric measure space satisfying (4.1), (4.2) and (4.3). Assume that the weighted volume has lower bound $\mathrm{V}_f(x,1) \geq v_0 > 0$ for all $x \in M$. Suppose $\psi \geq 0$ satisfies*

$$\Delta_f \psi \geq -c\psi^q$$

for some $q>1$, and

$$\lim_{x\to\infty}\psi(x)\,r^{\frac{1}{q-1}}(x)=0.$$

Then there exists $\delta>0$ and $C>0$ such that

$$\psi(x)\le C\,e^{-r^{\delta}(x)},$$

for all $x\in M$.

Proof. We first note that for $\phi\ge 0$ satisfying

$$\Delta_f\phi\ge -c\phi^q$$

and

$$\phi(x)\le\sigma(r(x))$$

for a non-increasing function $\sigma(t)$ with $\int_0^\infty\sigma^q(t)\,dt<\infty$, there exists $C_0>0$ and $\beta>0$ such that

$$\phi(x)\le C_0\left(\int_{\beta r(x)}^{\infty}\sigma^q(t)\,dt+e^{-\beta r(x)}\sigma^q(0)\right), \tag{4.19}$$

for all $x\in M$. Indeed, by Corollary 4.3 the equation $\Delta_f u=-c\phi^q$ has a solution $u\ge 0$ satisfying

$$u(x)\le C\left(\int_{\beta r(x)}^{\infty}\sigma^q(t)\,dt+e^{-\beta r(x)}\sigma^q(0)\right),$$

for all $x\in M$. In particular, u converges to zero at infinity. Since $\Delta_f\phi\ge -c\phi^q$ and $\Delta_f u=-c\phi^q$, by the maximum principle we get $\phi\le u$ on M. This proves (4.19).

Now let

$$m_0=\left[(q-1)^{-2}\right]+2.$$

Note that $q^m-m>0$ for $m\ge m_0$. Fix $\varepsilon>0$ small enough to be specified later, depending only on β and C_0 in (4.19) and on $q>1$. Fix also a large enough constant $B>0$, to be specified later. We prove by induction on $m\ge m_0$ that

$$\psi(x)\le\varepsilon^{q^m+m}\left(\beta^m r(x)+1\right)^{-\frac{1}{q-1}}+B^{q^m-m}e^{-\beta^m r(x)}, \tag{4.20}$$

for all $x\in M$.

First, note that (4.20) holds for $m=m_0$ by the assumption that

$$\lim_{x\to\infty}\psi(x)\,r^{\frac{1}{q-1}}(x)=0$$

and by adjusting the constant B if necessary.

We now assume (4.20) holds for m and prove

$$\psi(x)\le\varepsilon^{q^{m+1}+(m+1)}\left(\beta^{m+1}r(x)+1\right)^{-\frac{1}{q-1}}+B^{q^{m+1}-(m+1)}e^{-\beta^{m+1}r(x)}. \tag{4.21}$$

By the induction hypothesis we have $\psi(x)\le\sigma(r(x))$, where

$$\sigma(t):=\varepsilon^{q^m+m}\left(\beta^m t+1\right)^{-\frac{1}{q-1}}+B^{q^m-m}e^{-\beta^m t}$$

is decreasing and $\int_0^\infty\sigma^q(t)\,dt<\infty$. Applying (4.19) we get that

$$\psi(x)\le C_0\left(\int_{\beta r(x)}^{\infty}\sigma^q(t)\,dt+e^{-\beta r(x)}\sigma^q(0)\right). \tag{4.22}$$

Obviously,

$$\sigma^q(t)\le c\varepsilon^{q^{m+1}+qm}\left(\beta^m t+1\right)^{-\frac{q}{q-1}}+cB^{q^{m+1}-qm}e^{-q\cdot\beta^m t}, \tag{4.23}$$

for $c>0$ depending only on q. It follows that

$$\begin{aligned}\int_{\beta r(x)}^{\infty}\sigma^{q}(t)\,dt &\leq c\varepsilon^{q^{m+1}+qm}\int_{\beta r(x)}^{\infty}\left(\beta^{m}t+1\right)^{-\frac{q}{q-1}}dt \\ &\quad +cB^{q^{m+1}-qm}\int_{\beta r(x)}^{\infty}e^{-q\cdot\beta^{m}t}dt \\ &= c\beta^{-m}\varepsilon^{q^{m+1}+qm}\left(\beta^{m+1}r(x)+1\right)^{-\frac{1}{q-1}} \\ &\quad +c\beta^{-m}B^{q^{m+1}-qm}e^{-\beta^{m+1}r(x)}.\end{aligned} \tag{4.24}$$

Furthermore, we have by (4.23) that

$$\begin{aligned}e^{-\beta r(x)}\sigma^{q}(0) &\leq c\left(\varepsilon^{q^{m+1}+qm}+B^{q^{m+1}-qm}\right)e^{-\beta r(x)} \\ &\leq c\beta^{-m}B^{q^{m+1}-qm}e^{-\beta^{m+1}r(x)},\end{aligned} \tag{4.25}$$

where for the last line we used that $\beta<1$. Plugging (4.24) and (4.25) into (4.22) yields

$$\begin{aligned}\psi(x) &\leq \left(cC_0\beta^{-m}\varepsilon^{qm-(m+1)}\right)\varepsilon^{q^{m+1}+(m+1)}\left(\beta^{m+1}r(x)+1\right)^{-\frac{1}{q-1}} \\ &\quad +\left(cC_0\beta^{-m}B^{-qm+(m+1)}\right)B^{q^{m+1}-(m+1)}e^{-\beta^{m+1}r(x)}.\end{aligned} \tag{4.26}$$

