J. Appl. Prob. **54,** 710–719 (2017)
*doi:*10.1017/jpr.2017.29
© *Applied Probability Trust* 2017

OPTIMAL INVESTMENT WITH INTERMEDIATE CONSUMPTION UNDER NO UNBOUNDED PROFIT WITH BOUNDED RISK

HUY N. CHAU,* *Hungarian Academy of Sciences*
ANDREA COSSO,** *Politecnico di Milano*
CLAUDIO FONTANA,*** *Paris Diderot University*
OLEKSII MOSTOVYI,**** *University of Connecticut*

Abstract

We consider the problem of optimal investment with intermediate consumption in a general semimartingale model of an incomplete market, with preferences being represented by a utility stochastic field. We show that the key conclusions of the utility maximization theory hold under the assumptions of no unbounded profit with bounded risk and of the finiteness of both primal and dual value functions.

Keywords: Utility maximization; arbitrage of the first kind; local martingale deflator; duality theory; semimartingale; incomplete market

2010 Mathematics Subject Classification: Primary 91G10
Secondary 93E20

1. Introduction

Since the pioneering work of Harrison and Kreps [13], equivalent (local/sigma) martingale measures have played a prominent role in the problems of pricing and portfolio optimization. Their existence is equivalent to the absence of arbitrage in the sense of *no free lunch with vanishing risk* (NFLVR) (see [6] and [8]), and this represents the standard no-arbitrage type assumption in the classical duality approach to optimal investment problems; see, e.g. [17], [24], [25], and [32]. In a general semimartingale setting, necessary and sufficient conditions for the validity of the key assertions of the utility maximization theory (with the possibility of intermediate consumption) have been recently established in [29]. More specifically, such assertions were proven in [29] under the assumptions that the primal and dual value functions are finite and that there exists an *equivalent martingale deflator*. In particular, in a finite time horizon, the latter assumption is equivalent to the validity of NFLVR.

In this paper we consider a general semimartingale setting with an infinite time horizon where preferences are modelled via a utility stochastic field, allowing for intermediate consumption. Building on the abstract theorems of [29], in our main result we show that the standard assertions of the utility maximization theory hold as long as there is *no unbounded profit with bounded*

Received 28 June 2016; revision received 24 April 2017.
* Postal address: Alfred Renyi Institute of Mathematics, Hungarian Academy of Sciences, Realtanoda utca 13-15, H-1053 Budapest, Hungary. Email address: chau@renyi.hu
** Postal address: Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, 20133 Milano, Italy.
Email address: andrea.cosso@polimi.it
*** Postal address: Laboratoire de Probabilités et Modèles Aléatoires, Paris Diderot University, av. de France, 75205 Paris, France. Email address: fontana@math.univ-paris-diderot.fr
**** Postal address: Department of Mathematics, University of Connecticut, U1009, 341 Mansfield Road, Storrs, CT 06269-1009, USA. Email address: oleksii.mostovyi@uconn.edu

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29

risk (NUPBR) and the primal and dual value functions are finite. In general, NUPBR is weaker than NFLVR and can be shown to be equivalent to the existence of an *equivalent local martingale deflator*. Our results give a precise and general form to a widespread meta-theorem in the mathematical finance community stating that the key conclusions of the utility maximization theory hold under NUPBR. Even though such a result has been proven in some specific formulations of the utility maximization problem (see the discussion below), to the best of the authors' knowledge, it has not been justified in general semimartingale settings with an arbitrary consumption clock and a stochastic Inada utility.

The proofs rely on certain characterizations of the dual feasible set. Thus, in Lemma 1 we state a polarity description, show its closedness under countable convex combinations in Lemma 2, and demonstrate in Proposition 1 that nonemptyness of the set that generates the dual domain is equivalent to NUPBR. Upon that, we prove the bipolar relations between primal and dual feasible sets and apply the abstract theorems from [29]. As an implication of the bipolar relations, we also show how [24, Theorem 2.2] can be extended to hold under NUBPR (instead of NFLVR); see Remark 2 for details.

Neither NFLVR, nor NUPBR by itself, guarantee the existence of solutions to utility maximization problems; see [24, Example 5.2] and [5, Example 4.3] for counterexamples. This is why finiteness of the value functions is needed in the formulation of our main result. However, it was shown in [4] that NUPBR holds if and only if, for every sufficiently nice deterministic utility function, the problem of maximizing the expected utility from terminal wealth admits a solution under an equivalent probability measure, which can be chosen to be arbitrarily close to the original measure; see [4, Theorem 2.8] for details. Besides, NUPBR represents the minimal no-arbitrage type assumption that allows for the standard conclusions of the theory to hold for the problem of maximization of expected utility from terminal wealth. Indeed, by [16, Proposition 4.19], the failure of NUPBR implies that there exists a time horizon such that the corresponding utility maximization problem either does not have a solution, or has infinitely many. Our work complements these papers by providing the convex duality results under NUPBR, also allowing for stochastic preferences as well as intermediate consumption.

