QUADRATIC EXPANSIONS IN OPTIMAL INVESTMENT WITH RESPECT TO PERTURBATIONS OF THE SEMIMARTINGALE MODEL

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Abstract. We study the response of the optimal investment problem to small changes of the stock price dynamics. Starting with a multidimensional semimartingale setting of an incomplete market, we suppose that the perturbation process is also a general semimartingale. We obtain second-order expansions of the value functions, first-order corrections to the optimizers, and provide the adjustments to the optimal control that match the objective function up to the second order. We also give a characterization in terms of the risk-tolerance wealth process, if it exists, by reducing the problem to the Kunita-Watanabe decomposition under a change of measure and numéraire. Finally, we illustrate the results by examples of base models that allow for closed-form solutions, but where this structure is lost under perturbations of the model where our results allow for an approximate solution.

Contents

Date: November 27, 2021.

2020 *Mathematics Subject Classification.* 93E20, 91G10, 91G15, 60H30. *JEL Classification:* C61, G11.

Key words and phrases. asymptotic analysis, semimartingale, incomplete market, duality theory.

Oleksii Mostovyi has been supported by the National Science Foundation under grant No. DMS-1848339 (2019-2024), Mihai Sîrbu has been supported by the National Science Foundation under grant No. DMS-1908903 (2019-2022). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

1. Introduction

We study the response of the optimal investment problem to perturbations of the stock price dynamics up to the second-order for the value function and first-order for optimizers. We focus on the asymptotic analysis without a continuity assumption on the driving controlled process, which we assume to be a semimartingale. While the existing literature studies changes in the market price of risk or, closely related, volatility (in continuous process settings), our contribution allows for general perturbations, which cannot be treated with the existing methods even in continuous markets, *and* jumps, in one unified piece of analysis.

Compared to the existing second-order expansion results, our model allows for

(a) the perturbations of both the finite variation and the local martingale part,
(b) both the driving semimartingale and the perturbation processes to have jumps,
(c) both processes to be multidimensional.

We stress that mathematically and even on the heuristic level, establishing the asymptotic analysis results for processes with jumps and general perturbations is much harder than in continuous settings, which itself is challenging compared to only perturbing the market price of risk or volatility. The analysis below supports this point.

Our results include the second-order expansion of the value function, the first-order corrections to the optimizers, and explicit formulas for the corrections of the optimal strategies. The latter is particularly difficult to obtain. We also work with a general utility function and show that the response of the value function is linear up to the first order, whereas up to the second order, it is non-linear and governed by four auxiliary quadratic stochastic control problems. Their joint structure allows for the explicit second-order asymptotic expansion formulas for both value functions. Moreover, we give a characterization of the asymptotic expansions in terms of a *risk-tolerance wealth process*. In the case when it exists, the problem can be reduced to the *Kunita-Watanabe decomposition*.

One of the applications of our results is to study perturbations of models which allow for explicit solutions and typically lose this property under perturbations, which can result from, for example, statistical inference/calibration procedure. For models, which allow for closed-form solutions, we refer to [Kal00], [Zar01], [GK03], [HIM05], [Liu07], [KS06b], [KMK10], [GR12], [HHI$^+$14], [ST14], and [BK21]. A special structure is also exhibited by asymptotically complete models, see [Rob17] and [RSA17].

A general utility function framework is considered in [KS06b], [HHI$^+$14], and [ST14], and thus, it seems important to us to build *an approach for asymptotic analysis that also works for general utility functions.* Below, we provide examples involving the base model that allow for explicit solutions of the optimal investment problem.

Mathematically, to establish results, we have to deviate from the existing papers, where primal-dual expansions are used, and first to develop some *elements of the calculus of numéraire changes, for both primal and dual problems*, and where the key auxiliary processes can be represented in terms of the semimartingale characteristics of the driving processes as well as solutions to certain equations. Then we construct candidate nearly optimal processes explicitly. This allows building the corrections to the optimal strategies that match the value function up to the second order.

To obtain the quadratic minimization problems that govern the second-order corrections in ε in tractable forms, we had to make extra additional reformulation-like steps that were not needed for perturbations of the market price of risk or volatility in continuous process settings which allowed for a special structure. For our class of the base and perturbed models, these reformulation-like steps are at the core of our analysis, and they allow for considering the domains of these minimization problems as sets of martingales under an appropriate change of measure and numéraire. In turn these problems are related to the ones in [CLP98], [PRS98], [LP99], [KS06a], [KS06b], [ČK07], [Mon13], [CS13], [JMSS12], [MS19], and [Mos20]. A primal-dual approach for asymptotic analysis in mathematical finance has been introduced in [Hen02], [HH02], and [Kal02].

Further, our approach relies on *an increase of dimensionality of the value functions*, which is necessary to handle the general utility. The key process which governs the derivative in ε of the dual optimizer is characterized via *implicit differentiation* lemmas, which were pivotal in handling the multi-dimensional stock-price case. The integrability condition on the perturbation processes is related to *entropic submartingales* introduced in [BEK13] in continuous settings.

The asymptotic analysis of stochastic control problems under perturbations of the controlled processes is a challenging problem. The majority of the existing results are obtained under *continuity of the stock price assumptions*, and they only include particular changes of the driving controlled process, such as the *perturbations of the finite-variation part* of it, see, e.g., [VS18]. Also, the existing second-order results assume that the driving controlled process is *one-dimensional*, see [HMKS17] and [MS19].

More precisely, if one perturbs a driving semimartingale R by another one, $\widetilde{R}$, e.g.,

$$R^{\varepsilon} = R + \varepsilon \widetilde{R},$$

the answers to the following questions become non-obvious, even on the heuristic level, even in continuous settings:

(1) how does the value change up to the first order in ε, even harder to answer how does it change up to the second order,
(2) what are the derivatives of the optimal wealth process in ε, and do they exist,
(3) what are the corrections to the optimal strategy needed to match the indirect utility up to the second order.

As mentioned above, to the best of our knowledge, the existing answers are partial, and they are obtained under particular forms of both R and $\widetilde{R}$. The results in [HMKS17], [VS18], and [MS19] include neither the models where R or $\widetilde{R}$ allow for jumps nor they can handle the situation when the martingale part of $\widetilde{R}$ has a component orthogonal to the martingale part of R.

The remainder of the paper is organized as follows. In Section 2, we introduce the model, and in Section 3, we provide some elements of the calculus of the numéraire change, which is central to the analysis. Section 4 contains the expansion theorems, whose proofs are given in section 5. We build a connection of the expansion results to the Kunita-Watanabe decomposition in Section 6. In Section 7, we provide an example of perturbations of the Black-Scholes model and a Lévy process-based model and discuss connections to other models allowing for explicit solutions. Finally, in Appendix, we give an important for our analysis technical characterization of the approximating sets for the primal and dual domains.

2. The model

Let us consider a complete stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, where $T \in (0, T)$ is the time horizon, $\mathcal{F}$ satisfies the usual conditions, and $\mathcal{F}_0$ is trivial. We suppose that there are $d+1$ traded securities, d stocks, and a bank account with zero interest rate. For the base model, we suppose that the *returns* of these d stocks satisfy the structure condition from [FS10]; that is they are given by a special semimartingale R, whose dynamics is

$$R = M + \int_0^{\cdot} d\langle M\rangle_s \lambda_s, \quad R_0 = 0. \tag{1}$$

where M is a locally $\mathbb{P}$-square-integrable martingale and λ is a predictable process such that

$$\int_0^T \lambda_s^{\top} d\langle M\rangle_s \lambda_s < \infty, \quad \mathbb{P}\text{-a.s.} \tag{2}$$

Note that the absolute continuity of the finite variation with respect to $\langle M\rangle$ in the semimartingale decomposition for R is known as the *structure condition* from [FS10]. This condition goes back to [AS92] and [Sch94] and is needed to preclude arbitrage.

2.1. **Parametrization of perturbations.** The family of perturbed models has the returns of the form

$$R^{\varepsilon} = R + \varepsilon\widetilde{R}, \quad R^{\varepsilon}_0 = 0, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0), \tag{3}$$

where $\widetilde{R}$ is another semimartingale of the form (5) below and $\varepsilon_0 > 0$ is a constant.

2.1.1. *Preliminary discussion.* We start by assuming that

$$\widetilde{R} = \phi \cdot M + M^{\perp} + FV, \tag{4}$$

for some predictable matrix-valued process ϕ and some martingale $M^{\perp}$ orthogonal to M and an adapted process of finite variation FV. By the orthogonality in multidimensional settings, we mean that every component of M is orthogonal to every component of $M^{\perp}$. The form of $\widetilde{R}$ in (4) is consistent with the GKW decomposition.

In view of the no-arbitrage considerations, it is natural to suppose that

$$FV = \int_0^{\cdot} d\langle M\rangle_s \zeta_s + \int_0^{\cdot} \langle M^{\perp}\rangle_s d\zeta^{\perp}_s,$$

for some processes ζ and $\zeta^{\perp}$, which are predictable and integrable with respect to $\langle M\rangle$ and $\langle M^{\perp}\rangle$, respectively. Therefore, we can rewrite (3) as

$$R^{\varepsilon} = R + \varepsilon\left(\phi \cdot M + M^{\perp} + \int_0^{\cdot} d\langle M\rangle_s \zeta_s + \int_0^{\cdot} \langle M^{\perp}\rangle_s d\zeta^{\perp}_s\right).$$

In view of Assumption 4.5, which is key for the expansions below, we can further suppose that $\zeta^{\perp} \equiv 0$, without loss of generality.

Note that for a bounded ϕ and every ε sufficiently close to 0, on

$$\{0 = d\langle (I+\varepsilon\phi)\cdot M + \varepsilon M^{\perp}\rangle\} = \{0 = (I+\varepsilon\phi)d\langle M\rangle(I+\varepsilon\phi)^{\top} + \varepsilon^2 d\langle M^{\perp}\rangle\},$$

we have that $d\left\langle M\right\rangle = d\left\langle M^{\perp}\right\rangle = 0$, so

$$d\left(\int_0^{\cdot} d\langle M\rangle_s(\lambda + \varepsilon\zeta)_s + \varepsilon\int_0^{\cdot} d\langle M^{\perp}\rangle_s \zeta^{\perp}_s\right) = 0.$$

Therefore the *structure condition* from [FS10] holds, the finite variation of R^{ε} is absolutely continuous with respect to the predictable quadratic covariation of its martingale part.

2.1.2. *The exact form of $\widetilde{R}$.* To summarize, in view of the discussion above and Assumption 4.5 below, we suppose that

$$\widetilde{R} = \phi \cdot M + M^{\perp} + \int_0^{\cdot} d\langle M\rangle_s \zeta_s, \quad \widetilde{R}^{\varepsilon} = 0, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0), \tag{5}$$

where $M^{\perp}$ is a locally $\mathbb{P}$-square integrable martingale orthogonal to M, and ζ is a predictable process measured in the units of λ and such that, there exists a constant $C' > 0$, such that

$$|\zeta_t| \le C'|\lambda_t|, \quad t \in [0,T], \quad \mathbb{P}\text{-a.s..} \tag{6}$$

Note that (2) and (6) imply that

$$\int_0^T \zeta_s^{\top} d\langle M\rangle_s \zeta_s < \infty, \quad \mathbb{P}\text{-a.s.,}$$

2.2. **Primal problem.** The family of admissible wealth processes is defined, for $(x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0)$ (where $\varepsilon_0>0$ is specified in (3)) as

$$\mathcal{X}(x,\varepsilon):=\{X\geq 0:\ X=x+H\cdot R^{\varepsilon}, H\ is\ R^{\varepsilon}-integrable\}. \tag{7}$$

Assumption 2.1. The utility function U on $(0,\infty)$ is strictly increasing, strictly concave, twice continuously differentiable, and its relative risk aversion is bounded away from 0 and ∞, that is, there exist positive constants c_1 and c_2 such that

$$c_1\leq A(x):=-\frac{U''(x)x}{U'(x)}\leq c_2,\quad x>0. \tag{8}$$

A is the relative risk aversion. The family of the utility functions is given by

$$u(x,\varepsilon):=\sup_{X\in\mathcal{X}(x,\varepsilon)}\mathbb{E}\left[U(X_T)\right],\quad (x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0). \tag{9}$$

We use the convention

$$\mathbb{E}\left[U(X_T)\right]:=-\infty,\quad if\quad \mathbb{E}\left[U^-(X_T)\right]=\infty,$$

where U^- denotes the negative part of U.

2.3. **Dual problem.** For every $(y,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0)$, we first specify the dual feasible set as follows

$$\begin{aligned}\mathcal{Y}(y,\varepsilon):=\{\ &Y\geq 0:\ Y\ is\ a\ supermartingale\ such\ that\ Y_0=y\\ &and\ XY=(X_tY_t)_{t\in[0,T]}\ is\ a\ supermartingale,\\ &for\ every\ X\in\mathcal{X}(1,\varepsilon)\ \}.\end{aligned} \tag{10}$$

Note that, for every $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, this is the usual formulation of the dual domain as in [KS99], [KS03], etc.

Next, we define the convex conjugate of U as

$$V(y):=\sup_{x>0}(U(x)-xy),\quad y>0, \tag{11}$$

and we note that it follows from (8) that

$$B(y):=-\frac{V''(y)y}{V'(y)},\quad y>0, \tag{12}$$

is well-defined and satisfies

$$\frac{1}{c_2}\leq B(y)\leq\frac{1}{c_1},\quad y>0. \tag{13}$$

We also have

$$A(x)=\frac{1}{B(U'(x))},\quad x>0, \tag{14}$$

so $B(U'(x))$ is the relative risk-tolerance of U computed at x, and

$$V''(U'(x))=-\frac{1}{U''(x)},\quad x>0.$$

The dual value function is defined as

$$v(y,\varepsilon) := \inf_{Y\in\mathcal{Y}(y,\varepsilon)} \mathbb{E}\left[V(Y_T)\right], \quad (y,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0), \tag{15}$$

Here we use the convention

$$\mathbb{E}[V(Y_T)] := \infty \quad if \quad \mathbb{E}[V^+(Y_T)] = \infty,$$

where V^+ is the positive part of V.

2.4. **Assumptions on the base model.** As our main results provide asymptotics around $\varepsilon = 0$ and some $x > 0$, we need to impose conditions on the base model that will allow for the expansion results as well as some conditions on perturbations. In order for the base model to be well-defined, we need to ensure that

$$\mathcal{Y}(1,0) \neq \emptyset, \tag{16}$$

which follows from (1) and (2). We also need the following condition.

Assumption 2.2. The dual value function for the base model is finite-valued, that is

$$v(y,0) < \infty, \quad y > 0.$$

Condition (16) is necessary for the absence of arbitrage in the sense of [KK07] and in our context it follows from (1), whereas Assumption 2.2 is necessary for the standard conclusions of the utility maximization theory, as in [KS03]. We also refer to the abstract theorems in [Mos15] for the case when the existence of local martingale measures condition fails. Under these conditions, and with the utility function satisfying Assumption 2.1, we deduce the existence and uniqueness of the optimizers to (9) and (15), for $\varepsilon = 0$, denoted by $\widehat{X}(x,0)$ and $\widehat{Y}(y,0)$, respectively, for every positive x and y, continuous differentiability of $u(\cdot,0)$ and $v(\cdot,0)$ as well as the duality relation

$$\widehat{Y}_T(u_x(x,0),0) = U'\left(\widehat{X}_T(x,0)\right), \quad x > 0. \tag{17}$$

We note that $u_x(\cdot,0)$ is well-defined by the abstract theorems in [Mos15]. Also, $\widehat{X}(x,0)\widehat{Y}(u_x(x,0),0)$ is a martingale, which allows to define a new probability measure $\mathbb{R}(x,0)$, via

$$\frac{d\mathbb{R}(x,0)}{d\mathbb{P}} = \frac{\widehat{X}_T(x,0)\widehat{Y}_T(u_x(x,0),0))}{xu_x(x,0)}. \tag{18}$$

As usually $\mathbb{R}(x,0)$ plays an important role in the second-order expansions of the value function, see [KS06a], [KS06b], [MS19], [Mos20], among others. Below we fix $x > 0$, and set $y = u_x(x,0)$, which is well-defined in the present settings. We will also denote

$$\widehat{X}(x,0) = x\mathcal{E}(\pi\cdot R)$$

for some R-integrable process π. **As computations get technically involved, for brevity, below, we denote $\widehat{X}(x,0)$ by X, and $\widehat{Y}(u_x(x,0),0) = \widehat{Y}(y,0)$ by Y, where $x > 0$ is fixed, and $y = u_x(x,0)$. Likewise, we will denote $\mathbb{R}(x,0)$ by $\mathbb{R}$.**

Assumption 2.3. We assume that

$$\widehat{Y}(y,0) = Y = y\mathcal{E}(-\lambda \cdot M + \beta \cdot M^{\perp} + L), \tag{19}$$

for some predictable process β, such that $\int_0^T \beta_s^\top d\left\langle M^{\perp}\right\rangle_s \beta_s < \infty$, $\mathbb{P}$-a.s., and $L \in \mathcal{H}^2_{loc}(\mathbb{P})$, which is orthogonal to every component of both M and $M^{\perp}$. We also set

$$H := -\lambda \cdot M + \beta \cdot M^{\perp} + L, \tag{20}$$

so that $Y = y\mathcal{E}(H)$. We remark that the form of Y in (19) is fairly natural. It is consistent with most models in the literature from the introduction, whereas building a model where such a form of Y fails requires special treatment.

3. Elements of the change of numéraire calculus

The key role in the second-order analysis is played by the sets $\mathcal{M}^2$ and $\mathcal{N}^2$ or complementary and orthogonal martingales under $\mathbb{R}$. These sets are introduced in [KS06a] in a two-step procedure. First, while the original assets are $\left(1, \mathcal{E}(R^{\{1\}}), \dots, \mathcal{E}(R^{\{d\}})\right)$, we change numéraire to

$$S^X := \left(\frac{x}{X}, \frac{x\mathcal{E}(R^{\{1\}})}{X}, \dots, \frac{x\mathcal{E}(R^{\{d\}})}{X}\right), \tag{21}$$

and second, we define

$$\mathcal{M}^2 := \{\widetilde{M} \in \mathcal{H}^2_0(\mathbb{R}) : \ \widetilde{M} = \int H dS^X\},$$

where $\mathcal{H}^2_0(\mathbb{R})$ is the set of square-integrable martingales under $\mathbb{R}$ with the initial value 0. The complement of $\mathcal{M}^2$ in $\mathcal{H}^2_0(\mathbb{R})$ is denoted by $\mathcal{N}^2$, that is

$$\mathcal{N}^2 := \{N \in \mathcal{H}^2_0(\mathbb{R}) : \ \widetilde{M}N \textit{ is a } \mathbb{R} \textit{ martingale for every } \widetilde{M} \in \mathcal{M}^2\}.$$

Following [KS06a], let us denote by $\mathcal{M}^\infty$ the family of uniformly bounded wealth processes under the numéraire X with initial value 0, that is the family of semimartingales M such that for some $\delta = \delta(M) > 0$, we have

$$X(1+\delta M) \in \mathcal{X}(x,0) \quad \textit{and} \quad X(1-\delta M) \in \mathcal{X}(x,0).$$

By $\mathcal{N}^\infty$ we denote the family of semimartingales N such that for some $\delta = \delta(N) > 0$, we have

$$Y(1+\delta N) \in \mathcal{Y}(y,0) \quad \textit{and} \quad Y(1-\delta N) \in \mathcal{Y}(y,0).$$

The appendix provides a characterization of these sets in the present settings.

We need a representation of R in terms of its *predictable characteristics*. We follow [JS03, section II.2, p. 75 - 77] and fix a truncation function $h(x) : x \to x1_{\{|x|\leq 1\}}$ and denote by R^c the continuous martingale part of R, by B the predictable finite variation part of R corresponding to the truncation function h, by μ the jump measure of R, that is, a random counting measure on $[0,T]\times\mathbb{R}^d$ defined by

$$\mu\left([0,t]\times E\right) := \sum_{0\leq s\leq t} 1_{\{E\setminus\{0\}\}}(\Delta R_s), \quad t \in [0,T], E \subseteq \mathbb{R}^d,$$

where 1_E is the indicator function of a set E and ν is the predictable compensator of μ, that is, a predictable random measure on $[0,T]\times\mathbb{R}^d$, such that, in particular, $\left(x1_{\{|x|\leq 1\}}\right)*(\mu-\nu)$ is a purely discontinuous local martingale. Setting the quadratic covariation process $C:=[R^c,R^c]$ of R^c, we call (B,C,η) the triplet of predictable characteristics of R associated with the truncation function h. By [JS03, Theorem II.2.34, p. 84], a semimartingale R can be represented in terms of (B,C,η) as

$$R = R^c + B + \left(x1_{\{|x|\leq 1\}}\right)*(\mu-\nu) + \left(x1_{\{|x|>1\}}\right)*\mu.$$

Defining a predictable scalar-valued locally integrable increasing process process $\widetilde{A}$ as

$$\widetilde{A} := \sum_{i\leq d} Var(B^i) + \sum_{i\leq d} C^{i,i} + (\min(1,|x|^2))*\nu,$$

where $Var(B^i)$ denotes the variation process of B^i, $i=1,\dots,d$. Then B, C, and ν are absolutely continuous with respect to $\widetilde{A}$, and therefore we have

$$B = b\cdot\widetilde{A},\quad C = c\cdot\widetilde{A},\quad \text{and}\quad \nu = \eta\cdot\widetilde{A},$$

where b is a predictable $\mathbb{R}^d$-valued process, c is a predictable process with values in the set of positive semidefinite matrices, and ν is a predictable Lévy-measure-valued process. Further, since one can represent as $X = x\mathcal{E}(\pi\cdot R)$ for some R-integrable process π, following [KK07, p. 456], let us set

$$R^\pi = R - (c\pi)\cdot\widetilde{A} - \left(\frac{\pi^\top x}{1+\pi^\top x}x\right)*\mu.$$

One can see that R^π is a semimartingale. It is shown in [Mos20, Lemma 4.1] that every element of $\mathcal{M}^\infty$ can be represented as a stochastic integral with respect to R^π. Further, as one can represent every element of $\mathcal{M}^\infty$ as $\frac{\mathcal{E}(\alpha\cdot R)}{\mathcal{E}(\pi\cdot R)}-1$, for some R-integrable process α that can also be written as stochastic integrals with respect to R^π, which is given through

$$\frac{\mathcal{E}(\alpha\cdot R)}{\mathcal{E}(\pi\cdot R)} = \mathcal{E}\left((\alpha-\pi)\cdot R^\pi\right). \tag{22}$$

Equivalently, the process R^π can be described as follows. For a semimartingale K, let us consider the following transformation.

$$K^\pi = K - [K^c,\pi\cdot R^c] - \sum_{s\leq\cdot}\Delta K_s\frac{\pi_s\Delta R_s}{1+\pi_s\Delta R_s},$$

where, particularly important cases corresponds to $K=\pi'\cdot R$, for some predictable and R-integrable process π', in which case we have

$$\pi'\cdot R^\pi = \pi'\cdot R - [\pi'\cdot R^c,\pi\cdot R^c] - \sum_{s\leq\cdot}\pi'_s\Delta R_s\frac{\pi_s\Delta R_s}{1+\pi_s\Delta R_s},$$

and to $K=\widetilde{\pi}\cdot\widetilde{R}$, where $\widetilde{\pi}$ is predictable and $\widetilde{R}$-integrable, so that

$$\widetilde{\pi}\cdot\widetilde{R}^\pi = \widetilde{\pi}\cdot\widetilde{R} - [\widetilde{\pi}\cdot\widetilde{R}^c,\pi\cdot R^c] - \sum_{s\leq\cdot}\widetilde{\pi}_s\Delta\widetilde{R}_s\frac{\pi_s\Delta R_s}{1+\pi_s\Delta R_s}. \tag{23}$$

While the dual problem does not have a numéraire, a very similar transformation is needed, we call it the *dual numéraire change*. It can be described as follows. For a semimartingale K, with H being defined in (20), and where $\frac{Y}{y} = \mathcal{E}(H)$ is the dual numéraire, we set

$$K^H := K - [K^c, H^c] - \sum_{s\leq \cdot} \Delta K_s \frac{\Delta H_s}{1+\Delta H_s},$$

which is also a semimartingale.