Since $m\geq m_0$, it is easy to check that $m(q-1)\geq 2$. We then have

$$\begin{aligned}qm-(m+1) &= \frac{1}{4}m(q-1)+\frac{3}{4}m(q-1)-1 \\ &\geq \frac{1}{4}m(q-1)+\frac{1}{2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}cC_0\beta^{-m}\varepsilon^{qm-(m+1)} &\leq \left(cC_0\sqrt{\varepsilon}\right)\left(\varepsilon^{\frac{q-1}{4}}\beta^{-1}\right)^{m} \\ cC_0\beta^{-m}B^{-qm+(m+1)} &\leq \left(cC_0\sqrt{B^{-1}}\right)\left(B^{-\frac{q-1}{4}}\beta^{-1}\right)^{m}.\end{aligned}$$

Now take ε sufficiently small so that $\varepsilon^{\frac{q-1}{4}}\beta^{-1}\leq 1$ and $cC_0\sqrt{\varepsilon}\leq 1$, and B sufficiently large so that $B^{-\frac{q-1}{4}}\beta^{-1}\leq 1$ and $cC_0\sqrt{B^{-1}}\leq 1$. It follows by (4.26) that

$$\psi(x)\leq \varepsilon^{q^{m+1}+(m+1)}\left(\beta^{m+1}r(x)+1\right)^{-\frac{1}{q-1}}+B^{q^{m+1}-(m+1)}e^{-\beta^{m+1}r(x)},$$

for all $x\in M$. This proves (4.21). Hence,

$$\psi(x)\leq \varepsilon^{q^{m}+m}\left(\beta^{m}r(x)+1\right)^{-\frac{1}{q-1}}+B^{q^{m}-m}e^{-\beta^{m}r(x)} \tag{4.27}$$

for all $m\geq m_0$ and $x\in M$.

For $x\in M$ fixed, we take

$$m:=\left[\frac{\ln r(x)}{2\ln\left(q\beta^{-1}\right)}\right],$$

where $[\cdot]$ denotes the greatest integer function. Here we may assume that $r(x)$ is large enough so that $m\geq m_0$. This implies that

$$B^{q^{m}-m}e^{-\beta^{m}r(x)}\leq ce^{-c\sqrt{r}(x)}$$

and

$$\varepsilon^{q^m+m}\left(\beta^m r(x)+1\right)^{-\frac{1}{q-1}} \leq c e^{-r^{\delta}(x)}$$

for some $\delta > 0$ depending only on n, K, a and $\lambda_1(\Delta_f)$. Hence, from (4.27) we conclude that

$$\psi(x) \leq c e^{-r^{\delta}(x)} \quad \text{for all } x \in M.$$

□

We conclude this section by showing the following simple proposition. Note that the positivity of $\lambda_1(\Delta_f)$ implies that the weighted volume of M must be infinite and the bottom spectrum $\lambda_1^{M\setminus\Omega}(\Delta_f)$ of the weighted Laplacian Δ_f on $M\setminus\Omega$ subject to the Dirichlet boundary conditions on $\partial\Omega$ is positive as well. The proposition says the converse is also true.

Proposition 4.5. *Assume that $\lambda_1^{M\setminus\Omega}(\Delta_f) > 0$ for compact domain $\Omega \subset M$ and the weighted volume of M is infinite. Then M has positive spectrum $\lambda_1(\Delta_f) > 0$.*

Proof. Pick $r_0 > 0$ so that $\Omega \subset B(p, r_0)$. Following [14, 15], we first construct a non-constant bounded f-harmonic function on $M\setminus B(p, r_0)$. Indeed, for $R_i \to \infty$, consider the sequence $u_i : B(p, R_i)\setminus B(p, r_0) \to \mathbb{R}$ defined by solving the Dirichlet boundary value problem

$$\begin{aligned} \Delta_f u_i &= 0 \ \text{ on } B(p, R_i)\setminus B(p, r_0) \\ u_i &= 0 \ \text{ on } \partial B(p, R_i) \\ u_i &= 1 \text{ on } \partial B(p, r_0). \end{aligned}$$

Standard elliptic theory implies that u_i converges to an f-harmonic function $u : M\setminus B(p, r_0) \to \mathbb{R}$, such that $0 < u \leq 1$ and $u = 1$ on $\partial B(p, r_0)$. Applying Theorem 2.5 to the sequence u_i, we get that

$$\int_{B(p,R+1)\setminus B(p,R)} u_i^2 e^{-f} \leq C e^{-2\sqrt{\lambda_1(\Delta_f)}R} \int_{B(p,r_0+1)\setminus B(p,r_0)} u_i^2 e^{-f},$$

for all $R > 2(R_0 + 1)$. Taking $i \to \infty$, we obtain the same estimate for u. As M has infinite weighted volume, this proves that u is nonconstant. Define

$$w_0(x) := \begin{cases} 1 & \text{on } B(p, r_0) \\ u(x) & \text{on } M\setminus B(p, r_0). \end{cases}$$

Then $0 < w_0 \leq 1$ on M and $\Delta_f w_0 \leq 0$ in weak sense. Consider

$$w(x,t) := \int_M H(x,y,t)\, w_0(y)\, e^{-f(y)} dy,$$

where $H(x,y,t)$ is the heat kernel of Δ_f. Since w_0 is bounded, we know that $w(x,t)$ satisfies the heat equation $\left(\Delta_f - \frac{\partial}{\partial t}\right) w = 0$ with $w(0) = w_0$. Since $\Delta_f w_0 \leq 0$, we obtain from above that $\Delta_f w \leq 0$ and satisfies the heat equation as well. By the strong maximum principle, we conclude that $\Delta_f w < 0$ for $t > 0$. Hence, we obtain a positive, strictly f-superharmonic function w. Let $\rho := -w^{-1}\Delta_f w > 0$. Then

$$\Delta_f w = -\rho w.$$

This implies that a weighted Poincare inequality on M of the form

$$\int_M \rho \phi^2 e^{-f} \leq \int_M |\nabla \phi|^2 e^{-f} \tag{4.28}$$

is valid for all ϕ with compact support in M. Let $\eta > 0$ be a cut-off function so that $\eta = 0$ on $B(p, r_0)$ and $\eta = 1$ on $M \setminus B(p, 2r_0)$. For any function ϕ with compact support in M we have

$$\int_M \phi^2 e^{-f} \leq 2 \int_M (\phi\eta)^2 e^{-f} + 2 \int_M \phi^2 (1-\eta)^2 e^{-f}. \tag{4.29}$$