The problem of utility maximization without relying on the existence of martingale measures has already been addressed in the literature. In the very first paper [28] on expected utility maximization in continuous-time settings, an optimal investment problem was explicitly solved even though an equivalent martingale measure did not exist, in general, in the infinite time horizon case. In an incomplete Itô process setting under a finite time horizon, Karatzas *et al.* [18] considered the problem of maximization of expected utility from terminal wealth and established the existence results for an optimal portfolio via convex duality theory without the full strength of NFLVR; see also [15, Section 10.3] and [11, Section 4.6.3]. In particular, in view of [19, Theorem 4], [18, Assumption 2.3] is equivalent to the nonemptyness of the set of equivalent local martingale deflators. Passing from an Itô process to a continuous semimartingale setting, the results of [24] have been extended in [26] by weakening the NFLVR requirement (note that [26, Assumption 2.1] is equivalent to NUPBR). In a general semimartingale setting, Larsen and Žitković [27] established convex duality results for the problem of maximizing the expected utility from terminal wealth (for a deterministic utility function) in the presence of trading constraints without relying on the existence of martingale measures. In particular, in the absence of trading constraints, the no-arbitrage type requirement adopted in [27] turns out to be equivalent to NUPBR. Indeed, [27, Assumption 2.3] requires the $\boldsymbol{L}^0_+$-solid hull of the set of all terminal wealths generated by admissible strategies with initial wealth x, denoted by $\mathcal{C}(x)$, to be convexly compact for all $x \in \mathbb{R}$ and nonempty for some $x \in \mathbb{R}$. As usual, $\boldsymbol{L}^0$

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29

denotes the space of equivalence classes of real-valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the topology of convergence in probability; $\boldsymbol{L}^0_+$ is the positive orthant of $\boldsymbol{L}^0$. We recall that, by [33, Theorem 3.1], a closed convex subset of $\boldsymbol{L}^0_+$ is convexly compact if and only if it is bounded in $\boldsymbol{L}^0$. In the absence of trading constraints, Kardaras [19, Theorem 2] showed that the boundedness in $\boldsymbol{L}^0$ of $\mathcal{C}(x)$ already implies its closedness in $\boldsymbol{L}^0$, thus, in such a framework the convex compactness of $\mathcal{C}(x)$ holds if and only if the NUPBR condition does.

The paper is structured as follows. In Section 2 we begin with a description of the general setting (Subsection 2.1), introduce and characterize the NUPBR condition (Subsection 2.2), and then proceed with the statement of the main results (Subsection 2.3). Section 3 contains the proofs of our results.

2. Setting and main results

2.1. Setting

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ be a complete stochastic basis, with $\mathcal{F}_0$ being the completion of the trivial σ-algebra, and $S = (S_t)_{t\geq 0}$ an $\mathbb{R}^d$-valued semimartingale, representing the discounted prices of d risky assets. As explained in [29, Remark 2.2], there is no loss of generality in assuming that asset prices are discounted, since we allow for preferences represented by utility stochastic fields (see Section 2.3 below). We fix a *stochastic clock* $\kappa = (\kappa_t)_{t\geq 0}$, which is a nondecreasing, càdlàg adapted process such that

$$\kappa_0 = 0, \qquad \mathbb{P}(\kappa_\infty > 0) > 0 \quad \text{and} \quad \kappa_\infty \leq A \tag{1}$$

for some finite constant A. The stochastic clock κ represents the notion of time according to which consumption is assumed to occur. By suitably specifying the clock process κ, several different formulations of investment problems, with or without intermediate consumption, can be recovered from the present setting; see [29, Examples 2.5–2.9] and [32, Section 2.8].

A *portfolio* is defined by a triplet $\Pi = (x, H, c)$, where $x \in \mathbb{R}$ represents an initial capital, $H = (H_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued predictable S-integrable process representing the holdings in the d risky assets, and $c = (c_t)_{t\geq 0}$ is a nonnegative optional process representing the consumption rate. The discounted value process $V = (V_t)_{t\geq 0}$ of a portfolio $\Pi = (x, H, c)$ is defined as

$$V_t := x + \int_0^t H_u \,\mathrm{d}S_u - \int_0^t c_u \,\mathrm{d}\kappa_u, \qquad t \geq 0.$$

Let $\mathcal{X}$ be the collection of all nonnegative value processes associated to portfolios of the form $\Pi = (1, H, 0)$, i.e.

$$\mathcal{X} := \left\{ X \geq 0\colon X_t = 1 + \int_0^t H_u \,\mathrm{d}S_u,\ t \geq 0 \right\}.$$

For a given initial capital $x > 0$, a consumption process c is said to be *x-admissible* if there exists an $\mathbb{R}^d$-valued predictable S-integrable process H such that the value process V associated to the portfolio $\Pi = (x, H, c)$ is nonnegative. The set of x-admissible consumption processes corresponding to a stochastic clock κ is denoted by $\mathcal{A}(x)$. For brevity, we let $\mathcal{A} := \mathcal{A}(1)$.

