Asymptotic Analysis of the Expected Utility Maximization when Wealth Can Become Negative

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We consider an incomplete financial market where the stock price dynamics is modeled by a continuous, not necessarily Markovian semimartingale, and the wealth is allowed to be negative. In such settings, an explicit second-order expansion formula for the exponential investor's indirect utility as a function of the underlying market price of the risk process is established. This allows to provide first-order approximations of the value functions and establish nearly optimal strategies that match the indirect utility up to the second order. By change of measure, we also provide a characterization of the approximation problem in terms of the Galtchouk-Kunita-Watanabe decomposition. We also illustrate the results by an example.

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Section 1

Introduction

The project's main goal is to understand the impact of the perturbations of the dynamics of the stock price process, specifically the effect of small changes in the market price of risk, on the solution to the optimal investment from terminal wealth when wealth can become negative. Under a fairly natural parametrization of perturbations as below in (3.3), the goals are to obtain

- second-order expansions of the value functions,
- first-order expansions of the optimizers,
- corrections to the strategies that match the primal value function up to the second order,
- a characterization of the correction terms through the Kunita-Watanabe decomposition by a possible change of measure and numéraire and by relying on the risk-tolerance wealth process.

The existing literature only handles the case of utilities on the positive real line, thus prohibiting negative wealth processes, see, e.g., [2, 3, 4, 5]. Establishing results in the settings when wealth can become negative is the main contribution.

Section 2

The Optimal Investment Problem

Let us consider an investor in a security market who has to decide how to arrange his or her portfolio. As the payoffs of the investment are typically random, it is not very simple to formalize this problem in a manageable way. One way to do this is to consider the $\mathbb{E}[X]$, the expected value of the payoff, and then to try to maximize it over the portfolios X. This criterion, however, does not take into account that, typically, investors are willing to give up some profits to lower the risk. Most investors prefer a certain 5% return to a highly risky investment with a 5.5% return.

A classical approach is based on maximizing the expected utility, $\mathbb{E}[U(X)]$, where the function U is typically an increasing concave function, which represents the preferences of an investor. The monotonicity corresponds to preferring more wealth to less, and the concavity implies that the investor is risk-averse.

In [6], it is shown that the following axioms imply that the preference can be represented via the expected utility. We follow the discussion in [7].

1. (known laws on a finite state space) an investor can choose between random

payoffs taking values in some fixed and finite set S. The law of any payoff is assumed to be known;

2. (law invariance) he or she is indifferent between two payoffs with the same law;

3. (completeness) for any two payoffs, the investor either prefers one of them or he is indifferent between them;

4. (transitivity) if the investor likes Z as much or more than Y and Y as much or more than X, then he or she also likes Z as much or more than X;

5. (Archimedean property) if X, Y, and Z are the payoffs, such that the investor prefers Z to Y and Y to X. Consider a new payoff W, which equals (lucrative) Z with probability $p \in (0, 1)$, and (unlucrative) X with probability $1 - p$, decided by a previous independent Bernoulli trial. The (axiom states that the) investor prefers W to Y for a sufficiently large $p < 1$ and Y to W for a sufficiently small $p > 0$.

6. (independence) let X, Y, and Z be payoffs such that Z is independent of both X and Y. Consider payoffs W and V, that give payoff X and Y, respectively, with probability $p \in (0, 1)$ and payoff Z with probability $1 - p$, decided by an independent Bernoulli trial. The (axiom states that the) investor prefers W to V if he or she prefers Y to X.

Under these assumptions, by the results in [6], there exists a function $U : S \to \mathbb{R}$, such that you prefer any payoff Y to any other payoff X if and only if

$$\mathbb{E}[U(Y)] \geq \mathbb{E}[U(X)]. \tag{2.1}$$

While the prospect theory in behavioral economics, pioneered in [8], suggests that most investors are not rational in the sense of the axioms above. Below, we will consider rational investors and the associated utilities. Behavioral finance does not imply that it is unreasonable to base decisions on expected utility maximization.

In continuous-time settings, the first results for continuous-time models were obtained by Merton [9] and [10] in a Markovian setting via dynamic programming arguments. An alternative duality-based approach was developed among others by Cox and Huang [11] and [12], Karatzas, Lehoczky, and Shreve [13], and Karatzas and Shreve [14] and [15] for complete markets and by Karatzas, Lehoczky, Shreve, and Xu [16], He and Pearson [17] and [18], Kramkov and Schachermayer [19] and [20]. Stochastic utility is treated in [21], [22], [23], [24], and [25].

The utility from terminal wealth can be described as maximizing $\mathbb{E}[U(X_T)]$ (similarly to (2.1)), where X_T, however, are terminal values of the wealth processes obtained by so-called self-financing strategies starting from an initial wealth x, where the self-financing strategies can be described as the ones where no influx of withdrawal of cash is allowed, and changes in the value of the portfolio come from the changes in the prices of the securities, which constitute the portfolio. In this work, the main focus will be on the wealth processes that are allowed to take negative values. This is in contrast to a large amount of literature, including most of the references above, that focuses on the nonnegative wealth processes.

The more formal definition of the optimal investment problem, which includes the probability space and the exact dynamics of traded securities, as well as a parametrized family of perturbations is given below, see (4.2). Now, various formulations of this problem, including the ones with asymmetry of information, large markets, and beyond are considered in [14], [26], [27], [7], [28], and [29]. Utility-based approaches, at the heart of which is the optimal investment problem, also expanded intro pricing and hedging of the derivatives, see [30], [31], and [32].

Section 3

The Base Model and Parametrization of Perturbations

Let us consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $T \in (0,\infty)$ is the time horizon, the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions of right-continuity, and that $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible events in $\mathcal{F}$. We further suppose that $\mathcal{F}_0$ is trivial. We assume that there are two traded securities, a bank account with 0 interest rate and a stock S. We suppose that the evolution of the stock price is given by

$$S = M + \lambda \cdot \langle M \rangle, \tag{3.1}$$

where M is a *continuous* local martingale under $\mathbb{P}$ and λ is a predictable process, such that

$$\lambda^2 \cdot \langle M \rangle_T < \infty, \quad \mathbb{P} - a.s.. \tag{3.2}$$

Here and below, $\langle M \rangle$ denotes the quadratic variation of M. We note that the form of the dynamics of S is consistent with the semimartingale decomposition of a stock price, where the semimartingale property is necessary for the absence of arbitrage in the various forms, see, e.g., [29, 33, 34]. Further, the dynamics of S is consistent with the no-arbitrage characterizations in [35], which assert that (in continuous stock price process settings), the finite-variation part of the stock has to be absolutely continuous with respect to the quadratic variation for the martingale part of the stock, which is precisely captured by the dynamics in (3.1) together with the condition (3.2), which is closely related to the No Unbounded Profit with Bounded Risk (NUBPR) condition introduced in [34], we also refer to [29] for the overview of the recent developments of the market viability and no-arbitrage conditions. We also note that for the utility function defined on the positive real line, the existence and uniqueness results for the optimal investment problem in general settings, which might include intermediate consumption and stochastic preferences, follow from the abstract theorems in [22] and are formally established in [24].

Next, we introduce a parametric family of continuous semimartingales S^{ε}, $\varepsilon \in \mathbb{R}$, which correspond to perturbations of the finite-variation part of S, that is we set

$$S^{\varepsilon} = (\lambda + \varepsilon\nu) \cdot \langle M \rangle + M, \quad \varepsilon \in \mathbb{R}, \tag{3.3}$$

where ν is another predictable process, such that

$$\nu^2 \cdot \langle M \rangle_T < \infty, \quad \mathbb{P} - a.s., \tag{3.4}$$

we note that the latter condition, together with (3.2), ensures that the no-arbitrage condition in the sense of NUPBR holds for every $\varepsilon \in \mathbb{R}$.

The preferences of an economic agent are given by a utility function on $\mathbb{R}$, which we suppose to be the exponential utility with the risk-aversion $\gamma > 0$, that is

$$U(x) := -e^{-\gamma x}, \quad x \in \mathbb{R}. \tag{3.5}$$

Section 4

Existence and Uniqueness for the Base Model

4.1 Admissibility and the primal problem

We need to specify the admissible portfolios. For the ε's model, the portfolio processes are represented by the stochastic integrals with respect to S^{ε}. However, an admissibility condition is needed to rule out arbitrage, e.g., in the form of doubling strategies, etc. There are several admissibility conditions for the utility defined on the whole real line in the literature. We follow [1], and by admissible, we mean the wealth processes uniformly bounded from below, that is

$$\mathcal{X}(x,\varepsilon) = \{X \; bounded \; from \; below: \; X = x + H \cdot S^{\varepsilon}, \; H \; is \; S^{\varepsilon}-integrable\}, \quad (x,\varepsilon) \in \mathbb{R}^2. \tag{4.1}$$

Now, we are ready to set the family of the optimization problems as

$$u(x,\varepsilon) := \sup_{X \in \mathcal{X}(x,\varepsilon)} \mathbb{E}\left[U(X_T)\right], \quad (x,\varepsilon) \in \mathbb{R}^2. \tag{4.2}$$

here we use the convention

$$\mathbb{E}\left[U(X_T)\right] := -\infty, \quad if \quad \mathbb{E}\left[U^-(X_T)\right] = \infty,$$

where U^- is the negative part of U.

4.2 The duality approach

The investigation of the primal problem (4.2) is performed via the duality approach. A probability measure $\mathbb{Q}(\varepsilon) \sim \mathbb{P}(\varepsilon)$ (resp. $\mathbb{Q}(\varepsilon) \ll \mathbb{P}(\varepsilon)$) is called an equivalent (resp. absolutely continuous) local martingale measure if S^ε is a local martingale under $\mathbb{Q}(\varepsilon)$, $\varepsilon \in \mathbb{R}$. For $\varepsilon \in \mathbb{R}$, the family of equivalent (resp. absolutely continuous) local martingale measure will be denoted by $\mathcal{M}^e(\varepsilon)$ (resp. $\mathcal{M}^a(\varepsilon)$).

The dual to the utility function U is defined to be its convex conjugate, that is

$$V(y) := \sup_{x\in\mathbb{R}} (U(x) - xy) = \frac{y}{\gamma}\left(-1 + \log\frac{y}{\gamma}\right), \quad y > 0. \tag{4.3}$$

we set the dual value functions as

$$v(y,\varepsilon) := \inf_{\mathbb{Q}\in\mathcal{M}^a(\varepsilon)} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \quad (y,\varepsilon) \in (0,\infty)\times\mathbb{R}. \tag{4.4}$$

we use the convention

$$\mathbb{E}\left[V\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] := -\infty, \quad if \quad \mathbb{E}\left[V^+\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] = \infty,$$

where V^+ is the positive part of V.