Since $\phi\eta$ has support in $M \backslash \Omega$,

$$\begin{aligned} \lambda_1^{M\backslash\Omega}(\Delta_f) \int_M (\phi\eta)^2 e^{-f} &\leq \int_M |\nabla(\phi\eta)|^2 e^{-f} \\ &\leq 2 \int_M |\nabla\phi|^2 e^{-f} + 2 \int_M |\nabla\eta|^2 \phi^2 e^{-f}. \end{aligned}$$

Together with (4.29), it follows that there exists a constant $\alpha > 0$, depending on r_0 and $\lambda_1^{M\backslash\Omega}(\Delta_f)$, such that

$$\alpha \int_M \phi^2 e^{-f} \leq \int_M |\nabla\phi|^2 e^{-f} + \int_{B(p,2r_0)} \phi^2 e^{-f}. \tag{4.30}$$

As the weight $\rho > 0$ on M, there exists $\beta > 0$ so that $\rho \geq \beta$ on $B(p, 2r_0)$. Therefore, from (4.28), we get

$$\begin{aligned} \beta \int_{B(p,2r_0)} \phi^2 e^{-f} &\leq \int_M \rho\phi^2 e^{-f} \\ &\leq \int_M |\nabla\phi|^2 e^{-f}. \end{aligned}$$

Combining with (4.30), we conclude that

$$\alpha \left(1 + \beta^{-1}\right)^{-1} \int_M \phi^2 e^{-f} \leq \int_M |\nabla\phi|^2 e^{-f}$$

for all ϕ with compact support in M. This proves the result. □

5. Applications to steady Ricci solitons

In this section, we discuss some applications to steady gradient Ricci solitons. Recall that a complete manifold (M, g) is a steady gradient Ricci soliton if there exists a smooth potential function f such that

$$\mathrm{Ric} + \mathrm{Hess}(f) = 0. \tag{5.1}$$

Hamilton proved that the scalar curvature S satisfies $S + |\nabla f|^2 = C$ for some positive constant C. By scaling the metric if necessary, we may assume that $C = 1$ and

$$S + |\nabla f|^2 = 1. \tag{5.2}$$

It is known [6] that $S > 0$ on M unless the soliton is flat. In particular, $|\nabla f| \leq 1$ on M and f is of linear growth. The identity (5.2) together with $S + \Delta f = 0$ implies that

$$\Delta_f(f) = -1. \tag{5.3}$$

Therefore,

$$\begin{aligned}\Delta_f e^{\frac{f}{2}} &= \left(\frac{1}{2}\Delta_f(f) + \frac{1}{4}|\nabla f|^2\right)e^{\frac{f}{2}} \\ &\le -\frac{1}{4}e^{\frac{f}{2}}.\end{aligned}$$

It is known (see [15]) that the existence of a positive function $u > 0$ satisfying $\Delta_f u \le -\frac{1}{4}u$ implies $\lambda_1(\Delta_f) \ge \frac{1}{4}$. Hence, the weighted Laplacian on a steady gradient Ricci soliton has positive spectrum. In fact, in [24] it was shown that

$$\lambda_1(\Delta_f) = \frac{1}{4}. \tag{5.4}$$

In view of (5.1), (5.2) and (5.4) we see that our results developed in the previous sections are applicable to steady Ricci solitons. Let us also note that Theorem 1.8 is sharp on a large class of steady solitons. Indeed, assume for now that the steady soliton has positive Ricci curvature and its scalar curvature converges to zero at infinity. By [4], this implies that f is proper and there exist constants $c, c_1 > 0$ so that

$$-r(x) + c_1 \le f(x) \le -cr(x) + c_1. \tag{5.5}$$

In particular, by adding a constant to f if necessary we may assume that $-f > 0$. Using (5.3) we get for $k > 1$,

$$\Delta_f(-f)^{-k+1} = -(k-1)(-f)^{-k} + k(k-1)|\nabla f|^2(-f)^{-k-1}.$$

By (5.2) and (5.5) we see that

$$\begin{aligned}u &= (-f)^{-k+1} \\ \varphi &= (k-1)(-f)^{-k}\left(1 - k|\nabla f|^2(-f)^{-1}\right)\end{aligned}$$

satisfy $\Delta_f u = -\varphi$ and decay to zero at infinity precisely as in Theorem 1.8.

However, in the following we will study steady solitons with no assumptions of positivity on their curvature. Returning to this general setting, let us also recall some curvature identities on steady gradient Ricci solitons

$$\begin{aligned}\Delta_f S &= -2|\mathrm{Ric}|^2 \\ \Delta_f \mathrm{Rm} &= \mathrm{Rm} \star \mathrm{Rm}.\end{aligned}$$

The last identity implies a useful inequality

$$\Delta_f |\mathrm{Rm}| \ge -c|\mathrm{Rm}|^2.$$

As before, we fix $p \in M$ and denote

$$r(x) := r(p, x).$$

Also, denote by C_0, C, c constants depending only on the dimension n and possibly the geometry of $B(p, 2)$.

We now estimate the volume of unit balls in (M, g).