2.2. No unbounded profit with bounded risk

In this paper we shall assume the validity of the following no-arbitrage type condition:

(NUPBR) the set $\mathcal{X}_T := \{X_T : X \in \mathcal{X}\}$ is bounded in $\boldsymbol{L}^0$ for every $T \in \mathbb{R}_+$.

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29

For each $T \in \mathbb{R}_+$, the boundedness in probability of the set $\mathcal{X}_T$ was termed *no unbounded profit with bounded risk* in [16] and, as shown in [19, Proposition 1], is equivalent to the absence of *arbitrages of the first kind* on $[0, T]$. Hence, condition (NUPBR) above is equivalent to the absence of arbitrages of the first kind in the sense of [22, Definition 1].

We define the set of *equivalent local martingale deflators* as

$$\mathcal{Z} := \{Z > 0\colon Z \text{ is a càdlàg local martingale such that } Z_0 = 1 \text{ and } ZX = (Z_t X_t)_{t\geq 0} \text{ is a local martingale for every } X \in \mathcal{X}\}.$$

The following result is already known in the one-dimensional case in a finite time horizon; see [20, Theorem 2.1]. The extension to the multidimensional infinite horizon case relies on [31, Theorem 2.6]; see also [1, Proposition 2.3].

Proposition 1. *Condition (NUPBR) holds if and only if $\mathcal{Z} \neq \varnothing$.*

Remark 1. In [29], it was assumed that

$$\{Z \in \mathcal{Z}\colon Z \text{ is a martingale}\} \neq \varnothing, \tag{2}$$

which is stronger than (NUPBR) by Proposition 1. A classical example where (NUPBR) holds but (2) fails is provided by the three-dimensional Bessel process; see, e.g. [7], [16, Example 4.6], and [26, Example 2.2].

2.3. Optimal investment with intermediate consumption

We now proceed to show that the key conclusions of the utility maximization theory can be established under condition (NUPBR). We assume that preferences are represented by a *utility stochastic field* $U = U(t, \omega, x)\colon [0, \infty) \times \Omega \times [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ satisfying the following assumption.

Assumption 1. *For every $(t, \omega) \in [0, \infty) \times \Omega$, the function $x \mapsto U(t, \omega, x)$ is strictly concave, strictly increasing, continuously differentiable on $(0, \infty)$, and satisfies the Inada conditions*

$$\lim_{x\downarrow 0} U'(t, \omega, x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} U'(t, \omega, x) = 0,$$

with U' denoting the partial derivative of U with respect to its third argument. By continuity, at $x = 0$ we suppose that $U(t, \omega, 0) = \lim_{x\downarrow 0} U(t, \omega, x)$ (note that this value may be $-\infty$). Finally, for every $x \geq 0$, the stochastic process $U(\cdot, \cdot, x)$ is optional.

To a utility stochastic field U satisfying Assumption 1, we associate the *primal value function*, defined as

$$u(x) := \sup_{c\in\mathcal{A}(x)} \mathbb{E}\bigg[\int_0^\infty U(t, \omega, c_t)\,\mathrm{d}\kappa_t\bigg], \qquad x > 0, \tag{3}$$

with the convention $\mathbb{E}[\int_0^\infty U(t, \omega, c_t)\,\mathrm{d}\kappa_t] := -\infty$ if $\mathbb{E}[\int_0^\infty U^-(t, \omega, c_t)\,\mathrm{d}\kappa_t] = +\infty$.

In order to construct the dual value function, we define as follows the *stochastic field V conjugate to U*:

$$V(t, \omega, y) := \sup_{x>0}(U(t, \omega, x) - xy), \qquad (t, \omega, y) \in [0, \infty) \times \Omega \times [0, \infty).$$

We also introduce the following set of dual processes (where 'a.e.' is short for almost everywhere):

$$\mathcal{Y}(y) := \mathrm{cl}\{Y\colon Y \text{ is càdlàg adapted and } 0 \leq Y \leq yZ\ (\mathrm{d}\kappa \times \mathbb{P})\text{-a.e. for some } Z \in \mathcal{Z}\},$$

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29

where the closure is taken in the topology of convergence in measure ($\mathrm{d}\kappa \times \mathbb{P}$) on the space of real-valued optional processes. We write $\mathcal{Y} := \mathcal{Y}(1)$ for brevity. The value function of the dual optimization problem (*dual value function*) is then defined as

$$v(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[\int_0^\infty V(t, \omega, Y_t)\, \mathrm{d}\kappa_t\right], \qquad y > 0, \tag{4}$$

with the convention $\mathbb{E}[\int_0^\infty V(t, \omega, Y_t)\, \mathrm{d}\kappa_t] := +\infty$ if $\mathbb{E}[\int_0^\infty V^+(t, \omega, Y_t)\, \mathrm{d}\kappa_t] = +\infty$. We are now in a position to state the following theorem, which is the main result of this paper.