4.3 Technical assumptions

4.3.1 Absence of arbitrage for the base model

We will impose the conditions on the base model that are consistent with [1], and thus we suppose that the set of equivalent martingale measures is non-empty.

$$\mathcal{M}^e(0) \neq \emptyset, \tag{4.5}$$

this condition is equivalent to the No Free Lunch with Vanishing Risk condition, see [33]. Note that the Inada conditions

$$\lim_{x \downarrow -\infty} U'(x) = \infty \quad and \quad \lim_{x \uparrow \infty} U'(x) = 0,$$

are satisfied by the exponential utility in (3.5).

4.3.2 Finiteness of the value function for the base model

We need to impose finiteness of the value function in the following sense

$$u(x, 0) < U(\infty) := \lim_{x \uparrow \infty} U(x), \quad for\ some\ x \in \mathbb{R}, \tag{4.6}$$

and thus for every $x \in \mathbb{R}$, which is precisely equation (9) in [1]. For $\varepsilon = 0$, the results of [1, Theorem 2.2] imply the existence and uniqueness of a solution to both (4.2) and (4.4). We recall them here for the case of the exponential utility. First, following [1, Definition 1.3], we need to introduce the following sets.

Definition 4.3.1. For $(x,\varepsilon) \in \mathbb{R}^2$, define the set $C_U^b(x,\varepsilon)$ by

$$C_U^b(x,\varepsilon) = \Big\{G_T \in \mathbb{L}^0(\Omega, \mathcal{F}_T, \mathbb{P}) : G_T \le X_T \\ for\ some\ X \in \mathcal{X}(x,\varepsilon)\ and\ \mathbb{E}\left[|U(G_T)|\right] < \infty\Big\}. \tag{4.7}$$

and let $C_U(x,\varepsilon)$ denotes the set

$$C_U(x,\varepsilon) = \Big\{G_T \in \mathbb{L}^0(\Omega, \mathcal{F}_T, \mathbb{P} : \mathbb{R} \cup \{\infty\}) \ : \ U(F_T)\ is\ in \\ the\ \mathbb{L}^1(\mathbb{P}) - closure\ of\ \{U(G_T) :\ G_T \in C_U^b(x,\varepsilon)\}\Big\}. \tag{4.8}$$

4.4 Schachermayer's theorem from [1] for the base model

We state a version of [1, Theorem 2.2] in the settings of our base model.

Theorem 4.4.1 (Schachermayer, 2001, Theorem 2.2)**.** *Under the assumptions* (3.5)*,* (4.5)*, and* (4.6) *hold for the base model. Then, we have*

(i) The value functions $u(\cdot,0)$ *and* $v(\cdot,0)$ *are finitely valued, then (trivially under* (3.5)*) strictly concave (resp. convex), differentiable functions defined on* $\mathbb{R}$ *(resp.* $\mathbb{R}_+$*); they are conjugate and satisfy the Inada conditions*

$$\lim_{x\uparrow\infty} u_x(x,0) = 0, \quad \lim_{y\downarrow 0} v_y(y,0) = -\infty,$$

$$\lim_{x\downarrow -\infty} u_x(x,0) = \infty, \quad \lim_{y\uparrow\infty} v_y(y,0) = \infty.$$

Further, under (3.5)*,* $u(\cdot,0)$ *is of the exponential form* (3.5) *times a constant, and* $v(\cdot,0)$ *is its convex conjugate.*

(ii) For every $y > 0$*, the optimal solution* $\hat{\mathbb{Q}}(0) \in \mathcal{M}^a(0)$ *to the dual problem* (4.4) *exists for* $\varepsilon = 0$*.*

(iii) *For* $x \in \mathbb{R}$*, the optimal solution* $\hat{X}_T(x,0) \in C_U(x,0)$ *to the primal problem* (4.2) *exists, is unique and is given by*

$$\hat{X}_T(x,0) = I\left(y\frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}}\right),$$

where, $I = (U')^{-1}$*, and* y *is given by*

$$y = u_x(x,0) = -\gamma u(x,0) = -\gamma e^{-\gamma x} u(0,0). \tag{4.9}$$

(iv) *If* $\hat{\mathbb{Q}}(0) \in \mathcal{M}^{\varepsilon}(0)$ *and* $x = -v_y(y,0)$*, then* $\hat{X}_T(x,0)$ *equals the terminal value for a process of the form* $\hat{X}_t(x,0) = x + H \cdot S_t$, $t \in [0,T]$*, where* H *is predictable and* S*-integrable, such that* $\hat{X}$ *is a uniformly integrable martingale under* $\hat{\mathbb{Q}}(0)$*.*

(v) *We have the formulae*

$$v_y(y,0) = \mathbb{E}\left[\frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}} V'\left(y\frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}}\right)\right],$$

$$x u_x(x,0) = \mathbb{E}\left[\hat{X}_T(x,0) U'(\hat{X}_T(x,0))\right],$$

where the usual rule $0 \times \infty = 0$ *is applied, if the integrands are of this form.*

Section 5

Stability

5.1 An overview of the results on stability

The question of stability under parametrization (3.3) is among the central ones. We note that stability was investigated in [36] and, in dynamic settings, in [37]. In general, stability can be investigated under different parametrizations of perturbations, including more general ones, and under convergences of the market price of risk processes in the corresponding topologies.

As the focus of this work is on the sensitivity analysis, the results on stability follow as a byproduct of the asymptotic expansions, where even the first-order results on the value function u imply the stability. Conversely, under perturbations (3.3), one can show that the lack of the existence of the first-order expansion of u in ε implies instability. More precisely, one can show that stability will fail if Assumption 5.2.1 below does not hold. A counterexample can be constructed along the lines of [4, Example 7.1, p. 4467]. The discussion above motivates the following assumption central for the analysis below.

5.2 Assumption on perturbations

Only one integrability assumption should be imposed. Let us denote

$$F := \nu \cdot S_T \quad \textit{and} \quad G := \langle \nu \cdot S \rangle_T, \tag{5.1}$$

and, with $\mathcal{E}$ denoting the Doleans-Dade exponential, we set

$$L^{\varepsilon} := \mathcal{E}(-\varepsilon \nu \cdot S)_T = \exp\left(-\varepsilon F - \tfrac{1}{2}\varepsilon^2 G\right), \quad \varepsilon \in \mathbb{R}.$$

Assumption 5.2.1. There exists a constant $c > 0$, such that

$$\mathbb{E}^{\hat{\mathbb{Q}}(0)}\left[\exp\left((\gamma + c)(1 + |\hat{X}_T(x, 0)|)(1 + \bar{L}^c)\right)\right] < \infty,$$

where γ is given in (3.5),

$$\bar{L}^c = \exp(c(|F| + G)),$$

and $\hat{\mathbb{Q}}(0)$, the dual minimizer to the base model, is given by Theorem 4.4.1

Section 6

First-order Analysis

6.1 On Peano versus usual differentiability

We point out that below, the first-order derivatives of the value functions u and v in ε should be understood in the usual sense. On the other hand, the second-order derivates will be understood in the Peano sense, where the Peano differentiability is equivalent to the existence of the Taylor expansion on the appropriate order, with the remainder being given in the Peano form.

An illuminating example illustrating the subtle difference between the usual differentiability and the Peano differentiability is given in [38, p. 98]: the function

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right), & if \quad x \neq 0 \\ 0, & if \quad x = 0. \end{cases}$$

has quadratic expansion at $x = 0$ but is not two-times differentiable there. On the other hand, twice differentiability implies having quadratic expansion at x.

6.2 First-order expansion theorem

The first-order analysis can be conducted separately. However, the approach for the second-order analysis allows for the formulas for u_ε and v_ε. We recall that, under the assumptions of Theorem 4.4.1, the optimal solution to the dual problem $\hat{\mathbb{Q}}(0)$ is a density of an absolutely continuous martingale measure.

Theorem 6.2.1. *Let us consider the family of stock price returns given by* <u>*continuous*</u> *processes* (3.3)*. Let $x \in \mathbb{R}$ be fixed and assume that for $\varepsilon = 0$, the base model satisfies the assumptions of Theorem 4.4.1. For the perturbations, we suppose that Assumption 5.2.1 holds. Then, there exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have*

$$u(x,\varepsilon) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad \textit{and} \quad v(y,\varepsilon) \in \mathbb{R}, \quad y > 0.$$

Further, the partial derivatives of the primal and dual value functions at $\varepsilon = 0$ are well-defined and are given by

$$u_\varepsilon(x,0) = v_\varepsilon(y,0) = y\mathbb{E}^{\hat{\mathbb{Q}}(0)}\left[\hat{X}_T(x,0)F\right], \quad y = u_x(x,0),$$

where $\mathbb{Q}$ is the dual minimizer to (4.4) *at $(y,\varepsilon) = (u_x(x,0), 0)$.*

Section 7

Second-order Analysis

7.1 Quadratic minimization problem for the primal problem

We note that with the admissibility as in [1], an extra condition that the optimizer to the base (that is corresponding to $\varepsilon = 0$) dual problem, $\hat{\mathbb{Q}}(0)$ is a density of the *equivalent* (and not just absolutely conditions) martingale measure ensures that

$$\hat{X}(x, 0) = x + H \cdot S,$$

and such that $\hat{X}$ is a uniformly integrable martingale under $\hat{\mathbb{Q}}(0)$. This is due to the results of [1]. However, the abstract versions allow circumventing this condition.

Next, for the perturbed model, we look for an approximate nearly optimal wealth process in the form

$$X^{\varepsilon} = \left(\hat{X}(x, 0) + \varepsilon H^1 \cdot S\right)(\mathcal{E}(-\varepsilon\nu \cdot S))^{-1}, \tag{7.1}$$

where H^1 is yet to be determined. This leads to the following auxiliary optimization problem that determines H^1.

$$\begin{aligned} a_{\varepsilon\varepsilon} &= -y \min_{H^1\cdot S_T\in\mathbb{L}^2(\hat{\mathbb{Q}}(0))} \mathbb{E}^{\hat{\mathbb{Q}}(0)}\Big[A(\hat{X}_T(x,0))\Big(H^1\cdot S_T + \hat{X}(x,0)F\Big)^2 \\ &\qquad\qquad -2FH^1\cdot S_T - \hat{X}(x,0)(F^2+G)\Big] \\ &= -y \min_{H^1\cdot S_T\in\mathbb{L}^2(\hat{\mathbb{Q}}(0))} \mathbb{E}^{\hat{\mathbb{Q}}(0)}\Big[A(\hat{X}_T(x,0))\Big(H^1\cdot S_T + \hat{X}(x,0)F\Big)^2 \\ &\qquad\qquad -2F\Big(H^1\cdot S_T + \hat{X}(x,0)F\Big) + \hat{X}(x,0)(F^2-G)\Big], \end{aligned} \tag{7.2}$$

where $A(X_T^0) = \gamma$ is the absolute risk aversion of U. As Theorem 7.3.1 shows, (7.2) gives the second-order expansion term for u in ε.