Lemma 5.1. *Let (M, g, f) be a steady gradient Ricci soliton. Then there exists a constant $C > 0$ such that*

$$\mathrm{V}(x, 1) \ge C^{-1}\, e^{-C\sqrt{r(x)\ln(1+r(x))}}\mathrm{V}(p, 1)$$

for all $x \in M$.

Proof. The proof is inspired by [26], where a similar estimate was given for shrinking Ricci solitons. Let us denote the volume form in geodesic coordinates centered at x by

$$dV|_{\exp_x(r\xi)} = J(x,r,\xi)\, dr d\xi$$

for $r > 0$ and $\xi \in S_x M$, the unit tangent sphere at x. Let $R := r(p,x)$. Without loss of generality, we may assume that $R \geq 2$.

Let $\gamma(s)$ be a minimizing normal geodesic with $\gamma(0) = x$ and $\gamma(T) \in B(p,1)$ for some $T > 0$. By the triangle inequality, we know that

$$R - 1 \leq T \leq R + 1. \tag{5.6}$$

Along γ, according to the Laplace comparison theorem,

$$m'(r) + \frac{1}{n-1} m^2(r) \leq f''(r),$$

where $m(r) := \frac{d}{dr} \ln J(x,r,\xi)$.

For arbitrary $k \geq 2$, multiplying this by r^k and integrating from $r = 0$ to $r = t$, we have

$$\int_0^t m'(r)\, r^k dr + \frac{1}{n-1} \int_0^t m^2(r)\, r^k dr \leq \int_0^t f''(r)\, r^k dr. \tag{5.7}$$

After integrating the first term in (5.7) by parts and rearranging terms, we get

$$\begin{aligned} & m(t)\, t^k + \frac{1}{n-1} \int_0^t \left(m(r)\, r^{\frac{k}{2}} - (n-1) \frac{k}{2} r^{\frac{k}{2}-1} \right)^2 dr \\ \leq \; & \frac{(n-1)\, k^2}{4(k-1)} t^{k-1} + \int_0^t f''(r)\, r^k dr. \end{aligned}$$

In particular,

$$m(t) \leq \frac{(n-1)\, k^2}{4(k-1)} \frac{1}{t} + \frac{1}{t^k} \int_0^t f''(r)\, r^k dr.$$

Integrating this from $t = 1$ to $t = T$, we obtain for some constant c depending only on n,

$$\ln \frac{J(x,T,\xi)}{J(x,1,\xi)} \leq c\, k \, \ln T + A, \tag{5.8}$$

where the term A is given by

$$A := \int_1^T \frac{1}{t^k} \left(\int_0^t f''(r)\, r^k dr \right) dt.$$

We now estimate A in the right side of (5.8). Integrating by parts implies

$$\begin{aligned} A &= f(T) - f(1) - k \int_1^T \frac{1}{t^k} \int_0^t f'(r)\, r^{k-1} dr dt \\ &= -\frac{1}{k-1} (f(T) - f(1)) \\ & \quad + \frac{k}{k-1} \frac{1}{t^{k-1}} \left(\int_0^t f'(r)\, r^{k-1} dr \right) |_{t=1}^{t=T}. \end{aligned}$$

As $|f(1)| \leq T + c$ and $|f'(r)| \leq 1$, it follows that

$$A \leq \frac{cT}{k}.$$

We now choose

$$k := \sqrt{\frac{T}{\ln T}}.$$

It follows from (5.8) that

$$\ln \frac{J(x,T,\xi)}{J(x,1,\xi)} \leq ck \ln T + \frac{cT}{k} \leq c\sqrt{T \ln T}.$$

We have thus proved that

$$J(x,1,\xi) \geq e^{-c\sqrt{R \ln R}} J(x,T,\xi).$$

By integrating this over a subset of $S_x M$ consisting of all unit tangent vectors ξ so that $\exp_x(T\xi) \in B(p,1)$ for some T, it follows that

$$\mathrm{A}(\partial B(x,1)) \geq e^{-c\sqrt{R \ln R}} \mathrm{V}(p,1),$$

where $R = d(p,x) \geq 2$. Clearly, for $\frac{1}{2} \leq t \leq 1$, a similar estimate holds for $\mathrm{A}(\partial B(x,t))$. Therefore,

$$\mathrm{V}(x,1) \geq e^{-c\sqrt{R \ln R}} \mathrm{V}(p,1).$$

This proves the result. □

From now on, we assume in addition that the potential f is bounded above by a constant. By adding a constant if necessary, we may assume without loss of generality that

$$f \leq 0 \text{ on } M. \tag{5.9}$$

Following [7], we now establish a sharp lower bound for the scalar curvature.

Lemma 5.2. *Let (M, g, f) be a complete steady gradient Ricci soliton satisfying (5.9). Then there exists $c > 0$ so that*

$$S \geq c\, e^{f} \text{ on } M.$$

Proof. Using that $\Delta_f e^f = -Se^f$ we compute for $a > 0$,

$$\begin{aligned} \Delta_f \left(S - ae^f\right) &= -2\left|\mathrm{Ric}\right|^2 + aSe^f \\ &\leq -\frac{2}{n}S^2 + aSe^f \\ &\leq \frac{na^2}{8}e^{2f}. \end{aligned}$$

Note that

$$\begin{aligned} \Delta_f e^{2f} &= \left(2\Delta_f(f) + 4\left|\nabla f\right|^2\right) e^{2f} \\ &= 2(1-2S)\, e^{2f}. \end{aligned}$$

It follows that for $b > 0$,

$$\Delta_f \left(S - ae^f - be^{2f}\right) \leq \left(\frac{na^2}{8} - 2b + 4bS\right) e^{2f}.$$