Theorem 1. *Assume that conditions (1) and (NUPBR) hold and let U be a utility stochastic field satisfying Assumption 1. Let us also suppose that*

$$v(y) < \infty \quad \textit{for every } y > 0 \quad \textit{and} \quad u(x) > -\infty \quad \textit{for every } x > 0.$$

Then the primal value function u and the dual value function v defined in (3) and (4), respectively, satisfy the following properties:

(i) $u(x) < \infty$ *for every* $x > 0$, *and* $v(y) > -\infty$ *for every* $y > 0$. *The functions u and v are conjugate, i.e.*

$$v(y) = \sup_{x>0}(u(x) - xy), \quad y > 0, \qquad u(x) = \inf_{y>0}(v(y) + xy), \quad x > 0;$$

(ii) *the functions u and $-v$ are continuously differentiable on* $(0, \infty)$, *strictly concave, strictly increasing, and satisfy the Inada conditions*

$$\lim_{x\downarrow 0} u'(x) = +\infty, \qquad \lim_{y\downarrow 0} - v'(y) = +\infty,$$
$$\lim_{x\to+\infty} u'(x) = 0, \qquad \lim_{y\to+\infty} - v'(y) = 0.$$

Moreover, for every $x > 0$ and $y > 0$, the solutions $\hat{c}(x)$ to (3) and $\hat{Y}(y)$ to (4) exist and are unique and, if $y = u'(x)$, we have the dual relations

$$\hat{Y}_t(y)(\omega) = U'\big(t, \omega, \hat{c}_t(x)(\omega)\big), \quad \mathrm{d}\kappa \times \mathbb{P}\text{-a.e.}, \quad \textit{and} \quad \mathbb{E}\left[\int_0^\infty \hat{c}_t(x)\hat{Y}_t(y)\, \mathrm{d}\kappa_t\right] = xy.$$

Finally, the dual value function v can be represented as

$$v(y) = \inf_{Z\in\mathcal{Z}} \mathbb{E}\left[\int_0^\infty V(t, \omega, yZ_t)\, \mathrm{d}\kappa_t\right], \qquad y > 0.$$

Remark 2. For κ corresponding to the maximization of utility from terminal wealth, it can be checked that the sets $\mathcal{A}$ and $\mathcal{Y}$ satisfy the assumptions of [24, Proposition 3.1]. This implies that for a deterministic utility U satisfying the Inada conditions and such that $AE(U) < 1$ (in the terminology of [24]), under the additional assumption of finiteness of $u(x)$ for some $x > 0$, the assertions of [24, Theorem 2.2] hold under (NUPBR) (and possibly without NFLVR). This is a consequence of the abstract Theorems 3.1 and 3.2 of [24] that also apply under (NUPBR). Note also that the condition $u(x) > -\infty$ for all $x > 0$ trivially holds if U is a deterministic real-valued utility function. In particular, this is the case in the setting of [25], where it was shown that the finiteness of the dual function v acts as a necessary and sufficient condition for the validity of the key assertions of the theory.

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29

3. Proofs

Proof of Proposition 1. Suppose that (NUPBR) holds. Then, for every $n \in \mathbb{N}$, the set $\mathcal{X}_n$ is bounded in $\boldsymbol{L}^0$ and, by [31, Theorem 2.6], there exists a strictly positive càdlàg local martingale Z^n such that $Z_0^n = 1$ (since $\mathcal{F}_0$ is trivial) and the $\mathbb{R}^d$-valued process $Z^n S$ is a sigma-martingale on $[0, n]$. As a consequence of [2, Corollary 3.5] (see also [4, Remark 2.4]), it holds that $Z^n X$ is a local martingale on $[0, n]$ for every $X \in \mathcal{X}$ and $n \in \mathbb{N}$. For all $t \geq 0$, let then $n(t) := \min\{n \in \mathbb{N} \colon n > t\}$ and define the càdlàg process $Z = (Z_t)_{t \geq 0}$ via

$$Z_t := \prod_{k=1}^{n(t)} \frac{Z_{k \wedge t}^k}{Z_{(k-1)\wedge t}^k}, \qquad t \geq 0.$$

We now claim that $Z \in \mathcal{Z}$. Since $X \equiv 1 \in \mathcal{X}$ and in view of [14, Lemma I.1.35], it suffices to show that, for every $X \in \mathcal{X}$, the process ZX is a local martingale on $[0, m]$ for each $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$. Consider an arbitrary $X \in \mathcal{X}$ and let $\{\tau_k^n\}_{k \in \mathbb{N}}$ be a localizing sequence for the local martingale $Z^n X$ on $[0, n]$ for each $n \in \{1, \dots, m\}$. Let $\tau^j_{k\{\tau_k^j < j\}} := \tau_k^j \mathbf{1}_{\{\tau_k^j < j\}} + \infty \mathbf{1}_{\{\tau_k^j \geq j\}}$ for $j = 1, \dots, m$ and $k \in \mathbb{N}$, and define the stopping times

$$T_k^m := \min\{\tau^1_{k\{\tau_k^1 < 1\}}, \dots, \tau^m_{k\{\tau_k^m < m\}}, m\}, \qquad k \in \mathbb{N}.$$

Similarly as in [12, Theorem 4.10], it can be readily verified that the stopped process $(ZX)^{T_k^m}$ is a martingale on $[0, m]$ for all $k \in \mathbb{N}$. Since $\lim_{k \to +\infty} \mathbb{P}(T_k^m = m) = 1$, this shows that ZX is a local martingale on $[0, m]$. By the arbitrariness of m, this proves the claim.