7.2 Quadratic minimization problem for the dual problem

For the dual, we seek for the nearly optimal density of the absolutely continuous equivalent martingale measure in the form

$$Y^\varepsilon = y\frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}}L^\varepsilon(1+\varepsilon N^1), \tag{7.3}$$

where N^1 is a correction to the density process of $\frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}}$ with the following properties: $N_T^1 \in \mathbb{L}^\infty(\hat{\mathbb{Q}}(0))$, and $\frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}}(1+\varepsilon N_T^1)$ is a density of the absolutely continuous martingale measure for the base model for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some constant $\varepsilon_0 > 0$. We denote the set of such densities by $\mathcal{N}^\infty(\hat{\mathbb{Q}}(0))$. This set consists of a singleton if the base model is complete and is larger than a singleton if the base model is incomplete. Also, as we are working with the continuous stock prices process in (3.1), which is in particular sigma-bounded, where we refer to [39] for the discussion of sigma-boundedness and its connection to the structure the closure of $\mathcal{N}^{\infty}(\hat{\mathbb{Q}}(0))$ in $\mathbb{L}^2(\hat{\mathbb{Q}}(0))$. We denote this closure by $\mathcal{N}^2(\hat{\mathbb{Q}}(0))$. One can see that $\mathcal{N}^2(\hat{\mathbb{Q}}(0))$ is space orthogonal square-integrable $\hat{\mathbb{Q}}(0)$-martingales of the form $H \cdot S$, where H is S-integrable. We refer to Appendix A for the discussion of orthogonal martingales. Actually, in the present settings $\mathcal{N}^2(\hat{\mathbb{Q}}(0))$ is the orthogonal complement of the space of $\hat{\mathbb{Q}}(0)$ square-integrable wealth processes of the form $H \cdot S$. This result follows from [39, Lemma 6].

This leads to the following quadratic minimization problem that governs $v_{\varepsilon\varepsilon}(y, 0)$.

$$\begin{aligned} b_{\varepsilon\varepsilon} &= y \min_{N^1 \in \mathcal{N}^2(\hat{\mathbb{Q}}(0))} \mathbb{E}^{\hat{\mathbb{Q}}(0)}\left[V''(Y_T^0)Y_T^0\left(N_T^1 - F\right)^2 - \hat{X}_T(x,0)(F^2 - G - 2N_T^1 F)\right] \\ &= y \min_{N^1 \in \mathcal{N}^2(\hat{\mathbb{Q}}(0))} \mathbb{E}^{\hat{\mathbb{Q}}(0)}\left[B(Y_T^0)\left(N_T^1 - F\right)^2 + 2\left(N_T^1 - F\right)\hat{X}_T(x,0)F + \hat{X}_T(x,0)(F^2 + G)\right], \end{aligned} \tag{7.4}$$

where $Y_T^0 = y\frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}}$ and $B(y) = V''(y)y$, $y > 0$.

7.3 Second-order expansion theorem

The optimization problems (7.2) and (7.4) are the ones that define the second-order Peano derivatives of the value functions in ε at $\varepsilon = 0$ rigorously. This is the subject of the following theorem.

Theorem 7.3.1. *Let $x \in \mathbb{R}$ be fixed, and the conditions of Theorem 6.2.1 hold. Then, with $y = u_x(x, 0)$, we have*

$$u(x, \varepsilon) = u(x, 0) + \varepsilon u_\varepsilon(x, 0) + \tfrac{1}{2}a_{\varepsilon\varepsilon}\varepsilon^2 + o(\varepsilon^2),$$

$$v(y, \varepsilon) = v(y, 0) + \varepsilon v_\varepsilon(y, 0) + \tfrac{1}{2}b_{\varepsilon\varepsilon}\varepsilon^2 + o(\varepsilon^2),$$

where $a_{\varepsilon\varepsilon}$ *and* $b_{\varepsilon\varepsilon}$ *are defined in* (7.2) *and* (7.4)*, whereas* $u_\varepsilon(x, 0)$ *and* $v_\varepsilon(y, 0)$ *are given by Theorem 10.3.1*.

Section 8

Derivatives of the Optimizers to (4.2) and (4.4)

The main result of this section is the following theorem.

Theorem 8.0.1. *Let $x \in \mathbb{R}$ be fixed assume that the assumptions of Theorem 6.2.1 hold, and $y = u_x(x, 0)$. Then the solutions to* (7.2) *and* (7.4) *exist and are unique $\mathbb{P} - a.s.$, we denote these solutions $H^1 \cdot S_T$ and N^1_T, respectively. The derivatives of the optimizers to* (4.2) *and* (4.4) *are given by*

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{\hat{X}_T(x,\varepsilon) - \hat{X}_T(x,0)}{\varepsilon} &= H^1 \cdot S_T + \hat{X}_T(x,0)F, \\ \lim_{\varepsilon \to 0} \frac{\frac{d\hat{\mathbb{Q}}(\varepsilon)}{d\mathbb{P}} - \frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}}}{\varepsilon} &= \frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}}\left(N^1_T - F\right), \end{aligned} \tag{8.1}$$

where the limits are in probability $\mathbb{P}$.

The proof of this theorem is also given below through the abstract version.

Section 9

Corrections to the Optimal Trading Strategy

While differentiability of the optimal strategies in the usual (or even somewhat weaker) sense is difficult to establish and likely requires much more stringent assumptions, one still can construct corrections to the trading strategies that match the indirect utility u up to the second-order in ε. In view of the approximation procedure in (7.1), the corrections can be constructed by stopping H^1, where the stopping rules have to be formulated precisely, and here the parallel to [3] is natural.

To be more precise, let us consider H^1, such that $H^1 \cdot S_T$ is the minimizer to (7.2). As the process $H^1 \cdot S$ is continuous, we can use localization to make the approximating sequence bounded. Thus, let us define the following family of stopping times

$$\tau_\delta = \inf\left\{t \in [0, T] : \ |H^1 \cdot S_t| \geq \frac{1}{\delta}\right\}, \quad \delta > 0.$$

Next, we set

$$H^{1,\delta} = H^1 1_{[0,\tau_\delta]}, \quad \delta > 0.$$

Theorem 9.0.1. *Assume that $x \in \mathbb{R}$ is fixed and assumptions of Theorem 6.2.1 hold.*

For every $(\varepsilon, \delta) \in \mathbb{R} \times \mathbb{R}_+$, *let us define*

$$X^{\delta,\varepsilon} = \left(\widehat{X}(x,0) + \varepsilon H^{1,\delta} \cdot S\right) \frac{1}{\mathcal{E}(-\varepsilon \nu \cdot S)}.$$

Then there exists a function $\delta = \delta(\varepsilon)$, $\varepsilon \in \mathbb{R}$, *such that*

$$\mathbb{E}\left[U\left(X_T^{\delta,\varepsilon}\right)\right] = u(x,\varepsilon) - o(\varepsilon^2).$$

Section 10

Abstract Version of the Main Results

10.1 Abstract primal and dual problems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We defined by $\mathbb{L}^0$ the vector space of (equivalence classes of) real-valued random variables on this probability space and by $\mathbb{L}^0_+$ its positive orthant. Following the construction of $C_U(x,0)$ as subsets of $\mathbb{L}^0$, below, we also define $C_U(x,\varepsilon)$ and an abstract family of sets for the dual abstract problem. This can be done as follows. For some random variables F and $G > 0$, let us set

$$L^\varepsilon = \exp\left(-\varepsilon F - \tfrac{1}{2}\varepsilon^2 G\right), \quad \varepsilon \in \mathbb{R}.$$

Next, we define

$$C_U(x,\varepsilon) = C_U(x,0)\frac{1}{L^\varepsilon}, \quad (x,\varepsilon) \in \mathbb{R}^2. \tag{10.1}$$

For the dual domain, first, we set

$$D(1,0) = cl\{h \in \mathbb{L}^0_+ :\ 0 \le h \le \frac{d\mathbb{Q}}{d\mathbb{P}},\ \ for\ some\ \mathbb{Q} \in \mathcal{M}^a(0)\}, \tag{10.2}$$

where the closure is taken in $\mathbb{L}^0$. Next, we define

$$D(y,\varepsilon) = yD(1,0)L^\varepsilon, \quad (y,\varepsilon) \in \mathbb{R}_+ \times \mathbb{R}. \tag{10.3}$$

With these preparations, we define the abstract versions of the perturbed optimization problems as follows

$$u(x,\varepsilon) = \sup_{\xi\in C_U(x,\varepsilon)} \mathbb{E}\left[U(\xi)\right] = \sup_{\xi\in C_U(x,0)} \mathbb{E}\left[U\left(\frac{\xi}{L^\varepsilon}\right)\right], \quad (x,\varepsilon)\in\mathbb{R}^2,$$
$$v(y,\varepsilon) = \inf_{\eta\in D(y,\varepsilon)} \mathbb{E}\left[V(\eta)\right] = \inf_{\eta\in D(y,0)} \mathbb{E}\left[V\left(\eta L^\varepsilon\right)\right], \quad (y,\varepsilon)\in\mathbb{R}_+\times\mathbb{R}. \tag{10.4}$$

We note that both $C_U(x,0)$ and $D(y,0)$ contain strictly positive elements by their construction. For the abstract base model, we need to suppose that

$$u(x,0) < \infty, \quad for\ some\ x\in\mathbb{R}. \tag{10.5}$$

Now, we state the abstract version of Theorem 4.4.1.

Theorem 10.1.1. *Let $x\in\mathbb{R}$ be fixed. Then, under conditions* (3.5)*,* (4.5)*, and* (10.5)*, we have:*

(i) The value functions $u(\cdot,0)$ and $v(\cdot,0)$ are finitely valued, then (trivially under (3.5)*) strictly concave (resp. convex), differentiable functions defined on $\mathbb{R}$ (resp. $\mathbb{R}_+$); they are conjugate and satisfy the Inada conditions*

$$\lim_{x\uparrow\infty} u_x(x,0) = 0, \quad \lim_{y\downarrow 0} v_y(y,0) = -\infty,$$

$$\lim_{x\downarrow\infty} u_x(x,0) = \infty, \quad \lim_{y\uparrow\infty} v_y(y,0) = \infty.$$

Further, under (3.5), u(·, 0) is of the exponential form (3.5) times a constant, and v(·, 0) is its convex conjugate.