Remark 3.1. If $\mathcal{E}(H+K)$ is non-vanishing, K^H can be thought of as an excessive return of K^H under the (dual) numéraire $\mathcal{E}(H)$, that is

$$\mathcal{E}\left(K^H\right) = \frac{\mathcal{E}(K+H)}{\mathcal{E}(H)}.$$

Transformations $\cdot^H$ and $\cdot^\pi$ are central in our analysis. If K is a continuous process of finite variation, we have $K^H = K^\pi = K$. More characterizations are given below, in particular, with the joint primal-dual structure, as in Lemma 5.13, for which have to also establish other foundations, such as integrability and implicit differentiation results, among others.

4. Expansion theorems

4.1. **Assumptions on the perturbations.** We begin by introducing the remaining assumptions needed for the second-order asymptotics. With $\kappa := \sum\limits_{i=1}^d \left\langle M^i \right\rangle$, we have $\langle M\rangle = A\cdot\kappa$, for some process A.

Assumption 4.1. We suppose that A_t is invertible for every $t\in[0,T]$, $\mathbb{P}$-a.s..

Assumption 4.2. R^π, $\widetilde{R}^\pi$, M^H, and $M^{\perp,H}$ are sigma-bounded.

Remark 4.3. R^π being sigma bounded is exactly the sigma boundedness from [KS06a, Assumption 2], which is needed to ensure that $u(\cdot,0)$ allows for the second-order expansion in x.

Remark 4.4. A sufficient condition for Assumption 4.2, which ensures that every semimartingale on the probability space is sigma-bounded, see [KS06a, Theorem 3] can be formulated as follows. There is a d-dimensional local martingale $\bar{M}$ such that any bounded, purely discontinuous martingale N is a stochastic integral with respect to $\bar{M}$, that is,

$$N_t = N_0 + \int_0^t H_u d\bar{M}_u, \quad 0\leq t\leq T,$$

for some predictable and $\bar{M}$-integrable H. We recall that this condition was introduced in [KS06a, Assumption 4]. We note that it is invariant with respect to an equivalent choice of reference probability measure, see [KS06a, Remark 3].

We make a structure-type condition on the orthogonal local martingale $M^\perp$ of the perturbed model.

Assumption 4.5. We suppose that

$$\langle M^{\perp}\rangle = \nu \cdot \langle M\rangle, \quad \textit{for some bounded and predictable matrix} - \textit{valued process } \nu. \tag{24}$$

and that M (and therefore also $M^{\perp}$) is quasi left-continuous.

Remark 4.6. In [MS19] and the volatility uncertainty part in [HMKS17], Assumption 4.5 holds due to the special parameterizations of perturbations.

Assumption 4.7. We suppose that ϕ is bounded.

The following process plays a central role in our analysis.

$$g_t := A_t^{-1}\nu_t A_t \beta_t - A_t^{-1}\phi_t A_t \lambda_t + \zeta_t, \quad t \in [0, T]. \tag{25}$$

We characterize g in Lemma 5.2 below, see also Remark 5.3.

Remark 4.8. If $d = 1$, that is, if there is only one risky asset, g reduces to

$$g = \nu\beta - \phi\lambda + \zeta.$$

For the perturbations, we need to impose the following condition.

Assumption 4.9. We suppose that $\int_0^T \pi_s^\top d\langle M\rangle_s \pi_s < \infty$, so that under the assumptions above, $\pi \cdot \widetilde{R}^\pi$ and $g \cdot M^H$ are well-defined, and there exists $c > 0$, such that

$$\mathbb{E}_{\mathbb{R}}\left[\exp\left(c\left(|\pi \cdot \widetilde{R}^\pi_T| + [\pi \cdot \widetilde{R}^\pi]_T + [g \cdot M^H]_T\right)\right)\right] < \infty.$$

Assumption 4.10.

$$\textit{The jumps of } \pi \cdot \widetilde{R}^\pi \textit{ and of } g \cdot M^H \textit{ are bounded.}$$

We state below the first-order expansion theorem, Theorem 4.11, under the same assumptions as the second-order expansion theorem, Theorem 4.15. This is for brevity, even though some assumptions in Theorem 4.11 can be relaxed, such as the sigma-boundedness Assumption 4.2.

Theorem 4.11. *Let $x > 0$ be fixed, and suppose M and $M^{\perp}$ are in $\mathcal{H}^2_{loc}(\mathbb{P})$, (2) and (6), and Assumptions 2.1, 2.2, 2.3, 4.1, 4.2, 4.5, 4.7, 4.9, and 4.10 hold true. Let us denote $y = u_x(x, 0)$, which is well-defined by the abstract theorems in* [Mos15]. *Then, there exists $\bar\varepsilon_0 > 0$, such that for every $\varepsilon \in (-\bar\varepsilon_0, \bar\varepsilon_0)$, we have*

$$u(x, \varepsilon) \in \mathbb{R}, \quad x > 0, \quad \textit{and} \quad v(y, \varepsilon) \in \mathbb{R}, \quad y > 0.$$

Further, u and v are jointly differentiable (thus, continuous) at $(x, 0)$ and $(y, 0)$, respectively. We also have

$$\nabla u(x, 0) = \begin{pmatrix} y \\ u_\varepsilon(x, 0) \end{pmatrix}, \quad \nabla v(y, 0) = \begin{pmatrix} -x \\ v_\varepsilon(y, 0) \end{pmatrix}, \tag{26}$$

where

$$u_\varepsilon(x, 0) = xy\mathbb{E}_{\mathbb{R}}\left[\pi \cdot \widetilde{R}^\pi_T\right] = v_\varepsilon(y, 0) = xy\mathbb{E}_{\mathbb{R}}\left[g \cdot M^H_T\right], \tag{27}$$

and g is defined in (25).

Remark 4.12 (The excessive return characterization of $\pi \cdot \widetilde{R}^\pi_T$). Similarly to Remark 3.1, we can further characterize $\pi \cdot \widetilde{R}^\pi_T$ as the stochastic logarithm of $\frac{\mathcal{E}(\pi\cdot(\widetilde{R}+R))}{\mathcal{E}(\pi\cdot R)}$, provided that the latter exists. If $\Delta(\pi \cdot (\varepsilon\widetilde{R} + R)) \neq -1$, for some constant ε, we have

$$\varepsilon\pi \cdot \widetilde{R}^\pi_T = \mathcal{L}og\left(\frac{\mathcal{E}(\pi \cdot (\varepsilon\widetilde{R} + R))}{\mathcal{E}(\pi \cdot R)}\right)_T,$$

that is the return of $\mathcal{E}(\pi \cdot (\varepsilon\widetilde{R} + R))$ under the numeraire $\mathcal{E}(\pi \cdot R)$, where $\mathcal{L}og$ denotes the stochastic logarithm. If $\Delta(\pi \cdot (\widetilde{R} + R)) \neq -1$, (27) reads

$$u_\varepsilon(x, 0) = xy\mathbb{E}_{\mathbb{R}}\left[\mathcal{L}og\left(\frac{\mathcal{E}(\pi \cdot (\widetilde{R} + R))}{\mathcal{E}(\pi \cdot R)}\right)_T\right]. \tag{28}$$

Remark 4.13. Theorem 4.11 shows that the response of the utility maximization problem (9) to perturbations of the stock price is linear in the first order of the utility functions. Thus, in particular, perturbations of the drift, volatility, and the orthogonal martingale part, could be considered separately, and the proof of Theorem 4.11 could be implemented by matching the associated terms in the primal and dual representations for the first-order derivatives given by (27) (under sufficient integrability). Up to the second order, such linearity no longer holds, and the proof of Theorem 4.15 is significantly more involved.

4.2. **Minimization problems for u_{xx} and $u_{\varepsilon\varepsilon}$.** Having specified the structure of $\mathcal{M}^2(x)$, we are ready to state the auxiliary minimization problems that govern the second-order expansion terms for u:

$$a(x, x) := \inf_{\widetilde{M}\in\mathcal{M}^2} \mathbb{E}_{\mathbb{R}}\left[A(X_T)(1 + \widetilde{M}_T)^2\right] \tag{29}$$

$$\widetilde{a}(\varepsilon, \varepsilon) = \min_{\widetilde{M}\in\mathcal{M}^2} \mathbb{E}_{\mathbb{R}}\left[A(X^0_T)(\widetilde{M}_T + x\pi \cdot \widetilde{R}^\pi_T)^2 - 2\widetilde{M}_T xg \cdot M^H_T\right]. \tag{30}$$

and

(31)

$$T_1 := -\mathbb{E}_{\mathbb{R}}\left[\left(\int_0^T \pi_s^\top d\left\langle M\right\rangle_s g_s\right)^2 + 2\left(\int_0^T \pi_s^\top d\left\langle M\right\rangle_s g_s\right)\left(\pi \cdot \left(\phi \cdot R + M^\perp - \int_0^\cdot d\left\langle M^\perp\right\rangle_s \beta_s\right)^\pi_T\right)\right]$$

$$a(\varepsilon, \varepsilon) := \widetilde{a}(\varepsilon, \varepsilon) + x^2 T_1. \tag{32}$$

Using the standard techniques of the calculus of variation, one can show the existence and uniqueness of the solutions to (29) and (30), which we denote by M^x and M^ε, respectively. Let us now set

$$a(x, \varepsilon) := \mathbb{E}^{\mathbb{R}}\left[A(X_T)(M^\varepsilon_T + x\pi \cdot \widetilde{R}^\pi_T)(M^x_T + 1) - xg \cdot M^H_T(M^x_T + 1)\right]. \tag{33}$$

4.3. **Minimization problems for v_{yy} and $v_{\varepsilon\varepsilon}$.** We set

$$b(y,y) := \inf_{N\in\mathcal{N}^2} \mathbb{E}_{\mathbb{R}}\left[B(Y_T)(1+N_T)^2\right], \tag{34}$$

$$\widetilde{b}(\varepsilon,\varepsilon) = \min_{N\in\mathcal{N}^2} \mathbb{E}_{\mathbb{R}}\left[B(Y_T^0)\left(N_T - yg\cdot M_T^H\right)^2 + 2N_T(y\pi\cdot\widetilde{R}_T^\pi)\right]. \tag{35}$$

and with

$$T_2 := \mathbb{E}_{\mathbb{R}}\left[\left(\int_0^T \pi_s^\top d\left\langle M\right\rangle_s g_s\right)^2 - 2\left(\int_0^T \pi_s^\top d\left\langle M\right\rangle_s g_s\right) g\cdot M_T^H - 2\int_0^T \pi_s^\top \phi_s d\left\langle M\right\rangle_s g_s\right] \tag{36}$$

where g is given by (25), we set

$$b(\varepsilon,\varepsilon) := \widetilde{b}(\varepsilon,\varepsilon) + y^2 T_2. \tag{37}$$

With N^y and N^ε denoting the unique solutions to (34) and (35), respectively, we define

$$b(y,\varepsilon) := \mathbb{E}^{\mathbb{R}}\left[B(Y_T)(N_T^\varepsilon - yg\cdot M_T^H)(N_T^y+1) + y\pi\cdot\widetilde{R}_T^\pi(N_T^y+1)\right]. \tag{38}$$

4.4. **The joint structure of four auxiliary value functions and their optimizers.**

Theorem 4.14. *Let $x>0$ be fixed, the conditions of Theorem 4.11 hold and $y = u_x(x,0)$. With M^x, M^ε, N^y, and N^ε denoting the solutions to* (29), (32), (34), *and* (37), *respectively, we have*

$$\begin{pmatrix} a(x,x) & 0 \\ a(x,\varepsilon) & -\frac{x}{y}\end{pmatrix}\begin{pmatrix} b(y,y) & 0 \\ b(y,\varepsilon) & -\frac{y}{x}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{39}$$

and

$$\frac{y}{x}a(\varepsilon,\varepsilon) + \frac{x}{y}b(\varepsilon,\varepsilon) = a(x,\varepsilon)b(y,\varepsilon). \tag{40}$$

Further, we have

$$A(X_T)\begin{pmatrix} 1+M_T^x \\ x\pi\cdot\widetilde{R}_T^\pi + M_T^\varepsilon\end{pmatrix} = \begin{pmatrix} a(x,x) & 0 \\ a(x,\varepsilon) & -\frac{x}{y}\end{pmatrix}\begin{pmatrix} 1+N_T^y \\ -yg\cdot M_T^H + N_T^\varepsilon\end{pmatrix}, \tag{41}$$

or, equivalently,

$$B(Y_T)\begin{pmatrix} 1+N_T^y \\ -yg\cdot M_T^H + N_T^\varepsilon,\end{pmatrix} = \begin{pmatrix} b(y,y) & 0 \\ b(y,\varepsilon) & -\frac{y}{x}\end{pmatrix}\begin{pmatrix} 1+M_T^x \\ x\pi\cdot\widetilde{R}_T^\pi + M_T^\varepsilon\end{pmatrix}. \tag{42}$$

4.5. **Second-order expansions of the value functions.**

Theorem 4.15. *Let $x>0$ be fixed. Let us assume the conditions of Theorem 4.11, denote $y = u_x(x,0)$, and recall* (26). *Then, with*

$$H_u(x,0) := -\frac{y}{x}\begin{pmatrix} a(x,x) & a(x,\varepsilon) \\ a(x,\varepsilon) & a(\varepsilon,\varepsilon)\end{pmatrix}, \tag{43}$$

where $a(x,x)$, $a(x,\varepsilon)$, $a(\varepsilon,\varepsilon)$ are given by (29), (33), *and* (32), *respectively, we have*

$$\begin{aligned} u(x+\Delta x,\varepsilon) &= u(x,0) + (\Delta x \quad \varepsilon)\nabla u(x,0) \\ &\quad + \tfrac{1}{2}(\Delta x \quad \varepsilon)H_u(x,0)\begin{pmatrix}\Delta x \\ \varepsilon\end{pmatrix} + o(\Delta x^2 + \varepsilon^2). \end{aligned}$$

Likewise with

$$H_v(y,0) := \frac{x}{y}\begin{pmatrix} b(y,y) & b(y,\varepsilon) \\ b(y,\varepsilon) & b(\varepsilon,\varepsilon)\end{pmatrix}, \tag{44}$$

where $b(y,y)$, $b(y,\varepsilon)$, *and* $b(\varepsilon,\varepsilon)$ *are given by* (34), (38), *and* (37), *respectively, we have*

$$\begin{aligned} v(y+\Delta y,\varepsilon) &= v(y,0) + (\Delta y \quad \varepsilon)\nabla v(y,0) \\ &\quad + \tfrac{1}{2}(\Delta y \quad \varepsilon) H_v(y,0)\begin{pmatrix}\Delta y \\ \varepsilon\end{pmatrix} + o(\Delta y^2+\varepsilon^2). \end{aligned}$$

4.6. **Derivatives of the optimizers.**

Theorem 4.16.

$$\begin{aligned} \lim_{|\Delta x|+|\varepsilon|\to 0}\frac{1}{|\Delta x|+|\varepsilon|}\Big|&\widehat{X}_T(x+\Delta x,\varepsilon) \\ &-\frac{\widehat{X}_T(x,0)}{x}\left(x+\Delta x(1+M^x_T)+\varepsilon(M^\varepsilon_T + x\pi\cdot \widetilde{R}^\pi_T)\right)\Big| = 0, \end{aligned} \tag{45}$$

where the convergence takes place in $\mathbb{P}$*-probability. Likewise, with* N^y *and* N^ε *denoting the optimizers to* (34) *and* (37), *we have*

$$\begin{aligned} \lim_{|\Delta y|+|\varepsilon|\to 0}\frac{1}{|\Delta y|+|\varepsilon|}\Big|&\widehat{Y}_T(y+\Delta y,\varepsilon) \\ &-\frac{\widehat{Y}_T(y,0)}{y}\left(y+\Delta y(1+N^y_T)+\varepsilon(N^\varepsilon_T - yg\cdot M^H_T)\right)\Big| = 0, \end{aligned} \tag{46}$$

where the convergence takes place in $\mathbb{P}$*-probability.*

4.7. **Corrections to the optimal strategies.** With $M^x \in \mathcal{M}^2$ and $M^\varepsilon \in \mathcal{M}^2$ denoting the solutions to (29) and (32), respectively, we can approximate these optimizers by the bounded processes in $\mathcal{M}^\infty$, such that

$$\lim_{n\to\infty}\bar{M}^{x,n}_T = M^x_T \quad \textit{and} \quad \lim_{n\to\infty}\bar{M}^{\varepsilon,n}_T = M^\varepsilon_T, \quad \mathbb{P}\text{-a.s.},$$

we also refer to Lemma 5.10 below, where construction of this type is made explicitly. Without loss of generality, we may suppose that $\bar{M}^{x,n}$ is bounded by n, $n\in\mathbb{N}$, and then we can further localize $\bar{M}^{x,n}$ by taking

$$T^n_k = \inf\{t\geq 0:\ [\bar{M}^{x,n}]_t \geq k\}, \quad k\in\mathbb{N}, \quad n\in\mathbb{N}.$$

Then in particular, $[\bar{M}^{x,n}]_{t\wedge T^n_k} \leq k+4n^2$. Thus, we set

$$\widetilde{M}^{x,n}_t := \bar{M}^{x,n}_{t\wedge T^n_n}, \quad t\in[0,T], \quad n\in\mathbb{N}.$$

Then $\widetilde{M}^{x,n}_t$ is bounded by n, its quadratic variation is bounded by $n+4n^2$, and its jumps are bounded by $2n$. Further, by this construction, we have

$$\lim_{n\to\infty}\widetilde{M}^{x,n}_T = M^x_T, \quad \mathbb{P}\text{-a.s.}$$

We can construct a similar approximating sequence for M^ε, which we denote $\widetilde{M}^{\varepsilon,n}$, $n \in \mathbb{N}$. Now, [Mos20, Lemma 4.1] implies the existence of predictable R^π-integrable processes, $\pi^{x,n}$ and $\pi^{\varepsilon,n}$, such that

$$\pi^{x,n} \cdot R^\pi = \frac{\widetilde{M}^{x,n}}{x}, \quad \pi^{\varepsilon,n} \cdot R^\pi = \frac{\widetilde{M}^{\varepsilon,n}}{x}, \quad n \in \mathbb{N}.$$

Remark 4.17. The dimensions match in the sense that the elements of $\mathcal{M}^\infty$ are stochastic integrals with respect to the $(d+1)$-dimensional process S^X defined in (21) can be represented as stochastic integrals with respect to the d-dimensional process R^π, see [Mos20, Lemma 4.1].

With these preliminaries, we can set the family $(\widetilde{X}^{\Delta x,\varepsilon,n})_{(\Delta x,\varepsilon,n)\in(-x,\infty)\times(-\varepsilon_0,\varepsilon_0)\times\mathbb{N}}$ of semimartingales as follows

$$\widetilde{X}^{\Delta x,\varepsilon,n} = (x+\Delta x)\mathcal{E}\left((\pi + \Delta x \pi^{x,n} + \varepsilon\pi^{\varepsilon,n})\cdot R^\varepsilon\right). \tag{47}$$

Theorem 4.18. *Let $x>0$ be fixed, and assume the conditions of Theorem 4.11. Then we have*

(1) For every $n\in\mathbb{N}$, there exists $\varepsilon = \varepsilon(n) > 0$, such that

$$\widetilde{X}^{\Delta x,\varepsilon,n} \in \mathcal{X}(x+\Delta x,\varepsilon), \quad (\Delta x,\varepsilon)\in B_{\varepsilon(n)}(0,0),$$

where $B_{\varepsilon(n)}(0,0)$ denotes the ball of radius $\varepsilon(n)$ centered at $(0,0)$.

(2) There exists a function $n = n(\Delta x,\varepsilon)$: $(-x,\infty)\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{N}$, such that

$$\mathbb{E}\left[U(\widetilde{X}_T^{\Delta x,\varepsilon,n})\right] = u(x+\Delta x,\varepsilon) - o(\Delta x^2+\varepsilon^2).$$

(3) The process $\widetilde{X}^{\Delta x,\varepsilon,n}$ has the following proportions invested in the corresponding traded assets

$$\pi + \Delta x\pi^{x,n} + \varepsilon\pi^{\varepsilon,n}.$$

Remark 4.19. We note that the results of Theorem 4.18 are consistent with [MS19, Theorem 3.1], where

$$X^{\Delta x,\Delta,\varepsilon} = (x+\Delta x)\mathcal{E}\left((\pi + \Delta x\gamma^{0,\varepsilon} + \varepsilon(\nu+\gamma^{1,\varepsilon}))\cdot R^\varepsilon\right). \tag{48}$$

If we consider the perturbations as in [MS19] and a continuous one-dimensional stock (also as in [MS19]), the difference between (47) and (48) is due to the difference in the notations, and in particular, we can obtain that have $\pi^{\varepsilon,n} = \nu + \gamma^{1,\varepsilon_n}$ (modulo a slightly different localization procedure).

5. Proofs of the expansion theorems

5.1. Preliminary results.

Lemma 5.1. *Let us consider a local martingale $\widetilde{M}$ with $\widetilde{M}_0 = 0$, bounded jumps by a constant A, that is $|\Delta\widetilde{M}|\le A$, and whose quadratic variation has some exponential moments, that is there exists $c>0$, such that*

$$\mathbb{E}\left[e^{c[\widetilde{M}]_T}\right] < \infty.$$

Then, $\widetilde{M}$ is a martingale, and for $c' = \min\left(\frac{1}{4A}, \frac{\sqrt{c}}{2}\right) > 0$, we have

$$\mathbb{E}\left[e^{c'|\widetilde{M}_T|}\right] < \infty.$$

Proof. First, let us observe that the choice of c' allows to ensure that the jumps of $2c'\widetilde{M}$ are bounded by $\frac{1}{2}$, and thus we have both

$$\mathcal{E}\left(2c'\widetilde{M}\right) > 0, \quad \mathbb{P}\text{-a.s.,} \tag{49}$$

and

$$\sum_{s\leq T}\left|\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s\right| \leq \sum_{s\leq T}(2c'\Delta\widetilde{M}_s)^2, \quad \mathbb{P}\text{-a.s.} \tag{50}$$

Let us observe that

$$\mathbb{E}\left[e^{c'\widetilde{M}_T}\right] = \mathbb{E}\left[e^{c'\widetilde{M}_T - [c'\widetilde{M}^c]_T + \frac{1}{2}\sum\limits_{s\leq T}\{\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s\}}\right.$$
$$\left. e^{[c'\widetilde{M}^c]_T - \frac{1}{2}\sum\limits_{s\leq T}\{\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s\}}\right].$$

Applying the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\mathbb{E}\left[e^{c'\widetilde{M}_T}\right] &\leq \mathbb{E}\left[e^{2c'\widetilde{M}_T - \frac{1}{2}[2c'\widetilde{M}^c]_T + \sum\limits_{s\leq T}\{\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s\}}\right]^{\frac{1}{2}} \\ &\quad \times \mathbb{E}\left[e^{2[c'\widetilde{M}^c]_T - \sum\limits_{s\leq T}\{\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s\}}\right]^{\frac{1}{2}}.\end{aligned} \tag{51}$$

We observe that the first term in the right-hand side of (51) is bounded by 1, as

$$e^{2c'\widetilde{M}_T - \frac{1}{2}[2c'\widetilde{M}^c]_T + \sum\limits_{s\leq T}\{\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s\}} = \mathcal{E}\left(2c'\widetilde{M}\right)_T,$$

which, in view of (49), is a terminal value of a nonnegative local martingale, thus supermartingale, starting from 1. To bound the last term in (51), we use (50), to deduce that

$$\begin{aligned}\mathbb{E}\left[e^{2[c'\widetilde{M}^c]_T - \sum\limits_{s\leq T}\{\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s\}}\right] &\leq \mathbb{E}\left[e^{2[c'\widetilde{M}^c]_T + \sum\limits_{s\leq T}(2c'\Delta\widetilde{M}_s)^2}\right] \\ &\leq \mathbb{E}\left[e^{4(c')^2[\widetilde{M}]_T}\right] \\ &\leq \mathbb{E}\left[e^{c[\widetilde{M}]_T}\right] < \infty.\end{aligned}$$

We deduce that in (51), we have

$$\mathbb{E}\left[e^{c'\widetilde{M}_T}\right] \leq \mathbb{E}\left[e^{c[\widetilde{M}]_T}\right]^{\frac{1}{2}} < \infty.$$

Similarly, we can obtain

$$\mathbb{E}[e^{-c'\widetilde{M}_T}] < \infty.$$

We conclude that

$$\mathbb{E}[e^{c'|\widetilde{M}_T|}] \leq \mathbb{E}[e^{c'\widetilde{M}_T}] + \mathbb{E}[e^{-c'\widetilde{M}_T}] < \infty,$$

which also implies that $\widetilde{M}$ is a true martingale. $\square$

5.1.1. *Implicit differentiation.*

Lemma 5.2 (First-order implicit differentiation)**.** *Let $x > 0$ be fixed, the conditions of Theorem 4.11 hold and $y = u_x(x, 0)$. Then, there exists $\widetilde{\varepsilon} > 0$ such that, for every $\varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, the families of vector-valued processes λ^ε and matrix-valued processes G^ε are given (implicitly) via*

$$\begin{aligned} \int_0^\cdot d\langle M\rangle_s(\lambda_s + \varepsilon\zeta_s) &= \int_0^\cdot (I + \varepsilon\phi_s)d\langle M\rangle_s \lambda^\varepsilon_s, \\ \int_0^\cdot (I + \varepsilon\phi_s)d\langle M\rangle_s (G^\varepsilon_s)^\top &= \varepsilon\int_0^\cdot \nu_s d\langle M\rangle_s, \end{aligned} \tag{52}$$

are well-defined. Further, for every predictable process a, such that

$$\int_0^\cdot a_s^\top d\langle M\rangle_s\zeta_s, \quad \int_0^\cdot a_s^\top \phi_s d\langle M\rangle_s \lambda_s, \quad \textit{and} \quad \int_0^\cdot a_s^\top \nu_s d\langle M\rangle_s \beta_s$$

are well-defined and finite-valued, $\mathbb{P}$-a.s., we have

$$\begin{aligned} \int_0^\cdot a_s^\top d\langle M\rangle_s (\lambda^0_s)' &= \int_0^\cdot a_s^\top d\langle M\rangle_s\zeta_s - \int_0^\cdot a_s^\top \phi_s d\langle M\rangle_s \lambda_s, \\ \int_0^\cdot a_s^\top d\langle M\rangle_s \left((G^0_s)^\top\right)' \beta_s &= \int_0^\cdot a_s^\top \nu_s d\langle M\rangle_s \beta_s, \quad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{53}$$

where the derivatives above are given by

$$(\lambda^0_t)' = -(A_t)^{-1}\phi_t A_t\lambda_t + \zeta_t, \quad ((G^0_t)^\top)' = A_t^{-1}\nu_t A_t, \quad t \in [0, T], \tag{54}$$

and β is the process from (19)*. Further, we have*

$$\int_0^\cdot \pi_s^\top d\langle M\rangle_s \left((\lambda^0_s)' + ((G^0_s)^\top)'\beta_s\right) = \int_0^\cdot \pi_s^\top d\langle M\rangle_s g_s. \tag{55}$$

Remark 5.3. In view of Assumption 4.1, the assertions of Lemma 5.2, in particular (55), the process g can be characterized in the following (convenient) form

$$g_t = ((G^0_t)')^\top\beta_t + (\lambda^0_t)', \quad t \in [0, T],$$

where $(G^0_t)'$ and $(\lambda^0_t)'$ are given by (54).