To prove the converse implication, note that, for any $X \in \mathcal{X}$ and $Z \in \mathcal{Z}$, the process ZX is a supermartingale and, hence, for every $T \in \mathbb{R}_+$, it holds that $\mathbb{E}[Z_T X_T] \leq 1$. This shows that the set $Z_T \mathcal{X}_T$ is bounded in $\boldsymbol{L}^1$ and, hence, the set $\mathcal{X}_T$ is bounded in $\boldsymbol{L}^0$. □

Let us now turn to the proof of Theorem 1. Together with the abstract results established in [29, Section 3], the key step is represented by Lemma 1 below, which generalizes [29, Lemma 4.2] by relaxing the no-arbitrage type requirement into condition (NUPBR).

Lemma 1. *Let c be a nonnegative optional process and κ a stochastic clock. Under assumptions (1) and (NUPBR), the following conditions are equivalent:*

(i) $c \in \mathcal{A}$;

(ii) $\sup_{Z \in \mathcal{Z}} \mathbb{E}[\int_0^\infty c_t Z_t \,\mathrm{d}\kappa_t] \leq 1$.

Proof. If $c \in \mathcal{A}$, there exists an $\mathbb{R}^d$-valued predictable S-integrable process H such that

$$1 + \int_0^t H_u \,\mathrm{d}S_u \geq \int_0^t c_u \,\mathrm{d}\kappa_u \geq 0, \qquad t \geq 0.$$

We define $C_t := \int_0^t c_u \,\mathrm{d}\kappa_u$, $t \geq 0$, and observe that C is an increasing process. For an arbitrary $Z \in \mathcal{Z}$, the process $(\int_0^t C_{u-} \,\mathrm{d}Z_u)_{t \geq 0}$ is a local martingale and we let $\{\tau_n\}_{n \in \mathbb{N}}$ be a localizing sequence such that $(\int C_- \,\mathrm{d}Z)^{\tau_n}$ is a uniformly integrable martingale for every $n \in \mathbb{N}$. Using the supermartingale property of $Z(1 + \int H \,\mathrm{d}S)$, we obtain, for every $n \in \mathbb{N}$,

$$1 \geq \mathbb{E}\bigg[Z_{\tau_n}\bigg(1 + \int_0^{\tau_n} H_u \,\mathrm{d}S_u\bigg)\bigg] \geq \mathbb{E}[Z_{\tau_n} C_{\tau_n}] = \mathbb{E}\bigg[\int_0^{\tau_n} Z_u \,\mathrm{d}C_u + \int_0^{\tau_n} C_{u-} \,\mathrm{d}Z_u\bigg],$$

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29

where the last equality follows from integration by parts. Since $\{\tau_n\}_{n\in\mathbb{N}}$ is a localizing sequence for $\int C_- \,\mathrm{d}Z$, it holds that $\mathbb{E}[\int_0^{\tau_n} C_{u-}\,\mathrm{d}Z_u] = 0$ for every $n \in \mathbb{N}$. Hence,

$$1 \geq \mathbb{E}\bigg[\int_0^{\tau_n} Z_u \,\mathrm{d}C_u\bigg] \quad \text{for every } n \in \mathbb{N}.$$

By the monotone convergence theorem, we obtain

$$1 \geq \lim_{n\to\infty} \mathbb{E}\bigg[\int_0^{\tau_n} Z_u \,\mathrm{d}C_u\bigg] = \mathbb{E}\bigg[\int_0^{\infty} Z_u \,\mathrm{d}C_u\bigg].$$

Since $Z \in \mathcal{Z}$ is arbitrary, this proves the implication (i)⇒(ii).

Suppose now that $\sup_{Z\in\mathcal{Z}} \mathbb{E}[\int_0^\infty c_t Z_t \,\mathrm{d}\kappa_t] \leq 1$. Take an arbitrary $Z \in \mathcal{Z}$ and let $\{\varrho_n\}_{n\in\mathbb{N}}$ be a sequence of bounded stopping times increasing to infinity, $\mathbb{P}$-a.s., such that Z^{ϱ_n} is a uniformly integrable martingale for each $n \in \mathbb{N}$. Denoting

$$\mathcal{M}_\sigma(S) := \{\mathbb{Q} \sim \mathbb{P} \colon S \text{ is a } \mathbb{Q}\text{-sigma-martingale}\},$$