(ii) For every $y > 0$ and $\varepsilon = 0$, the optimal solution $\eta(y,0)$ to the dual problem (4.4) *exists and*

$$\mathbb{E}[\eta(y,0)] = y. \tag{10.6}$$

(iii) For $x \in \mathbb{R}$, the optimal solution $\xi(x,0) \in C_U(x,0)$ to the primal problem in (10.4) *exists is unique, and is given by*

$$\xi(x,0) = I\left(\eta(y,0)\right),$$

where $I = (U')^{-1}$ and y is given by

$$y = u_x(x,0) = -\gamma u(x,0) = -\gamma e^{-\gamma x} u(0,0). \tag{10.7}$$

(iv) We have the formulae

$$yv_y(y,0) = \mathbb{E}\left[\eta(y,0)V'\left(\eta(y,0)\right)\right],$$

$$xu_x(x,0) = \mathbb{E}\left[\xi(x,0)(x,0)U'(\xi(x,0))\right],$$

where the usual rule $0 \times \infty = 0$ is applied, if the integrands are of this form.

Under the integrability conditions specified below, for ε in a neighborhood of 0, the existence and uniqueness of solutions to both problems in (10.4) can be established as in [1], and the case $\varepsilon = 0$ is already specified above. For the dual, as pointed out by [1, p. 699 and below], by passing to the solid hull of $D(1,0)$, does not change the optimizer, and the optimizer satisfies $\eta(y,0) = y\frac{d\mathbb{Q}}{d\mathbb{P}}$ for some probability measure $\mathbb{Q}$, equivalently that (10.6) holds. This ensures consistency with the concrete versions of the main results.

10.1.1 Condition on perturbations

For a fixed $x \in \mathbb{R}$, and with $y = u_x(x,0)$, which is well-defined by the argument of Theorem 10.1.1 above, assuming (10.6), let us define a probability measure $\mathbb{Q}$ via its

Radon-Nikodym derivative with respect to $\mathbb{P}$ as follows

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\eta(y,0)}{y},$$

where $\eta(y,0)$ is the optimizer to the dual problem in (10.4) corresponding to $y = u_x(x,0)$ and $\varepsilon = 0$. Next, let us set the abstract version of the integrability assumption 5.2.1. As in item 4 of Theorem 4.4.1, we need to suppose that the dual minimizer in (10.4), for $\varepsilon = 0$, satisfies

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\eta(y,0)}{y} > 0, \quad \mathbb{P} - a.s., \tag{10.8}$$

which is the way of ensuring that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is an equivalent and not just an absolutely continuous measure. Next, we state the abstract version of the integrability condition.

Assumption 10.1.2. There exists a constant $c > 0$, such that

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left((\gamma + c)(1 + |\xi(x,0)|)(1 + \bar{L}^c)\right)\right] < \infty,$$

where γ is given in (3.5), $\xi(x,0)$ is the primal optimizer in (10.4) at $(x,0)$ and

$$\bar{L}^c = \exp(c(|F| + G)).$$

The following assumption corresponds to the abstract version of the NFLVR for perturbed models, so that we could get the assertions of Theorem 10.1.1 in some neighborhood of $\varepsilon = 0$.

Assumption 10.1.3. There exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\varepsilon F - \frac{\varepsilon^2}{2}G\right)\right] = 1.$$

10.1.2 Auxiliary sets $\mathcal{A}$ and $\mathcal{B}$

Let us consider the following sets

$$\begin{aligned} \mathcal{A}^\infty(x,0) &= \{\alpha \in \mathbb{L}^\infty : \ \xi(x,0) \pm c\alpha \in C_U(x,0) \ for\ some\ c > 0\}, \quad x \in \mathbb{R} \\ \mathcal{B}^\infty(y,0) &= \{\beta \in \mathbb{L}^\infty : \ \eta(y,0)(1 \pm c\beta) \in D(y,0) \ for\ some\ c > 0\}, \quad y > 0. \end{aligned} \tag{10.9}$$

Then $\mathcal{A}^\infty(x,0)$ and $\mathcal{B}^\infty(u_x(x,0),0)$ are orthogonal linear subspaces of

$$\mathbb{L}^2(\mathbb{Q}) = \{\zeta \in \mathbb{L}^0 : \ \mathbb{E}^{\mathbb{Q}}[\zeta] = 0 \quad and \quad \mathbb{E}^{\mathbb{Q}}\left[\zeta^2\right] < \infty\}.$$

By $\mathcal{A}^2(x,0)$ and $B^2(y,0)$, we denote the closures of $\mathcal{A}^\infty(x,0)$ and $\mathcal{B}^\infty(y,0)$ in $\mathbb{L}^2(\mathbb{Q})$.

Assumption 10.1.4. For $x \in \mathbb{R}$ and $y = u_x(x,0)$ and $\varepsilon = 0$, $\mathcal{A}^2(x,0)$ and $\mathcal{B}^2(y,0)$ are *complementary* linear subspaces of $\mathbb{L}^2(\mathbb{Q})$, that is

$$\alpha \in \mathcal{A}^2(x,0) \ \ if\ and\ only\ if \ \ \alpha \in \mathbb{L}^2(\mathbb{Q}) \ \ and \ \ \mathbb{E}^{\mathbb{Q}}[\alpha\beta] = 0, \ \ for\ every\ \ \beta \in \mathcal{B}^2(y,0),$$

$$\beta \in \mathcal{B}^2(y,0) \ \ if\ and\ only\ if \ \ \beta \in \mathbb{L}^2(\mathbb{Q}) \ \ and \ \ \mathbb{E}^{\mathbb{Q}}[\alpha\beta] = 0, \ \ for\ every\ \ \alpha \in \mathcal{A}^2(x,0).$$

10.2 First-order expansion

The following theorem shows the differentiability of the value functions u and v in ε at $\varepsilon = 0$ and is a consequence of the second-order expansion results. This is why Assumption 10.1.4 is stated above Theorem 10.2.1. The proof of the following theorem could be given without Assumption 10.1.4.

Theorem 10.2.1. *Let $x \in \mathbb{R}$ be fixed. Suppose that the assumptions of Theorem 10.1.1, and Assumptions 10.1.2, 10.1.3, and 10.1.4 hold.*

Then, there exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have

$$u(x,\varepsilon) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad \textit{and} \quad v(y,\varepsilon) \in \mathbb{R}, \quad y > 0.$$

Further

$$u_\varepsilon(x,0) = v_\varepsilon(y,0) = y\mathbb{E}^{\mathbb{Q}}\left[\xi(x,0)F\right], \quad x \in \mathbb{R},\ y = u_x(x,0),$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\eta(y,0)}{y}$.

10.3 Second-order results

To state the second-order expansion theorems, we start with abstract versions of the auxiliary minimization problems first. Let us consider

$$a_{\varepsilon\varepsilon} = -y \inf_{\alpha \in \mathcal{A}^2(x,0)} \mathbb{E}^{\mathbb{Q}}\left[A(\xi(x,0))\left(\alpha + \xi(x,0)F\right)^2 - 2F\left(\alpha + \xi(x,0)F\right) + \xi(x,0)(F^2 - G)\right], \tag{10.10}$$

where $A(\xi(x,0)) = \gamma$ is the absolute risk aversion of U.

$$b_{\varepsilon\varepsilon} = y \min_{\beta \in \mathcal{B}^2(y,0)} \mathbb{E}^{\mathbb{Q}}\left[B(\eta(y,0))\left(\beta - F\right)^2 + 2\left(\beta - F\right)\xi(x,0)F + \xi(x,0)(F^2 + G)\right], \tag{10.11}$$

where $B(y) = V''(y)y$, $y > 0$. Now, we are ready to state the second-order expansion results for the abstract u and v. We note that the quadratic optimization problems are investigated in the literature extensively, and we refer to [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], see also an overview of several approaches to quadratic problems in [50] and [7].

Theorem 10.3.1. *Let $x \in \mathbb{R}$ be fixed and the conditions of Theorem 10.2.1 hold. Then, with $y = u_x(x, 0)$, we have*

$$u(x, \varepsilon) = u(x, 0) + \varepsilon u_\varepsilon(x, 0) + \tfrac{1}{2} a_{\varepsilon\varepsilon}\varepsilon^2 + o(\varepsilon^2), \tag{10.12}$$

$$v(y, \varepsilon) = v(y, 0) + \varepsilon v_\varepsilon(y, 0) + \tfrac{1}{2} b_{\varepsilon\varepsilon}\varepsilon^2 + o(\varepsilon^2), \tag{10.13}$$

where $a_{\varepsilon\varepsilon}$ and $b_{\varepsilon\varepsilon}$ are defined in (10.10)*, whereas $u_\varepsilon(x, 0)$ and $v_\varepsilon(y, 0)$ are given by Theorem 10.3.1.*

Now, we can state the abstract version of the theorem, which specifies the derivatives of the optimizers.

Theorem 10.3.2. *Let $x \in \mathbb{R}$ be fixed and the conditions of Theorem 10.2.1 hold. Then, with $y = u_x(x, 0)$, we have*

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{\xi(x, \varepsilon) - \xi(x, 0)}{\varepsilon} &= \alpha + \xi(x, 0) F, \\ \lim_{\varepsilon \to 0} \frac{\eta(y, \varepsilon) - \eta(y, 0)}{\varepsilon} &= \eta(y, 0)\,(\beta - F)\,, \end{aligned} \tag{10.14}$$

where α and β are optimizers to (10.10) *and* (10.11)*, respectively, and the convergence takes place in probability $\mathbb{P}$.*

10.4 Proofs

10.4.1 Obtaining a lower bound for the expansion of u

We begin with the following lemma.

Lemma 10.4.1. *Let $x \in \mathbb{R}$ be fixed and the assumptions of Theorem 10.2.1 hold, and $y = u_x(x, 0)$. For an arbitrary random variable $\alpha \in \mathcal{A}^\infty(x, 0)$, let us define*

$$\psi(s) = (\xi(x, 0) + s\alpha)\frac{1}{L^s},$$

and

$$w(s) = \mathbb{E}\left[U(\psi(s))\right], \quad s \in \mathbb{R}.$$

Then w admits the following second-order expansion at $s = 0$:

$$w(s) = w(0) + sw'(0) + \tfrac{1}{2}s^2 w''(0) + o(s^2),$$

where the derivative of w at 0 are given by

$$w'(0) = y\mathbb{E}^{\mathbb{Q}}\left[\xi(x, 0)F\right],$$

$$w''(0) = -y\mathbb{E}^{\mathbb{Q}}\Big[A(\xi(x, 0))\left(\alpha + \xi(x, 0)F\right)^2 - 2F\left(\alpha + \xi(x, 0)F\right) + \xi(x, 0)(F^2 - G)\Big].$$

Proof. The proof is very close to the one in [3, proof of Lemma 5.14]. It is omitted for brevity. □

As the partial derivatives in x are trivial for U given by (3.5), we can state the following corollary.