Proof of Lemma 5.2. First, we observe that at $\varepsilon = 0$,

$$\lambda^0 \equiv \lambda \quad \textit{and} \quad G^0 \equiv 0,$$

satisfy (52). From the boundedness of ϕ and the Gershgorin Theorem, we deduce that there exists $\widetilde{\varepsilon} > 0$, such that for every $\varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, $(I + \varepsilon\phi)$ is invertible and that $(I + \varepsilon\phi)^{-1}$ is bounded. Using Assumption 4.1 or the vector and matrix-valued versions of the Radon-Nikodym Theorem, see e.g., [RR68, Theorem 5.1], ensure that, for every $\varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, the vector-valued process λ^ε and the matrix-valued process G^ε given through (52), are well-defined, $\mathbb{P}$-a.s. Now, fro (52) using Assumption 4.1 we deduce (54).

Next, by invertibility of $(I+\varepsilon\phi)$, for every $\varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, one can rewrite (52) as

$$\begin{aligned}\int_0^{\cdot} (I+\varepsilon\phi_s)^{-1} d\langle M\rangle_s(\lambda_s+\varepsilon\zeta_s) &= \int_0^{\cdot} d\left\langle M\right\rangle_s \lambda^{\varepsilon}_s,\\ \int_0^{\cdot} d\left\langle M\right\rangle_s (G^{\varepsilon}_s)^{\top} &= \varepsilon \int_0^{\cdot} (I+\varepsilon\phi_s)^{-1}\nu_s d\left\langle M\right\rangle_s.\end{aligned} \tag{56}$$

Let ε_n, $n\in\mathbb{N}$, be a sequence convergent to 0. Considering the quotients of the form

$$\int_0^{\cdot} a_s^{\top} d\left\langle M\right\rangle_s \frac{\lambda^{\varepsilon_n}_s-\lambda_s}{\varepsilon_n} \quad and \quad \int_0^{\cdot} a_s^{\top} d\left\langle M\right\rangle_s \frac{(G^{\varepsilon_n}_s)^{\top}}{\varepsilon_n}, \tag{57}$$

and using (56), we obtain (53). (55) can be obtained similarly, where we note that $\mathbb{P}$-a.s. finiteness of the right-hand side of (55) follows from Assumption 4.9 and Lemma 5.6. □

The proof of the following lemma is similar to the proof of Lemma 5.2; it is skipped for brevity.

Lemma 5.4 (Second-order implicit differentiation)**.** *Let $x>0$ be fixed, the conditions of Theorem 4.11 hold and $y=u_x(x,0)$. Then, for every predictable process a, such that*

$$\int_0^{\cdot} a_s^{\top}\phi_s d\langle M\rangle_s\zeta_s, \quad \int_0^{\cdot} a_s^{\top}\phi_s\phi_s d\left\langle M\right\rangle_s \lambda_s, \quad and \quad \int_0^{\cdot} a_s^{\top}\phi_s\nu_s d\left\langle M\right\rangle_s\beta_s$$

are well-defined and finite-valued, $\mathbb{P}$-a.s., and with λ^{ε} and G^{ε} given (implicitly) via (52), we have

$$\begin{aligned}\int_0^{\cdot} a_s^{\top} d\left\langle M\right\rangle_s (\lambda^0_s)'' &= -2\int_0^{\cdot} a_s^{\top}\phi_s d\langle M\rangle_s\zeta_s + 2\int_0^{\cdot} a_s^{\top}\phi_s\phi_s d\left\langle M\right\rangle_s\lambda_s,\\ \int_0^{\cdot} a_s^{\top} d\left\langle M\right\rangle_s \left((G^0)^{\top}\right)''_s\beta_s &= -2\int_0^{\cdot} a_s^{\top}\phi_s\nu_s d\left\langle M\right\rangle_s\beta_s, \quad \mathbb{P}\text{-}a.s.,\end{aligned} \tag{58}$$

and thus

$$\int_0^{\cdot} a_s^{\top} d\left\langle M\right\rangle_s \left(((G^0_s)'')^{\top}\beta_s + (\lambda^0_s)''\right) = \int_0^{\cdot} a_s^{\top}\phi_s d\left\langle M\right\rangle_s g_s.$$

Lemma 5.5. *Under the conditions of Lemma 5.2, we have*

$$\begin{aligned}\int_0^T \pi_s^{\top} d\left\langle M\right\rangle_s \left(((G^0_s)'')^{\top}\beta_s + (\lambda^0_s)''\right) &= -2\int_0^T \pi_s^{\top}\phi_s\left(\nu_s d\left\langle M\right\rangle_s\beta_s + d\left\langle M\right\rangle_s\zeta_s - \phi_s d\left\langle M\right\rangle_s\lambda_s\right)\\ &= -2\int_0^T \pi_s^{\top}\phi_s d\left\langle M\right\rangle_s g_s.\end{aligned} \tag{59}$$

As a consequence, we have

$$\mathbb{E}_{\mathbb{R}}\left[\left((\lambda^0)''+\beta\cdot(G^0)''\right)\cdot M^H_T\right] = -2\mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^{\top}\phi_s d\left\langle M\right\rangle_s g_s\right]. \tag{60}$$

Proof. (59) can be proven similary to Lemma 5.2. For the (60), we observe that under the conditions of this lemma, one can show the integrability of $\left((\lambda^0)''+\beta\cdot(G^0)''\right)\cdot M^H_T$ and that $\left((\lambda^0)''+\beta\cdot(G^0)''\right)\cdot\left(M-\int_0^{\cdot}\left\langle M\right\rangle_s\pi_s\right)^H$ is an $\mathbb{R}$-martingale. Then, (60) follows from completing $\left((\lambda^0)''+\beta\cdot(G^0)''\right)\cdot M^H$ to an $\mathbb{R}$-martingale. □

Lemma 5.6. *Let $x > 0$ be fixed. Under the conditions of Theorem 4.11, with $y = u_x(x, 0)$, we have*

$$\pi \cdot \widetilde{R}^{\pi} = \pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s \right)^{\pi} + \int_0^{\cdot} \pi_s^{\top} d \left\langle M \right\rangle_s g_s, \tag{61}$$

where each term is well-defined, $\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s\right)^{\pi}$ is an $\mathbb{R}$-martingale, and there exists $\bar{c}'' > 0$, such that

(62)

$$\mathbb{E}_{\mathbb{R}} \left[\exp \left(\bar{c}'' \left(\left| \int_0^T g_s^{\top} d \left\langle M \right\rangle_s \pi_s \right| + \left| g \cdot M_T^H \right| + \left| \pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s \right)_T^{\pi} \right| \right) \right) \right] < \infty,$$

and $\left|g \cdot \left(M - \int_0^{\cdot} d \left\langle M \right\rangle_s \pi_s\right)_T^H \right|$ also has exponential moments under $\mathbb{R}$.

Remark 5.7. Assumption 4.9 and Lemma 5.6 allow characterizing $c\pi \cdot R^{\pi}$ and $c\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s\right)^{\pi}$ as $\mathbb{R}$-*entropic submartingales*, for a sufficiently small and positive constant c, in the terminology of [BEK13, Section 3]. We recall that a process X that has sufficient exponential integrability is called an entropic submartingale, if for every stopping times σ and τ, such that $\sigma \leq \tau \leq T$, we have $X_{\sigma} \leq \rho_{\sigma}(X_{\tau})$, $\mathbb{P}$-a.s., where $\rho_s(X_{\tau}) = \log \mathbb{E}^{\mathbb{R}} \left[\exp(X_{\tau}) | \mathcal{F}_{\sigma} \right]$. The operator $\rho_{\cdot}$ is known as the entropic process and is studied in the context of risk measures; see the references in [BEK13]. Further, following [BEK13], our integrability assumption can be restated in terms of the key driving entropic (sub)martingales being in the U_{exp} class, that is, the class of $\mathbb{R}$ martingales, whose stochastic exponential is a uniformly integrable martingale. In turn, this can also be characterized through the class $\mathbb{L}^1_{\exp}(\mathbb{R})$ of random variables Z, such that $\exp(|Z|) \in \mathbb{L}^1(\mathbb{R})$, that the random variables appearing in Assumption 4.9 have to be in $\mathbb{L}^1_{\exp}(\mathbb{R})$. Again, we refer to [BEK13, Section 3] for more details.

Proof of Lemma 5.6 . Using Lemma 5.2, by direct computations, $\pi \cdot \widetilde{R}^{\pi}$ can be represented as in (61), where each term is well-defined. Thus, with quasi-left continuity of $\langle M \rangle$, we have

$$\left[\pi \cdot \widetilde{R}^{\pi} \right] = \left[\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s \right)^{\pi} \right], \tag{63}$$

and together with Assumption 4.10, we deduce that the jumps of $\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s\right)^{\pi}$ are bounded. By Assumption 4.2, $\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s\right)^{\pi}$ can be written as a stochastic integral with respect to a sigma-bounded local martingale. Therefore, together with the boundedness of the jumps as above, via the Ansel-Stricker theorem, see [DS06, Theorem 7.3.7], we deduce that $\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s\right)^{\pi}$ is a local martingale. As a result Assumption 4.9, (63), and [Pro04, Corollary 3, p. 73] imply that the process $\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s\right)^{\pi}$ is an $\mathbb{R}$-martingale. As $[\pi \cdot \widetilde{R}^{\pi}]_T$ has exponential moments under $\mathbb{R}$ by Assumption 4.9, Assumption 4.10, (63), and Lemma 5.1 imply that $\left|\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d \left\langle M^{\perp} \right\rangle_s \beta_s\right)_T^{\pi}\right|$ also have exponential moments under $\mathbb{R}$. This further implies, via Hölder's inequality, from Assumption 4.9 and (61), that $\left| \int_0^T g_s^{\top} d \left\langle M \right\rangle_s \pi_s \right|$ has exponential moments under $\mathbb{R}$.

Similarly, representing

$$g \cdot M^H = g \cdot \left(M - \int_0^{\cdot} d \left\langle M \right\rangle_s \pi_s \right)^H + \int_0^{\cdot} g_s^{\top} d \left\langle M \right\rangle_s \pi_s$$

and using Assumption 4.9, 4.10, and Lemma 5.1 we deduce that $|g \cdot \left(M - \int_0^{\cdot} d\left\langle M\right\rangle_s \pi_s\right)_T^H|$ has exponential moments under $\mathbb{R}$. Consequently, via Assumption 4.9 and Hölder's inequality, we deduce that $|g \cdot M_T^H|$ also has exponential moments under $\mathbb{R}$. Thus, (62) holds.

$\square$

Lemma 5.8. *Let $x > 0$ be fixed. Under the conditions of Theorem 4.11, with $y = u_x(x, 0)$, we have*

$$\mathbb{E}_{\mathbb{R}}\left[\pi \cdot \widetilde{R}_T^{\pi}\right] = \mathbb{E}_{\mathbb{R}}\left[g \cdot M_T^H\right]. \tag{64}$$

Proof. Let us observe that

$$\pi \cdot \widetilde{R}^{\pi} = \pi \cdot \phi \cdot R^{\pi} + \pi \cdot \left(M^{\perp} - \int_0^{\cdot} d\left\langle M^{\perp}\right\rangle_s \beta_s\right)^{\pi} + \int_0^{\cdot} \pi_s^{\top} d\left\langle M\right\rangle_s g_s,$$

Assumption 4.9 implies that

$$\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d\left\langle M^{\perp}\right\rangle_s \beta_s\right)^{\pi} \in \mathcal{H}^2(\mathbb{R}) \qquad \int_0^T \pi_s^{\top} d\left\langle M\right\rangle_s g_s \in \mathbb{L}^2(\mathbb{R}).$$

We deduce that

$$\begin{aligned} \mathbb{E}_{\mathbb{R}}\left[\pi \cdot \widetilde{R}_T^{\pi}\right] &= \mathbb{E}_{\mathbb{R}}\left[\pi \cdot \left(\phi \cdot R + M^{\perp} - \int_0^{\cdot} d\left\langle M^{\perp}\right\rangle_s \beta_s\right)_T^{\pi}\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^{\top} d\left\langle M\right\rangle_s g_s\right] \\ &= \mathbb{E}_{\mathbb{R}}\left[\int_0^T g_s^{\top} d\left\langle M\right\rangle_s \pi_s\right]. \end{aligned} \tag{65}$$

Now, as $g \cdot M_T^H \in \mathbb{L}^2(\mathbb{R})$ by Lemma 5.6, we can rewrite the latter expression as

$$\mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^{\top} d\left\langle M\right\rangle_s g_s\right] = \mathbb{E}_{\mathbb{R}}\left[g \cdot \left(\int_0^{\cdot} d\left\langle M\right\rangle_s \pi_s - M\right)_T^H + g \cdot M_T^H\right] = \mathbb{E}_{\mathbb{R}}\left[g \cdot M_T^H\right], \tag{66}$$

where we have used that $g \cdot \left(\int_0^{\cdot} \left\langle M\right\rangle_s \pi_s - M\right)^H \in \mathcal{H}^2(\mathbb{R})$, which follows from Assumption 4.9 and Lemma 5.1. Comparing (65) and (66), we deduce (64). $\square$

Remark 5.9. The equality of $u_{\varepsilon}(x, 0) = v_{\varepsilon}(x, 0)$ will follow from the proof of Theorem 4.15. Here we discuss the implications of the two matching representations in (27), in particular, the linear structure of $u_{\varepsilon}(x, 0) = v_{\varepsilon}(x, 0)$ with respect to the individual components entering the process $\widetilde{R}$, that is, $\phi \cdot M$, $M^{\perp}$, and the finite variation term $\int_0^{\cdot} d\left\langle M\right\rangle_s \zeta_s$.

From Lemma 5.6, we have

$$\mathbb{E}_{\mathbb{R}}\left[\left|\int_0^T g_s^{\top} d\left\langle M\right\rangle_s \pi_s\right|\right] < \infty.$$

and that $g \cdot \left(M - \int_0^{\cdot} d\langle M\rangle_s \pi_s\right)^H$ is a true martingale under $\mathbb{R}$. This allows to rewrite $v_{\varepsilon}(y, 0)$ as follows.

$$\begin{aligned} \frac{v_{\varepsilon}(y, 0)}{xy} &= \mathbb{E}_{\mathbb{R}}\left[g \cdot \left(M - \int_0^{\cdot} d\langle M\rangle_s \pi_s\right)_T^H\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T g_s^{\top} d\langle M\rangle_s \pi_s\right] \\ &= \mathbb{E}_{\mathbb{R}}\left[\int_0^T g_s^{\top} d\langle M\rangle_s \pi_s\right] = \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^{\top} d\langle M\rangle_s \left((\lambda_s^0)' + \left((G_s^0)^{\top}\right)' \beta_s\right)\right], \end{aligned} \tag{67}$$

where, in the last equality, we have used Lemma 5.2, see (55).

For the purpose of giving the intuition in this remark, assuming the needed integrability of the individual components of g, using Lemma 5.2, we can rewrite the last expression in (67) as

$$\begin{aligned}\frac{v_\varepsilon(y,0)}{xy} &= \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s \left((G_s^0)^\top\right)' \beta_s\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s (\lambda_s^0)'\right]. \\ &= \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top \nu_s d\langle M\rangle_s \beta_s\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s \zeta_s\right] - \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top \phi_s d\langle M\rangle_s \lambda_s\right].\end{aligned}\tag{68}$$

Likewise, assuming the needed integrability, we can rewrite $u_\varepsilon(x,0)$ as follows.

$$\begin{aligned}&\frac{u_\varepsilon(x,0)}{xy} = \mathbb{E}_{\mathbb{R}}\left[\pi\cdot \widetilde{R}_T^\pi\right] = \mathbb{E}_{\mathbb{R}}\left[\pi\cdot\phi\cdot M_T^\pi + \int_0^T \pi_s^\top d\langle M\rangle_s\zeta_s + \pi\cdot M_T^{\perp,\pi}\right] \\ &= \mathbb{E}_{\mathbb{R}}\left[\pi\cdot\phi\cdot\left(M + \int_0^\cdot d\langle M\rangle_s\lambda_s\right)_T^\pi\right] - \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top\phi_s d\langle M\rangle_s\lambda_s\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s\zeta_s\right] \\ &+ \mathbb{E}_{\mathbb{R}}\left[\pi\cdot\left(M^\perp - \int d\langle M^\perp\rangle_s\beta_s\right)_T^\pi\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M^\perp\rangle_s\beta_s\right] \\ &= \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top \nu_s d\langle M\rangle_s\beta_s\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s\zeta_s\right] - \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top\phi_s d\langle M\rangle_s\lambda_s\right].\end{aligned}\tag{69}$$

Comparing the last expressions in (68) and (69), we conclude that, heuristically, we can get $v_\varepsilon(y,0) = u_\varepsilon(x,0)$ matching terms corresponding to three different types of perturbations, represented by ν - orthogonal martingale term, ϕ - perturbations of the volatility term, and ζ - perturbations of the finite variation term. The effects of these terms on $v_\varepsilon(y,0)$ and $u_\varepsilon(x,0)$, on a heuristic level, can be matched separately.

5.2. **Constructing a second-order bound for the primal problem.** For bounded and predictable π^0 and π', let us set

$$\begin{aligned}K^{\Delta x,\varepsilon} &:= (\pi + \Delta x\pi^0 + \varepsilon\pi')\cdot R^\varepsilon - \pi\cdot R \\ &= (\Delta x\pi^0 + \varepsilon\pi')\cdot R + \varepsilon\pi\cdot\widetilde{R} + \varepsilon(\Delta x\pi^0 + \varepsilon\pi')\cdot\widetilde{R}.\end{aligned}\tag{70}$$

We will drop the subscript $\Delta x,\varepsilon$ in $K^{\Delta x,\varepsilon}$ and write K instead for brevity of notations.

$$\begin{aligned}&\frac{\partial K}{\partial\Delta x} = \pi^0\cdot R^\varepsilon, \quad \frac{\partial K}{\partial\varepsilon} = (\pi'\cdot R + \pi\cdot\widetilde{R}) + \Delta x\pi^0\cdot\widetilde{R} + 2\varepsilon\pi'\cdot\widetilde{R}, \\ &\frac{\partial^2 K}{\partial\Delta x^2} = 0, \qquad \frac{\partial^2 K}{\partial\Delta x\partial\varepsilon} = \pi^0\cdot\widetilde{R}, \qquad \textit{and} \qquad \frac{\partial^2 K}{\partial\varepsilon^2} = 2\pi'\cdot\widetilde{R}.\end{aligned}\tag{71}$$

We observe that $\pi^0\cdot\widetilde{R}^\pi$ and $\pi'\cdot\widetilde{R}^\pi$ are well-defined.

Lemma 5.10. *Let $x>0$ be fixed. Under the conditions of Theorem 4.11, with $y = u_x(x,0)$, let us consider bounded and predictable processes π^0 and π', such that the following processes are bounded:*

(1) $\pi^0\cdot R^\pi$ and $\pi'\cdot R^\pi$;
(2) $[\pi^0\cdot R^\pi]$ and $[\pi'\cdot R^\pi]$;
(3) $\pi^0\cdot\widetilde{R}^\pi$, $\pi'\cdot\widetilde{R}^\pi$, $\pi'\cdot\phi\cdot R^\pi$, $\pi'\cdot(M^\perp - \int_0^\cdot d\langle M^\perp\rangle\beta_s)^\pi$, $\int_0^\cdot \zeta_s^\top d\langle M\rangle_s\pi_s'$, $\int_0^\cdot(\pi_s')^\top\phi_s d\langle M\rangle_s\lambda_s$, and $\int_0^\cdot(\pi_s')^\top d\langle M^\perp\rangle_s\beta_s$;
(4) $[\pi^0\cdot\widetilde{R}^\pi]$, $[\pi'\phi\cdot R^\pi]$, $[\pi'\cdot(M^\perp - \int_0^\cdot d\langle M^\perp\rangle\beta_s)^\pi]$;

(5) $[\pi \cdot \widetilde{R}^{\pi}, \pi^0 \cdot R^{\pi}]$ *and* $[\pi \cdot \widetilde{R}^{\pi}, \pi' \cdot R^{\pi}]$*;*
(6) $[\pi \cdot \widetilde{R}^{\pi}, \pi^0 \cdot \widetilde{R}^{\pi}]$ *and* $[\pi \cdot \widetilde{R}^{\pi}, \pi' \cdot \widetilde{R}^{\pi}]$*.*

Then there exists a constant $\delta > 0$*, such that for every* $(\Delta x, \varepsilon) \in B_\delta(0,0)$*, we have*

$$\frac{X}{x}\mathcal{E}\left(K^{\Delta x, \varepsilon, \pi}\right) \in \mathcal{X}(1, \varepsilon),$$

where K *is given by* (70) *and similarly to* (23)*,* $K^{\Delta x,\varepsilon,\pi}$ *is given by*

$$K^{\Delta x,\varepsilon,\pi} = K^{\Delta x,\varepsilon} - [K^{\Delta x,\varepsilon,c}, \pi \cdot R^c] - \sum_{s \leq \cdot} \Delta K_s^{\Delta x,\varepsilon} \frac{\pi_s \Delta R_s}{1 + \pi_s \Delta R_s}. \tag{72}$$

Proof. Follows from the calculus of stochastic exponentials, we observe that

$$\frac{\mathcal{E}((\pi + \Delta x \pi^0 + \varepsilon \pi') \cdot R^{\varepsilon})}{\mathcal{E}(\pi \cdot R)} = \mathcal{E}\left(K^{\Delta x,\varepsilon,\pi}\right), \tag{73}$$

where the positivity of $\mathcal{E}\left(K^{\Delta x,\varepsilon,\pi}\right)$ follows from Assumption 4.10 and the choice of δ in $B_\delta(0,0)$. □

Now, under the transformation (72), for π^0 and π' as in Lemma 5.10, we have

$$K^{\Delta x,\varepsilon,\pi} = (\Delta x \pi^0 + \varepsilon \pi') \cdot R^{\pi} + \varepsilon(\Delta x \pi^0 + \varepsilon \pi') \cdot \widetilde{R}^{\pi} + \varepsilon \pi \cdot \widetilde{R}^{\pi},$$

where $(\Delta x \pi^0 + \varepsilon \pi') \cdot R^{\pi} \in \mathcal{M}^{\infty}$, $(\Delta x \pi^0 + \varepsilon \pi') \cdot \widetilde{R}^{\pi}$ is bounded, and $\pi \cdot \widetilde{R}^{\pi}$ has exponential moments under $\mathbb{R}$ and with bounded jumps, by Assumptions 4.9 and 4.10. Note that the partial derivatives of K^{π} with respect to Δx and ε are determined by (71), and only a notational change is needed there.