Let ϕ be a smooth cut-off function so that $\phi = 1$ on $B(p, R)$ and $\phi = 0$ on $M \backslash B(p, 2R)$. We may assume that

$$\begin{aligned} -\frac{c}{R} &\leq \phi' \leq 0 \\ |\phi''| &\leq \frac{c}{R^2}. \end{aligned}$$

Let

$$G := \left(S - ae^f - be^{2f}\right)\phi^2. \tag{5.10}$$

If G achieves its minimum at x_0 with $G(x_0) < 0$, then $x_0 \in B(p, 2R)$. By the Laplacian comparison theorem [31] we have on the support of ϕ that

$$\begin{aligned} \Delta_f r(x) &\leq \frac{n-1}{r(x)} + 1 \\ &\leq c. \end{aligned}$$

Since

$$\begin{aligned} \Delta_f G &= \phi^2 \Delta_f \left(S - ae^f - be^{2f}\right) + 2\phi^{-1}(\Delta_f \phi) G \\ &\quad + 4\langle \nabla G, \nabla \ln \phi \rangle - 6|\nabla \phi|^2 \phi^{-2} G, \end{aligned}$$

it follows that at x_0,

$$\begin{aligned} 0 &\leq \Delta_f G \\ &\leq \left(\frac{na^2}{8} - 2b + 4bS\right) e^{2f}\phi^2 - 6|\nabla\phi|^2 \phi^{-2} G \\ &\quad + 2\phi^{-1}(\Delta_f \phi) G. \end{aligned} \tag{5.11}$$

Let us note that as $f \leq 0$ and $S > 0$, we have $0 > G(x_0) > -(a+b)$. By (5.11), there exists $c > 0$ depending on a and b so that

$$0 \leq \left(\frac{na^2}{8} - 2b + 4bS(x_0)\right) e^{2f(x_0)}\phi^2(x_0) + \frac{c}{R}.$$

However, since $G(x_0) < 0$ and $f \leq 0$, we have

$$\begin{aligned} S(x_0) &< ae^{f(x_0)} + be^{2f(x_0)} \\ &\leq a + b. \end{aligned}$$

Hence, we get that

$$0 \leq \left(\frac{na^2}{8} - 2b + 4b(b+a)\right) e^{2f(x_0)}\phi^2(x_0) + \frac{c}{R}. \tag{5.12}$$

Now let $b = \frac{1-2a}{4}$ and note that

$$\frac{na^2}{8} - 2b + 4b(b+a) = -\frac{1}{4} + \frac{1}{8}\left((n-8)a^2 + 8a\right).$$

Hence, for $a > 0$ sufficiently small, (5.12) implies that

$$e^{2f(x_0)}\phi^2(x_0) \leq \frac{c}{R}, \tag{5.13}$$

for some $c > 0$ depending only on a. We claim that

$$G(x_0) \geq -cR^{-\frac{1}{4}}. \tag{5.14}$$

Indeed, if $e^{2f(x_0)} \leq R^{-\frac{1}{2}}$, then (5.10) implies $G(x_0) \geq -cR^{-\frac{1}{4}}$ as claimed. On the other hand, if $e^{2f(x_0)} > R^{-\frac{1}{2}}$, then (5.13) implies $\phi^2(x_0) \leq c\,R^{-\frac{1}{2}}$. So from (5.10), $G(x_0) \geq -c\,R^{-\frac{1}{2}}$. In either case, (5.14) is proved.

Certainly, this is true as well if $G(x_0) \geq 0$. In conclusion, we have proved that

$$G \geq -cR^{-\frac{1}{4}} \text{ on } M.$$

As $\phi = 1$ on $B(p,R)$, one has

$$\begin{aligned} \inf_{B(p,R)} \left(S - ae^f - be^{2f}\right) &\geq \inf_M G \\ &\geq -cR^{-\frac{1}{4}}. \end{aligned}$$

Letting $R \to \infty$, we conclude that $S - ae^f - be^{2f} \geq 0$ on M. This proves the result. □

We now prove the main result of this section.

Theorem 5.3. *Let (M, g, f) be a complete steady gradient Ricci soliton satisfying (5.9). If*

$$\lim_{x\to\infty} |\mathrm{Rm}|(x)\, r(x) = 0,$$

then there exists $c > 0$ such that

$$|\mathrm{Rm}|(x) \leq c\,(1 + r(x))^{3(n+1)}\, e^{-r(x)} \quad \text{on } M.$$

Proof. Since $S \leq c\,(1 + r(x))^{-1}$, by Lemma 5.2, f is proper and

$$f(x) \leq -c_1 \ln(1 + r(x)) + c_2. \tag{5.15}$$

Let $\sigma > 8$ be an absolute constant to be specified later.

Recall that $\Delta_f |\mathrm{Rm}| \geq -c\,|\mathrm{Rm}|^2$and $\Delta_f(-f) = 1$. As in [10] for expanding solitons, we combine these two formulas and obtain that

$$\begin{aligned} \Delta_f\left(|\mathrm{Rm}|(-f+\sigma)\right) &\geq -c\,|\mathrm{Rm}|^2(-f+\sigma) + |\mathrm{Rm}| \\ &\quad +2\left\langle \nabla|\mathrm{Rm}|, \nabla(-f+\sigma)\right\rangle \\ &= |\mathrm{Rm}|\left(1 - c\,|\mathrm{Rm}|(-f+\sigma)\right) \\ &\quad +2\left\langle \nabla\left(|\mathrm{Rm}|(-f)\right), \nabla \ln(-f+\sigma)\right\rangle \\ &\quad -2\,|\mathrm{Rm}|(-f+\sigma)^{-1}|\nabla f|^2. \end{aligned} \tag{5.16}$$