Corollary 10.4.2. *Let $x \in \mathbb{R}$ be fixed and the assumptions of Theorem 10.2.1 hold, and $y = u_x(x, 0)$. For $a_{\varepsilon\varepsilon}$ being defined in* (10.10) *and α being the minimizer to* (10.10)*, let us set*

$$a_{x\varepsilon} = -y\mathbb{E}^{\mathbb{Q}}\left[A(\xi(x, 0))(\xi(x, 0)F + \alpha) - F\right] \quad \text{and} \quad a_{xx} = u_{xx}(x, 0), \tag{10.15}$$

as well as

$$H_u = \begin{pmatrix} a_{xx} & a_{x\varepsilon} \\ a_{x\varepsilon} & a_{\varepsilon\varepsilon} \end{pmatrix}. \tag{10.16}$$

Then, we have

$$u(x + \Delta x, \varepsilon) \geq u(x, 0) + \Delta x y + \varepsilon y \mathbb{E}^{\mathbb{Q}}\left[\xi(x, 0)F\right] + \tfrac{1}{2}(\Delta x \;\; \varepsilon) H_u \begin{pmatrix} \Delta x \\ \varepsilon \end{pmatrix} + o(\Delta x^2 + \varepsilon^2).$$

10.4.2 Obtaining an upper bound for the expansion of v

Similarly to Lemma 10.4.1, we state the following expansion for an upper bound for the abstract dual value function. The proof is omitted for brevity.

Lemma 10.4.3. *Let $x \in \mathbb{R}$ be fixed and the assumptions of Theorem 10.2.1 hold, and $y = u_x(x, 0)$. For an arbitrary random variable $\beta \in \mathcal{B}^\infty(y, 0)$, let us define*

$$\phi(t) = \frac{\eta(y, 0)}{y}(y + t\beta) L^t,$$

and

$$\tilde{w}(t) = \mathbb{E}\left[V(\phi(t))\right], \quad t \in \mathbb{R}.$$

The the function $\tilde{w}$ admits the following expansion at 0.

$$\tilde{w}(t) = \tilde{w}(0) + t\tilde{w}'(0) + \tfrac{1}{2}t^2\tilde{w}''(0) + o(t^2),$$

where the derivatives of $\tilde{w}$ at 0 are given by

$$\tilde{w}'(0) = y\mathbb{E}^{\mathbb{Q}}\left[\xi(x, 0)F\right],$$

$$\tilde{w}''(0) = y\mathbb{E}^{\mathbb{Q}}\left[B(\eta(y, 0))(\beta - F)^2 + 2(\beta - F)\xi(x, 0)F + \xi(x, 0)(F^2 + G)\right].$$

Corollary 10.4.4. *Let $x \in \mathbb{R}$ be fixed and the assumptions of Theorem 10.2.1 hold, and $y = u_x(x,0)$. Then, with $b_{\varepsilon\varepsilon}$ is defined in* (10.11)*, and β being the minimizer to* (10.11)*, with*

$$b_{y\varepsilon} = \mathbb{E}^{\mathbb{Q}}\left[B(\eta(y,0))(\beta - F) + \xi(x,0)F\right] \quad \textit{and} \quad b_{yy} = v_{yy}(y,0), \tag{10.17}$$

as well as

$$H_v = \begin{pmatrix} b_{yy} & b_{y\varepsilon} \\ b_{y\varepsilon} & b_{\varepsilon\varepsilon} \end{pmatrix}, \tag{10.18}$$

we have

$$v(y+\Delta y, \varepsilon) \leq v(y,0) - x\Delta y + \varepsilon y \mathbb{E}^{\mathbb{Q}}\left[\xi(x,0)F\right] + \tfrac{1}{2}(\Delta y \;\; \varepsilon) H_v \begin{pmatrix} \Delta y \\ \varepsilon \end{pmatrix} + o(\Delta y^2 + \varepsilon^2).$$

10.4.3 Closing the duality gap

Lemma 10.4.5. *Let $x \in \mathbb{R}$ be fixed, and the assumptions of Theorem 10.2.1 hold. Then, with $y = u_x(x,0)$, for the coefficients $a_{\varepsilon\varepsilon}$, $b_{\varepsilon\varepsilon}$, $a_{x\varepsilon}$, and $b_{y\varepsilon}$ being defined in* (10.10), (10.11), (10.15)*, and* (10.17)*, respectively, we have*

$$a_{\varepsilon\varepsilon} - b_{\varepsilon\varepsilon} = a_{x\varepsilon} b_{y\varepsilon}. \tag{10.19}$$

Proof. Let us consider the optimizer to (10.10), α. As for every $\alpha' \in \mathcal{A}^2(x,0)$, $\alpha + \varepsilon\alpha'$ for (10.10) is optimal at $\varepsilon = 0$, we deduce that

$$A(\xi(x,0))(\alpha + \xi(x,0)F) - F = c + \tilde{\beta}, \tag{10.20}$$

for some $\tilde{\beta} \in \mathcal{B}^2(y, 0)$ and a constant c. Similarly, with β being the minimizer to (10.11), we get

$$B(\eta(x,0))(\beta - F) + \xi(x,0)F = d + \tilde{\alpha}, \tag{10.21}$$

for some $\alpha \in \mathcal{A}^2(x, 0)$ and a constant d. As $A(\xi(x, 0))B(\eta(y, 0)) = 1$, from (10.21) that

$$\begin{aligned} \beta - F &= A(\xi(x,0))\left(d + \tilde{\alpha} - \xi(x,0)F\right) \\ &= -d\frac{a_{xx}}{y} + A(\xi(x,0))\left(\tilde{\alpha} - \xi(x,0)F\right). \end{aligned} \tag{10.22}$$

Rearranging the terms, we get

$$A(\xi(x,0))\left(-\tilde{\alpha} + \xi(x,0)F\right) - F = -d\frac{a_{xx}}{y} - \beta.$$

Comparing this equation to (10.20) and using the uniqueness characterization of the optimizer to (10.10) given by (10.20), we deduce that

$$\tilde{\alpha} = -\alpha, \quad \beta = -\tilde{\beta}, \quad \textit{and} \quad c = -d\frac{a_{xx}}{y}.$$

Equivalently, we have

$$\tilde{\alpha} = -\alpha, \quad \tilde{\beta} = -\beta, \quad \textit{and} \quad d = cb_{yy}y. \tag{10.23}$$

This allows to restate (10.20) as

$$\begin{aligned} A(\xi(x,0))\left(\alpha + \xi(x,0)F\right) - F &= c - \beta, \\ B(\eta(y,0))\left(\beta - F\right) + \xi(x,0)F &= d - \alpha. \end{aligned} \tag{10.24}$$

To determine the constants c and d, let us take the expectations under $\mathbb{Q}$ in (10.24). This gives

$$\begin{aligned} c &= \mathbb{E}^{\mathbb{Q}}\left[A(\xi(x,0))\left(\alpha + \xi(x,0)F\right) - F\right], \\ d &= \mathbb{E}^{\mathbb{Q}}\left[B(\eta(y,0))\left(\beta - F\right) + \xi(x,0)F\right]. \end{aligned} \tag{10.25}$$

Comparing (10.25) to (10.15) and (10.17), we deduce that

$$c = -\frac{1}{y} a_{x\varepsilon} \quad and \quad d = b_{y\varepsilon}. \tag{10.26}$$

Further, from (10.22) and (10.23), we get

$$\beta - F = -d\frac{a_{xx}}{y} - A(\xi(x,0))\,(\alpha + \xi(x,0)F)\,. \tag{10.27}$$

Rearranging terms, we further get

$$A(\xi(x,0))\,(\alpha + \xi(x,0)F) = -(\beta - F) - d\frac{a_{xx}}{y} \tag{10.28}$$

Multiplying (10.27) by $B(\eta(y,0))$, we obtain

$$\begin{aligned} B(\eta(y,0))\,(\beta - F) &= -\,(\alpha + \xi(x,0)F) - d\frac{a_{xx}}{y} B(\eta(y,0)) \\ &= -\,(\alpha + \xi(x,0)F) + d. \end{aligned} \tag{10.29}$$

Now, using (10.28) and (10.29), we can rewrite $a_{\varepsilon\varepsilon}$ and $b_{\varepsilon\varepsilon}$ defined in (10.10) and (10.11), respectively, as

$$\begin{aligned} -a_{\varepsilon\varepsilon}\frac{1}{y} &= \mathbb{E}^{\mathbb{Q}}\left[(\alpha + \xi(x,0)F)\left(-(\beta - F) - d\frac{a_{xx}}{y}\right) - 2F\,(\alpha + \xi(x,0)F) + \xi(x,0)(F^2 - G)\right] \\ b_{\varepsilon\varepsilon}\frac{1}{y} &= \mathbb{E}^{\mathbb{Q}}\left[(\beta - F)\,(-\,(\alpha + \xi(x,0)F) + d) + 2(\beta - F)\xi(x,0)F + \xi(x,0)(F^2 + G)\right] \end{aligned}$$

In order to simplify $\frac{b_{\varepsilon\varepsilon} - a_{\varepsilon\varepsilon}}{y}$, let us set

$$\begin{aligned} J_1 &= \mathbb{E}^{\mathbb{Q}}\left[(\alpha + \xi(x,0)F)\left(-d\frac{a_{xx}}{y}\right) + d(\beta - F)\right], \\ J_2 &= \mathbb{E}^{\mathbb{Q}}\left[(\alpha + \xi(x,0)F)\,(F - \beta) - 2F\,(\alpha + \xi(x,0)F)\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}\left[(F - \beta)(\alpha + \xi(x,0)F) + 2(\beta - F)\xi(x,0)F + 2\xi(x,0)F^2\right], \end{aligned}$$

so that

$$\frac{b_{\varepsilon\varepsilon} - a_{\varepsilon\varepsilon}}{y} = J_1 + J_2. \tag{10.30}$$