We will need the following lemma from [MS19].

Lemma 5.11 (Mostovyi, Sirbu, 2019)**.** *Under Assumption 2.1, for every* $z > 0$ *and* $x > 0$*, we have*

$$\begin{aligned} U'(zx) &\leq \max\left(z^{-c_2}, 1\right) U'(x) \leq \left(z^{-c_2} + 1\right) U'(x), \\ -V'(zx) &\leq \max\left(z^{-\frac{1}{c_1}}, 1\right)(-V'(x)) \leq \left(z^{-\frac{1}{c_1}} + 1\right)(-V'(x)). \end{aligned}$$

Lemma 5.12. *Let* $x > 0$ *be fixed. Under the conditions of Theorem 4.11, with* $y = u_x(x, 0)$*, let both* π^0 *and* π' *satisfy the assumption of Lemma 5.10. Then, we have*

$$\begin{aligned} &\mathbb{E}\Bigg[U''(X_T)\left(X_T\left(\frac{1}{x} + \pi^0 \cdot R_T^{\pi}\right)\right)^2 \\ &\quad + U'(X_T)X_T\left(\left(\frac{1}{x} + \pi^0 \cdot R_T^{\pi}\right)^2 - \frac{1}{x^2} - \left[\pi^0 \cdot R^{\pi}\right]_T\right)\Bigg] \\ &= -\frac{y}{x}\mathbb{E}\left[A(X_T)\left(1 + x\pi^0 \cdot R_T^{\pi}\right)^2\right]. \end{aligned} \tag{74}$$

as well as

$$
\begin{aligned}
&\mathbb{E}\Big[U''(X_T)\left(\pi'\cdot R^\pi_T+\pi\cdot\widetilde{R}^\pi_T\right)^2\\
&+U'(X_T)X_T\left(\pi'\cdot R^\pi_T+\pi\cdot\widetilde{R}^\pi_T\right)^2+2\pi'\cdot\widetilde{R}^\pi_T-\left[\pi'\cdot R^\pi+\pi\cdot\widetilde{R}^\pi\right]_T\Big]\\
=&-xy\mathbb{E}_{\mathbb{R}}\left[(A(X^0_T)-1)(\pi'\cdot R^\pi_T+\pi\cdot\widetilde{R}^\pi_T)^2-2\pi'\cdot\widetilde{R}^\pi_T+[\pi\cdot\widetilde{R}^\pi+\pi'\cdot R^\pi]_T\right].
\end{aligned}
\tag{75}
$$

Proof. As $\pi^0\cdot R^\pi$ is a bounded martingale under $\mathbb{R}$, we deduce that

$$
\begin{aligned}
&\mathbb{E}\left[U'(X_T)X_T\left(\left(\frac{1}{x}+\pi^0\cdot R^\pi_T\right)^2-\frac{1}{x^2}-[\pi^0\cdot R^\pi]_T\right)\right]=\\
&xy\mathbb{E}_{\mathbb{R}}\left[\frac{2}{x}\pi^0\cdot R^\pi_T+(\pi^0\cdot R^\pi_T)^2-[\pi^0\cdot R^\pi]_T\right]=0.
\end{aligned}
\tag{76}
$$

Using the definition of $\mathbb{R}$, we also get

$$
\mathbb{E}\left[U''(X_T)\left(X_T\left(\frac{1}{x}+\pi^0\cdot R^\pi_T\right)\right)^2\right]=-\frac{y}{x}\mathbb{E}_{\mathbb{R}}\left[A(X_T)\left(1+x\pi^0\cdot R^\pi_T\right)^2\right].
\tag{77}
$$

Combining (76) and (77), we deduce (74). (75) can be obtained similarly. □

5.3. Preliminaries for representing $a(\varepsilon,\varepsilon)$ and $a(x,\varepsilon)$.

To obtain the representations in (32) and (33), we will need some auxiliary results. First, we observe that

Lemma 5.13. *Let $x>0$ be fixed. Under the conditions of Theorem 4.11, with $y=u_x(x,0)$, let π' satisfy the assumption of Lemma 5.10. Then, we have*

$$
\mathbb{E}_{\mathbb{R}}\left[\int_0^T\pi'^{\top}_s d\langle M\rangle_s g_s\right]=\mathbb{E}_{\mathbb{R}}\left[\pi'\cdot R^\pi_T g\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H_T\right].
\tag{78}
$$

Proof. First, let us observe that $\pi'\cdot M$ and $g\cdot M$ are in $\mathcal{H}^2_{loc}(\mathbb{P})$. As $\frac{X_-Y_-}{xy}$ is left-continuous, by [Pro04, Theorem III.29], the following process is a $\mathbb{P}$-local martingale

$$
\frac{X_{s-}Y_{s-}}{xy}\cdot\left(\int_0^\cdot\pi'^{\top}_s d\left[M^d\right]_s g_s-\int_0^\cdot\pi'^{\top}_s d\left\langle M^d\right\rangle_s g_s\right),
\tag{79}
$$

where $M=M^c+M^d$ is a decomposition of M into a continuous and purely discontinuous parts, see, e.g., [JS03, Theorem I.4.18].

Let τ_n, $n\in\mathbb{N}$ is a localizing sequence for the local $\mathbb{P}$-martingale appearing in (79) as well as for $\int_0^\cdot\pi'^{\top}_s d\langle M^c\rangle_s g_s$, and $\int_0^\cdot\pi'^{\top}_s d\left\langle M^d\right\rangle_s g_s$. Let us fix $n\in\mathbb{N}$, consider an arbitrary stopping time σ, and set $\tau:=\sigma\wedge\tau_n$. Next, recalling the definitions of the transformations $\cdot^\pi$ and $\cdot^H$, using the Kunita-Watanabe inequality, we get

(80)

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{R}}\left[\pi' \cdot R^{\pi}_{\tau} g \cdot \left(M - \int_0^{\cdot} d\langle M\rangle_s \pi_s\right)^H_{\tau}\right] \\
= &\mathbb{E}_{\mathbb{R}}\Bigg[\Bigg[\pi' \cdot R - [\pi' \cdot R^c, \pi \cdot R^c] - \sum_{s\leq \cdot} \pi'_s \Delta M_s \frac{\pi_s \Delta R_s}{1+\pi_s \Delta R_s}, \\
&\qquad g \cdot \left(M - \int_0^{\cdot} d\langle M\rangle_s \pi_s\right) - \left[g \cdot \left(M - \int_0^{\cdot} d\langle M\rangle_s \pi_s\right)^c, H^c\right] - \sum_{s\leq \cdot} g_s \Delta M_s \frac{\Delta H_s}{1+\Delta H_s}\Bigg]_{\tau}\Bigg] \\
= &\mathbb{E}_{\mathbb{R}}\left[\left[\pi' \cdot M - \sum_{s\leq \cdot} \pi'_s \Delta M_s \frac{\pi_s \Delta R_s}{1+\pi_s \Delta R_s}, g\cdot M - \sum_{s\leq \cdot} g_s \Delta M_s \frac{\Delta H_s}{1+\Delta H_s}\right]_{\tau}\right].
\end{aligned}
$$

one can rewrite the latter expectation in (80) as

$$
\mathbb{E}_{\mathbb{R}}\left[\int_0^{\tau} \pi'^{\top}_s d\left\langle M^c\right\rangle_s g_s + \sum_{s\leq \tau} \frac{\pi'_s \Delta M_s}{1+\pi_s \Delta R_s} \frac{g_s \Delta M_s}{1+\Delta H_s}\right].
$$

Denoting

$$
\widetilde{T}_1 := \mathbb{E}_{\mathbb{R}}\left[\int_0^{\tau} \pi'^{\top}_s d\left\langle M^c\right\rangle_s g_s\right] \quad \textit{and} \quad \widetilde{T}_2 := \mathbb{E}_{\mathbb{R}}\left[\sum_{s\leq \tau} \frac{\pi'_s \Delta M_s}{1+\pi_s \Delta R_s} \frac{g_s \Delta M_s}{1+\Delta H_s}\right],
$$

the computations above show that the right-hand side in (78) equals to $\widetilde{T}_1 + \widetilde{T}_2$, where both $\widetilde{T}_1$ and $\widetilde{T}_2$ are finite. Let us observe that, by Lemma 5.6, both $\pi' \cdot R^{\pi}$ and $g \cdot \left(M - \int_0^{\cdot} d\langle M\rangle_s \pi_s\right)^H$ are in $\mathcal{H}^2(\mathbb{R})$, and thus we have

$$
\mathbb{E}_{\mathbb{R}}\left[\sum_{s\leq \tau}\left(\frac{\pi'_s \Delta M_s}{1+\pi_s \Delta R_s}\right)^2\right] \leq \mathbb{E}_{\mathbb{R}}\left[\left[\pi' \cdot R^{\pi}\right]_{\tau}\right] < \infty, \tag{81}
$$

and

$$
\mathbb{E}_{\mathbb{R}}\left[\sum_{s\leq \tau}\left(\frac{g_s \Delta M_s}{1+\Delta H_s}\right)^2\right] \leq \mathbb{E}_{\mathbb{R}}\left[\left[g \cdot \left(M - \int_0^{\cdot} d\langle M\rangle_s \pi_s\right)^H\right]_{\tau}\right] < \infty, \tag{82}
$$

Further, since

$$
1+\pi_s \Delta R_s = \frac{X_s}{X_{s-}} \quad \textit{and} \quad 1+\Delta H_s = \frac{Y_s}{Y_{s-}},
$$

and using (81), (82), localization, and integration by parts, one can rewrite $\widetilde{T}_2$ as

$$
\widetilde{T}_2 = \mathbb{E}\left[\sum_{s\leq \tau} \frac{X_{s-}Y_{s-}}{xy} \pi'_s \Delta M_s g_s \Delta M_s\right]. \tag{83}
$$

By [Pro04, Theorem 28, p.75], we have

$$
\int_0^{\cdot} \pi'^{\top}_s d\left[M^d\right]_s g_s = \left[\pi' \cdot M^d, g \cdot M^d\right] = \sum_{s\leq \cdot} \pi'_s \Delta M_s g_s \Delta M_s. \tag{84}
$$

As the process in (79) is a true $\mathbb{P}$-martingale on $[0,\tau]$, from (84), one can further rewrite $\widetilde{T}_2$ appearing in (83) as

$$\widetilde{T}_2 = \mathbb{E}\left[\int_0^\tau \frac{X_{s-}Y_{s-}}{xy}\pi_s'^\top d\left\langle M^d\right\rangle_s g_s\right]. \tag{85}$$

Next, using localization, [JS03, Theorem I.4.49] (as $\int_0^\tau \pi_s'^\top d\left\langle M^d\right\rangle_s g_s$ is $\mathbb{P}$-integrable under the construction of τ), one can further rewrite $\widetilde{T}_2$ as

$$\widetilde{T}_2 = \mathbb{E}_{\mathbb{R}}\left[\int_0^\tau \pi_s'^\top d\left\langle M^d\right\rangle_s g_s\right].$$

We recapitulate that $\mathbb{E}_{\mathbb{R}}\left[\pi'\cdot R^\pi_\tau g\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H_\tau\right]$ appearing in (80), can be rewritten as

(86)

$$\begin{aligned}\mathbb{E}_{\mathbb{R}}\left[\pi'\cdot R^\pi_\tau g\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H_\tau\right] &= \widetilde{T}_1+\widetilde{T}_2\\ &= \mathbb{E}_{\mathbb{R}}\left[\int_0^\tau \pi_s'^\top d\left\langle M^c\right\rangle_s g_s\right]+\mathbb{E}_{\mathbb{R}}\left[\int_0^\tau \pi_s'^\top d\left\langle M^d\right\rangle_s g_s\right]\\ &= \mathbb{E}_{\mathbb{R}}\left[\int_0^\tau \pi_s'^\top d\left\langle M\right\rangle_s g_s\right].\end{aligned}$$

As τ is an arbitrary stopping time on $[0,\tau_n]$, and τ_n, $n\in\mathbb{N}$, is a localizing sequence, we conclude that $\int_0^\tau \pi_s'^\top d\left\langle M\right\rangle_s g_s$ is a predictable quadratic covariation of the pair $\left(\pi'\cdot R^\pi, g\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H\right)$ (under $\mathbb{R}$).

Now, using Lemma 5.6, we observe that both $\pi'\cdot R^\pi$ and $g\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H$ are in $\mathcal{H}^2(\mathbb{R})$. Therefore, [JS03, Theorem I.4.2, p.38], asserts that $(\pi'\cdot R^\pi)\left(g\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H\right)-\int_0^\cdot \pi_s'^\top d\langle M\rangle_s g_s$ is a true martingale under $\mathbb{R}$, which implies (78). ☐

Lemma 5.14. *Let $x>0$ be fixed. Under the conditions of Theorem 4.11, with $y=u_x(x,0)$, let π' be as in Lemma 5.10. Then we have*

$$\mathbb{E}_{\mathbb{R}}\left[\pi'\cdot\widetilde{R}^\pi_T\right] = \mathbb{E}_{\mathbb{R}}\left[\left(\pi'\cdot R^\pi_T\right)\left(g\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H_T\right)\right]. \tag{87}$$

Proof. As Lemma 5.6 (see (61)), using Lemma 5.2, we get

$$\mathbb{E}_{\mathbb{R}}\left[\pi'\cdot\widetilde{R}^\pi_T\right] = \mathbb{E}_{\mathbb{R}}\left[\pi'\cdot\left(\phi\cdot R^\pi+(M^\perp-\beta\cdot\langle M^\perp\rangle\right)^\pi_T+\int_0^T \pi_s'^\top d\left\langle M\right\rangle_s g_s\right]. \tag{88}$$

The construction of π' implies that $\pi'\cdot(\phi\cdot R^\pi+(M^\perp-\beta\cdot\langle M^\perp\rangle)^\pi\in\mathcal{H}^2(\mathbb{R})$, and thus

$$\mathbb{E}_{\mathbb{R}}\left[\pi'\cdot\left(\phi\cdot R^\pi+(M^\perp-\beta\cdot\langle M^\perp\rangle\right)^\pi_T\right]=0.$$

Consequently, in (88), we obtain

$$\mathbb{E}_{\mathbb{R}}\left[\pi'\cdot\widetilde{R}^\pi_T\right] = \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s'^\top d\left\langle M\right\rangle_s g_s\right].$$

In turn, using Lemma 5.13, we deduce that the latter expression is equal to (87), and this completes the proof of the lemma. ☐

Lemma 5.15. *Let $x > 0$ be fixed. Under the conditions of Theorem 4.11, with $y = u_x(x, 0)$, let π' be as in Lemma 5.10. Then, we have*

$$\begin{aligned} &- \mathbb{E}_{\mathbb{R}} \left[(1 - A(X_T^0))(\pi' \cdot R_T^\pi + \pi \cdot \widetilde{R}_T^\pi)^2 + 2\pi' \cdot \widetilde{R}_T^\pi - [\pi \cdot \widetilde{R}^\pi + \pi' \cdot R^\pi]_T \right], \\ &= \mathbb{E}_{\mathbb{R}} \left[A(X_T^0)(\pi' \cdot R_T^\pi + \pi \cdot \widetilde{R}_T^\pi)^2 - 2\pi' \cdot R_T^\pi g \cdot M_T^H \right] + T_1, \end{aligned} \tag{89}$$

where T_1 is given in (31)*, and g is given by* (25)*. For π^0 as in Lemma 5.10 , we also have*[1]

$$\begin{aligned} 0 = \mathbb{E}^{\mathbb{R}} \Big[& xg \cdot M_T^H + (x\pi^0 \cdot R_T^\pi) \left(x \int_0^T g_s^\top d \langle M \rangle_s \pi_s \right) \\ & - (1 + x\pi^0 \cdot R_T^\pi)(x\pi' \cdot R_T^\pi + x\pi \cdot \widetilde{R}_T^\pi) \\ & + \left[1 + x\pi^0 \cdot R^\pi, x\pi' \cdot R^\pi + x\pi \cdot \widetilde{R}^\pi \right]_T \Big]. \end{aligned} \tag{90}$$

Proof. First, using the representation in Lemma 5.6 and the square-integrability of

$$\left(\pi' \cdot R_T^\pi + \pi \cdot \left(\phi \cdot R + M^\perp - \int_0^\cdot d \langle M \rangle_s^\perp \beta_s \right)_T^\pi \right)$$

under $\mathbb{R}$, we get

$$\begin{aligned} &\mathbb{E}_{\mathbb{R}} \left[2\pi' \cdot R_T^\pi \int_0^T g_s^\top d \langle M \rangle_s \pi_s - \left(\pi' \cdot R_T^\pi + \pi \cdot \widetilde{R}_T^\pi \right)^2 + \left[\pi' \cdot R^\pi + \pi \cdot \widetilde{R}^\pi \right]_T \right] \\ =&\mathbb{E}_{\mathbb{R}} \Bigg[2\pi' \cdot R_T^\pi \int_0^T g_s^\top d \langle M \rangle_s \pi_s - \left(\int_0^T g_s^\top d \langle M \rangle_s \pi_s \right)^2 \\ & -2 \int_0^T g_s^\top d \langle M \rangle_s \pi_s \left(\pi' \cdot R_T^\pi + \pi \cdot \left(\phi \cdot R + M^\perp - \int_0^\cdot d \langle M \rangle_s^\perp \beta_s \right)_T^\pi \right) \Bigg] \\ =& - \mathbb{E}_{\mathbb{R}} \Bigg[\left(\int_0^T g_s^\top d \langle M \rangle_s \pi_s \right)^2 \\ & +2 \left(\int_0^T g_s^\top d \langle M \rangle_s \pi_s \right) \left(\pi \cdot \left(\phi \cdot R + M^\perp - \int_0^\cdot d \langle M \rangle_s^\perp \beta_s \right)_T^\pi \right) \Bigg] \\ =&T_1, \end{aligned} \tag{91}$$

where, we recall that T_1 is defined in (31). We Note that the representation in (91) does not depend on the choice of π' (which still has to satisfy Lemma 5.10). Next, let us consider the left-hand side in (89). Using Lemma 5.14, we can restate it as follows.

$$\begin{aligned} &- \mathbb{E}_{\mathbb{R}} \left[(1 - A(X_T^0))(\pi' \cdot R_T^\pi + \pi \cdot \widetilde{R}_T^\pi)^2 + 2\pi' \cdot \widetilde{R}_T^\pi - [\pi \cdot \widetilde{R}^\pi + \pi' \cdot R^\pi]_T \right] \\ =&\mathbb{E}_{\mathbb{R}} \left[A(X_T^0)(\pi' \cdot R_T^\pi + \pi \cdot \widetilde{R}_T^\pi)^2 - 2\pi' \cdot R_T^\pi g \cdot M_T^H \right] \\ &+ \mathbb{E}_{\mathbb{R}} \left[2\pi' \cdot R_T^\pi \int_0^T g_s^\top d \langle M \rangle_s \pi_s - \left(\pi' \cdot R_T^\pi + \pi \cdot \widetilde{R}_T^\pi \right)^2 + \left[\pi' \cdot R^\pi + \pi \cdot \widetilde{R}^\pi \right]_T \right] \\ =&\mathbb{E}_{\mathbb{R}} \left[A(X_T^0)(\pi' \cdot R_T^\pi + \pi \cdot \widetilde{R}_T^\pi)^2 - 2\pi' \cdot R_T^\pi g \cdot M_T^H \right] + T_1, \end{aligned}$$

where, in the second equality, we have used (91). The computations behind (90) are similar. □

[1](90) is used later for the representation of $a(x, \varepsilon)$.

Lemma 5.16. *Let $x > 0$ be fixed. Under the conditions of Theorem 4.11, with $y = u_x(x, 0)$, let π^0 and π' be as in the statement of Lemma 5.10, and the associated K is defined in* (70)*. Then, let us consider*

$$\psi(\Delta x, \varepsilon) := \left(1 + \frac{\Delta x}{x}\right) \mathcal{E}\left(K^{\Delta x, \varepsilon, \pi}\right)_T, \ (\Delta x, \varepsilon) \in B_\delta(0, 0),$$

where $\delta > 0$ is chosen to be sufficiently close to 0, so that for every $(\Delta x, \varepsilon) \in B_\delta(0, 0)$, $\frac{X}{x}\mathcal{E}\left(K^{\Delta x,\varepsilon,\pi}\right) \in \mathcal{X}(1, \varepsilon)$ by Lemma 5.10 and the jumps of $K^{\Delta x,\varepsilon,\pi}$ take values in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Let us define

$$w(\Delta x, \varepsilon) := \mathbb{E}\left[U\left(X_T \psi(\Delta x, \varepsilon)\right)\right], \quad (\Delta x, \varepsilon) \in B_\delta(0, 0).$$

Then w admits the following expansion at $(0, 0)$:

$$w(\Delta x, \varepsilon) = w(0, 0) + (\Delta x \quad \varepsilon)\nabla w(0, 0) + \frac{1}{2}(\Delta x \quad \varepsilon) H_w \begin{pmatrix} \Delta x \\ \varepsilon \end{pmatrix} + o(\Delta x^2 + \varepsilon^2),$$

where

$$\begin{aligned} w_{\Delta x}(0, 0) &= u_x(x, 0), \\ w_\varepsilon(0, 0) &= xy\mathbb{E}_{\mathbb{R}}\left[\pi \cdot \widetilde{R}^\pi_T\right]. \end{aligned}$$

and

$$H_w := \begin{pmatrix} w_{\Delta x \Delta x}(0, 0) & w_{\Delta x \varepsilon}(0, 0) \\ w_{\Delta x \varepsilon}(0, 0) & w_{\varepsilon\varepsilon}(0, 0), \end{pmatrix}$$

and where

$$\begin{aligned} w_{\Delta x \Delta x}(0, 0) &= -\frac{y}{x}\mathbb{E}_{\mathbb{R}}\left[A(X_T)\left(1 + x\pi^0 \cdot R^\pi_T\right)^2\right], \\ w_{\Delta x \varepsilon}(0, 0) &= -\frac{y}{x}\mathbb{E}_{\mathbb{R}}\left[A(X_T)\left(1 + x\pi^0 \cdot R^\pi_T\right)\left(x\pi^1 \cdot R^\pi_T + x\pi \cdot \widetilde{R}^\pi_T\right) - xg \cdot M^H_T (x\pi^0 \cdot R^\pi_T + 1)\right], \\ w_{\varepsilon\varepsilon}(0, 0) &= -\frac{y}{x}\mathbb{E}_{\mathbb{R}}\left[A(X_T)\left(x\pi^1 \cdot R^\pi_T + x\pi \cdot \widetilde{R}^\pi_T\right)^2 - 2x\pi^1 \cdot R^\pi_T xg \cdot M^H_T + x^2 T_1\right]. \end{aligned}$$

Proof. Let us consider

$$\mathcal{E}\left(K^{\Delta x,\varepsilon,\pi}\right)_T = \exp\left(K^{\Delta x,\varepsilon,\pi}_T - \tfrac{1}{2}[K^{\Delta x,\varepsilon,\pi}]^c_T + \sum_{s \le T}\left(\log(1 + \Delta K^{\Delta x,\varepsilon,\pi}_s) - \Delta K^{\Delta x,\varepsilon,\pi}_s\right)\right). \tag{92}$$

As

$$|\log(1 + x) - x| \le x^2, \quad \textit{for every } x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right],$$

we observe that, in (92), the series $\sum\limits_{s \le T}\left(\log(1 + \Delta K^{\Delta x,\varepsilon,\pi}_s) - \Delta K^{\Delta x,\varepsilon,\pi}_s\right)$ converges absolutely for every $(\Delta x, \varepsilon) \in B_\delta(0, 0)$, $\mathbb{P} - a.s.$, and we have

$$\sum_{s \le T}\left(\log(1 + \Delta K^{\Delta x,\varepsilon,\pi}_s) - \Delta K^{\Delta x,\varepsilon,\pi}_s\right) \le [K^{\Delta x,\varepsilon,\pi}]_T.$$