we can show that $\mathcal{M}_\sigma(S^{\varrho_n}) \neq \varnothing$ for every $n \in \mathbb{N}$. Let $\mathbb{Q} \in \mathcal{M}_\sigma(S^{\varrho_n})$ and denote by $M = (M_t)_{t\geq 0}$ its càdlàg density process (i.e. $M_t = \mathrm{d}\mathbb{Q}|_{\mathcal{F}_t}/\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}$, $t \geq 0$). Letting $Z' := M^{\varrho_n} Z (Z^{\varrho_n})^{-1}$, [30, Lemma 2.3] implies that $Z' \in \mathcal{Z}$. Therefore, for any stopping time τ,

$$\mathbb{E}^{\mathbb{Q}}[C_{\tau\wedge\varrho_n}] = \mathbb{E}[M_{\tau\wedge\varrho_n} C_{\tau\wedge\varrho_n}] = \mathbb{E}[Z'_{\tau\wedge\varrho_n} C_{\tau\wedge\varrho_n}] \leq 1,$$

where the last inequality follows from the assumption that $\sup_{Z\in\mathcal{Z}} \mathbb{E}[\int_0^\infty c_t Z_t \,\mathrm{d}\kappa_t] \leq 1$ by the same arguments used in the first part of the proof together with an application of Fatou's lemma. As a consequence, we have

$$\sup_{\mathbb{Q}\in\mathcal{M}_\sigma(S^{\varrho_n})} \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_{\tau\wedge\varrho_n}] \leq 1,$$

where $\mathcal{T}$ is the set of all stopping times. An application of [10, Proposition 4.2] then yields the existence of an adapted càdlàg process V^n such that $V^n_t \geq C_{t\wedge\varrho_n}$ for every $t \geq 0$ and admitting a decomposition of the form

$$V^n_t = V^n_0 + \int_0^t H^n_u \,\mathrm{d}S^{\varrho_n}_u - A^n_t, \qquad t \geq 0,$$

where H^n is an $\mathbb{R}^d$-valued predictable S^{ϱ_n}-integrable process, A^n is an adapted increasing process with $A^n_0 = 0$, and $V^n_0 = \sup_{\mathbb{Q}\in\mathcal{M}_\sigma(S^{\varrho_n}),\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_{\tau\wedge\varrho_n}] \leq 1$. Therefore, for every $n \in \mathbb{N}$, we obtain

$$1 + \int_0^t H^n_u \,\mathrm{d}S_u \geq V^n_0 + \int_0^t H^n_u \,\mathrm{d}S_u = V^n_t + A^n_t \geq V^n_t \geq C_t, \qquad 0 \leq t \leq \varrho_n.$$

Let $\bar{H}^n := H^n \mathbf{1}_{[[0,\varrho_n]]}$ for all $n \in \mathbb{N}$. By [10, Lemma 5.2], we can construct a sequence of processes $\{Y^n\}_{n\in\mathbb{N}}$, with $Y^n \in \operatorname{conv}(1 + \int \bar{H}^n \,\mathrm{d}S, 1 + \int \bar{H}^{n+1} \,\mathrm{d}S, \ldots)$, $n \in \mathbb{N}$, and a càdlàg process Y such that $\{ZY^n\}_{n\in\mathbb{N}}$ is Fatou convergent to a supermartingale ZY for every strictly positive càdlàg local martingale Z such that ZX is a supermartingale for every $X \in \mathcal{X}$. Note that $Y_t \geq C_t$ for all $t \geq 0$ and $Y_0 \leq 1$. Similarly as above, applying [10, Theorem 4.1] to the stopped process Y^{ϱ_n} for $n \in \mathbb{N}$, we obtain the decomposition

$$Y^{\varrho_n}_t = Y_0 + \int_0^t G^n_u \,\mathrm{d}S^{\varrho_n}_u - B^n_t, \qquad t \geq 0,$$

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29

where G^n is an $\mathbb{R}^d$-valued predictable S^{ϱ_n}-integrable process and B^n is an adapted increasing process with $B^n = 0$ for $n \in \mathbb{N}$. Letting

$$G := G^1 + \sum_{n=1}^{\infty} (G^{n+1} - G^n)\, \mathbf{1}_{[[\varrho_n, +\infty]]} = G^1\, \mathbf{1}_{[[0,\varrho_1]]} + \sum_{n=1}^{\infty} G^{n+1}\, \mathbf{1}_{[[\varrho_n, \varrho_{n+1}]]},$$

it follows that $1 + \int_0^t G_u \, \mathrm{d}S_u \geq C_t$ for all $t \geq 0$, thus establishing the implication (ii)⇒(i) and completing the proof. □

We are now in a position to complete the proof of Theorem 1, which generalizes the results of [29, Theorems 2.3 and 2.4] to the case where only (NUPBR) is assumed to hold.