By direct computations, we have

$$\begin{aligned}
\frac{J_2}{2} &= \mathbb{E}^{\mathbb{Q}}\left[(\alpha + \xi(x,0)F)(F - \beta) - F(\alpha + \xi(x,0)F) + (\beta - F)\xi(x,0)F + \xi(x,0)F^2\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\alpha F + \xi(x,0)F^2 - \beta\xi(x,0)F - \alpha F \right. \\
&\qquad \left. -\xi(x,0)F^2 + \beta\xi(x,0)F - \xi(x,0)F^2 + \xi(x,0)F^2\right] \\
&= 0.
\end{aligned}$$

Finally, let us rewrite J_1 as

$$J_1 = d\mathbb{E}^{\mathbb{Q}}\left[\left(-\frac{a_{xx}}{y}\right)(\alpha + \xi(x,0)F) + (\beta - F)\right] = dc \tag{10.31}$$

where, in the last equality, we have used (10.26). As $J_2 = 0$ and J_1 is given by (10.31), we conclude from (10.30) that (10.19) holds. □

Lemma 10.4.6. *Let $x \in \mathbb{R}$ be fixed, and the assumptions of Theorem 10.2.1 hold. Then, with $y = u_x(x,0)$ and*

$$\Delta y = a_{xx}\Delta x + a_{x\varepsilon}\varepsilon, \tag{10.32}$$

we have

$$(\Delta y \;\; \varepsilon)H_v \begin{pmatrix} \Delta y \\ \varepsilon \end{pmatrix} + 2\Delta x \Delta y = (\Delta x \;\; \varepsilon)H_u \begin{pmatrix} \Delta x \\ \varepsilon \end{pmatrix}, \tag{10.33}$$

Proof. First, similarly to the proof of Lemma 10.4.5, one can establish the following relations between the components of H_u and H_v, summarized in the matrix form

$$\begin{pmatrix} a_{xx} & 0 \\ a_{x\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} -b_{yy} & 0 \\ -b_{y\varepsilon} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here, for example, $a_{xx}b_{yy} = -1$ represents the well-known result stating that for the convexly conjugate two-times differentiable functions u and v, satisfying the assertions of Theorem 10.1.1, $u_{xx}(x,0)v_{yy}(y,0) = -1$. With these preliminaries, the assertion follows from Lemma 10.4.5 by direct substitution of Δy from (10.32) into (10.33) and the definitions of H_u and H_v in (10.16) and (10.18), respectively. □

Proof of Theorem 10.3.1. For Δx and ε being sufficiently close to 0 and Δy being given by (10.32), and from the convex conjugacy of $u(\cdot,\varepsilon)$ and $v(\cdot,\varepsilon)$, we get

$$\begin{aligned} u(x+\Delta x,\varepsilon) &\le v(y+\Delta y,\varepsilon) + (x+\Delta x)(y+\Delta y) \\ &\le v(y,0) - xy + \varepsilon y \mathbb{E}^{\mathbb{Q}}\left[\xi(x,0)F\right] + \tfrac{1}{2}(\Delta y \;\; \varepsilon) H_v \begin{pmatrix} \Delta y \\ \varepsilon \end{pmatrix} \\ &\quad + xy + x\Delta y + y\Delta x + \Delta x \Delta y + o(\Delta y^2 + \varepsilon^2), \end{aligned} \tag{10.34}$$

where in the second inequality, we have used the assertion of Corollary 10.4.4, and where H_v is given by (10.18). Using the identities $y = u_x(x,0)$, $x = -v_y(y,0)$, $u(x,0) = v(y,0) - xy$ from Theorem 10.1.1, and collecting terms in (10.34), we get

$$\begin{aligned} u(x+\Delta x,\varepsilon) &\le u(x,0) + \Delta x y + \varepsilon y \mathbb{E}^{\mathbb{Q}}\left[\xi(x,0)F\right] \\ &\quad + \Delta x \Delta y + \tfrac{1}{2}(\Delta y \;\; \varepsilon) H_v \begin{pmatrix} \Delta y \\ \varepsilon \end{pmatrix} + o(\Delta x^2 + \varepsilon^2). \end{aligned} \tag{10.35}$$

Similarly, from Corollary 10.4.2, we get

$$\begin{aligned} u(x+\Delta x,\varepsilon) &\ge u(x,0) + \Delta x y + \varepsilon y \mathbb{E}^{\mathbb{Q}}\left[\xi(x,0)F\right] \\ &\quad + \tfrac{1}{2}(\Delta x \;\; \varepsilon) H_u \begin{pmatrix} \Delta x \\ \varepsilon \end{pmatrix} + o(\Delta x^2 + \varepsilon^2). \end{aligned} \tag{10.36}$$

Comparing (10.35) and (10.36) and using (10.33) from Lemma 10.4.6, we deduce that

$$u(x + \Delta x, \varepsilon) = u(x, 0) + \Delta xy + \varepsilon y \mathbb{E}^{\mathbb{Q}} [\xi(x, 0)F] + \frac{1}{2}(\Delta x \ \ \varepsilon) H_u \begin{pmatrix} \Delta x \\ \\ \varepsilon \end{pmatrix} + o(\Delta x^2 + \varepsilon^2), \tag{10.37}$$

which completes the proof of (10.12) and, in particular, shows that

$$u_\varepsilon(x, 0) = y \mathbb{E}^{\mathbb{Q}} [\xi(x, 0)F] .$$

In turn, (10.13) can be obtained similarly. □

Finally, with the assertions of Theorem 10.3.1 established, the proof of Theorem 10.3.2 is a line-by-line adaptation of the proof of [3, Theorem 5.8].

Section 11

Proofs of the Main Theorems

We start from the following proposition.

Lemma 11.0.1. *Under the conditions of Theorem 7.3.1, we have*

$$C_U(1,\varepsilon) = C_U(1,0)\frac{1}{\mathcal{E}(-\varepsilon\nu\cdot S^0)_T}, \quad \varepsilon\in\mathbb{R}, \tag{11.1}$$

$$D(1,\varepsilon) = D(1,0)\mathcal{E}(-\varepsilon\nu\cdot S^0)_T, \quad \varepsilon\in\mathbb{R}. \tag{11.2}$$

where the families of sets C_U's and D's are defined in (10.1) *and* (10.3)*, respectively.*

Proof. Let us fix $\varepsilon\in\mathbb{R}$, and observe that, by an application of Ito's lemma, one can show that for any S^0 integrable process H, there exists a process H', such that

$$\frac{x + H\cdot S^0}{\mathcal{E}(-\varepsilon\nu\cdot S^0)} = x + H'\cdot S^\varepsilon. \tag{11.3}$$

Further, the converse is true as well. That is, for any S^ε-integrable process H', there exists a process H such that (11.3) holds. As any process of the form $x + H'\cdot S^\varepsilon$ can be approximated by bounded stochastic integrals by stopping, we deduce that (11.1) holds.

Next, we show (11.2). Let us recall the definition of a polar of a set in the present settings. We say that a polar of a set $A \subseteq \mathbb{L}^0_+$ is

$$A^o = \{h \in \mathbb{L}^0_+ : \ \mathbb{E}[gh] \leq 1, \quad for\ every \quad g \in A\}.$$

Here it is convenient to use the bipolar theorem of Brannath and Schachermayer [51], which asserts that, A^{oo}, the bipolar of a set A, is the smallest convex, solid, and closed set containing A.

Now, let us recall the definitions of the sets of nonnegative wealth processes and supermartingale deflators. For every $\varepsilon \in \mathbb{R}$, we set

$$\mathcal{X}(1,\varepsilon) = \{X \geq 0 : \ X = 1 + H \cdot S^{\varepsilon}, \ \ for\ some\ S^{\varepsilon} - integrable\ H\},$$

$$\mathcal{Y}(1,\varepsilon) = \{Y \geq 0 : \ XY = (X_tY_t)_{t\in[0,T]}\ is\ a\ supermartingale\ for\ every\ X \in \mathcal{X}(1,\varepsilon)\}.$$

Following [19, Proof of Proposition 3.1], one can see that

$$D(1,\varepsilon)^{oo} = \{h \in \mathbb{L}^0_+ : \ 0 \leq h \leq Y_T : \ for\ some\ Y \in \mathcal{Y}(1,\varepsilon)\}.$$

Therefore, as by the Ito's formula, we have $\mathcal{X}(1,\varepsilon) = \mathcal{X}(1,0)\frac{1}{\mathcal{E}(-\varepsilon\nu\cdot S^0)}$. Therefore, by the argument in [19, Proof of Proposition 3.1], one sees that (11.2) holds. □

Proof of Theorem 6.2.1. To establish the first item, first, we show that there exists ε_0, such that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, there exists an equivalent local martingale measure for S^{ε}. This can be proven as follows. For $y = u_x(x,0)$, with $\frac{\hat{Y}(y,0)}{y}$ being a density of an equivalent martingale measure for the base model, it is enough to show that $\mathcal{E}(-\varepsilon\nu \cdot S)\hat{Y}(y,0)$ is a true martingale under $\mathbb{P}$. This is equivalent to proving that

$$\mathbb{E}^{\hat{\mathbb{Q}}(0)}[\mathcal{E}(-\varepsilon\nu\cdot S)_T] = 1, \tag{11.4}$$

as $\mathcal{E}(-\varepsilon\nu \cdot S)$ is an nonnegative local martingale, thus supermartingale under $\hat{\mathbb{Q}}(0)$. (11.4) now follows from Assumption 5.2.1 and (a version of) the Novikov condition, as regardless of the choice of $c > 0$ in Assumption 5.2.1, we have

$$c(e^{cx} + 1) - \frac{x}{2} > 0, \quad x > 0.$$

Given this family of inequalities, a tedious yet straightforward verification of whether $\mathbb{E}^{\hat{\mathbb{Q}}(0)}\left[\exp\left(\frac{1}{2}\langle\varepsilon\nu \cdot S\rangle_T\right)\right] < \infty$ gives an affirmative answer for every ε being sufficiently close to 0. So [15, Proposition 5.12, p. 198] implies that $\mathcal{E}(-\varepsilon\nu \cdot S)$ is a true martingale under $\hat{\mathbb{Q}}(0)$, thus $\mathcal{E}(-\varepsilon\nu \cdot S)\frac{\hat{Y}(y,0)}{y}$ is a true martingale under $\mathbb{P}$, in view of its positivity, we conclude that $\mathcal{E}(-\varepsilon\nu \cdot S)\frac{\hat{Y}(y,0)}{y}$ is a density of an equivalent local martingale measure, for every ε in some interval centered at 0. In particular, the argument above shows that the (abstract) Assumption 10.1.3 is satisfied by the concrete model.