Using the latter bounds, one can find a constant $C > 0$, such that for every $(\Delta x, \varepsilon) \in B_\delta(0, 0)$, we have

$$\mathcal{E}\left(K^{\Delta x,\varepsilon,\pi}\right)_T \le C\exp\left(|\varepsilon| C\left(|\pi \cdot \tilde{R}^\pi_T| + [\pi \cdot \tilde{R}^\pi]_T\right)\right), \quad \mathbb{P}\text{-a.s.}. \tag{93}$$

We observe that the series of term by term partial derivatives of

$$\sum_{s\leq T}\left(\log(1+\Delta K^{\Delta x,\varepsilon,\pi}_s)-\Delta K^{\Delta x,\varepsilon,\pi}_s\right)$$

convergence uniformly in $(\Delta x,\varepsilon)\in B_\delta(0,0)$, $\mathbb{P}$-a.s., where additionally the term by term partial derivatives of $\left(\log(1+\Delta K^{\Delta x,\varepsilon,\pi}_s)-\Delta K^{\Delta x,\varepsilon,\pi}_s\right)$ are continuous in $(\Delta x,\varepsilon)\in B_\delta(0,0)$, $\mathbb{P}$-a.s. We deduce that

$$\begin{aligned}
&\frac{\partial}{\partial \Delta x}\sum_{s\leq T}\left(\log(1+\Delta K^{\Delta x,\varepsilon,\pi}_s)-\Delta K^{\Delta x,\varepsilon,\pi}_s\right) = -\sum_{s\leq T}\frac{\Delta K^{\Delta x,\varepsilon,\pi}_s}{1+\Delta K^{\Delta x,\varepsilon,\pi}_s}\pi^0_s\Delta R^{\varepsilon,\pi}_s;\\
&\frac{\partial}{\partial \varepsilon}\sum_{s\leq T}\left(\log(1+\Delta K^{\Delta x,\varepsilon,\pi}_s)-\Delta K^{\Delta x,\varepsilon,\pi}_s\right)\\
&=-\sum_{s\leq T}\frac{\Delta K^{\Delta x,\varepsilon,\pi}_s}{1+\Delta K^{\Delta x,\varepsilon,\pi}_s}\left(\pi'_s\Delta R^\pi_s+\pi_s\Delta\tilde R^\pi_s+\Delta x\pi^0_s\Delta\tilde R^\pi_s+2\varepsilon\pi'_s\Delta\tilde R^\pi_s\right).
\end{aligned}$$

By the results of Lemma 5.10, we can rewrite $\psi(\Delta x,\varepsilon)$ as

For a fixed Δx and ε, let us denote $I:=K^{\Delta x,\varepsilon,\pi}$. Since, by direct computations, we have

$$\frac{\psi_{\Delta x}(\Delta x,\varepsilon)}{\psi(\Delta x,\varepsilon)}=\left(\frac{1}{x+\Delta x}+\frac{\partial I}{\partial\Delta x}-\left[I^c,\frac{\partial I^c}{\partial \Delta x}\right]-\sum_{s\leq\cdot}\frac{\frac{\partial\Delta I_s}{\partial\Delta x}\Delta I_s}{1+\Delta I_s}\right)_T,$$

as well as

$$\frac{\psi_{\varepsilon}(\Delta x,\varepsilon)}{\psi(\Delta x,\varepsilon)}=\left(\frac{\partial I}{\partial\varepsilon}-\left[I^c,\frac{\partial I^c}{\partial \varepsilon}\right]-\sum_{s\leq\cdot}\frac{\frac{\partial\Delta I_s}{\partial\varepsilon}\Delta I_s}{1+\Delta I_s}\right)_T,$$

we deduce that

$$\begin{aligned}
\psi_{\Delta x}(\Delta x,\varepsilon)|_{(\Delta x,\varepsilon)=(0,0)}&=\frac{1}{x}+\pi^0\cdot R^\pi_T,\\
\psi_{\varepsilon}(\Delta x,\varepsilon)|_{(\Delta x,\varepsilon)=(0,0)}&=\pi'\cdot R^\pi_T+\pi\cdot\tilde R^\pi_T.
\end{aligned}$$

Likewise, one can show that the series of term by term second-order partial derivatives of

$$\sum_{s\leq T}\left(\log(1+\Delta K^{\Delta x,\varepsilon,\pi}_s)-\Delta K^{\Delta x,\varepsilon,\pi}_s\right)$$

convergence uniformly in $(\Delta x,\varepsilon)\in B_\delta(0,0)$, where additionally the term by term second-order partial derivatives of $\left(\log(1+\Delta K^{\Delta x,\varepsilon,\pi}_s)-\Delta K^{\Delta x,\varepsilon,\pi}_s\right)$ are continuous in $(\Delta x,\varepsilon)\in B_\delta(0,0)$. Therefore, with I denoting $K^{\Delta x,\varepsilon,\pi}$, we get

$$\begin{aligned}
\frac{\psi_{\Delta x\Delta x}(\Delta x,\varepsilon)}{\psi(\Delta x,\varepsilon)}=&\left(\left(\frac{1}{x+\Delta x}+\frac{\partial I}{\partial\Delta x}-\left[I^c,\frac{\partial I^c}{\partial \Delta x}\right]-\sum_{s\leq\cdot}\frac{\frac{\partial\Delta I_s}{\partial\Delta x}\Delta I_s}{1+\Delta I_s}\right)_T\right)^2\\
&-\frac{1}{(x+\Delta x)^2}+\frac{\partial^2 I}{\partial\Delta x^2}-\left[I^c,\frac{\partial^2 I^c}{\partial\Delta x^2}\right]-\left[\frac{\partial I^c}{\partial\Delta x}\right]\\
&-\sum_{s\leq\cdot}\frac{\frac{\partial^2\Delta I_s}{\partial\Delta x^2}\Delta I_s+\left(\frac{\partial\Delta I_s}{\partial\Delta x}\right)^2}{1+\Delta I_s}+\sum_{s\leq\cdot}\left(\frac{\frac{\partial\Delta I_s}{\partial\Delta x}}{1+\Delta I_s}\right)^2\Delta I_s,
\end{aligned}$$

and thus

$$\psi_{\Delta x\Delta x}|_{(\Delta x,\varepsilon)=(0,0)} = \left(\frac{1}{x} + \pi^0 \cdot R^\pi_T\right)^2 - \frac{1}{x^2} - \left[\pi^0 \cdot R^\pi\right]^c_T - \sum_{s\leq T}\left(\pi^0_s \Delta R^\pi_s\right)^2$$
$$= \left(\frac{1}{x} + \pi^0 \cdot R^\pi_T\right)^2 - \frac{1}{x^2} - [\pi^0 \cdot R^\pi]_T.$$

Similarly, we obtain

$$\begin{aligned}\frac{\psi_{\Delta x\varepsilon}}{\psi} = &\left(\frac{1}{x+\Delta x} + \frac{\partial I}{\partial \Delta x} - \left[I^c, \frac{\partial I^c}{\partial \Delta x}\right] - \sum_{s\leq \cdot}\frac{\frac{\partial \Delta I_s}{\partial \Delta x}\Delta I_s}{1+\Delta I_s}\right)_T \\ &\times \left(\frac{\partial I}{\partial \varepsilon} - \left[I^c, \frac{\partial I^c}{\partial \varepsilon}\right] - \sum_{s\leq \cdot}\frac{\frac{\partial \Delta I_s}{\partial \varepsilon}\Delta I_s}{1+\Delta I_s}\right)_T + \frac{\partial^2 I_T}{\partial \Delta x \partial \varepsilon} - \left[I^c, \frac{\partial^2 I^c}{\partial \Delta x \partial \varepsilon}\right]_T - \left[\frac{\partial I^c}{\partial \varepsilon}, \frac{\partial I^c}{\partial \Delta x}\right]_T \\ &- \sum_{s\leq T}\frac{\frac{\partial^2 \Delta I_s}{\partial \Delta x \partial \varepsilon}\Delta I_s + \frac{\partial \Delta I_s}{\partial \Delta x}\frac{\partial \Delta I_s}{\partial \Delta x}}{1+\Delta I_s} + \sum_{s\leq T}\frac{\frac{\partial \Delta I_s}{\partial \Delta x}\frac{\partial \Delta I_s}{\partial \varepsilon}\Delta I_s}{(1+\Delta I_s)^2},\end{aligned}$$

and thus

$$\begin{aligned}\psi_{\Delta x\varepsilon}|_{(\Delta x,\varepsilon)=(0,0)} = &\left(\frac{1}{x} + \pi^0 \cdot R^{0,\pi}_T\right)\left(\pi' \cdot R^{0,\pi}_T + \pi \cdot \tilde{R}^\pi_T\right) \\ &+ \pi^0 \cdot \tilde{R}^\pi_T - \left[\frac{1}{x} + \pi^0 \cdot R^{0,\pi}, \pi' \cdot R^{0,\pi} + \pi \cdot \tilde{R}^\pi\right]_T.\end{aligned}$$

Continuing in such a manner, we get

$$\begin{aligned}\frac{\psi_{\varepsilon\varepsilon}(\Delta x,\varepsilon)}{\psi(\Delta x,\varepsilon)} = &\left(\frac{\partial I}{\partial \varepsilon} - \left[I^c, \frac{\partial I^c}{\partial \varepsilon}\right] - \sum_{s\leq \cdot}\frac{\frac{\partial \Delta I_s}{\partial \varepsilon}\Delta I_s}{1+\Delta I_s}\right)^2_T + \frac{\partial^2 I_T}{\partial \varepsilon^2} - \left[I^c, \frac{\partial^2 I^c}{\partial \varepsilon^2}\right]_T - \left[\frac{\partial I^c}{\partial \varepsilon}\right]_T \\ &- \sum_{s\leq T}\frac{\frac{\partial^2 \Delta I_s}{\partial \varepsilon^2}\Delta I_s + \left(\frac{\partial \Delta I_s}{\partial \varepsilon}\right)^2}{1+\Delta I_s} + \sum_{s\leq T}\left(\frac{\frac{\partial \Delta I_s}{\partial \varepsilon}}{1+\Delta I_s}\right)^2\Delta I_s,\end{aligned}$$

and consequently, we obtain

$$\psi_{\varepsilon\varepsilon}(\Delta x,\varepsilon)|_{(\Delta x,\varepsilon)=(0,0)} = \left(\pi' \cdot R^\pi_T + \pi \cdot \widetilde{R}^\pi_T\right)^2 + 2\pi' \cdot \widetilde{R}^\pi_T - \left[\pi' \cdot R^\pi + \pi \cdot \widetilde{R}^\pi\right]_T.$$

Let us now fix $(\Delta x, \varepsilon) \in B_\delta(0,0)$ and define

$$\widetilde{\psi}(z) := \psi(z\Delta x, z\varepsilon), \quad z \in (-1,1).$$

We observe that

$$\widetilde{\psi}'(z) = \psi_{\Delta x}(\Delta x,\varepsilon)\Delta x + \psi_\varepsilon(\Delta x,\varepsilon)\varepsilon, \tag{94}$$

$$\begin{aligned}\widetilde{\psi}''(z) =&\psi_{\Delta x\Delta x}(z\Delta x, z\varepsilon)\Delta x^2 + 2\psi_{\Delta x\varepsilon}(z\Delta x, z\varepsilon)\Delta x\varepsilon \\ &+ \psi_{\varepsilon\varepsilon}(z\Delta x, z\varepsilon)\varepsilon^2.\end{aligned} \tag{95}$$

Let us set

$$W(z) := U(X_T\widetilde{\psi}(z)), \quad z \in (-1,1).$$

By direct computations, we get

$$W'(z) = U'(X_T\widetilde{\psi}(z))X_T\widetilde{\psi}'(z),$$
$$W''(z) = U''(X_T\widetilde{\psi}(z))(X_T\widetilde{\psi}'(z))^2 + U'(X_T\widetilde{\psi}(z))X_T\widetilde{\psi}''(z).$$

Let us define

$$J := 1 + |\pi\cdot\widetilde{R}^\pi_T| + [\pi\cdot\widetilde{R}^\pi]_T.$$

Now one can deduce the existence of a constant $b_1 > 0$, such that

$$|\widetilde{\psi}'(z)| \leq b_1 J\exp\left(b_1\delta J\right), \quad \text{and} \quad \widetilde{\psi}(z)^{-c_2} + 1 \leq b_1\exp\left(b_1\delta J\right), \quad z\in(-1,1).$$

Using Lemma 5.11, we from the computations above, we get

$$\begin{aligned}\sup_{z\in(-1,1)}\left|W'(z)\right| &\leq \sup_{z\in(-1,1)} U'(X_T)X_T\left(\widetilde{\psi}(z)^{-c_2}+1\right)|\widetilde{\psi}'(z)| \\ &\leq U'(X_T)X_Tb_1^2J\exp(2b_1\delta J).\end{aligned} \tag{96}$$

Likewise, from Assumption 2.1, using Lemma 5.11, we conclude the existence of a constant $b_2 > 0$, such that

$$\sup_{z\in(-1,1)}\left|W''(z)\right| \leq b_2U'(X_T)X_TJ^2\exp\left(b_2\delta J\right). \tag{97}$$

From (96) and (97), we get

$$\begin{aligned}&\sup_{z\in(-1,1)}\left|W'(z)\right| + \sup_{z\in(-1,1)}\left|W''(z)\right| \\ \leq &U'(X_T)X_Tb_1^2J\exp(2b_1\delta J) + b_2U'(X_T)X_TJ^2\exp\left(b_2\delta J\right).\end{aligned}$$

As $1 \leq J \leq J^2$, we deduce the existence of a constant $b > 0$, such that for every z_1 and z_2 in $(-1,1)$, we get

$$\left|\frac{W(z_1)-W(z_2)}{z_1-z_2}\right| + \left|\frac{W'(z_1)-W'(z_2)}{z_1-z_2}\right| \leq U'(X_T)X_TbJ^2\exp\left(b\delta J\right). \tag{98}$$

By passing to a smaller δ, if needed, via Hölder's inequality, we deduce that the right-hand side in (98) is integrable. Further, as the bound in (98) is uniform in $(\Delta x,\varepsilon)\in B_\delta(0,0)$, we deduce the assertion of the lemma from the dominated convergence theorem and the representation formulas from Lemmas 5.12 and 5.15. □

Lemma 5.17. *Let $x > 0$ be fixed, the conditions of Theorem 4.15 hold, and $y = u_x(x,0)$. Then, we have*

$$\begin{aligned}u(x+\Delta x,\varepsilon) \geq u(x,0) &+ \Delta xy + \varepsilon xy\mathbb{E}_{\mathbb{R}}\left[\pi\cdot\widetilde{R}^\pi_T\right] \\ &+ \tfrac{1}{2}(\Delta x \quad \varepsilon)H_u(x,0)\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix} + o(\Delta x^2+\varepsilon^2).\end{aligned}$$

Proof. Using Assumption 4.2, we can approximate the optimizers to (29) and (32) by the elements of $\mathcal{M}^\infty(x)$. By [Mos20, Lemma 4.1, p. 4453], these elements of $\mathcal{M}^\infty$ can be represented as stochastic integrals with respect to R^π. In turn, by stopping, we can assume that the strategies satisfy the assumptions of Lemma 5.10. The result now follows from Lemma 5.16 via the approximation of elements of $\mathcal{M}^2$ by the ones in $\mathcal{M}^\infty$ as in the appendix. □

5.4. **Constructing a second-order bound for the dual problem.** For every $\varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, where $\widetilde{\varepsilon} > 0$ is given in Lemma 5.2, we recall that λ^ε and G^ε are given by the solutions to (52), where we also note that Assumption 4.1 allows for the explicit forms of λ^ε's and G^ε's. For bounded and predictable β^0 and β', we set

$$\begin{aligned} J^{\Delta y, \varepsilon} := &-(\lambda^\varepsilon - \lambda^0) \cdot M - \beta \cdot G^\varepsilon \cdot M + (\Delta y \beta^0 + \varepsilon \beta') \cdot (-G^\varepsilon \cdot M + M^\perp) \\ &+ (\varepsilon + \Delta y \widetilde{\phi}) \cdot \widetilde{L} + \Delta y \bar{L}, \quad (\varepsilon, \Delta y) \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon}) \times \mathbb{R}, \end{aligned} \tag{99}$$

where $\bar{L}$ and $\widetilde{L}$ are in $\mathcal{H}^2_{loc}(\mathbb{P})$ that are orthogonal to every component of both M and $M^\perp$ and to each other and are such that $\bar{L}^H$ and $\widetilde{L}^H$ are bounded; $\widetilde{\phi}$ is a predictable and such that $\int_0^T \widetilde{\phi}^2_s d\left\langle \widetilde{L} \right\rangle_s < \infty$, $\mathbb{P}$-a.s., $\widetilde{\varepsilon} > 0$ is given in Lemma 5.2. We note that the stochastic integrals in (99) are well-defined.

The following lemma including the proof is similar to Lemma 5.10. Therefore, we skip the proof for brevity of the exposition.

Lemma 5.18. *Let $x > 0$ be fixed, the conditions of Theorem 4.11 hold, $y = u_x(x, 0)$. Let β^0 and β' be bounded, predictable (thus both are M and $M^\perp$-integrable) processes, such that the following processes are bounded:*

(1) $\beta^0 \cdot M^{\perp, H}$ *and* $\beta' \cdot M^{\perp, H}$*;*
(2) $[\beta^0 \cdot M^{\perp, H}]$ *and* $[\beta' \cdot M^{\perp, H}]$*;*
(3) $\beta^0 \cdot \nu \cdot M^H$, $\beta' \cdot \nu \cdot M^H$, $\nu^\top \beta' \cdot \left(M - \int_0^\cdot d\langle M \rangle_s \pi_s\right)^H$*, and* $\int_0^\cdot (\beta'_s)^\top \nu_s d\langle M \rangle_s \pi_s$*;*
(4) $[\beta^0 \nu \cdot M^H]$ *and* $[\beta' \nu \cdot M^H]$, $\int_0^\cdot (\beta^0_s)^\top d\langle M \rangle_s \beta^0$, $\int_0^\cdot (\beta'_s)^\top d\langle M \rangle_s \beta'$*;*
(5) $[g \cdot M^H, \beta^0 \nu \cdot M^H]$ *and* $[g \cdot M^H, \beta' \nu \cdot M^H]$*;*
(6) $[g \cdot M^H, \beta^0 \cdot M^{\perp, H}]$ *and* $[g \cdot M^H, \beta' \cdot M^{\perp, H}]$*.*

Then, there exists a constant $\delta > 0$, such that for every $(\Delta y, \varepsilon) \in B_\delta(0,0)$, we have

$$\frac{Y}{y} \mathcal{E}\left(J^{\Delta y, \varepsilon, H}\right) \in \mathcal{Y}(1, \varepsilon).$$

Lemma 5.19. *Let $x > 0$ be fixed, the conditions of Theorem 4.11 hold, $y = u_x(x, 0)$, and let β' satisfy the assumptions of Lemma 5.18, and $\widetilde{L} \in \mathcal{H}^2_{loc}(\mathbb{P})$ is orthogonal to every component of both M and $M^\perp$, and such that $\widetilde{L}^H$ is bounded. With $N := \beta' \cdot M^{\perp, H} + \widetilde{L}^H$, we have*

$$\begin{aligned} \mathbb{E}_{\mathbb{R}}\left[\beta' \cdot (G^0)' \cdot M^H_T\right] &= \mathbb{E}_{\mathbb{R}}\left[N_T \left(\pi \cdot \left(\phi \cdot R + M^\perp - \int_0^\cdot d\left\langle M^\perp \right\rangle_s \beta_s\right)^\pi_T\right)\right] \\ &= \mathbb{E}_{\mathbb{R}}\left[N_T\left(\pi \cdot \widetilde{R}^\pi_T\right)\right] - \mathbb{E}_{\mathbb{R}}\left[N_T \int_0^T \pi_s^\top d\left\langle M \right\rangle_s g_s\right]. \end{aligned} \tag{100}$$

Proof. Since β' satisfies the assumption of Lemma 5.18, we get

$$\mathbb{E}_{\mathbb{R}}\left[\beta' \cdot (G^0)' \cdot \left(M - \int_0^\cdot d\left\langle M \right\rangle_s \pi_s\right)^H_T\right] = 0.$$

Therefore, using a completion to a martingale, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{R}}\left[\beta'\cdot(G^0)'\cdot M^H_T\right] &= \mathbb{E}_{\mathbb{R}}\left[\beta'\cdot(G^0)'\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H_T\right]+\mathbb{E}_{\mathbb{R}}\left[\int_0^T(\beta'_s)^\top(G^0_s)'d\langle M\rangle_s\pi_s\right]\\ &=\mathbb{E}_{\mathbb{R}}\left[\int_0^T(\beta'_s)^\top(G^0_s)'d\langle M\rangle_s\pi_s\right]\\ &=\mathbb{E}_{\mathbb{R}}\left[\int_0^T\pi_s^\top d\langle M\rangle_s((G^0_s)')^\top\beta'_s\right]\\ &=\mathbb{E}_{\mathbb{R}}\left[\int_0^T\pi_s^\top\nu_s d\langle M\rangle_s\beta'_s\right],\end{aligned}$$

where the latter equality can be established along the lines of the proof of Lemma 5.2. Further, using Assumption 4.5, we get

$$\mathbb{E}_{\mathbb{R}}\left[\beta'\cdot(G^0)'\cdot M^H\right]=\mathbb{E}_{\mathbb{R}}\left[\int_0^T\pi_s^\top\nu_s d\langle M\rangle_s\beta'_s\right]=\mathbb{E}_{\mathbb{R}}\left[\int_0^T\pi_s^\top d\left\langle M^\perp\right\rangle_s\beta'_s\right]. \tag{101}$$

By Lemma 5.6, $\pi\cdot\widetilde{R}^\pi_T$ admits the representation (61), where $|\pi\cdot\left(\phi\cdot R+M^\perp-\int_0^\cdot d\left\langle M^\perp\right\rangle_s\beta_s\right)^\pi_T|$, and $|\int_0^T\pi_s^\top d\langle M\rangle_s g_s|$ have exponential moments under $\mathbb{R}$, that is satisfy (62).

Next, with N as above, similarly to Lemma 5.13, as $\pi\cdot\left(\phi\cdot R+M^\perp-\int_0^\cdot d\left\langle M^\perp\right\rangle_s\beta_s\right)^\pi$ and N are in $\mathcal{H}^2(\mathbb{R})$ by Lemma 5.6 and the assumption of this lemma, and $\pi\cdot M^\perp$ and $\beta'\cdot M^\perp$ in $\mathcal{H}^2_{loc}(\mathbb{P})$, one can show that the last expression in (101) can be represented as

$$\mathbb{E}_{\mathbb{R}}\left[\int_0^T\pi_s^\top d\left\langle M^\perp\right\rangle_s\beta'_s\right]=\mathbb{E}_{\mathbb{R}}\left[\left(\pi\cdot\left(\phi\cdot R+M^\perp-\int_0^\cdot d\left\langle M^\perp\right\rangle_s\beta_s\right)^\pi_T\right)N_T\right].$$

Combining this with (101), we conclude that (100) holds.

□

Lemma 5.20. *Let $x>0$ be fixed, the conditions of Theorem 4.11 hold, $y=u_x(x,0)$, let β' satisfy the assumptions of Lemma 5.18, $L\in\mathcal{H}^2_{loc}(\mathbb{P})$ is orthogonal to every component of both M and $M^\perp$, and such that L^H is bounded. Then with $N:=\beta'\cdot M^{\perp,H}+L^H$, we have*

$$\begin{aligned}&\mathbb{E}_{\mathbb{R}}\Big[\left(B(Y^0_T)-1\right)\left(-g\cdot M^H_T+N_T\right)^2\\ &\qquad+\left((\lambda^0)''+\beta\cdot(G^0)''+2\beta'\cdot(G^0)'\right)\cdot M^H_T+\left[-g\cdot M^H+N\right]_T\Big]\\ &=\mathbb{E}_{\mathbb{R}}\left[B(Y^0_T)\left(-g\cdot M^H_T+N_T\right)^2+2N_T\left(\pi\cdot\widetilde{R}^\pi_T\right)\right]+T_2,\end{aligned}\tag{102}$$

where T_2 is defined in (36).