Lemma 2. *Under (NUPBR), the set $\mathcal{Z}$ is closed under countable convex combinations. If, in addition, (1) holds then, for every $c \in \mathcal{A}$, we have*

$$\sup_{Z \in \mathcal{Z}} \mathbb{E}\bigg[\int_0^{\infty} c_t Z_t \, \mathrm{d}\kappa_t\bigg] = \sup_{Y \in \mathcal{Y}} \mathbb{E}\bigg[\int_0^{\infty} c_t Y_t \, \mathrm{d}\kappa_t\bigg] \leq 1. \tag{5}$$

Proof. Let $\{Z^n\}_{n\in\mathbb{N}}$ be a sequence of processes belonging to $\mathcal{Z}$ and $\{\lambda^n\}_{n\in\mathbb{N}}$ a sequence of positive numbers such that $\sum_{n=1}^{\infty} \lambda^n = 1$. Letting $Z := \sum_{n=1}^{\infty} \lambda^n Z^n$, we need to show that $Z \in \mathcal{Z}$. For each $N \in \mathbb{N}$, define $\widetilde{Z}^N := \sum_{n=1}^{N} \lambda^n Z^n$. For every $X \in \mathcal{X}$, $\{\widetilde{Z}^N X\}_{N\in\mathbb{N}}$ is an increasing sequence of nonnegative local martingales (i.e. $\widetilde{Z}_t^{N+1} X_t \geq \widetilde{Z}_t^N X_t$ for all $N \in \mathbb{N}$ and $t \geq 0$), such that $\widetilde{Z}_t^N X_t$ converges almost surely to $Z_t X_t$ as $N \to +\infty$ for every $t \geq 0$ and $Z_0 X_0 = 1$. The local martingale property of ZX then follows from [23, Proposition 5.1] (note that its proof carries over without modifications to the infinite horizon case), whereas [9, Theorem VI.18] implies that ZX is a càdlàg process. Since $X \in \mathcal{X}$ is arbitrary and $X \equiv 1 \in \mathcal{X}$, this proves the claim. Relation (5) follows by the same arguments used in [29, Lemma 4.3]. □

We denote by $\boldsymbol{L}^0(\mathrm{d}\kappa \times \mathbb{P})$ the linear space of equivalence classes of real-valued optional processes on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$, equipped with the topology of convergence in measure $(\mathrm{d}\kappa \times \mathbb{P})$. Let $\boldsymbol{L}^0_+(\mathrm{d}\kappa \times \mathbb{P})$ be the positive orthant of $\boldsymbol{L}^0(\mathrm{d}\kappa \times \mathbb{P})$.

Proof of Theorem 1. The sets $\mathcal{A}$ and $\mathcal{Y}$ are convex solid subsets of $\boldsymbol{L}^0_+(\mathrm{d}\kappa \times \mathbb{P})$. By definition, $\mathcal{Y}$ is closed in the topology of convergence in measure $(\mathrm{d}\kappa \times \mathbb{P})$. A simple application of Fatou's lemma together with Lemma 1 allows us to show that $\mathcal{A}$ is also closed in the same topology. Moreover, by the same arguments used in [29, Proposition 4.4(ii)], Lemma 1 and the bipolar theorem of [3] imply that $\mathcal{A}$ and $\mathcal{Y}$ satisfy the bipolar relations

$$c \in \mathcal{A} \iff \mathbb{E}\bigg[\int_0^{\infty} c_t Y_t \, \mathrm{d}\kappa_t\bigg] \leq 1 \quad \text{for all } Y \in \mathcal{Y},$$

$$Y \in \mathcal{Y} \iff \mathbb{E}\bigg[\int_0^{\infty} c_t Y_t \, \mathrm{d}\kappa_t\bigg] \leq 1 \quad \text{for all } c \in \mathcal{A}.$$

Since $X \equiv 1 \in \mathcal{X}$ and $\mathcal{Z} \neq \varnothing$, both $\mathcal{A}$ and $\mathcal{Y}$ contain at least one strictly positive element. In view of Lemma 2, Theorem 1 then follows directly from [29, Theorems 3.2 and 3.3]. □

Remark 3. We want to mention that Theorem 1 can also be proved by means of a change of numéraire argument. Indeed, one can consider the market where quantities are denominated in units of the *numéraire portfolio* (whose existence is equivalent to NUPBR; see [16]) and apply

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29

[29, Theorems 2.3 and 2.4] directly in that market, for which the set (2) is nonempty. In this regard, see [16, Section 4.7] and [21] in the case of maximization of expected (deterministic) utility from terminal wealth.

Acknowledgements

Huy N. Chau's work is supported by the Natixis Foundation for Quantitative Research and by the Lendület grant LP2015-6 of the Hungarian Academy of Sciences. Oleksii Mostovyi's work is supported by the National Science Foundation under award number DMS-1600307.

References

[1] Aksamit, A., Choulli, T., Deng, J. and Jeanblanc, M. (2014). Non-arbitrage up to random horizon for semimartingale models. Preprint. Available at https://arxiv.org/abs/1310.1142.