Now, the assertions of Theorem 6.2.1 follow from Lemma 11.0.1 and (abstract) Theorem 10.2.1, where we note that Assumption 10.1.4 holds in view of the continuity of the stock price process and the results from [39, Lemma 6]. □

Proof of Theorem 7.3.1. Similarly, the assertions of Theorem 7.3.1 follow from Lemma 11.0.1 and (abstract) Theorem 10.3.1. □

Proof of Theorem 8.0.1. As above, the assertions of Theorem 8.0.1 follow from Lemma 11.0.1 and (abstract) Theorem 10.3.2. □

Section 12

Connection to the Risk-Tolerance Wealth Process and the Galtchouk-Kunita-Watanabe Decomposition

For a fixed $x \in \mathbb{R}$ (and $\varepsilon = 0$), the risk-tolerance wealth process is defined as a wealth process $R(x,0)$ such that

$$R_T(x,0) = -\frac{U'(\hat{X}_T(x,0))}{U''(\hat{X}_T(x,0))}. \tag{12.1}$$

one can see that $R_T(x,0) > 0$, $\mathbb{P}-a.s.$. The existence of the risk-tolerance wealth process for the general utility function is not apparent. As it is established, see, e.g., [3], that for the general utility defined on the positive real line, if the risk-tolerance wealth process exists (where the definition is identical to (12.1), one can change the measure to the probability measure $\mathbb{R}$, whose Radon-Nykodym derivative is given by

$$\frac{d\mathbb{R}}{d\mathbb{P}} = \frac{\hat{Y}(y,0)}{y}\frac{R_T(x,0)}{R_0(x,0)},$$

where $\frac{\hat{Y}(y,0)}{y}$ corresponds to $\frac{d\hat{\mathbb{Q}}(0)}{d\mathbb{P}}$ in the present settings, and numéraire to $\frac{R(x,0)}{R_0(x,0)}$, that is, by measuring the treaded securities in the (normalized) units of $R(x,0)$, so that the traded securities become

$$\left(\frac{R_0(x,0)}{R(x,0)}, \frac{R_0(x,0)S}{R(x,0)}\right).$$

Then the joint structure of the solutions to (7.2) and (7.4) can be reduced to the Galtchouk-Kunita-Watanabe (GKW) decomposition of a certain random variable $P_T \in \mathbb{L}^2(\mathbb{R})$ embedded into an $\mathbb{R}$-martingale as follows

$$P_t := \mathbb{E}^{\mathbb{R}}\left[P_T|\mathcal{F}_t\right], \quad t \in [0,T].$$

We note that in the power utility case $U(x) = \frac{x^p}{p}$ with $p \in (-\infty, 0)$, the GKW approach was implemented in [2] and for the general utility on $\mathbb{R}_+$, in [3]. We refer to [7, Theorem 3.6.3, p. 238] and, in the Brownian case, to [15, Proposition 4.14, p.181] for the precise statements of the GKW decomposition.

In the present settings, the risk-tolerance wealth process exists and is equal to the constant $\frac{1}{\gamma}$, where γ is the absolute risk aversion of U in (3.5). Next, with F defined in (5.1), we set

$$P_t = \mathbb{E}^{\hat{\mathbb{Q}}(0)}\left[F\left(\frac{1}{\gamma} - \hat{X}_T(x,0)\right)|\mathcal{F}_t\right], \quad t \in [0,T]. \tag{12.2}$$

Now, we can state the main result of this chapter.

Proposition 12.0.1. *Let $x \in \mathbb{R}$ be fixed, the assumptions of Theorem 6.2.1 hold, and $y = u_x(x,0)$. Then the optimizers to* (7.2) *and* (7.4)*, $H^1 \cdot S$ and N^1, respectively, satisfy*

$$P = P_0 + H^1 \cdot S + \frac{N^1}{\gamma}, \tag{12.3}$$

where P is given by (12.2).

Proof. Let us consider (7.2). By direct computations, we get

$$\frac{a_{\varepsilon\varepsilon}}{-y} = \gamma\mathbb{E}^{\hat{\mathbb{Q}}(0)}\left[(H^1 \cdot S_T - P_T)^2\right] + \mathbb{E}^{\hat{\mathbb{Q}}(0)}\left[X_T(x,0)\left(F^2 - G\right) - \frac{F^2}{\gamma}\right].$$

Likewise, rewriting (7.4), we obtain

$$\frac{b_{\varepsilon\varepsilon}}{y} = \gamma \mathbb{E}^{\hat{\mathbb{Q}}(0)} \left[\left(\frac{N_T^1}{\gamma} - P_T \right)^2 \right] + \mathbb{E}^{\hat{\mathbb{Q}}(0)} \left[X_T(x,0) \left(F^2 + G \right) - \gamma (X_T(x,0)F)^2 \right]$$

We see that minimization problems (7.2) and (7.4) are reduced to projections of $(P - P_0)$ on the orthogonal and complementary subspaces of the space of square-integrable $\mathbb{Q}$-martingales starting from 0. The GKW decomposition (12.3) follows. □

Section 13

Example

One can illustrate the expansion results by the models, where the base one (corresponding to $\varepsilon = 0$) admits a closed-form solution, whereas the perturbed models do not. Let us suppose that B and W are independent Brownian motions on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where the filtration is generated by B and W. Let us suppose that the dynamics of the stock price process S for the base model is given by

$$S_t = \mu t + \sigma W_t, \quad t \in [0, T],$$

where μ and $\sigma > 0$ are constants. We also suppose that there is riskless security whose price equals 1 at all times. Other models admitting the closed-form solutions are also possible to consider. We refer to [3] for an overview of models that admit closed-form solutions. One can show that the minimal martingale measure, see Appendix B for an overview of the topic of minimal martingale measures has the density process given by

$$Y = \mathcal{E}\left(-\frac{\mu}{\sigma} W\right),$$

and is the dominating element in the set of the supermartingale deflators in the sense of the second-order, and thus every order up to infinity, stochastic dominance; see Appendix C for the overview of this topic and the results in [52]. We stress that $Y \in \mathcal{M}^e(0)$, that is, the extra condition in item 4 of Theorem 4.4.1 is satisfied.

To obtain the optimal strategy for the base model, we observe that

$$Y_T = \exp\left(-\frac{\mu}{\sigma^2}S_T + \frac{1}{2}\frac{\mu^2}{\sigma^2}T\right) = f(S_T),$$

where f is a deterministic function given by

$$f(x) = \exp\left(-\frac{\mu}{\sigma^2}x + \frac{1}{2}\frac{\mu^2}{\sigma^2}T\right), \quad x \in \mathbb{R}.$$

Therefore, one can see the replication argument as in the derivation of the Black-Scholes-Merton formula to replicate $(U')^{-1}(Y_T) = (U')^{-1}(f(S_T))$, which is the primal optimizer for an appropriate x. This leads to the following representation of the optimal total wealth invested in the risky asset

$$\frac{1}{\gamma}\frac{\mu}{\sigma^2},$$

where γ is given by (3.5). This gives the optimal strategy for the base model

$$H^0_t = 1_{\{S_t \neq 0\}}\frac{1}{S_t}\frac{1}{\gamma}\frac{\mu}{\sigma^2}, \quad t \in [0, T].$$

So that, for a fixed $x \in \mathbb{R}$, the primal optimizer to (4.2) is given by

$$\hat{X}(x, 0) = x + H^0 \cdot S.$$

Now, let us consider perturbations of the dynamics of S given by

$$dS^\varepsilon = (\mu + \varepsilon\nu_t)dt + \sigma dW_t, \quad t \in [0, T], \quad \varepsilon \in \mathbb{R},$$

where ν is a process satisfying (3.4) and Assumption 5.2.1. We stress that, unlike for the base model with $\varepsilon = 0$, for a fixed $\varepsilon \neq 0$ and a general ν, the closed-form solution to (4.2) does not exist, in general.

To construct the approximate solution, which is asymptotically optimal in the sense of the expansion theorems above, let us consider

$$P_t = \mathbb{E}^{\hat{\mathbb{Q}}(0)}\left[(\nu \cdot S_T)\left(\frac{1}{\gamma} - \hat{X}_T(x,0)\right)|\mathcal{F}_t\right], \quad t \in [0,T].$$

In the settings of this example, P can be decomposed into

$$P = P_0 + H^1 \cdot S + \beta \cdot B, \tag{13.1}$$

which is also the Föllmer-Schweizer decomposition of P_T, see [53], and it can be thought of as the replication of P_T in the market fictitiously completed by a risky security with a price process B. Thus, for H^1 appearing in (13.1), let define the following family of stopping times

$$\tau_\delta = \inf\left\{t \in [0,T] : \ |H^1 \cdot S_t| \geq \frac{1}{\delta}\right\}, \quad \delta > 0.$$

and truncate H^1 as follows

$$H^{1,\delta} = H^1 1_{[0,\tau_\delta]}, \quad \delta > 0.$$

This family specifies the corrections to the optimal strategy in the sense of Theorem 9.0.1. In turn, the second-order expansion of the value function is given by Theorem 7.3.1.

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Appendix A

Orthogonality of Martingales

This part follows the discussion in [54]. Let us consider a complete filtered probability space satisfying the usual hypotheses. We focus here on square-integrable martingales, that is, martingales M, such that $\mathbb{E}[M_t^2] < \infty$, $t \geq 0$. We start from the following definition.

Definition A.0.1. The space $\mathbb{M}^2$ of square-integrable martingales is all martingales M, such that $\sup_t \mathbb{E}[M_t^2] < \infty$ and $M_0 = 0$, $\mathbb{P} - a.s.$.

We notice that $\lim_{t\to\infty} \mathbb{E}[M_t^2] = \mathbb{E}[M_\infty^2] < \infty$, and $M_t = \mathbb{E}[M_\infty|\mathcal{F}_t]$, $t \geq 0$. Thus, every $M \in \mathbb{M}^2$ can be identified with its terminal value M_∞. One can endow $\mathbb{M}^2$ with the norm

$$||M|| = \mathbb{E}[M_\infty^2]^{\frac{1}{2}},$$

and with the inner product

$$(M, N) = \mathbb{E}[M_\infty N_\infty].$$

for $M, N \in \mathbb{M}^2$. One can see that $\mathbb{M}^2$ is a Hilbert space, whose dual is itself.

Once the Hilbert space structure is present, the notion of orthogonality readily pops up. Yet, we have to be careful to distinguish weak and strong orthogonality.

Definition A.0.2. Let M and N be some elements of $\mathbb{M}^2$. We say that M is weakly orthogonal to N if

$$\mathbb{E}[M_\infty N_\infty] = 0.$$

Definition A.0.3. Two martingales N and M in $\mathbb{M}^2$ are strongly orthogonal, if their product MN is a uniformly integrable martingale.