Proof. Using Lemmas 5.5 and 5.19, we have

$$\begin{aligned}&\mathbb{E}_{\mathbb{R}}\left[\left((\lambda^0)''+\beta\cdot(G^0)''+2\beta'\cdot(G^0)'\right)\cdot M^H_T\right]\\ &=-2\mathbb{E}_{\mathbb{R}}\left[\int_0^T\pi_s^\top\phi_s d\langle M\rangle_s g_s\right]\\ &\quad+2\mathbb{E}_{\mathbb{R}}\left[N_T\left(\pi\cdot\widetilde{R}^\pi_T\right)\right]-2\mathbb{E}_{\mathbb{R}}\left[N_T\int_0^T\pi_s^\top d\langle M\rangle_s g_s\right].\end{aligned}\tag{103}$$

This allows to rewrite the left-hand side in (102) as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{R}}\Big[\big(B(Y^0_T)-1\big)\left(-g\cdot M^H_T+N_T\right)^2\\
&\qquad+\left((\lambda^0)''+\beta\cdot(G^0)''+2\beta'\cdot(G^0)'\right)\cdot M^H_T+\left[-g\cdot M^H+N\right]_T\Big]\\
&=\mathbb{E}_{\mathbb{R}}\Big[B(Y^0_T)\left(-g\cdot M^H_T+N_T\right)^2+2N_T\left(\pi\cdot\widetilde{R}^\pi_T\right)\\
&\qquad-\left(-g\cdot M^H_T+N_T\right)^2+\left[-g\cdot M^H+N\right]_T\\
&\qquad-2\int_0^T\pi_s^\top\phi_s d\langle M\rangle_s g_s-2N_T\int_0^T\pi_s^\top d\langle M\rangle_s g_s\Big]\\
&=\mathbb{E}_{\mathbb{R}}\Big[B(Y^0_T)\left(-g\cdot M^H_T+N_T\right)^2+2N_T\left(\pi\cdot\widetilde{R}^\pi_T\right)\\
&\qquad+\left(\int_0^T\pi_s^\top d\langle M\rangle_s g_s\right)^2-2\left(\int_0^T\pi_s^\top d\langle M\rangle_s g_s\right)\left(g\cdot M^H_T-N_T\right)\\
&\qquad-2\int_0^T\pi_s^\top\phi_s d\langle M\rangle_s g_s-2N_T\int_0^T\pi_s^\top d\langle M\rangle_s g_s\Big]\\
&=\mathbb{E}_{\mathbb{R}}\Big[B(Y^0_T)\left(-g\cdot M^H_T+N_T\right)^2+2N_T\left(\pi\cdot\widetilde{R}^\pi_T\right)\\
&\qquad+\left(\int_0^T\pi_s^\top d\langle M\rangle_s g_s\right)^2-2\left(\int_0^T\pi_s^\top d\langle M\rangle_s g_s\right)\left(g\cdot M^H_T\right)\\
&\qquad-2\int_0^T\pi_s^\top\phi_s d\langle M\rangle_s g_s\Big]\\
&=\mathbb{E}_{\mathbb{R}}\Big[B(Y^0_T)\left(-g\cdot M^H_T+N_T\right)^2+2N_T\left(\pi\cdot\widetilde{R}^\pi_T\right)\Big]+T_2,
\end{aligned}$$

where we recall that T_2 is defined (36). □

Lemma 5.21. *Let $x>0$ be fixed, the conditions of Theorem 4.11 hold and $y=u_x(x,0)$. Let us set*

$$h^\varepsilon:=\lambda^\varepsilon-\lambda^0+(G^\varepsilon)^\top\beta,\quad \varepsilon\in(-\bar{\varepsilon}'_0,\widetilde{\varepsilon}),$$

where $\widetilde{\varepsilon}$ is as in Lemma 5.2. Then, there exists $c''>0$ and $\bar{\varepsilon}>0$, such that for every $\varepsilon\in(-\bar{\varepsilon},\bar{\varepsilon})$, we have

$$\mathbb{E}_{\mathbb{R}}\left[\exp\left(c''\left|h^\varepsilon\cdot M^H_T\right|+c''\left[h^\varepsilon\cdot M^H\right]_T\right)\right]<\infty. \tag{104}$$

Proof. For a sufficiently small positive c, with $\widetilde{M}:=cg\cdot(M-\int_0^\cdot d\langle M\rangle_s\pi_s)^H$, Lemma 5.1, applied under the measure $\mathbb{R}$, implies that

$$\mathbb{E}_{\mathbb{R}}\left[\exp\left(c\left|g\cdot(M-\int_0^\cdot d\langle M\rangle_s\pi_s)^H_T\right|\right)\right]<\infty. \tag{105}$$

Further, from boundedness of ν and ϕ, we observe that there exists a constant $\widetilde{c}'>0$ that does not depend on $\varepsilon\in(-\bar{\varepsilon},\bar{\varepsilon})$, such that

$$\left|\int_0^T(h^\varepsilon_s)^\top d\langle M\rangle_s\pi_s\right|\leq\widetilde{c}'\left|\int_0^T g_s^\top d\langle M\rangle_s\pi_s\right|,\quad \mathbb{P}\text{-a.s.}, \tag{106}$$

which, in view of Lemma 5.6 (see (62)), implies that there exists $c'>0$, such that

$$\mathbb{E}^{\mathbb{R}}\left[e^{c'\left|\int_0^T(h^\varepsilon_s)^\top d\langle M\rangle_s\pi_s\right|}\right]<\infty,\quad \varepsilon\in(-\bar{\varepsilon},\bar{\varepsilon}). \tag{107}$$

Similarly, from the assumptions of this lemma we have

$$[h^\varepsilon \cdot M^H] = [h^\varepsilon \cdot M^{H,c}] + \sum_{s\leq\cdot}(h^\varepsilon_s \Delta M^H_s)^2 \leq \bar{c}[g\cdot M^H], \quad \mathbb{P}\text{-a.s.}, \tag{108}$$

for some $\bar{c} > 0$ and every $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$. Therefore, from (108) and Assumption 4.9, we deduce that there exists $\bar{c}' > 0$, such that

$$\mathbb{E}^{\mathbb{R}}\left[e^{\bar{c}'[h^\varepsilon\cdot M^H]_T}\right] < \infty, \quad \varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon}). \tag{109}$$

Next, from (109), using Lemma 5.1 with $\widetilde{M} = h^\varepsilon \cdot (M - \int_0^\cdot d\langle M\rangle_s \pi_s)^H$ (under $\mathbb{R}$), we conclude from (109) that there exists $\bar{c}'' > 0$, such that

$$\mathbb{E}^{\mathbb{R}}\left[e^{\bar{c}''\left|h^\varepsilon\cdot (M - \int_0^\cdot d\langle M\rangle_s \pi_s)^H_T\right|}\right] < \infty, \quad \varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon}). \tag{110}$$

Now, from (107) and (110), using Hölder's inequality, we assert that existence of $\widetilde{c}'' > 0$, such that

$$\mathbb{E}^{\mathbb{R}}\left[e^{\widetilde{c}''\left|h^\varepsilon\cdot M^H_T\right|}\right] < \infty, \quad \varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon}).$$

The latter inequality and (109) imply (104).

□

Lemma 5.22. *Let $x > 0$ be fixed, the conditions of Theorem 4.11 hold, $y = u_x(x, 0)$, and β^0 and β' be as in the statement of Lemma 5.18. Then, let us consider*

$$\widetilde{\psi}(\Delta x, \varepsilon) := \left(1 + \frac{\Delta y}{y}\right)\mathcal{E}\left(J^{\Delta y,\varepsilon,H}\right)_T, \quad (\Delta y, \varepsilon) \in B_\delta(0,0),$$

where $\delta > 0$ is chosen to be sufficiently close to 0, so that for every $(\Delta y, \varepsilon) \in B_\delta(0, 0)$, the jumps of $J^{\Delta y,\varepsilon,H}$, defined in (99), take values in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and let us define

$$\widetilde{w}(\Delta x, \varepsilon) := \mathbb{E}\left[V\left(Y_T\widetilde{\psi}(\Delta y, \varepsilon)\right)\right], \quad (\Delta x, \varepsilon) \in B_\delta(0, 0).$$

Then w admits the following expansion at $(0, 0)$:

$$\widetilde{w}(\Delta x, \varepsilon) = \widetilde{w}(0,0) + (\Delta y \quad \varepsilon)\nabla\widetilde{w}(0,0) + \frac{1}{2}(\Delta y \quad \varepsilon)H_{\widetilde{w}}\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix} + o(\Delta y^2 + \varepsilon^2),$$

where

$$\begin{aligned}\widetilde{w}_{\Delta y}(0,0) &= v_y(y,0),\\ \widetilde{w}_\varepsilon(0,0) &= xy\mathbb{E}_{\mathbb{R}}\left[g\cdot M^H_T\right].\end{aligned}$$

and

$$H_{\widetilde{w}} := \begin{pmatrix}\widetilde{w}_{\Delta y\Delta y}(0,0) & \widetilde{w}_{\Delta y\varepsilon}(0,0)\\ \widetilde{w}_{\Delta y\varepsilon}(0,0) & \widetilde{w}_{\varepsilon\varepsilon}(0,0),\end{pmatrix},$$

whose components are

$$\begin{aligned}\widetilde{w}_{\Delta y\Delta y}(0,0) &= \frac{x}{y}\mathbb{E}_{\mathbb{R}}\left[B(Y_T)\left(1+\bar{N}^y_T\right)^2\right],\\ \widetilde{w}_{\Delta y\varepsilon}(0,0) &= \frac{x}{y}\mathbb{E}_{\mathbb{R}}\left[B(Y_T)\left(1+\bar{N}^y_T\right)\left(\bar{N}^\varepsilon_T - yg\cdot M^H_T\right) + y\pi\cdot\widetilde{R}^\pi_T\left(1+\bar{N}^y_T\right)\right],\\ \widetilde{w}_{\varepsilon\varepsilon}(0,0) &= \frac{x}{y}\mathbb{E}_{\mathbb{R}}\left[B(Y_T)\left(\bar{N}^\varepsilon_T - yg\cdot M^H_T\right)^2 + 2\bar{N}^\varepsilon_T\left(y\pi\cdot\widetilde{R}^\pi_T\right) + y^2T_2\right],\end{aligned} \tag{111}$$

where

$$\bar{N}^y = y(\beta^0 \cdot M^{\perp,H} + \bar{L}^H) \quad \text{and} \quad \bar{N}^\varepsilon = y(\beta' \cdot M^{\perp,H} + \widetilde{L}^H),$$

$\bar{L}$ and $\widetilde{L}$ are in $\mathcal{H}^2_{loc}(\mathbb{P})$ and are orthogonal to every component of both M and $M^\perp$, and such that $\bar{N}^y$ and $\bar{N}^\varepsilon$ are bounded[2].

Proof. The proof parallels the one of Lemma 5.16. In the context of the dual problem, there are some computational differences, and we have to establish the appropriate admissibility, some representations, and (exponential) integrability. The admissibility of the approximating dual elements is established in Lemma 5.18; the integrability is provided by Lemma 5.21. Following the expansion, which is similar to the one in Lemma 5.16, to represent $\widetilde{w}_{\varepsilon\varepsilon}(0,0)$ as in (111), one should use Lemma 5.20. For the representation of $\widetilde{w}_{\Delta y\varepsilon}(0,0)$ in (111), similarly to the proof of Lemma 5.20, one can show that

$$\begin{aligned}&\mathbb{E}_{\mathbb{R}}\left[-(1+\bar{N}^y_T)(-yg\cdot M^H_T + \bar{N}^\varepsilon_T) + y^2\beta^0\cdot(G^0)'\cdot M^H_T + \left[1+\bar{N}^y, -yg\cdot M^H + \bar{N}^\varepsilon\right]_T\right]\\ &= \mathbb{E}_{\mathbb{R}}\left[y\pi\cdot \widetilde{R}^\pi_T(1+\bar{N}^y_T)\right].\end{aligned}$$

With these comments, the details are omitted for brevity as they are similar to the proof of Lemma 5.16.

□

Lemma 5.23. *Let $x>0$ be fixed, the conditions of Theorem 4.11 hold and $y = u_x(x,0)$. Then, we have*

$$\begin{aligned}v(y+\Delta y,\varepsilon) \le\ & v(y,0) - \Delta y x + \varepsilon x y \mathbb{E}_{\mathbb{R}}\left[g\cdot M^H_T\right]\\ &+ \tfrac{1}{2}(\Delta y \quad \varepsilon) H_v(y,0)\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix} + o(\Delta y^2+\varepsilon^2),\end{aligned}$$

where $\nabla v(y,0) = (y\ v_\varepsilon(y,0))^\top$, $v_\varepsilon(y,0)$ is given by Theorem 4.11, and

$$H_v(y,0) = \frac{x}{y}\begin{pmatrix} b(y,y) & b(y,\varepsilon)\\ b(y,\varepsilon) & b(\varepsilon,\varepsilon)\end{pmatrix},$$

where $b(y,y)$, $b(y,\varepsilon)$, and $b(\varepsilon,\varepsilon)$ are given by (34), (38), and (37), respectively.

The proof of the lemma is skipped as it follows the structure of the proof of Lemma 5.17 with the corresponding modifications based on Lemmas 5.21 and 5.18.

5.5. **Closing the duality gap up to the second order.**

Lemma 5.24. *Let $x>0$ be fixed, the conditions of Theorem 4.11 hold and $y = u_x(x,0)$. Then, T_1 and T_2 defined in (31) and (36), respectively, satisfy*

$$\mathbb{E}_{\mathbb{R}}\left[\tfrac{1}{2}(T_1+T_2) + \left(g\cdot M^H_T\right)\left(\pi\cdot \widetilde{R}^\pi_T\right)\right] = 0. \tag{112}$$

[2]The appendix contains more explanation behind this construction.

Proof. With $c := \int_0^T \pi_s^\top d\langle M\rangle_s g_s$, by direct computations, we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{R}}\left[\tfrac{1}{2}(T_1+T_2) + g\cdot M_T^H \pi\cdot \widetilde{R}_T^\pi\right]\\
=&\mathbb{E}_{\mathbb{R}}\left[-\tfrac{1}{2}c^2 - c\left(\pi\cdot\left(\phi\cdot R + M^\perp - \int_0^\cdot d\langle M\rangle^\perp \beta\right)^\pi_T\right)\right.\\
&\left.+\tfrac{1}{2}c^2 - cg\cdot M_T^H - \int_0^T \pi_s^\top \phi_s d\langle M\rangle_s g_s + g\cdot M_T^H\pi\cdot\widetilde{R}_T^\pi\right].
\end{aligned}$$

Cancelling the $\frac{1}{2}c^2$ terms and collecting $-cg\cdot M_T^H$ and $g\cdot M_T^H\pi\cdot\widetilde{R}_T^\pi$ terms, we can rewrite the latter expectation as follows.

$$\begin{aligned}
&\mathbb{E}_{\mathbb{R}}\left[-c\left(\pi\cdot\left(\phi\cdot R + M^\perp - \int_0^\cdot d\langle M\rangle^\perp\beta\right)^\pi_T\right)\right.\\
&\left.-\int_0^T \pi_s^\top\phi_s d\langle M\rangle_s g_s + g\cdot M_T^H\left(\pi\cdot\left(\phi\cdot R+M^\perp - \int_0^\cdot d\langle M\rangle^\perp\beta\right)^\pi_T\right)\right].
\end{aligned}\tag{113}$$

Next, adding $g\cdot M_T^H\left(\pi\cdot\left(\phi\cdot R+M^\perp-\int_0^\cdot d\langle M\rangle^\perp\beta\right)^\pi_T\right)$ and $-c\left(\pi\cdot\left(\phi\cdot R+M^\perp-\int_0^\cdot d\langle M\rangle^\perp\beta\right)^\pi_T\right)$, we can further rewrite (113) as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{R}}\left[\left(g\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H_T\right)\left(\pi\cdot\left(\phi\cdot R+M^\perp-\int_0^\cdot d\langle M\rangle^\perp\beta\right)^\pi_T\right)\right.\\
&\left.-\int_0^T\pi_s^\top\phi_s d\langle M\rangle_s g_s\right]\\
=&0,
\end{aligned}$$

where the proof of the later equality is entirely similar to the one of Lemma 5.13, and both $\left(g\cdot\left(M-\int_0^\cdot d\langle M\rangle_s\pi_s\right)^H_T\right)$ and $\left(\pi\cdot\left(\phi\cdot R+M^\perp-\int_0^\cdot d\langle M\rangle^\perp\beta\right)^\pi_T\right)$ are in $\mathcal{H}^2(\mathbb{R})$ by Lemma 5.6 and $\pi\cdot M$ and $g\cdot M$ are in $\mathcal{H}^2_{loc}(\mathbb{P})$, which follows from the assumptions of this lemma. It is therefore skipped for brevity. □

Proof of Theorem 4.14. Using the standard technique of the calculus of variations, we get

$$\begin{aligned}
A(X_T)(M_T^\varepsilon + x\pi\cdot\widetilde{R}_T^\pi) - xg\cdot M_T^H &= c+\widetilde{N}_T,\\
B(Y_T)(N_T^\varepsilon - yg\cdot M_T^H) + y\pi\cdot\widetilde{R}_T^\pi &= d+\widetilde{M}_T,
\end{aligned}\tag{114}$$

for some constants c and d and some $\widetilde{M}\in\mathcal{M}^2(x)$, and $\widetilde{N}\in\mathcal{N}^2(y)$. To compute c and d, we multiply the equations in (114) by (M_T^x+1) and (N_T^y+1), respectively, and take the expectation under $\mathbb{R}$, to deduce that

$$\begin{aligned}
c &= \mathbb{E}^{\mathbb{R}}\left[A(X_T)(M_T^\varepsilon + x\pi\cdot\widetilde{R}_T^\pi)(M_T^x+1) - xg\cdot M_T^H(M_T^x+1)\right] = a(x,\varepsilon),\\
d &= \mathbb{E}^{\mathbb{R}}\left[B(Y_T)(N_T^\varepsilon - yg\cdot M_T^H)(N_T^y+1) + y\pi\cdot\widetilde{R}_T^\pi(N_T^y+1)\right] = b(y,\varepsilon).
\end{aligned}\tag{115}$$

From the second equation in (114), we deduce that

$$\begin{aligned}
N_T^\varepsilon - yg\cdot M_T^H &= A(X_T)\left(-y\pi\cdot\widetilde{R}_T^\pi + d + \widetilde{M}_T\right)\\
&= dA(X_T)(1+M_T^x) + A(X_T)\left(\widetilde{M}_T - dM_T^x - y\pi\cdot\widetilde{R}_T^\pi\right)
\end{aligned}\tag{116}$$

Let us recall that

$$\begin{aligned} A(X_T)(M^x_T+1) &= a(x,x)(N^y_T+1), \\ B(Y_T)(N^y_T+1) &= b(y,y)(M^x_T+1), \end{aligned} \tag{117}$$

which was proven in [KS06a]. This and (116) allow to obtain

$$A(X_T)\left(dM^x_T - \widetilde{M}_T + y\pi\cdot\widetilde{R}^\pi_T\right) - yg\cdot M^H_T = da(x,x) + da(x,x)N^y_T - N^\varepsilon_T. \tag{118}$$

Comparing to (114) and since the only element in $\mathcal{M}^2(x)$ that satisfies (114) is M^ε, we deduce from (118) that

$$M^\varepsilon_T = \frac{x}{y}\left(dM^x_T - \widetilde{M}_T\right),\quad c = d\frac{x}{y}a(x,x),\quad \mathit{and}\quad \widetilde{N}_T = d\frac{x}{y}a(x,x)N^y_T - \frac{x}{y}N^\varepsilon_T$$

Comparing with (115), we get

$$a(x,\varepsilon) = a(x,x)\frac{x}{y}b(y,\varepsilon). \tag{119}$$

Since $a(x,x)b(y,y)=1$ by [KS06a], we deduce (39). Next, from (118), plugging the expressions for c and $\widetilde{N}_T$ back in (114), we get

$$\begin{aligned} A(X_T)(M^\varepsilon_T + x\pi\cdot\widetilde{R}^\pi_T) &= xg\cdot M^H_T + a(x,\varepsilon) + a(x,\varepsilon)N^y_T - \frac{x}{y}N^\varepsilon_T, \\ &= a(x,\varepsilon)\left(1+N^y_T\right) - \frac{x}{y}\left(N^\varepsilon_T - yg\cdot M^H_T\right), \end{aligned}$$

combining which with (117), we get (41), which, in view of (39), also implies (42).

Let us now prove (40). In order to do this, let us denote

$$\begin{aligned} \bar{M}^x_T &:= M^x + 1, \quad \bar{N}^y_T := N^y_T + 1, \\ \bar{M}^\varepsilon_T &:= M^\varepsilon_T + x\pi\cdot\widetilde{R}^\pi_T, \quad \bar{N}^\varepsilon_T := N^\varepsilon_T - yg\cdot M^H_T, \end{aligned} \tag{120}$$

This and (41), allow to write

$$\begin{aligned} \frac{y}{x}a(\varepsilon,\varepsilon) &= \mathbb{E}^{\mathbb{R}}\left[\frac{y}{x}a(x,\varepsilon)\bar{N}^y_T\bar{M}^\varepsilon_T - \bar{N}^\varepsilon_T\bar{M}^\varepsilon_T + 2M^\varepsilon_T\bar{N}^\varepsilon_T + xyT_1\right], \\ \frac{x}{y}b(\varepsilon,\varepsilon) &= \mathbb{E}^{\mathbb{R}}\left[\frac{x}{y}b(y,\varepsilon)\bar{M}^x_T\bar{N}^\varepsilon_T - \bar{M}^\varepsilon_T\bar{N}^\varepsilon_T + 2N^\varepsilon_T\bar{M}^\varepsilon_T + xyT_2\right] \end{aligned} \tag{121}$$

Let us denote

$$\begin{aligned} J_1 &:= \mathbb{E}^{\mathbb{R}}\left[\frac{y}{x}a(x,\varepsilon)\bar{N}^y_T\bar{M}^\varepsilon_T + \frac{x}{y}b(y,\varepsilon)\bar{M}^x_T\bar{N}^\varepsilon_T\right], \\ J_2 &:= \mathbb{E}^{\mathbb{R}}\left[-\bar{N}^\varepsilon_T\bar{M}^\varepsilon_T + 2M^\varepsilon_T\bar{N}^\varepsilon_T + xyT_1 - \bar{M}^\varepsilon_T\bar{N}^\varepsilon_T + 2N^\varepsilon_T\bar{M}^\varepsilon_T + xyT_2\right]. \end{aligned} \tag{122}$$

Then, we have

$$\frac{y}{x}a(\varepsilon,\varepsilon) + \frac{x}{y}b(\varepsilon,\varepsilon) = J_1 + J_2. \tag{123}$$