[2] Ansel, J.-P. and Stricker, C. (1994). Couverture des actifs contingents et prix maximum. *Ann. Inst. H. Poincaré Prob. Statist.* **30,** 303–315.

[3] Brannath, W. and Schachermayer, W. (1999). A bipolar theorem for $L^0_+(\Omega, \mathcal{F}, \mathrm{P})$. In *Séminaire de Probabilités, XXXIII* (Lecture Notes Math. **1709**). Springer, Berlin, pp. 349–354.

[4] Choulli, T., Deng, J. and Ma, J. (2015). How non-arbitrage, viability and numéraire portfolio are related. *Finance Stoch.* **19,** 719–741.

[5] Christensen, M. M. and Larsen, K. (2007). No arbitrage and the growth optimal portfolio. *Stoch. Anal. Appl.* **25,** 255–280.

[6] Delbaen, F. and Schachermayer, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300,** 463–520.

[7] Delbaen, F. and Schachermayer, W. (1995). Arbitrage possibilities in Bessel processes and their relations to local martingales. *Prob. Theory Relat. Fields* **102,** 357–366.

[8] Delbaen, F. and Schachermayer, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.* **312,** 215–250.

[9] Dellacherie, C. and Meyer, P.-A. (1982). *Probabilities and Potential B*. North-Holland, Amsterdam.

[10] Föllmer, H. and Kramkov, D. (1997). Optional decomposition under constraints. *Prob. Theory Relat. Fields* **109,** 1–25.

[11] Fontana, C. and Runggaldier, W. (2013). Diffusion-based models for financial markets without martingale measures. In *Risk Measures and Attitudes*, Springer, London, pp. 45–81.

[12] Fontana, C., Øksendal, B. and Sulem, A. (2015). Market viability and martingale measures under partial information. *Methodol. Comput. Appl. Prob.* **17,** 15–39.

[13] Harrison, J. M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *J. Econom. Theory* **20,** 381–408.

[14] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd edn. Springer, Berlin.

[15] Karatzas, I. and Fernholz, R. (2009). Stochastic portfolio theory: an overview. In *Handbook of Numerical Analysis*, Vol. 15, North-Holland, Amsterdam, pp. 89–167.

[16] Karatzas, I. and Kardaras, C. (2007). The numéraire portfolio in semimartingale financial models. *Finance Stoch.* **11,** 447–493.

[17] Karatzas, I. and Žitković, G. (2003). Optimal consumption from investment and random endowment in incomplete semimartingale markets. *Ann. Prob.* **31,** 1821–1858.

[18] Karatzas, I., Lehoczky, J. P., Shreve, S. E. and Xu, G.-L. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control Optimization* **29,** 702–730.

[19] Kardaras, C. (2010). Finitely additive probabilities and the fundamental theorem of asset pricing. In *Contemporary Quantitative Finance*, Springer, Berlin, pp. 19–34.

[20] Kardaras, C. (2012). Market viability via absence of arbitrage of the first kind. *Finance Stoch.* **16,** 651–667.

[21] Kardaras, C. (2013). On the closure in the Emery topology of semimartingale wealth-process sets. *Ann. Appl. Prob.* **23,** 1355–1376.

[22] Kardaras, C. (2014). A time before which insiders would not undertake risk. In *Inspired by Finance*, Springer, Cham, pp. 349–362.

[23] Klein, I., Lépinette, E. and Perez-Ostafe, L. (2014). Asymptotic arbitrage with small transaction costs. *Finance Stoch.* **18,** 917–939.

[24] Kramkov, D. and Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Prob.* **9,** 904–950.

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29

[25] Kramkov, D. and Schachermayer, W. (2003). Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Prob.* **13,** 1504–1516.
[26] Larsen, K. (2009). Continuity of utility-maximization with respect to preferences. *Math. Finance* **19,** 237–250.
[27] Larsen, K. and Žitković, G. (2013). On utility maximization under convex portfolio constraints. *Ann. Appl. Prob.* **23,** 665–692.
[28] Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: the continuous-time case. *Rev. Econom. Statist.* **51,** 247–257.
[29] Mostovyi, O. (2015). Necessary and sufficient conditions in the problem of optimal investment with intermediate consumption. *Finance Stoch.* **19,** 135–159.
[30] Stricker, C. and Yan, J. A. (1998). Some remarks on the optional decomposition theorem. In *Séminaire de Probabilités, XXXII* (Lecture Notes Math. **1686**), Springer, Berlin, pp. 56–66.
[31] Takaoka, K. and Schweizer, M. (2014). A note on the condition of no unbounded profit with bounded risk. *Finance Stoch.* **18,** 393–405.
[32] Žitković, G. (2005). Utility maximization with a stochastic clock and an unbounded random endowment. *Ann. Appl. Prob.* **15,** 748–777.
[33] Žitković, G. (2010). Convex compactness and its applications. *Math. Financial Econom.* **3,** 1–12.

Downloaded from https://www.cambridge.org/core. University of Connecticut, on 14 Jun 2019 at 20:18:26, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/jpr.2017.29