We note that if martingales N and M are strongly orthogonal, then

$$\mathbb{E}[M_\infty N_\infty] = \mathbb{E}[M_\infty] = \mathbb{E}[N_\infty] = 0,$$

so strong orthogonality implies weak orthogonality. The opposite is not true in general: let us consider $M \in \mathbb{M}^2$ and let $Y \in \mathcal{F}_0$ be a Bernoulli random variable independent from M, with $\mathbb{P}[Y = 1] = \mathbb{P}[Y = -1] = \frac{1}{2}$. Let us set $N = YM$, that is

$$N_t = YM_t, \quad t \geq 0.$$

Then $N \in \mathbb{M}^2$ and

$$\mathbb{E}[M_\infty N_\infty] = \mathbb{E}[YM_\infty^2] = \mathbb{E}[Y]\mathbb{E}[M_\infty^2] = 0,$$

so M and N are orthogonal. Nevertheless, $MN = YM^2$ is not a martingale, unless $M \equiv 0$, as

$$\mathbb{E}[YM_t^2|\mathcal{F}_0] = Y\mathbb{E}[M_t^2|\mathcal{F}_0] \neq 0 = YM_0^2, \quad \mathbb{P} - a.s..$$

Appendix B

Minimal Martingale Measure

This discussion follows [55]. The minimal martingale measure plays an important role in multiple problems related to the pricing and optimal investment. It also allows for a particular structure of the dual element (4.4) for the base model.

If the returns of the discounted asset prices in a financial market are given by a $\mathbb{P}$-semimartingale of the form

$$S = S_0 + M + A$$

The minimal martingale measure for S is characterized by the property that it turns S into a local martingale, and the martingale property is preserved by any local $\mathbb{P}$-martingale strongly orthogonal to M under the probability measure $\mathbb{P}$, where strong orthogonality can be understood in the sense of Appendix A.

It plays an important role in finding locally risk-minimizing strategies, and it appears in multiple other contexts as well. Importantly, in particular, for the Example above, its density process can be written explicitly in terms of M and A. This explicit structure makes the minimal martingale measure very useful in applications. In some specific settings, such as the ones in the example in section 13 above, it also has other optimality properties.

The *structure condition* for S appearing in (3.1) says that M is locally $\mathbb{P}$-square-integrable and A has the form $A = \int_0^{\cdot} d\langle M\rangle \lambda$, for a predictable process λ, such that $\int_0^{\cdot} \lambda^{\top} d\langle M\rangle \lambda$ is finite-valued. It it proven in [35] that this structure condition is necessary for the absence of arbitrage in the sense of No Unbounded Profit with Bounded Risk (NUPBR), in the sense of [34]. Now, we can define the minimal martingale measure rigorously.

Definition B.0.1. Suppose S satisfies the structure condition. An equivalent local martingale measure $\tilde{\mathbb{P}}$ for R with $\mathbb{P}$-square-integrable density $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ is called minimal martingale measure (for S) if $\tilde{\mathbb{P}} = \mathbb{P}$ on $\mathcal{F}_0$ and if every local $\mathbb{P}$-martingale L, which is locally $\mathbb{P}$-square-integrable and strongly $\mathbb{P}$-orthogonal to M, is also a local $\tilde{\mathbb{P}}$-martingale.

We refer to Appendix A for a short overview of the orthogonality of martingales. Below, we complete the discussion with the following example. Let us consider a filtration generate by a two-dimensional Brownian motion (B^1, B^2). Let us suppose that the evolution of S is given by

$$dS_t = \alpha_t dt + \sigma_t dB^1_t, \quad t \geq 0,$$

for some predictable and sufficiently integrable processes α and $\sigma > 0$. Then every density of an equivalent local martingale measure whose density is in $\mathbb{L}^2(\mathbb{P})$ has the form

$$Z = Z_0 \mathcal{E}\left(-\frac{\alpha}{\sigma} \cdot B^1 + L\right),$$

for some locally $\mathbb{P}$-locally square integrable $\mathbb{P}$-local martingale L, $\mathcal{E}$ denotes the Doleans-Dade stochastic exponential. The minimal martingale measure exists, and its density process corresponds to $Z_0 = 1$ and $L \equiv 0$, that is

$$\tilde{Z} = \mathcal{E}\left(-\frac{\alpha}{\sigma} \cdot B^1\right).$$

Appendix C

Stochastic Dominance

The following discussion follows the results in [52], which are used in section 13. We recall stochastic dominance and various orders, including the infinite one, and state some recent results regarding the stochastic dominance in its connection to the domain of the dual minimization problem (4.4). We note that stochastic dominance, when it exists, allows comparison of distribution functions.

C.1 Stochastic dominance of a finite order

Let F and G be two cumulative distribution functions with supports on $\mathbb{R}_+ = [0, \infty)$, where we choose this support as this is the support of the elements of the dual domain for (4.4). We recall that F stochastically dominates G in the first order if

$$F(y) \leq G(y), \quad y \geq 0.$$

For the higher orders, stochastic dominance is defined recursively as follows, one can set

$$F_1 = F,$$

and then

$$F_i(y) = \int_0^y F_{i-1}(z)dz, \quad y \geq 0, \quad i = 2, 3, \dots.$$

Since $0 \leq F \leq 1$, the integrals are well defined, yet they can potentially be infinite-valued, which is not a problem. G_i's are defined similarly.

Definition C.1.1. For any $n \geq 1$, we say that the cumulative distribution function F stochastically dominates G in the sense of the n-th order and denote $F \succeq_n G$, if $F_n(y) \leq G_n(y)$, $y \geq 0$. For two random variables $\xi, \eta \geq 0$, we say that $\xi \succeq_n \eta$ if $F_\xi \succeq_n F_\eta$.

It is well-known that stochastic dominance of n-th order can be characterized by the test functions and invoke the following result. First, we recall that a function $f : (0, \infty) \to \mathbb{R}$ is called monotonic of (finite) order n, denoted by $f \in CM(n)$, if it has derivatives of order $k = 1, 2, \dots, n$ and $(-1)^k f^{(k)}(x) \geq 0, x > 0$, and $k = 0, 1, 2, \dots, n$. Whenever needed, we extend f to $[0, \infty)$ by $f(0) := \lim_{x \downarrow 0} f(x) \leq \infty$, where $f(0) \leq \infty$. We introduce the following definition.

Definition C.1.2. For $n \geq 1$, we define $D(n)$ to be the class of functions $W : [0, \infty) \to \mathbb{R}$, which satisfy

1. $-W' \in CM(n-1)$,
2. $W'(\infty) = 0$.

Proposition C.1.3 (Mostovyi, Sîrbu, and Zariphopoulou, 2021, Proposition 3.6)**.** *Consider two non-negative random variables ξ and η. Fix $n \geq 2$. Then, the following conditions are equivalent*

1. $\xi \succeq_n \eta$,
2. $\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$ *for every function $W \in D(n)$, such that $W(\infty) > -\infty$, (that is, W is bounded from below),*
3. $\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$ *for every function $W \in D(n)$ such that $\mathbb{E}[W_-(\xi)] < \infty$ and $\mathbb{E}[W_-(\eta)] < \infty$.*

C.2 Stochastic dominance of the infinite order

For the infinite order, the definition is as follows. Again, we follow [52].

Definition C.2.1. Consider two cumulative distributions F and G on $[0, \infty)$. We say that F dominates G in infinite degree stochastic dominance, and denote by $F \succeq_\infty G$, if

$$\int_0^\infty e^{-zy} dF(y) \leq \int_0^\infty e^{-zy} dG(y), \quad z > 0.$$

For nonnegative random variables ξ and η, we say that ξ dominates η in infinite-order stochastic dominance, and denote $\xi \succeq_\infty \eta$ if $F_\xi \succeq_\infty F_\eta$, that is

$$\mathbb{E}[e^{-z\xi}] \leq \mathbb{E}[e^{-z\eta}], \quad z > 0.$$

To characterize the infinite-order stochastic dominance via test functions, we recall that a function $f : (0, \infty) \to \mathbb{R}$ is called completely monotonic, denoted by $f \in CM$, if it has derivatives of order $k = 1, 2, \dots$ and $(-1)^k f^{(k)}(x) \geq 0, x > 0$, and $k = 0, 1, 2, \dots$. Whenever needed, we extend f to $[0, \infty)$ by $f(0) := \lim_{x \downarrow 0} f(x) \leq \infty$, where $f(0) \leq \infty$. Now, we state the following definition.

Definition C.2.2. We define D to be the class of functions $W : [0, \infty) \to \mathbb{R}$, which satisfy

1. $-W' \in CM$,
2. $W'(\infty) = 0$.

Proposition C.2.3. *Consider two non-negative random variables ξ and η. Then, the following conditions are equivalent:*

1. $\xi \geq_\infty \eta$,

2. $\mathbb{E}[W(\xi)] \leq E[W(\eta)]$ *for every function* $W \in D$*, such that* $W(\infty) > -\infty$*, i.e.,* W *is bounded from below,*

3. $\mathbb{E}[W(\xi)] \leq E[W(\eta)]$ *for every function* $W \in D$ *such that*

$$E[W_-(\xi)] < \infty \quad and \quad \mathbb{E}[W_-(\eta)] < \infty.$$

C.3 Stochastic dominance for supermartingale deflators

The next result, established in [52], states that, for the dual domain, stochastic dominance of different orders higher than the second one agrees. Let us recall that the set of supermartingale deflators is defined as the set $\mathcal{Y}$ of nonnegative supermartingales starting at 1, such that XY is a supermartingale for every nonnegative wealth process of a self-financing portfolio. We also recall that $\mathcal{Y} \neq \emptyset$ is equivalent to the absence of arbitrage in the sense of No Unbounded Profit with Bounded Risk (NUBPR), see [34].

Proposition C.3.1 (Mostovyi, Sîrbu, and Zariphopoulou, 2021, Proposition 7.2)**.** *Let us consider a financial model such that the associated set of supermartingale deflators* $\mathcal{Y} \neq \emptyset$*. Then, the following conditions are equivalent:*

1. *There exists* $\hat{Y} \in \mathcal{Y}$*, such that*

$$\hat{Y}_T \geq_\infty Y_T, \quad for\ every\ Y \in \mathcal{Y},$$

2. *There exists* $\hat{Y} \in \mathcal{Y}$*, such that*

$$\mathbb{E}[Y_T | \sigma(\hat{Y}_T)] \leq \hat{Y}_T, \quad for\ every\ Y \in \mathcal{Y},$$

3. *There exists* $\hat{Y} \in \mathcal{Y}$*, such that*

$$\hat{Y}_T \geq_2 Y_T, \quad for\ every\ Y \in \mathcal{Y},$$

where $T \in (0, \infty)$ *is a time horizon.*