Using , we can rewrite J_1 as

$$\begin{aligned}J_1 &= a(x,\varepsilon)\mathbb{E}^{\mathbb{R}}\left[\frac{y}{x}\bar{N}^y_T\bar{M}^\varepsilon_T + \frac{x}{y}\frac{b(y,\varepsilon)}{a(x,\varepsilon)}\bar{M}^x_T\bar{N}^\varepsilon_T\right] \\ &= a(x,\varepsilon)\mathbb{E}^{\mathbb{R}}\left[\frac{y}{x}\bar{N}^y_T\bar{M}^\varepsilon_T + b(y,y)\bar{M}^x_T\bar{N}^\varepsilon_T\right] \\ &= a(x,\varepsilon)\mathbb{E}^{\mathbb{R}}\left[\frac{y}{x}\bar{N}^y_T\bar{M}^\varepsilon_T + B(Y_T)\bar{N}^y_T\bar{N}^\varepsilon_T\right],\end{aligned}$$

where, in second equality, we have used (119), and in the third one, (117). As, $b(y,\varepsilon) = \mathbb{E}^{\mathbb{R}}\left[\frac{y}{x}\bar{N}^y_T\bar{M}^\varepsilon_T + B(Y_T)\bar{N}^y_T\bar{N}^\varepsilon_T\right]$, we now can rewrite J_1 as

$$J_1 = a(x,\varepsilon)b(y,\varepsilon). \tag{124}$$

Let us consider J_2. Using Lemma 5.24, we get

$$\begin{aligned}J_2 &= \mathbb{E}^{\mathbb{R}}\left[-\bar{N}^\varepsilon_T\bar{M}^\varepsilon_T + 2M^\varepsilon_T\bar{N}^\varepsilon_T - \bar{M}^\varepsilon_T\bar{N}^\varepsilon_T + 2N^\varepsilon_T\bar{M}^\varepsilon_T - 2xy\left(g\cdot M^H_T\right)\left(\pi\cdot\widetilde{R}^\pi_T\right)\right] \\ &= 2\mathbb{E}^{\mathbb{R}}\left[\left(-\bar{N}^\varepsilon_T\bar{M}^\varepsilon_T + M^\varepsilon_T\bar{N}^\varepsilon_T\right) + N^\varepsilon_T\bar{M}^\varepsilon_T - xy\left(g\cdot M^H_T\right)\left(\pi\cdot\widetilde{R}^\pi_T\right)\right] \\ &= 2\mathbb{E}^{\mathbb{R}}\left[-\bar{N}^\varepsilon_T x\pi\cdot\widetilde{R}^\pi_T - xy\left(g\cdot M^H_T\right)\left(\pi\cdot\widetilde{R}^\pi_T\right) + N^\varepsilon_T\bar{M}^\varepsilon_T\right] \\ &= 2\mathbb{E}^{\mathbb{R}}\left[-x\pi\cdot\widetilde{R}^\pi_T\left(\bar{N}^\varepsilon_T + g\cdot M^H_T\right) + N^\varepsilon_T\bar{M}^\varepsilon_T\right] \\ &= 2\mathbb{E}^{\mathbb{R}}\left[-x\pi\cdot\widetilde{R}^\pi_T N^\varepsilon_T + N^\varepsilon_T\bar{M}^\varepsilon_T\right] \\ &= 2\mathbb{E}^{\mathbb{R}}\left[\left(\bar{M}^\varepsilon_T - x\pi\cdot\widetilde{R}^\pi_T\right)N^\varepsilon_T\right] \\ &= 2\mathbb{E}^{\mathbb{R}}\left[M^\varepsilon_T N^\varepsilon_T\right] = 0.\end{aligned}$$

As $J_2 = 0$, from (123) and (124), we get (40). ☐

Lemma 5.25. *Let $x > 0$ be fixed, conditions of Theorem 4.15 hold, and $y = u_x(x,0)$. Then for*

$$\Delta y = -\frac{y}{x}a(x,x)\Delta x - \frac{b(y,\varepsilon)}{b(y,y)}\varepsilon, \tag{125}$$

we have

$$2\Delta x\Delta y + \begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}^\top H_v\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix} = \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^\top H_u\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}. \tag{126}$$

Proof. First, we recall that from Theorem 4.14, we have

$$b(\varepsilon,\varepsilon) = \frac{y^2}{x^2}\left(\frac{(a(x,\varepsilon))^2}{a(x,x)} - a(\varepsilon,\varepsilon)\right) \quad \textit{and} \quad b(y,\varepsilon) = \frac{a(x,\varepsilon)}{a(x,x)}\frac{y}{x}. \tag{127}$$

Therefore, we get

$$\begin{aligned}\frac{x}{y}\left(b(\varepsilon,\varepsilon) - \frac{(b(y,\varepsilon))^2}{b(y,y)}\right) &= \frac{x}{y}\left(\frac{y^2}{x^2}\left(\frac{(a(x,\varepsilon))^2}{a(x,x)} - a(\varepsilon,\varepsilon)\right) - \frac{(b(y,\varepsilon))^2}{b(y,y)}\right) \\ &= -\frac{y}{x}a(\varepsilon,\varepsilon) + \frac{x}{y}\left(\frac{y^2}{x^2}\frac{(a(x,\varepsilon))^2}{a(x,x)} - \frac{(b(y,\varepsilon))^2}{b(y,y)}\right) \\ &= -\frac{y}{x}a(\varepsilon,\varepsilon) + \frac{x}{y}\left(\frac{y^2}{x^2}\frac{(a(x,\varepsilon))^2}{a(x,x)} - \frac{y^2}{x^2}\frac{(a(x,\varepsilon))^2}{(a(x,x))^2 b(y,y)}\right) \\ &= -\frac{y}{x}a(\varepsilon,\varepsilon),\end{aligned} \tag{128}$$

where in the last equality, we have used that $a(x,x)b(y,y) = 1$.

Now, let us consider the left-hand side in (126) and rewrite it as follows

$$\begin{aligned}
&2\Delta x\Delta y + \begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}^{\top} H_v \begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}\\
&=\frac{x}{y}b(y,y)\Delta y^2 + 2\frac{x}{y}b(y,\varepsilon)\Delta y\varepsilon + \frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2 + 2\Delta x\Delta y\\
&=\Delta y\left(\frac{x}{y}b(y,y)\Delta y + 2\frac{x}{y}b(y,\varepsilon)\varepsilon + 2\Delta x\right) + \frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2.
\end{aligned}\tag{129}$$

Plugging the expression for Δy from (125), we can rewrite the latter equality in (129) as

$$\Delta y\left(\frac{x}{y}b(y,y)\left(-\frac{y}{x}a(x,x)\Delta x - \frac{b(y,\varepsilon)}{b(y,y)}\varepsilon\right) + 2\frac{x}{y}b(y,\varepsilon)\varepsilon + 2\Delta x\right) + \frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2.\tag{130}$$

As $a(x,x)b(y,y) = 1$, we can simplify (130) to

$$\begin{aligned}
&\Delta y\left(-\Delta x - \frac{x}{y}b(y,\varepsilon)\varepsilon + 2\frac{x}{y}b(y,\varepsilon)\varepsilon + 2\Delta x\right) + \frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2\\
&= -\frac{y}{x}\left(a(x,x)\Delta x + \frac{x}{y}\frac{b(y,\varepsilon)}{b(y,y)}\varepsilon\right)\left(\Delta x + \frac{x}{y}b(y,\varepsilon)\varepsilon\right) + \frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2.
\end{aligned}\tag{131}$$

Opening brackets and applying (127) again, we rewrite (131) as

$$\begin{aligned}
&-\frac{y}{x}\left(a(x,x)\Delta x^2 + 2a(x,x)\frac{x}{y}b(y,\varepsilon)\Delta x\varepsilon + \frac{x^2}{y^2}\frac{(b(y,\varepsilon))^2}{b(y,y)}\varepsilon^2\right) + \frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2\\
&= -\frac{y}{x}\left(a(x,x)\Delta x^2 + 2a(x,\varepsilon)\Delta x\varepsilon\right) + \frac{x}{y}\left(-\frac{(b(y,\varepsilon))^2}{b(y,y)} + b(\varepsilon,\varepsilon)\right)\varepsilon^2.
\end{aligned}$$

Now, applying (128), we can finally restate the latter expression as

$$-\frac{y}{x}\left(a(x,x)\Delta x^2 + 2a(x,\varepsilon)\Delta x\varepsilon + a(\varepsilon,\varepsilon)\varepsilon^2\right),$$

which is precisely the right-hand side of (126). □

Proof of Theorem 4.15. The biconjugacy relations imply that

$$\begin{aligned}
&u(x+\Delta x,\varepsilon) - u(x,0) - \Delta xy - \varepsilon xy\mathbb{E}_{\mathbb{R}}\left[\pi\cdot\widetilde{R}^{\pi}_T\right] - \frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top} H_u \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\
&\leq v(y+\Delta y,\varepsilon) + (x+\Delta x)(y+\Delta y) - v(y,0) - xy\\
&\quad -\Delta xy - \varepsilon xy\mathbb{E}_{\mathbb{R}}\left[\pi\cdot\widetilde{R}^{\pi}_T\right] - \frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top} H_u \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\
&= v(y+\Delta y,\varepsilon) - v(y,0) + x\Delta y - \varepsilon xy\mathbb{E}_{\mathbb{R}}\left[\pi\cdot\widetilde{R}^{\pi}_T\right]\\
&\quad + \Delta x\Delta y - \frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top} H_u \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}.
\end{aligned}\tag{132}$$

Further, since $-v_y(y,0) = x$ and using Lemma 5.8, we have

$$x\Delta y - \varepsilon xy\mathbb{E}_{\mathbb{R}}\left[\pi\cdot\widetilde{R}^{\pi}_T\right] = -v_y(y,0)\Delta y - \varepsilon xy\mathbb{E}_{\mathbb{R}}\left[g\cdot M^H_T\right],$$

we can rewrite (132) as

$$
\begin{aligned}
&v(y+\Delta y,\varepsilon)-v(y,0)+x\Delta y-\varepsilon xy\mathbb{E}_{\mathbb{R}}\left[\pi\cdot\widetilde{R}^{\pi}_{T}\right]\\
&\quad+\Delta x\Delta y-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top}H_u\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\
&=v(y+\Delta y,\varepsilon)-v(y,0)+\Delta yx-\varepsilon xy\mathbb{E}_{\mathbb{R}}\left[g\cdot M^{H}_{T}\right]+\Delta x\Delta y-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top}H_u\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}.
\end{aligned}\tag{133}
$$

Now, using Lemma 5.25 and picking Δy as in this lemma, we obtain

$$
\begin{aligned}
&v(y+\Delta y,\varepsilon)-v(y,0)-\left(-\Delta yx+\varepsilon xy\mathbb{E}_{\mathbb{R}}\left[g\cdot M^{H}_{T}\right]\right)+\Delta x\Delta y-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top}H_u\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\
&=v(y+\Delta y,\varepsilon)-v(y,0)-\left(-\Delta yx+\varepsilon xy\mathbb{E}_{\mathbb{R}}\left[g\cdot M^{H}_{T}\right]\right)-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top}H_v\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix},
\end{aligned}\tag{134}
$$

and thus (132), (133), and (134) give

$$
\begin{aligned}
&u(x+\Delta x,\varepsilon)-u(x,0)-\Delta xy-\varepsilon xy\mathbb{E}_{\mathbb{R}}\left[\pi\cdot\widetilde{R}^{\pi}_{T}\right]-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top}H_u\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\
&\le v(y+\Delta y,\varepsilon)-v(y,0)-\left(-\Delta yx+\varepsilon xy\mathbb{E}_{\mathbb{R}}\left[g\cdot M^{H}_{T}\right]\right)-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top}H_v\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}.
\end{aligned}\tag{135}
$$

Lemma 5.23 implies that

$$
v(y+\Delta y,\varepsilon)-v(y,0)-\left(-\Delta yx+\varepsilon xy\mathbb{E}_{\mathbb{R}}\left[g\cdot M^{H}_{T}\right]\right)-\frac{1}{2}\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}^{\top}H_v\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}+o(\Delta y^2+\varepsilon^2)\le 0,\tag{136}
$$

and, from Lemma 5.17, we get

$$
u(x+\Delta x,\varepsilon)-u(x,0)-\Delta xy-\varepsilon xy\mathbb{E}_{\mathbb{R}}\left[\pi\cdot\widetilde{R}^{\pi}_{T}\right]-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top}H_u\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}+o(\Delta x^2+\varepsilon^2)\ge 0.\tag{137}
$$

Finally, (135), (136), and (137) imply that inequalities (136) and (137) are equalities. □

Proof of Theorem 4.16. The proof of Theorem 4.16 parallels the proof of [MS19, Theorem 4.8], see [MS19, Section 4] where L^{ε_n} in [MS19, Section 4] being replaced by $\mathcal{E}(J^{\Delta y_n,\varepsilon_n,H})$, where $J^{\Delta y_n,\varepsilon_n,H}$'s are defined in (99), and using nearly optimal $\frac{y+\Delta y}{y}Y\mathcal{E}(J^{\Delta y_n,\varepsilon_n,H})$. We omit the proof for the brevity of the exposition. □

Proof of Theorem 4.18. The proof parallels the proof in [MS19, Theorem 3.1] (see also the proof of [Mos20, Theorem 4.2]). It is skipped for the brevity of the exposition. □

6. Characterization of M^ε and N^ε through the Kunita-Watanabe decomposition

Instead of minimizing (32) and (37), one can characterize M^ε and N^ε through a Kunita-Watanable decomposition, in the case when a risk-tolerance wealth process exists. We recall that, for the base model at a fixed $x > 0$, the risk-tolerance wealth process is a maximal wealth process, such that

$$\mathcal{R}_T = -\frac{U'(X_T)}{U''(X_T)}. \tag{138}$$

again, similarly to the notations of the previous sections, we will drop $(x, 0)$ and write $\mathcal{R}$ as in (138), which represents a more cumbersome, yet more precise expression

$$\mathcal{R}_T(x,0) = -\frac{U'(\widehat{X}_T(x,0))}{U''(\widehat{X}_T(x,0))}.$$

We refer to [KS06b, Theorem 4] for the equivalent conditions for its existence. We also point out that, even though in [KS06b], NFLVR was imposed, the proof of [KS06b, Theorem 4] goes through also under NUPBR.

If the risk-tolerance wealth process for the base model exists, one can change measure and numéraire as follows

$$\frac{d\widetilde{\mathbb{R}}}{d\mathbb{P}} := \frac{\mathcal{R}_T Y_T}{\mathcal{R}_0 y} \quad and \quad S^{\mathcal{R}} := \left(\frac{\mathcal{R}_0}{\mathcal{R}}, \frac{\mathcal{R}_0 \mathcal{E}\left(R^{\{1\}}\right)}{\mathcal{R}}, \dots \frac{\mathcal{R}_0 \mathcal{E}\left(R^{\{d\}}\right)}{\mathcal{R}}\right),$$

which leads to defining the orthogonal and complementary sets of square-integrable martingales under the measure $\widetilde{\mathbb{R}}$ and numéraire $S^{\mathcal{R}}$ as

$$\widetilde{\mathcal{M}}^2 := \left\{M \in \mathcal{H}_0^2(\widetilde{\mathbb{R}}) :\ M = H \cdot S^{\mathcal{R}}\ for\ some\ S^{\mathcal{R}} - integrable\ H\right\}, \tag{139}$$

$$\widetilde{\mathcal{N}}^2\ is\ the\ orthogonal\ complement\ of\ \widetilde{\mathcal{M}}^2\ in\ \mathcal{H}_0^2(\widetilde{\mathbb{R}}). \tag{140}$$

Similarly to [MS19, Lemma 9.1], one can prove the following lemma.

Lemma 6.1. *Let us suppose that $x > 0$ is fixed and assume that* (2) *and Assumptions 2.1 and 2.2 hold, $y = u_x(x, 0)$, and the risk-tolerance wealth process $\mathcal{R}$ for the base model at x exists. Then we have*

$$M \in \mathcal{M}^2 \quad if\ and\ only\ if \quad M\frac{X}{\mathcal{R}} \in \widetilde{\mathcal{M}}^2, \tag{141}$$

and

$$\widetilde{\mathcal{N}}^2 = \mathcal{N}^2. \tag{142}$$

The following proposition characterizes the solutions to (32) and (37) through a Kunita-Watanabe decomposition obtained by embedding the random variable

$$g \cdot M_T^H - \pi \cdot \widetilde{R}_T^\pi \frac{X_T}{\mathcal{R}_T}$$

into a square-integrable martingale under the measure $\widetilde{\mathbb{R}}$ (and numéraire $\mathcal{R}$).

Proposition 6.2. *Let us suppose that $x > 0$ is fixed and assume that the assumptions of Theorem 4.11 hold, $y = u_x(x, 0)$, and the risk-tolerance wealth process $\mathcal{R}$ for the base model at x exists. Then*

$$P_t := x\mathbb{E}^{\widetilde{\mathbb{R}}}\left[\left(g \cdot M^H_T - \pi \cdot \widetilde{R}^\pi_T \frac{X_T}{\mathcal{R}_T}\right)|\mathcal{F}_t\right], \quad t \in [0, T], \tag{143}$$

satisfies $P \in \mathcal{H}^2(\widetilde{\mathbb{R}})$. Consider the Kunita-Watanabe decomposition of P, given by

$$P = -a(x, \varepsilon) + \widetilde{M}^\varepsilon + \widetilde{N}^\varepsilon, \quad \textit{where} \quad \widetilde{M}^\varepsilon \in \widetilde{\mathcal{M}}^2 \quad \textit{and} \quad \widetilde{N}^\varepsilon \in \widetilde{N}^2. \tag{144}$$

Then, the optimal solutions to (32) *and* (37) *can be obtained by reverting to the original numérarie according to Lemma 6.1 through the identities*

$$M^\varepsilon_t = \widetilde{M}^\varepsilon_t \frac{\mathcal{R}_t}{X_t}, \quad N^\varepsilon_t = \frac{y}{x}\widetilde{N}^\varepsilon_t, \quad t \in [0, T]. \tag{145}$$

Proof. Let us consider $\widetilde{a}(\varepsilon, \varepsilon)$ given by (30). By direct computations, we have

$$\begin{aligned} &\mathbb{E}_{\mathbb{R}}\left[A(X_T)\left(M^\varepsilon_T + x\pi \cdot \widetilde{R}^\pi_T\right)^2 - 2M^\varepsilon_T x g \cdot M^H_T\right] \\ =&\frac{\mathcal{R}_0}{x}\mathbb{E}_{\widetilde{\mathbb{R}}}\left[(P_T - \widetilde{M}^\varepsilon_T)^2\right] + \frac{\mathcal{R}_0}{x}\mathbb{E}_{\widetilde{\mathbb{R}}}\left[x^2 g \cdot M^H_T\left(2\pi \cdot \widetilde{R}^\pi_T \frac{X_T}{\mathcal{R}_T} - g \cdot M^H_T\right)\right]. \end{aligned} \tag{146}$$

Similarly, we can rewrite $\widetilde{b}(\varepsilon, \varepsilon)$ in (35) as

$$\begin{aligned} \widetilde{b}(\varepsilon, \varepsilon) &= \mathbb{E}_{\mathbb{R}}\left[B(Y^0_T)\left(N^\varepsilon_T - yg \cdot M^H_T\right)^2 + 2N_T y(\pi \cdot \widetilde{R}^\pi_T)\right] \\ &= \frac{\mathcal{R}_0}{x}\mathbb{E}_{\widetilde{\mathbb{R}}}\left[\left(N^\varepsilon_T - \frac{y}{x}P_T\right)^2\right] + \frac{\mathcal{R}_0}{x}\mathbb{E}_{\widetilde{\mathbb{R}}}\left[y^2\pi \cdot \widetilde{R}^\pi_T \frac{X_T}{\mathcal{R}_T}\left(2g \cdot M^H_T - \pi \cdot \widetilde{R}^\pi_T \frac{X_T}{\mathcal{R}_T}\right)\right]. \end{aligned} \tag{147}$$

Since $A(X_T) = \frac{X_T}{\mathcal{R}_T}$, taking the expectation in (41) under $\widetilde{\mathbb{R}}$ and using Lemma 6.1, we get

$$P_0 = -a(x, \varepsilon). \tag{148}$$

From (146), (147), and (148), we deduce (144). In turn, (145) follows from (146), (147), and Lemma 6.1. $\square$

7. Examples

We begin with perturbations of the Black-Scholes model with a general utility function.

7.1. **Perturbations of the Black-Scholes model (with a general utility function).**

We suppose that the vector-valued discounted return process R for the base model is given by

$$R_t = \mu t + \sigma^T B_t, \tag{149}$$

for some $\mu \in \mathbb{R}^d$ and an invertible matrix σ. B is a d-dimensional Brownian motion. With $\widetilde{\lambda} := (\sigma^T)^{-1}\mu$, one can express B_T in terms of R_T as follows

$$B_T = (\sigma^T)^{-1}(R_T - \mu T) = (\sigma^T)^{-1}R_T - \widetilde{\lambda}T. \tag{150}$$

Then, we can represent the density of the minimal martingale measure as

$$
\begin{aligned}
Z_T &= \exp\left(-\widetilde{\lambda}^T B_T - \frac{1}{2}||\widetilde{\lambda}||^2 T\right) \\
&= \exp\left(-\widetilde{\lambda}^T (\sigma^T)^{-1} R_T + \frac{1}{2}||\widetilde{\lambda}||^2 T\right).
\end{aligned} \tag{151}
$$

The latter formula suggests that instead of a d-dimensional state process R we can use a 1-dimensional mutual fund

$$
\bar{R} := \widetilde{\lambda}^T (\sigma^T)^{-1} R.
$$

Then we can represent $\bar{R}$ as

$$
\bar{R}_t = \widetilde{\lambda}^T (\sigma^T)^{-1} \left(\mu t + \sigma^T B_t\right) = \widetilde{\lambda}^T \left(\widetilde{\lambda} t + B_t\right) = \widetilde{\lambda}^T B^{\mathbb{Q}}_t, \quad t \in [0, T],
$$

where $B^{\mathbb{Q}}_t$ is the Brownian motion under the minimal martingale measure. In particular, $\bar{R}$ is a martingale under this measure. For a utility function satisfying Assumption 2.1, with $y = u_x(x, 0)$, we aim to replicate

$$
-V'(yZ_T) = -V'\left(y \exp\left(-\bar{R}_T + ||\widetilde{\lambda}||^2 T/2\right)\right).
$$

Let introduce

$$
\bar{g}(z) = -V'\left(y \exp\left(-z + ||\widetilde{\lambda}||^2 T/2\right)\right), \quad z \in \mathbb{R},
$$

and denote by Φ the one-dimensional heat kernel, that is

$$
\Phi(z, t) := \frac{1}{\sqrt{4\pi s}} \exp\left(-\frac{z^2}{4s}\right), \quad z \in \mathbb{R}, s > 0.
$$

With Φ_x denoting the partial derivate of Φ with respect to the first argument, the dynamics of $\hat{X}(x, 0)$ is given by

$$
\begin{aligned}
dX_T(x, 0) &= \left(\int_{\mathbb{R}} \Phi_x\left(\frac{\sqrt{2}\bar{R}_t}{||\widetilde{\lambda}||} - z, T - t\right) \frac{\sqrt{2}}{||\widetilde{\lambda}||} \bar{g}\left(\frac{||\widetilde{\lambda}||}{\sqrt{2}} z\right) dz\right) d\bar{R}_t \\
&= \widetilde{\lambda}^T (\sigma^T)^{-1} \left(\int_{\mathbb{R}} \Phi_x\left(\frac{\sqrt{2}\widetilde{\lambda}^T (\sigma^T)^{-1} R_t}{||\widetilde{\lambda}||} - z, T - t\right) \frac{\sqrt{2}}{||\widetilde{\lambda}||} \bar{g}\left(\frac{||\widetilde{\lambda}||}{\sqrt{2}} z\right) dz\right) dR_t
\end{aligned}
$$

In turn, the optimal proportions invested in risky assets are

$$
\pi^\top_t = \frac{\widetilde{\lambda}^T (\sigma^T)^{-1}}{X_T(x, 0)} \left(\int_{\mathbb{R}} \Phi_x\left(\frac{\sqrt{2}\widetilde{\lambda}^T (\sigma^T)^{-1} R_t}{||\widetilde{\lambda}||} - z, T - t\right) \frac{\sqrt{2}}{||\widetilde{\lambda}||} \bar{g}\left(\frac{||\widetilde{\lambda}||}{\sqrt{2}} z\right) dz\right) 1_{[0,T)}(t).
$$