Now define

$$F := f + 2\ln(-f+\sigma).$$

Since $|\nabla f|^2 \leq 1$ and $\sigma > 8$, by (5.16) the function $w := |\mathrm{Rm}|(-f+\sigma)$ satisfies

$$\Delta_F w \geq |\mathrm{Rm}|\left(\frac{1}{2} - c\,|\mathrm{Rm}|(-f+\sigma)\right),$$

for a constant $c > 0$ depending only on dimension. As $|\mathrm{Rm}|(x)\, r(x) = o(1)$, there exists $R_0 > 0$ so that

$$\Delta_F w \geq 0 \text{ on } M \backslash B(p, R_0). \tag{5.17}$$

Moreover, by hypothesis w converges to zero at infinity. To get a decay rate for w, we want to construct a barrier function in (5.17). As (5.15) is not sharp, using f to construct a barrier function does not seem to yield optimal results. Instead, we will use the Green's function of the operator Δ_F, and apply the decay estimates obtained in Section 3. In the following, we prove that the metric space $\left(M, g, e^{-F}dx\right)$ satisfies the conditions (4.1), (4.2) and (4.3).

One checks directly that

$$\Delta_F (f) = -1 + 2\left|\nabla f\right|^2 (-f+\sigma)^{-1}.$$

Hence, since $\left|\nabla f\right|^2 \leq 1$, we get

$$\begin{aligned} \Delta_F e^{\frac{f}{2}} &= \frac{1}{2}\left(-1 + 2\left|\nabla f\right|^2 (-f+\sigma)^{-1} + \frac{1}{2}\left|\nabla f\right|^2\right) e^{\frac{f}{2}} \\ &\leq -\rho e^{\frac{f}{2}}, \end{aligned}$$

where

$$\begin{aligned} \rho &= \frac{1}{4} - (-f+\sigma)^{-1} \\ &\geq \frac{1}{4} - \frac{1}{\sigma}. \end{aligned} \tag{5.18}$$

It is well known (see [14, 24]) that this implies an estimate for the bottom of spectrum of the weighted Laplacian $\Delta_F := \Delta - \langle \nabla F, \nabla \rangle$ of the form

$$\begin{aligned} \lambda_1(\Delta_F) &\geq \frac{1}{4} - \frac{1}{\sigma} \\ &> \frac{1}{8}. \end{aligned} \tag{5.19}$$

By (5.2) we get

$$\begin{aligned} |\nabla F| &= \left(1 - \frac{2}{(-f+\sigma)}\right)|\nabla f| \\ &\leq 1. \end{aligned} \tag{5.20}$$

We note that the Bakry-Emery tensor associated to the weight F is

$$\begin{aligned} \mathrm{Ric}_F &= \mathrm{Ric}_f + \mathrm{Hess}\left(2\ln(-f+\sigma)\right) \\ &= -2(-f+\sigma)^{-1}\mathrm{Hess}(f) - 2(-f+\sigma)^2 \nabla f \otimes \nabla f. \end{aligned}$$

As M has bounded Ricci curvature $|\mathrm{Ric}| \leq c$, it is clear from above that

$$\mathrm{Ric}_F \geq -(n-1)\bar{K} \tag{5.21}$$

for some constant $\bar{K} > 0$ independent of σ. Applying (3.1) to the Green's function $\bar{G}(x,y)$ of Δ_F we get

$$\int_{B(p,R+1)\backslash B(p,R)} \bar{G}^2(p,y)\, e^{-F(y)} dy \leq C e^{-2\sqrt{\lambda_1(\Delta_F)}R}, \tag{5.22}$$

for any $R \geq 1$. For any $x \in M\backslash B(p,2)$ we have by the triangle inequality that

$$B(x,1) \subset B(p, r(x)+1)\backslash B(p, r(x)-1).$$

It follows from (5.22) and (5.19) that

$$\int_{B(x,1)} \bar{G}^2(p,y)\, e^{-F(y)} dy \leq C e^{-\left(1-\frac{4}{\sigma}\right) r(x)} \tag{5.23}$$

for any $x \in M\backslash B(p,2)$. Since $\Delta_F \bar{G}(p,\cdot) = 0$ on $B(x,2)$, by (3.7) we get a gradient estimate of the form

$$\sup_{y\in B(x,1)} \left|\nabla \ln \bar{G}\right|(p,y) \leq c$$

for a constant $c>0$ depending only on n and $\bar{K}$. Integrating this estimate, one sees that

$$\bar{G}(p,x)\leq c\bar{G}(p,y)$$

for all $y\in B(x,1)$. Hence, (5.23) implies that

$$\bar{G}(p,x)\leq C\mathrm{V}^{-\frac{1}{2}}(x,1)\,e^{-\left(\frac{1}{2}-\frac{2}{\sigma}\right)r(x)+\frac{1}{2}F(x)}.$$

Since $F=f+2\ln(-f+\sigma)$, we obtain

$$\bar{G}(p,x)\leq C\mathrm{V}^{-\frac{1}{2}}(x,1)\,e^{-\left(\frac{1}{2}-\frac{2}{\sigma}\right)r(x)+\frac{1}{2}f(x)}\left(-f(x)+\sigma\right) \tag{5.24}$$

for all $x\in M\backslash B(p,2)$. In particular, by (5.9), (5.24) and Lemma 5.1 we get that $\bar{G}(p,x)$ converges to zero as $x\to\infty$.