Then, the associated wealth process is given by

$$
\hat{X}(x, 0) = x\mathcal{E}(\pi \cdot R).
$$

Further, the risk-tolerance wealth process exists. With

$$
\widetilde{h}(z) := y \exp\left(-z + ||\widetilde{\lambda}||^2 T/2\right) V''\left(y \exp\left(-z + ||\widetilde{\lambda}||^2 T/2\right)\right), \quad z \in \mathbb{R},
$$

the dynamics of the risk tolerance wealth process can be represented as

$$
\begin{aligned}
d\mathcal{R}_t &= \left(\int_{\mathbb{R}} \Phi_x \left(\frac{\sqrt{2}\bar{R}_t}{||\widetilde{\lambda}||} - z, T-t\right) \frac{\sqrt{2}}{||\widetilde{\lambda}||}\bar{h}\left(\frac{||\widetilde{\lambda}||}{\sqrt{2}}z\right) dz\right) d\bar{R}_t \\
&= \widetilde{\lambda}^T(\sigma^T)^{-1}\left(\int_{\mathbb{R}} \Phi_x \left(\frac{\sqrt{2}\widetilde{\lambda}^T(\sigma^T)^{-1}R_t}{||\widetilde{\lambda}||} - z, T-t\right) \frac{\sqrt{2}}{||\widetilde{\lambda}||}\bar{h}\left(\frac{||\widetilde{\lambda}||}{\sqrt{2}}z\right) dz\right) dR_t,
\end{aligned}
$$

and the proportions invested in risky assets are

$$
\rho_t^\top = \frac{\widetilde{\lambda}^T(\sigma^T)^{-1}}{\mathcal{R}_t}\left(\int_{\mathbb{R}} \Phi_x \left(\frac{\sqrt{2}\widetilde{\lambda}^T(\sigma^T)^{-1}R_t}{||\widetilde{\lambda}||} - z, T-t\right) \frac{\sqrt{2}}{||\widetilde{\lambda}||}\bar{h}\left(\frac{||\widetilde{\lambda}||}{\sqrt{2}}z\right) dz\right) 1_{[0,T)}(t),
$$

so that, we have

$$
\mathcal{R} = \mathcal{R}_0\mathcal{E}(\rho \cdot R),
$$

For the sensitivity analysis, we consider the general perturbations of R that have the form

$$
\widetilde{R} = \varphi \cdot B + M^\perp + \int_0^\cdot \xi_s ds,
$$

for some predictable and sufficiently integrable processes φ and ξ, where φ is $d \times d$-matrix-valued and ξ is $\mathbb{R}^d$-valued, and for some martingale $M^\perp$, which is orthogonal to B. To make the dynamics of $\widetilde{R}$ more consistent with the previous sections, we set $\phi := \varphi\sigma^{-1}$ and $\zeta := (\sigma^\top\sigma)^{-1}\xi$. Then, we rewrite $\widetilde{R}$ as

$$
\widetilde{R} = \phi \cdot (\sigma^\top B) + M^\perp + \sigma^\top\sigma \int_0^\cdot \zeta_s ds,
$$

and suppose that the assumptions of Theorem 4.11 hold. In Proposition 6.2, the key role is played by the random variable

$$
P_T = x\left(g \cdot M_T^H - \pi \cdot \widetilde{R}_T^\pi \frac{X_T}{\mathcal{R}_T}\right),
$$

where, in the present settings, we have $\kappa_t = tr(\sigma^\top\sigma)t$, $A_t = \frac{1}{tr(\sigma^\top\sigma)}\sigma^\top\sigma$, which is invertible, and thus, Assumption 4.1 holds. By direct computations, we get

$$
(\lambda_t^0)' = \zeta_t - (\sigma^\top\sigma)^{-1}\phi_t\mu = (\sigma^\top\sigma)^{-1}\left(\xi_t - \varphi_t\sigma^{-1}\mu\right),
$$

As $Y(y,0) = yZ$, $y > 0$, we deduce that $\beta = 0$. Consequently, we have

$$
g_t = (\lambda_t^0)' = \zeta_t - (\sigma^\top\sigma)^{-1}\phi_t\mu, \quad t \in [0,T].
$$

$$
M_t^H = \sigma^\top B_t - [\sigma^\top B, -\widetilde{\lambda}^\top B]_t = \sigma^\top B_t + \sigma^\top(\sigma^\top)^{-1}\mu t = R_t, \quad t \in [0,T].
$$

$$
\widetilde{R}^\pi = \widetilde{R} - [\widetilde{R}^c, \pi \cdot R^c] = \widetilde{R} - \int_0^\cdot \phi_s\sigma^\top\sigma\pi_s ds,
$$

and thus, we can rewrite P_T as

$$
P_T = x\left((\zeta_t - (\sigma^\top\sigma)^{-1}\phi_t\mu) \cdot R_T - \pi \cdot \widetilde{R}_T^\pi \frac{X_T}{\mathcal{R}_T}\right).
$$

Similarly to the proof of [Mos20, Lemma 4.1], one can show that the process

$$
R^\rho := R - [R^c, \rho \cdot R^c] - \sum_{s \le \cdot} \Delta R_s \frac{\rho_s \Delta R_s}{1 + \rho_s \Delta R_s}
$$

can be used to represent the elements of $\widetilde{\mathcal{M}}^2$, where $\widetilde{\mathcal{M}}^2$ is defined in (139). We note that in the settings of this example the evolution of R^ρ reduces to

$$R^\rho = R - [R, \rho \cdot R].$$

The decomposition (144) can be written as

$$P = P_0 + \alpha \cdot R^\rho + N,$$

for some process α and $N \in \widetilde{N}^2$. Using (145) in Proposition 6.2, we can represent π^ε as a solution to

$$\pi^\varepsilon \cdot R^\pi = \frac{1}{x}\alpha \cdot R^\rho \frac{\mathcal{R}_0 \mathcal{E}(\rho \cdot R)}{x\mathcal{E}(\pi \cdot R)} = \frac{\mathrm{R}}{x^2}\alpha \cdot R^\rho \mathcal{E}((\rho - \pi) \cdot R^\pi).$$

$$\pi_t^\varepsilon = \frac{R}{x^2}\mathcal{E}((\rho - \pi) \cdot R^\pi)_t \left(\alpha_t + (\alpha \cdot R_t^\rho)(\rho_t - \pi_t)\right), \quad t \in [0, T].$$

In turn, π^x can be represented as

$$\pi_t^x = \frac{\mathcal{R}_t}{\mathcal{R}_0 X_T}(\rho_t - \pi_t), \quad t \in [0, T],$$

and, for the nearly optimal processes in the sense of Theorem 4.18, we need to truncate π^x and π^ε, that is to use $\pi^x 1_{[0,\tau_n]}$ and $\pi^\varepsilon 1_{[0,\sigma_n]}$ for appropriate localizing sequences of stopping times τ_n, $n \in \mathbb{N}$, and σ_n, $n \in \mathbb{N}$.

In particular, by assuming particular forms of $\widetilde{R}$, we obtain the corrections to perturbations of the finite variation part (as in [MS19]), or the martingale part, or the orthogonal martingale part, which does not have to be continuous.

7.2. **Distortions of the exponential Lévy models.** Closed-form solutions to the optimal investment problem with exponential Levy models have been obtained in [Kal00]. However, once the dynamics of the underlying stock price processes is perturbed, the closed-form solutions cease to exist, in general. We recall that the dynamics of the returns of d discounted risky assets in [Kal00] is given by a Levy process R, with the characteristic triplet (b, c, F) relative to some truncation function h: $\mathbb{R}^d \to \mathbb{R}^d$. Below we will use $h(x) = x1_{\{||x|| \leq 1\}}$. With $U(x) = \frac{x^p}{p}$, $x > 0$, where $p \in (-\infty, 0) \bigcup (0, 1)$, and assuming the conditions of [Kal00, Theorem 3.2], the optimal π is characterized there by a solution to

$$b - (1-p)c\pi + \int_{\mathbb{R}^d} \left(\frac{x}{(1 + \pi^\top x)^{1-p}} - h(x) \right) F(dx) = 0,$$

where we note that additional conditions are needed to ensure finiteness of the value function and admissibility of $\pi \cdot R$ (see conditions 1 and 2 in [Kal00, Theorem 3.2]). Further, the structure of the dual problem is investigated in [JKM07] and the proof of [Kal00, Theorem 3.2].

With power utility, the risk-tolerance wealth process exists for every $x > 0$. Up to a multiplicative constant, $\mathcal{R}$ is equal to $\widehat{X}(1, 0)$, where, in turn, $\widehat{X}(x, 0) = x\widehat{X}(1, 0)$, $x > 0$, that is the optimizer trivially depends on the initial wealth. Therefore, assuming the conditions of Theorem 4.11, one can see that $\widetilde{\mathbb{R}} = \mathbb{R}$ that do not depend on $x > 0$. Below, we suppose that $x = 1$, for simplicity of notations. In this case, we have

$$\frac{X_T}{\mathcal{R}_T} = (1 - p).$$

Writing R as in (1) and assuming that the perturbation process has the dynamics

$$\widetilde{R} = \phi \cdot M + M^{\perp} + \int_0^{\cdot} d\langle M\rangle_s \zeta_s,$$

for some predictable bounded matrix-valued process ϕ, and orthogonal martingale $M^{\perp}$ and a bounded process ζ, to obtain the Kunita-Watanabe decomposition as in Proposition 6.2, under the conditions of Proposition 6.2, we represent P_T as

$$P_T = g \cdot M^H_T - \pi \cdot \widetilde{R}^{\pi}_T(1-p), \tag{152}$$

where

$$\pi \cdot \widetilde{R}^{\pi} = \pi \cdot \widetilde{R} - [\pi \cdot \phi \cdot (\sigma B), \pi \cdot (\sigma B)] - \sum_{s\leq \cdot} \pi_s \Delta \widetilde{R}_s \frac{\pi_s \Delta R_s}{1 + \pi_s \Delta R_s}. \tag{153}$$

As by the structural condition, (local) martingale M is locally $\mathbb{P}$-square integrable, then M is $\mathbb{P}$-square integrable and $\int_{\mathbb{R}^d} x^{\top} x F(dx) < \infty$. Further, we have

$$\langle M\rangle_t = \left(c + \int_{\mathbb{R}^d} x x^{\top} F(dx)\right) t,$$

and thus

$$R_t = \left(\sigma B_t + x * (\mu^R - \nu^{\mathbb{P}})_t\right) + \left(b + \int_{\mathbb{R}^d} (x - h(x)) F(dx)\right) t = M_t + \int_0^t \langle M\rangle_s \lambda_s,$$

where $\nu^{\mathbb{P}}(dx, dt) = F(dx)dt$ is the predictable $\mathbb{P}$-compensator of μ^R, which is the random measure associated with the jumps of R. Therefore, assuming that $\Sigma := \left(c + \int_{\mathbb{R}^d} x x^{\top} F(dx)\right)$ is invertible, λ can be represented as

$$\lambda = \Sigma^{-1} \left(b + \int_{\mathbb{R}^d} (x - h(x)) F(dx)\right). \tag{154}$$

The form of H comes from [JKM07, Theorem 2.7, p.1621], which asserts that

$$H = \bar{\beta} \cdot \sigma B + (\bar{Y} - 1) * (\mu^R - \nu^{\mathbb{P}}),$$

where $\bar{\beta} \in \mathbb{R}^d$ and $\bar{Y} : \mathbb{R}^d \to \mathbb{R}_+$, solving the constraint minimization problem from [JKM07, p.1621]. $\bar{\beta}$ and $\bar{Y}$ are also characterized in the proof of [Kal00, Theorem 3.2, p. 364]. Note we do not suppose that the jumps of M and $M^{\perp}$ are not simultaneous, as was in some of the early works on orthogonality of martingales, see [Pro79, Theorem 5.2]. Next, we have $\kappa_t = (tr\Sigma)\, t$, and $\langle M\rangle_t = \Sigma t = A \cdot \kappa_t = (tr\Sigma)\, t$ produces $A = \frac{\Sigma}{tr\Sigma}$, therefore, Assumption 4.1 holds. With $M^{\perp}$ allowing for bounded $(\nu_t \beta_t)_{t\in[0,T]}$, where β is given through Assumption 2.3 and ν through Assumption 4.5, we get the following representation for g.

$$g_t = -\Sigma^{-1} \phi_t \Sigma \lambda_t + \zeta_t + \Sigma^{-1} \nu_t \Sigma \beta_t, \quad t \in [0, T].$$

Therefore, using the characterization of H in proof of [Kal00, Theorem 3.2, p. 364], we get

$$g \cdot M^H = g \cdot M - [g \cdot (\sigma B), \bar{\beta} \cdot \sigma B] - \sum_{s\leq \cdot} g_s \Delta M_s \left(1 - (1 + \pi_s \Delta R_s)^{1-p}\right). \tag{155}$$

Having bounded ζ, ϕ, $\nu\beta$, and λ, where the latter being given by (154), to ensure that Assumption 4.10 holds, it suffices to suppose that the jumps of both M and $M^{\perp}$ are bounded, plus that $1+\pi\Delta R \geq \delta > 0$.

As $\widetilde{\mathbb{R}} = \mathbb{R}$, with P_T given in (152), with (153) and (155), assuming sufficient exponential moments, we can reduce the decomposition to

$$P = P_0 + \alpha \cdot R^{\pi} + N,$$

and, using Proposition 6.2 and Theorem 4.18, we have

$$\pi^{\varepsilon} = \frac{\alpha}{1-p}, \quad \textit{and} \quad \pi^{x} = 0,$$

where the latter follows from the power utility. To specify the nearly optimal wealth processes in the sense of Theorem 4.18, we need to approximate π^{ε} along the lines of Section 4.7.

We conclude this section by pointing out that a similar analysis can be performed for the models driven by processes with conditionally independent increments, relying on [KMK10]. Further, as mentioned in the introduction, many more models of the stock price process allow for solutions that can be characterized explicitly. They are developed in [Zar01], [GK03], [HIM05], [Liu07], [KS06b], [KMK10], [GR12], [Rob17], [RSA17], [HHI+14], and [ST14], among others, and in some cases, working with general utility utility settings. Our results provide an approach for approximating solutions to these models even when perturbations include jumps.

Appendix: the structure of $\mathcal{M}^{\infty}$ and $\mathcal{N}^{\infty}$

We recall that [Mos20, Lemma 4.1] shows that every element of $\mathcal{M}^{\infty}$ can be represented as a stochastic integral with respect to R^{π}. The following lemma establishes the opposite direction.

Lemma A.1. *Let $x > 0$ be fixed, and suppose that $M \in \mathcal{H}^2_{loc}(\mathbb{P})$, (2), and Assumptions 2.1 and 2.2, hold true, and R^{π} is sigma-bounded. Let us denote $y = u_x(x, 0)$. Then, we have*

$$\mathcal{M}^{\infty} = \{\alpha \cdot R^{\pi} : \alpha \textit{ is predictable and } R^{\pi} - \textit{integrable, such that } \alpha \cdot R^{\pi} \textit{ is bounded}\}.$$

Remark A.2. The proof goes through without the sigma-boundedness assumption. It is imposed to ensure that $\mathcal{M}^{\infty}$ is non-degenerate and that the closure of $\mathcal{M}^{\infty}$ in $\mathcal{H}^2_0(\mathbb{R})$ is equal to $\mathcal{M}^2$. Also, the proof goes through with general NUPBR instead of $M \in \mathcal{H}^2_{loc}(\mathbb{P})$ and (2), and Inada conditions instead of Assumptions 2.1, such that, together with Assumptions 2.2, the standard assertions of the utility maximization theory hold.

Proof of Lemma A.1. Let α be a predictable and R^{π}-integrable process, such that $\alpha \cdot R^{\pi}$ is bounded. Then there exists a constant $C > 0$, such that $C + \alpha \cdot R^{\pi}$ is strictly positive. By [JS03, II.8.3, p. 134], there exists a predictable R^{π}-integrable process $\widetilde{\alpha}$, such that

$$C + \alpha \cdot R^{\pi} = C\mathcal{E}\left(\widetilde{\alpha} \cdot R^{\pi}\right) = C\frac{\mathcal{E}\left((\pi + \widetilde{\alpha}) \cdot R\right)}{\mathcal{E}\left(\pi \cdot R\right)},$$

where, in the second equality, we have used (22). We deduce that the bounded process $\alpha \cdot R^\pi$ admits the representation

$$\alpha \cdot R^\pi = C \frac{\mathcal{E}\left((\pi + \widetilde{\alpha}) \cdot R\right) - \mathcal{E}\left(\pi \cdot R\right)}{\mathcal{E}\left(\pi \cdot R\right)},$$

which is the element of $\mathcal{M}^\infty$, by the definition of $\mathcal{M}^\infty$. As the argument above holds for every predictable and R^π-integrable α, such that $\alpha \cdot R^\pi$ is bounded, the proof is complete. $\square$

Lemma A.3. *Let $x > 0$ be fixed, and suppose that assumptions of Lemma A.1 hold true, both $M \in \mathcal{H}^2_{loc}(\mathbb{P})$ and*[3] *$H \in \mathcal{H}^2_{loc}(\mathbb{P})$, and $y = u_x(x, 0)$. Then, we have*

$$\mathcal{N}^\infty = \{N^H : \ N^H \text{ is bounded}, \ N \in \mathcal{H}^2_{loc}(\mathbb{P}) \text{ and orthogonal to each component of } M\}.$$

Proof. Let us consider a process $N \in \mathcal{H}^2_{loc}(\mathbb{P})$, such that N^H is bounded, let us fix $\widetilde{M} \in \mathcal{M}^\infty$. By [Mos20, Lemma 4.1], $\widetilde{M} = \alpha \cdot R^\pi$, for some predictable and R^π-integrable process α. By sigma-boundedness, we may approximate α by M-integrable processes

$$\alpha^n := -n \vee \alpha \wedge n 1_{[0,\tau_n]}, \quad n \in \mathbb{N}, \tag{156}$$

where we mean that every component of α is truncated from above by n and below by $-n$, and where $(\tau_n)_{n \in \mathbb{N}}$ is a localizing sequence for both M and N. Then, for a fixed $n \in \mathbb{N}$, and every stopping time τ, similarly to Lemma 5.13, we get

$$\mathbb{E}_{\mathbb{R}}\left[\alpha^n \cdot R^\pi_\tau N^H_\tau\right] = \mathbb{E}_{\mathbb{R}}\left[\langle \alpha^n \cdot M, N \rangle_{\tau_n \wedge \tau}\right] = 0. \tag{157}$$

As N^H is bounded and $\alpha^n \cdot R^\pi$, $n \in \mathbb{N}$, converges to $\alpha \cdot R^\pi$ in $\mathcal{H}^2(\mathbb{R})$, we deduce from (157) that N^H is orthogonal to $\alpha \cdot R^\pi$. Now, from Lemma A.1, we deduce that N^H is orthogonal to $\mathcal{M}^\infty$. Since additionally, the closure of $\mathcal{M}^\infty$ in $\mathcal{H}^2_0(\mathbb{R})$ is equal to $\mathcal{M}^2$ by [KS06a, Lemma 6, p. 1370], we get

$$\mathcal{N}^\infty \supseteq \{N^H : \ N^H \text{ is bounded}, \ N \in \mathcal{H}^2_{loc}(\mathbb{P}) \text{ and orthogonal to each component of } M\}.$$

To show the opposite inclusion, we proceed as follows. Let us consider $K \in \mathcal{N}^\infty$. First, we show that $N := K + [K, H](= K^{-H}) \in \mathcal{H}^2_{loc}(\mathbb{P})$. As K is bounded and $H \in \mathcal{H}^2_{loc}(\mathbb{P})$, we deduce that N is locally square-integrable under $\mathbb{P}$. We will suppose that $\mathcal{E}(K) > 0$, as otherwise we may multiply K by a sufficiently small constant ε, and conduct the proof for εK.

For $\alpha = -\pi$, as $\mathcal{E}(\alpha \cdot R^\pi) = \frac{1}{\mathcal{E}(\pi \cdot R)} > 0$, in view of the sigma-boundedness of R^π and by the Ansel-Stricker theorem (as in [DS06, Corollary 7.3.8, p. 128]), we deduce that $\mathcal{E}(\alpha \cdot R^\pi)$ is a local martingale under $\mathbb{R}$. Now, by [Pro04, Theorem IV. 51, p. 194], $\mathcal{E}(\alpha \cdot R^\pi) \in \mathcal{H}^1_{loc}(\mathbb{R})$. Let σ_n, $n \in \mathbb{N}$, be a localizing sequence for $\mathcal{E}(\alpha \cdot R^\pi)$ (such that, on $[0, \sigma_n]$, $\mathcal{E}(\alpha \cdot R^\pi)$ is in $\mathcal{H}^1(\mathbb{R})$), where by boundedness of K, we also suppose that $\mathcal{E}(K)$ is bounded on $[0, \sigma_n]$. Further, as R^π is sigma-bounded, by [KS06a, Theorem 4, p. 1383], we can approximate in $\mathcal{H}^1(\mathbb{R})$ the stopped

[3] We recall that H is defined in Assumption 2.3 and is such that $Y = y\mathcal{E}(H)$.

process $\mathcal{E}(\alpha \cdot R^{\pi})$ by some bounded stochastic integrals with respect to R^{π}. These integrals are in $\mathcal{M}^{\infty}$ by Lemma A.1. By convergence in $\mathcal{H}^1(\mathbb{R})$ and boundedness of $\mathcal{E}(K)$ on $[0, \sigma_n]$, we obtain

$$\mathbb{E}_{\mathbb{R}}\left[\mathcal{E}(\alpha \cdot R^{\pi})_{t\wedge\sigma_n}\mathcal{E}(K)_{t\wedge\sigma_n}|\mathcal{F}_{s\wedge\sigma_n}\right] = \mathcal{E}(\alpha \cdot R^{\pi})_{s\wedge\sigma_n}\mathcal{E}\left(K\right)_{s\wedge\sigma_n} = \frac{\mathcal{E}\left(N+H\right)_{s\wedge\sigma_n}}{\mathcal{E}(H)_{s\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{s\wedge\sigma_n}}, \tag{158}$$

and, by change of measure, we have

(159)

$$\begin{aligned}
\mathbb{E}_{\mathbb{R}}\left[\mathcal{E}(\alpha \cdot R^{\pi})_{t\wedge\sigma_n}\mathcal{E}\left(K\right)_{t\wedge\sigma_n}|\mathcal{F}_{s\wedge\sigma_n}\right] &= \mathbb{E}\left[\frac{\mathcal{E}(H)_{t\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{t\wedge\sigma_n}}{\mathcal{E}(H)_{s\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{s\wedge\sigma_n}}\mathcal{E}(\alpha \cdot R^{\pi})_{t\wedge\sigma_n}\mathcal{E}\left(N^H\right)_{t\wedge\sigma_n}|\mathcal{F}_{s\wedge\sigma_n}\right] \\
&= \mathbb{E}\left[\frac{\mathcal{E}(H)_{t\wedge\sigma_n}}{\mathcal{E}(H)_{s\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{s\wedge\sigma_n}}\frac{\mathcal{E}\left(N+H\right)_{t\wedge\sigma_n}}{\mathcal{E}(H)_{t\wedge\sigma_n}}|\mathcal{F}_{s\wedge\sigma_n}\right] \\
&= \frac{1}{\mathcal{E}(H)_{s\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{s\wedge\sigma_n}}\mathbb{E}\left[\mathcal{E}\left(N+H\right)_{t\wedge\sigma_n}|\mathcal{F}_{s\wedge\sigma_n}\right].
\end{aligned}$$

Comparing (158) and (159), we deduce that $\mathcal{E}(N+H)$ is a local martingale under $\mathbb{P}$. As $\mathcal{E}(N+H)$ and $\mathcal{E}(N+H)_-$ are non-vanishing by construction ($\mathcal{E}(N+H) = \mathcal{E}(K)\mathcal{E}(H)$), the stochastic logarithm of $\mathcal{E}(N+H)$ is well defined by [JS03, Theorem II.8.3, p. 134] and is equal to $N+H$ by [JS03, Corollary II.8.7, p. 135]. Further, from [JS03, Theorem II.8.3, p. 134] and [Pro04, Theorem III.29, p. 128], we conclude that $N+H$ is a local martingale under $\mathbb{P}$, which is additionally locally square-integrable by the previous paragraph. As $H \in \mathcal{H}^2_{loc}(\mathbb{P})$ by assumption of the lemma, we conclude that $N \in \mathcal{H}^2_{loc}(\mathbb{P})$.

Second, we show that N is orthogonal to M. For this, let us consider an element of $\mathcal{M}^{\infty}$ of the form $\alpha \cdot R^{\pi}$, where, by sigma-boundedness, we suppose that each component of α takes values in $(0, 1]$. Let us choose a stopping time σ in a way that $\mathbb{E}\left[\int_0^{\sigma}\alpha_s^{\top} d\langle M\rangle_s \alpha_s\right] < \infty$ and $\mathbb{E}\left[\langle N\rangle_{\sigma}\right] < \infty$. Then as $K = N^H \in \mathcal{N}^{\infty}$, and with $Z = \frac{XY}{xy}$, similarly to the proof of Lemma 5.13, we have

$$0 = \mathbb{E}_{\mathbb{R}}\left[\left[\alpha \cdot R^{\pi}, N^H\right]_{\sigma}\right] = \mathbb{E}_{\mathbb{R}}\left[\langle\alpha \cdot M, N\rangle_{\sigma}\right]. \tag{160}$$

From (160), we deduce that $\langle\alpha \cdot M, N\rangle$ is $\mathbb{R}$ martingale on $[0, \sigma]$. Further, since $\langle\alpha \cdot M, N\rangle$ is predictable and is of finite variation, we deduce that $\langle\alpha \cdot M, N\rangle \equiv 0$ on $[0, \sigma]$.

As M and N are locally square-integrable, and each component of α in the previous paragraph was $(0, 1]$-valued, thus, is non-vanishing, by localization, we can deduce that each component of M is orthogonal to N on $[0, T]$. □

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