We now use $\bar{G}(p,x)$ as barrier function in (5.17). Fix $A>0$ large enough with

$$w(x)-A\bar{G}(p,x)<0 \text{ for all } x\in\partial B(p,R_0).$$

Then the function $w-A\bar{G}(p,\cdot)$ is F-subharmonic on $M\backslash B(p,R_0)$, converges to zero at infinity and is negative on $\partial B(p,R_0)$. The maximum principle implies that

$$w(x)-A\bar{G}(p,x)<0 \text{ for all } x\in M\backslash B(p,R_0).$$

Combining with (5.24) we get

$$|\mathrm{Rm}|(x)\leq C\mathrm{V}^{-\frac{1}{2}}(x,1)\,e^{-\left(\frac{1}{2}-\frac{2}{\sigma}\right)r(x)+\frac{1}{2}f(x)}.$$

Lemma 5.2 implies $e^{f(x)}\leq|\mathrm{Rm}|(x)$. Hence we get from above that there exists $C>0$ so that

$$|\mathrm{Rm}|(x)\leq C\mathrm{V}^{-1}(x,1)\,e^{-\left(1-\frac{4}{\sigma}\right)r(x)},$$

for all $x\in M$. Together with Lemma 5.1 this proves that for given $\varepsilon>0$ there exists $C(\varepsilon)>0$ so that

$$|\mathrm{Rm}|(x)\leq C(\varepsilon)\,e^{-(1-\varepsilon)r(x)}. \tag{5.25}$$

To finish the proof of the theorem, we use Theorem 4.2. Note that by (5.25) we have $|\mathrm{Rm}|(x)\leq ce^{-\frac{3}{4}r(x)}$ for all $x\in M$.

Recall that $\Delta_f|\mathrm{Rm}|\geq -c|\mathrm{Rm}|^2$, for some constant c depending only on dimension. Solving the Poisson equation $\Delta_f u=-c|\mathrm{Rm}|^2$ by choosing $\alpha=1$ in Theorem 4.2 and noticing that $K=0$ and $a=b=1$ due to our normalization, we obtain a solution u such that

$$\begin{aligned}|u|(x)\leq\;& C\int_{\frac{3}{4}r(x)}^{\infty}\omega(t)\,dt\\ &+C\left(1+r(x)\right)^{n+1}e^{-\frac{1}{2}r(x)+\frac{1}{2}f(x)}\mathrm{V}^{-\frac{1}{2}}(x,1)\int_0^{r(x)}\omega(t)\,e^t dt,\end{aligned} \tag{5.26}$$

where

$$\omega(t)=ce^{-\frac{3}{2}t}.$$

By (5.25) and standard comparison geometry we know that

$$\mathrm{V}(x,1)\geq c^{-1}\left(1+r(x)\right)^{-n-1}.$$

Therefore (5.26) implies that

$$\begin{aligned}|u|(x)&\leq C\left(1+r(x)\right)^{\frac{3}{2}(n+1)}e^{-\frac{1}{2}r(x)+\frac{1}{2}f(x)}\\ &\leq C\left(1+r(x)\right)^{\frac{3}{2}(n+1)}e^{-\frac{1}{2}r(x)}\sqrt{|\mathrm{Rm}|(x)},\end{aligned} \tag{5.27}$$

where in the last line we have used Lemma 5.2. We have $\Delta_f |\mathrm{Rm}| \geq -c|\mathrm{Rm}|^2$ and $\Delta_f u = -c|\mathrm{Rm}|^2$, where both $|\mathrm{Rm}|$ and u converge to zero at infinity. By the maximum principle, we have that $|\mathrm{Rm}| \leq u$ on M. In conclusion, (5.27) implies that

$$|\mathrm{Rm}|(x) \leq C\,(1+r(x))^{3(n+1)}\,e^{-r(x)}.$$

This proves the theorem. □

Finally we point out that stronger results can be obtained by assuming the sectional curvature is non-negative. First, we recall a result from [9] and [5]. For completeness, we supply a more direct proof here.

Proposition 5.4. *Let (M^n, g, f) be an n-dimensional complete non-flat steady gradient Ricci soliton with non-negative sectional curvature. Assume that the scalar curvature is integrable on M. Then (M, g) is isometric to a quotient of $\mathbb{R}^{n-2}\times\Sigma$, where Σ is the cigar soliton.*

Proof. We may assume $n \geq 3$ as otherwise the result is known. We note that $2|\mathrm{Ric}|^2 \leq S^2$. Indeed, the fact that M has nonnegative sectional curvature implies that the eigenvalues λ_i of the Ricci curvature satisfy $\sum_{j\neq i}\lambda_j \geq \lambda_i$. So $S \geq 2\lambda_i$ and

$$2|\mathrm{Ric}|^2 = 2\sum_i \lambda_i^2 \leq \sum_i (\lambda_i S) = S^2. \tag{5.28}$$

Hence, using the cut-off function $\phi := \left(\frac{R-r(p,x)}{R}\right)_+$ with support in $B(p,R)$ we get

$$\begin{aligned}
0 &\leq \int_M \left(S^2 - 2|\mathrm{Ric}|^2\right)\phi^2 \\
&= -\int_M S(\Delta f)\phi^2 + 2\int_M R_{ij}f_{ij}\phi^2 \\
&= \int_M \langle \nabla S, \nabla f\rangle \phi^2 - 2\int_M (\nabla_j R_{ij}) f_i\phi^2 \\
&\quad + \int_M \langle \nabla f, \nabla\phi^2\rangle S - 2\int_M R_{ij}f_i\left(\phi^2\right)_j \\
&\leq \frac{c}{R}\int_{B(p,R)} S,
\end{aligned}$$

where in the last line we have used that $2\nabla_j R_{ij} = \nabla_i S$ by the Bianchi identity. Therefore, by letting $R\to\infty$, we conclude $2|\mathrm{Ric}|^2 = S^2$ on M. In particular, from (5.28) we see that either $\lambda_i = 0$ or $\lambda_i = \frac{1}{2}S$. If $\lambda_i = \frac{1}{2}S$ for all i at all points, then M is Einstein and flat. So $\lambda_i = 0$ for some i at some point. Applying Hamilton's strong maximum principle, the result follows. □

Combining the proposition with Theorem 5.3 we get the following.

Corollary 5.5. *Let (M, g, f) be a complete non-flat steady gradient Ricci soliton with non-negative sectional curvature. Assume that the scalar curvature decays faster than linear, that is,*

$$\lim_{x\to\infty} r(x)\,S(x) = 0.$$

Then (M, g) is isometric to a quotient of $\mathbb{R}^{n-2}\times\Sigma$, where Σ is the cigar soliton.

Proof. By [4], the function f satisfies

$$-r(x) + c_1 \leq f(x) \leq -cr(x) + c_1.$$

Theorem 5.3 implies that the curvature decays exponentially, that is,

$$|\mathrm{Rm}|(x) \leq c(1 + r(x))^{3(n+1)} e^{-r(x)}.$$

This shows that $S \in L^1(M)$ and the result follows from Proposition 5.4. ☐

References

[1] S. Agmon, Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-Body Schrödinger operators, Mathematical Notes, vol. 29, Princeton University Press, Princeton, NJ, 1982.

[2] S. Brendle, Rotational symmetry of self-similar solutions to the Ricci flow, Invent. Math., 194 (2013), 731-764.

[3] K. Brighton, A Liouville-type theorem for smooth metric measure spaces, J. Geom. Anal. 23 (2013), 562-570.

[4] J. Carillo and L. Ni, Sharp logarithmic Sobolev inequalities on gradient solitons and applications, Comm. Anal. Geom. 17 (2009) 721-753.

[5] G. Catino, P. Mastrolia and D. Monticelli, Classification of expanding and steady Ricci solitons with integral curvature decay, Geom. Topol. 20 (2016), 2665–2685.

[6] B.L. Chen, Strong uniqueness of the Ricci flow, J. Differential Geom. 82 (2009), no. 2, 362-382.

[7] B. Chow, P. Lu and B. Yang, A lower bound for the scalar curvature of noncompact nonflat Ricci shrinkers, Comptes Rendus Mathematique 349 (2011), no. 23-24, 1265-1267.

[8] Y. Deng and X. Zhu, 3d steady gradient Ricci solitons with linear curvature decay, to appear in Int. Math. Res. Not.

[9] A. Deruelle, Steady gradient Ricci soliton with curvature in L1, Comm. Anal. Geom. 20 (2012) 31-53.

[10] A. Deruelle, Asymptotic estimates and compactness of expanding gradient Ricci solitons, Ann. Sc. Norm. Super. Pisa 17 (2017), 485-530.

[11] D. Ganguly and Y. Pinchover, On Green functions of second-order elliptic operators on Riemannian manifolds: the critical case, J. Funct. Anal. 274 (2018), 2700-2724.

[12] A. Grigor'yan, Heat kernel and analysis on manifolds, AMS/IP Studies in Advanced Mathematics 2009.

[13] P. Li and L.F. Tam, Symmetric Green's functions on complete manifolds, Amer. J. Math. 109 (1987), 1129-1154.

[14] P. Li and J. Wang, Complete manifolds with positive spectrum, J. Differential Geom. 58 (2001), 501–534.

[15] P. Li and J. Wang, Weighted Poincare inequality and rigidity of complete manifolds, Ann. Sci. Ecole Norm. Sup. 39 (2006), 921–982.

[16] P. Li and J. Wang, Counting cusps on complete manifolds of finite volume, Math. Res. Lett. 17 (2010), 675-688.

[17] P. Li and S.T. Yau, On the Parabolic Kernel of the Schrodinger Operator, Acta Math. 156 (1986), 153-201.

[18] P. Li, Geometric Analysis, Cambridge University Press 2012.

[19] G. Liu, Gromov-Hausdorff limit of Kahler manifolds and the finite generation conjecture, Ann. Math. 184 (2016), 775-815.

[20] B. Malgrange, Existence et approximation des solutions de equations aux derivees partielles et des equations de convolution, Annales de l'Inst. Fourier 6 (1955), 271-355.

[21] N. Mok, Y-T. Siu and S.T. Yau, The Poincare-Lelong equation on complete Kahler manifolds, Compositio Math. 44 (1981) 183–218.

[22] O. Munteanu and N. Sesum, The Poisson equation on complete manifolds with positive spectrum and applications, Adv. Math. 223 (2010), 198-219.

[23] O. Munteanu and N. Sesum, On gradient Ricci solitons, J. Geom. Anal. 23 (2013), no. 2, 539-561.

[24] O. Munteanu and J. Wang, Smooth metric measure spaces with nonnegative curvature, Comm. Anal. Geom. 19 (2011), 451–486.

[25] O. Munteanu and J. Wang, Analysis of weighted Laplacian and applications to Ricci solitons, Comm. Anal. Geom. 20 (2012), 55–94.
[26] O. Munteanu and J. Wang, Geometry of manifolds with densities, Adv. Math. 259 (2014), 269-305.
[27] L. Ni, Y. Shi and L.-F. Tam, Poisson equation, Poincare-Lelong equation and the curvature decay on complete Kahler manifolds, J. Differential Geom. 57 (2001), 339-388.
[28] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159.
[29] R. Strichartz, Analysis of the Laplacian on a complete Riemannian manifold, J. Funct. Anal. 52 (1983), 48-79.
[30] C. J. Sung, L. F. Tam, and J. Wang, Spaces of harmonic functions, J. London Math. Soc. 61 (2000), 789-806.
[31] G. Wei and W. Wylie, Comparison geometry for the Bakry-Emery Ricci tensor, J. Differential Geom 83 (2009), 377-405.

E-mail address: ovidiu.munteanu@uconn.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06268, USA

E-mail address: cjsung@math.nthu.edu.tw

Department of Mathematics, National Tsing Hua University, Hsin-Chu, Taiwan

E-mail address: jiaping@math.umn.edu

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA