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OPTIMAL CONSUMPTION OF MULTIPLE GOODS IN INCOMPLETE MARKETS

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Abstract

We consider the problem of optimal consumption of multiple goods in incomplete semimartingale markets. We formulate the dual problem and identify conditions that allow for the existence and uniqueness of the solution, and provide a characterization of the optimal consumption strategy in terms of the dual optimizer. We illustrate our results with examples in both complete and incomplete models. In particular, we construct closed-form solutions in some incomplete models.

Keywords: Optimal consumption; multiple goods; utility maximization; no unbounded profit with bounded risk; arbitrage of the first kind; local martingale deflator; duality theory; semimartingale; incomplete market; optimal investment

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1. Introduction

The problem of optimal consumption of multiple goods was investigated by Breeden [2] and Fischer [5]. For a single consumption good in continuous-time settings, it was first formulated by Merton [17]. Since then, this problem has received much attention in both complete and incomplete settings with a range of techniques based on Hamilton–Jacobi–Bellman equations, backward stochastic differential equations, and convex duality being used for its analysis.

In the present paper we formulate a problem of optimal consumption of multiple goods in a general incomplete semimartingale model of a financial market. We construct the dual problem and characterize optimal consumption policies in terms of the solution to the dual problem. We also identify mathematical conditions that allow for the existence and uniqueness of the solution and a dual characterization. We illustrate our results by examples where, in particular, we obtain closed-form solutions in incomplete markets.

Our proofs rely on certain results on weakly measurable correspondences for Carathéodory functions, multidimensional convex-analytic techniques, and some recent advances in stochastic analysis in mathematical finance, in particular, the characterization of the 'no unbounded profit with bounded risk' condition in terms of nonemptiness of the set of equivalent local martingale deflators (see [3] and [9]), and sharp conditions for solvability of the expected utility maximization problem in a single consumption good setting; see [19]. Measurability-wise the price processes of consumption goods are needed only to be optional. Strict positivity is also required, but no boundedness away from 0 or ∞ is supposed. Comparing to [3] and [19], apart from the conditions on the price process of the consumption goods that are unequivocal in single-good settings, one of the leading challenges in the present work is handling the

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multidimensionality of the utility process in the spatial variable. The key step is the introduction of an auxiliary utility process (see (2.7)) and its representation as a pointwise image function (in the convex-analytic sense) of the original multivalued utility process under the corresponding linear transformation, which is also identified. Further, a challenging point in the proofs, which only emerges in the case of multiple goods but not in single-good settings, is the measurability of the candidate optimizer. We prove this via establishing the weak measurability of a certain correspondence in the spirit of [1]. Here the model assumptions, in particular the aforementioned optionality and strict positivity of the consumption goods processes, play an important role. Finally, we provide a general framework for analyzing numerous questions related to stability and asymptotics in the multiple goods settings from both mathematical and economic viewpoints.

The remainder of this paper is organized as follows. In Section 2 we specify the model setting, formulate the problem, and state the main results (Theorem 2.1). In Section 3 we discuss various specific cases. In particular, we present the structure of the solution for complete models and the additive utility case, as well as closed-form solutions in some incomplete models (with and without an additive structure of the utility). We conclude the paper with Section 4, which contains the proofs.

2. Setting and main results

2.1. Setting

Let $\widetilde{S} = (\widetilde{S}_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued semimartingale, representing the discounted prices of d risky assets on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$, with $\mathcal{F}_0$ being the trivial σ-algebra. Since we allow preferences to be stochastic (see the definition below), there is no loss of generality in assuming that asset prices are discounted; see [19, Remark 2.2] for a more detailed explanation of this observation. We fix a *stochastic clock* $\kappa = (\kappa_t)_{t\geq 0}$, which is a nondecreasing, càdlàg, adapted process such that

$$\kappa_0 = 0, \qquad \mathbb{P}(\kappa_\infty > 0) > 0, \quad \text{and} \quad \kappa_\infty \leq \bar{A}, \tag{2.1}$$

where $\bar{A}$ is a positive constant. The stochastic clock κ specifies times at which consumption is assumed to occur. Various optimal investment–consumption problems can be recovered from the present general setting by suitably specifying the clock process κ. For example, the problem of maximizing the expected utility of terminal wealth at some finite investment horizon $T < \infty$ can be recovered by simply letting $\kappa \triangleq \mathbf{1}_{[T,\infty]}$. Likewise, maximization of the expected utility from consumption only up to a finite horizon $T < \infty$ can be obtained by letting $\kappa_t \triangleq \min(t, T)$ for $t \geq 0$. Other specifications include maximization of the utility from lifetime consumption, from consumption at a finite set of stopping times, and from terminal wealth at a random horizon; see, for example, [19, Examples 2.5–2.9] for a description of possible standard choices of the clock process κ.

We suppose that there are m different consumption goods, where S^k_t denotes the discounted price of commodity k at time t. We assume that for each $k \in \{1, \dots, m\}$, $S^k = (S^k_t)_{t\geq 0}$ is a *strictly* positive optional processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$. Denote $S = (S^1, \dots, S^m)$.

A *portfolio* is defined by a triplet $\Pi = (x, H, c)$, where $x \in \mathbb{R}$ represents an initial capital, $H = (H_t)_{t\geq 0}$ is a d-dimensional $\widetilde{S}$-integrable process, H^j_t represents the holdings in the jth risky asset at time t, $j \in \{1, \dots, d\}$, $t \geq 0$, and c is an m-dimensional *consumption process*, whose every component $(c^k_t)_{t\geq 0}$ is a nonnegative optional process representing the consumption *rate* of commodity k, $k = \{1, \dots, m\}$. The wealth process $X = (X_t)_{t\geq 0}$ of a portfolio $\Pi = (x, H, c)$ is defined as

$$X_t \triangleq x + \int_0^t H_u \, \mathrm{d}\widetilde{S}_u - \int_0^t c_u \cdot S_u \, \mathrm{d}\kappa_u, \qquad t \geq 0, \tag{2.2}$$

where '$\cdot$' denotes the dot product in $\mathbb{R}^m$.

2.2. Absence of arbitrage

Our main objective in this subsection is to specify the no-arbitrage-type condition *no unbounded profit with bounded risk* (NUPBR). As is common in the literature (see, for example, [16]), we begin by defining $\mathcal{X}$ to be the collection of all nonnegative wealth processes associated to portfolios of the form $\Pi = (1, H, 0)$, i.e.

$$\mathcal{X} \triangleq \left\{ X \geq 0 \colon X_t = 1 + \int_0^t H_u \, \mathrm{d}\widetilde{S}_u, \; t \geq 0 \right\}.$$

In this paper we define the following no-arbitrage-type condition:

(NUPBR) the set $\mathcal{X}_T \triangleq \{X_T \colon X \in \mathcal{X}\}$ is bounded in probability for every $T \in \mathbb{R}_+$.

This condition was originally introduced in [10]. In [12, Proposition 1], it was proved that NUPBR is equivalent to another (weak) no-arbitrage condition; namely, the absence of *arbitrages of the first kind* on $[0, T]$; see [14, Definition 1].

A useful characterization of NUPBR is given via the set of *equivalent local martingale deflators* defined as

$$\begin{aligned} \mathcal{Z} \triangleq \{Z > 0 \colon & Z \text{ is a càdlàg local martingale such that } Z_0 = 1 \text{ and} \\ & ZX = (Z_t X_t)_{t \geq 0} \text{ is a local martingale for every } X \in \mathcal{X}\}. \end{aligned} \tag{2.3}$$

In [3, Proposition 2.1] (see also [9]), it was shown that condition (NUPBR) holds if and only if $\mathcal{Z} \neq \varnothing$. This result was previously established for the one-dimensional case in the finite-time horizon in [13, Theorem 2.1]. Also, [24, Theorem 2.6] contains a closely related result (in a finite-time horizon) in terms of *strict σ-martingale densities*; see [24] for the corresponding definition and details.

Remark 2.1. Condition (NUPBR) is weaker than the existence of an equivalent martingale measure (see, for example, [4, p. 463] for the definition an equivalent martingale measure), another classical no-arbitrage-type assumption, which in the infinite-time horizon is even stronger than

$$\{Z \in \mathcal{Z} \colon Z \text{ is a martingale}\} \neq \varnothing. \tag{2.4}$$

Note that in the *finite-time horizon* setting, (2.4) is equivalent to the existence of an equivalent martingale measure. In addition, (2.4) is apparently stronger than (NUPBR) (by comparison of (2.3) and (2.4) combined with [3, Proposition 2.1]). We also point out that (2.4) holds in every original formulation of [17], where the problem of optimal consumption from investment (in a single consumption good setting) was introduced, including the infinite-time horizon case. In general, (2.4) can be stronger than (NUPBR). A classical example, where (NUPBR) holds but (2.4) fails, corresponds to the three-dimensional Bessel process driving the stock price; see, for example, [10, Example 4.6].

2.3. Admissible consumptions

For a given initial capital $x > 0$, an m-dimensional optional consumption process c is said to be x-*admissible* if there exists an $\mathbb{R}^d$-valued predictable $\widetilde{S}$-integrable process H such that the wealth process X in (2.2), corresponding to the portfolio $\Pi = (x, H, c)$, is nonnegative; the set of x-admissible consumption processes corresponding to a stochastic clock κ is denoted by $\mathcal{A}(x)$. For brevity, we denote $\mathcal{A} \triangleq \mathcal{A}(1)$.

2.4. Preferences of a rational economic agent

Building from the formulation of [18], we assume that preferences of a rational economic agent are represented by a *optional utility-valued process* (or simply a *utility process*) $U = U(t, \omega, x)\colon [0,\infty) \times \Omega \times [0,\infty)^m \to \mathbb{R} \cup \{-\infty\}$, where for every $(t, \omega) \in [0,\infty) \times \Omega$, $U(t, \omega, \cdot)$ is an Inada-type utility function, i.e. $U(t, \omega, \cdot)$ satisfies the next (technical) assumption.

Assumption 2.1. *For every $(t, \omega) \in [0,\infty) \times \Omega$, the function*

$$\mathbb{R}^m_+ \ni x \mapsto U(t, \omega, x) \in \mathbb{R} \cup \{-\infty\}$$

is strictly concave, strictly increasing in every component, finite-valued and continuously differentiable in the interior of the positive orthant, and satisfies the Inada conditions

$$\lim_{x_i \downarrow 0} \partial_{x_i} U(t, \omega, x) = \infty \quad \text{and} \quad \lim_{x_i \uparrow \infty} \partial_{x_i} U(t, \omega, x) = 0, \qquad i = 1, \dots, m,$$

where $\partial_{x_i} U(t, \omega, \cdot)\colon \mathbb{R}^m_{++} \mapsto \mathbb{R}$ is the partial derivative of $U(t, \omega, \cdot)$ with respect to the ith spatial variable. (For the results below, we need to specify only the gradient of $U(t, \omega, \cdot)$ in the interior *of the first orthant, i.e. at the points $x \in \mathbb{R}^m$ where $U(t, \omega, x)$ is (finite-valued and) differentiable.) On the boundary of the first orthant, by upper semicontinuity, we suppose that $U(t, \omega, x) = \limsup_{x' \to x} U(t, \omega, x')$ (note that some of these values may be $-\infty$ and that $U(t, \omega, x) = \lim_{t \downarrow 0} U(t, \omega, x + t(x' - x))$, where x' is an arbitrary element in the interior of the first orthant; see [7, Proposition B.1.2.5]). Finally, for every $x \in \mathbb{R}^m_+$, we assume that the stochastic process $U(\cdot, \cdot, x)$ is optional.*

Remark 2.2. The Inada conditions in Assumption 2.1 were introduced in [8]. These are technical assumptions that have natural economic interpretations and allow for a deeper tractability of the problem; see, for example, [16]. Likewise, the semicontinuity of U is imposed for regularity purposes; see, for example, [22] and [23].

In particular, modeling the preferences via a utility process allows us to take into account utility maximization problems under a change of numéraire; see, for example, [20, Example 4.2]. This is the primary reason why we suppose that the prices of the traded stocks are discounted; it allows us to simplify the notation without any loss of generality. Note also that Assumption 2.1 does not make any requirement on the *asymptotic elasticity* of U, introduced in [16].

To a utility process U satisfying Assumption 2.1 we associate the *primal value function*

$$u(x) \triangleq \sup_{c \in \mathcal{A}(x)} \mathbb{E}\bigg[\int_0^\infty U(t, \omega, c_t(\omega))\, \mathrm{d}\kappa_t\bigg], \qquad x > 0, \tag{2.5}$$

where $c = (c^1, \dots, c^m)$. To ensure that the integral above is well defined, we use the convention

$$\mathbb{E}\bigg[\int_0^\infty U(t, \omega, c_t(\omega))\, \mathrm{d}\kappa_t\bigg] \triangleq -\infty \quad \text{if } \mathbb{E}\bigg[\int_0^\infty U^-(t, \omega, c_t(\omega))\, \mathrm{d}\kappa_t\bigg] = \infty, \tag{2.6}$$

where $U^{-}(t,\omega,\cdot)$ is the negative part of $U(t,\omega,\cdot)$. Note that (2.5) is a generalization of the formulation of [18, p. 205]. In the form (2.5), we allow for stochastic preferences and include several standard formulations as particular cases.

2.5. Dual problem

In order to specify the assumptions of the model to ensure the existence and uniqueness of solutions to (2.5), and to provide a characterization of this solution, we need to formulate an appropriate dual problem. Define

$$U^*(t,\omega,x) \triangleq \sup_{c\in\mathbb{R}^m_+:\, c\cdot S_t(\omega)\le x} U(t,\omega,c), \qquad (t,\omega,x)\in[0,\infty)\times\Omega\times[0,\infty). \tag{2.7}$$

Define a family of transformations $A\colon [0,\infty)\times\Omega\times\mathbb{R}^m \mapsto \mathbb{R}$ as

$$A(t,\omega,c) \triangleq c\cdot S_t(\omega), \qquad (t,\omega,c)\in[0,\infty)\times\Omega\times[0,\infty)^m.$$

Note that for every $(t,\omega)\in[0,\infty)\times\Omega$, $A(t,\omega,\cdot)$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}$, and $U^*(t,\omega,\cdot)$ is the image of $U(t,\omega,\cdot)$ under $A(t,\omega,\cdot)$; see, for example, [7, p. 96] for the definition and properties of the image of a function under a linear mapping. (Equivalently, see [21, Theorem 5.2], where $U^*(t,\omega,\cdot)$ was called the image of $U(t,\omega,\cdot)$ under the linear transformation $A(t,\omega,\cdot)$, $(t,\omega)\in[0,\infty)\times\Omega$.) We define a stochastic field V^* as the pointwise conjugate of U^* (equivalently, as the pointwise conjugate of the image function of U under A) in the sense that

$$V^*(t,\omega,y) \triangleq \sup_{x>0}(U^*(t,\omega,x)-xy), \qquad (t,\omega,y)\in[0,\infty)\times\Omega\times[0,\infty),$$

where $\sup_{x>0}$ and $\sup_{x\ge 0}$ coincide, thanks to the continuity of U^* established in Lemma 4.1. We also introduce the following set of dual processes:

$$\mathcal{Y}(y) \triangleq \mathrm{cl}\{Y\colon Y \text{ is càdlàg adapted and } 0\le Y\le yZ,\ (\mathrm{d}\kappa\times\mathbb{P})\text{-a.e. for some } Z\in\mathcal{Z}\},$$

where 'cl' is the closure, taken in the topology of the convergence in measure $(\mathrm{d}\kappa\times\mathbb{P})$ on the measure space of real-valued optional processes $([0,\infty)\times\Omega, \mathcal{O}, \mathrm{d}\kappa\times\mathbb{P})$, where $\mathcal{O}$ is the optional sigma-field. We write $\mathcal{Y}\triangleq\mathcal{Y}(1)$ for brevity and abbreviate almost everywhere to a.e. Note that $\mathcal{Y}$ is closely related to—but different from—the set with the same name as in [16]. The value function of the dual optimization problem or, equivalently, the *dual value function* is then defined as

$$v(y) \triangleq \inf_{Y\in\mathcal{Y}(y)} \mathbb{E}\biggl[\int_0^\infty V^*(t,\omega,Y_t(\omega))\,\mathrm{d}\kappa_t\biggr], \qquad y>0, \tag{2.8}$$

with the convention $\mathbb{E}[\int_0^\infty V^*(t,\omega,Y_t(\omega))\,\mathrm{d}\kappa_t]\triangleq\infty$ if $\mathbb{E}[\int_0^\infty V^{*+}(t,\omega,Y_t(\omega))\,\mathrm{d}\kappa_t]=\infty$, where $V^{*+}(t,\omega,\cdot)$ is the positive part of $V^*(t,\omega,\cdot)$. Note that, in the single-good but otherwise similar setting, properties of the dual value function were investigated in [3] and [19]. We are now in a position to state our first result.

Theorem 2.1. *Assume that conditions (2.1) and (NUPBR) hold, and let U satisfy Assumption 2.1. We also suppose that*

$$v(y)<\infty \quad \textit{for every } y>0 \quad \textit{and} \quad u(x)>-\infty \quad \textit{for every } x>0. \tag{2.9}$$

Then we have

(i) *$u(x)<\infty$ for every $x>0$ and $v(y)>-\infty$ for every $y>0$, i.e. the value functions are finite-valued.*

(ii) *The functions u and $-v$ are continuously differentiable on $(0, \infty)$, strictly concave, strictly increasing, and satisfy the Inada conditions*

$$\left.\begin{matrix}\lim_{x\downarrow 0} u'(x) \\ \lim_{y\downarrow 0} -v'(y)\end{matrix}\right\} = \infty, \qquad \left.\begin{matrix}\lim_{x\to\infty} u'(x) \\ \lim_{y\to\infty} -v'(y)\end{matrix}\right\} = 0. \tag{2.10}$$

(iii) *For every $x > 0$ and $y > 0$, the solutions $\widehat{c}(x)$ to (2.5) and $\widehat{Y}(y)$ to (2.8) exist and are unique, and, if $y = u'(x)$, we have the optimality characterizations*

$$\widehat{Y}_t(y)(\omega) = \frac{\partial_{x_i} U(t, \omega, \widehat{c}_t(x)(\omega))}{S^i_t(x)(\omega)}, \qquad (\mathrm{d}\kappa \times \mathbb{P})\text{-a.e.}, \ i = 1, \ldots, m, \tag{2.11}$$

and

$$\widehat{Y}_t(y)(\omega) = U^*_x(t, \omega, \widehat{c}_t(x)(\omega) \cdot S_t(\omega)), \qquad (\mathrm{d}\kappa \times \mathbb{P})\text{-a.e.}, \tag{2.12}$$

with U^*_x denoting the partial derivative of U^* with respect to its third argument.

(iv) *For every $x > 0$, the constraint x is binding in the sense that*

$$\mathbb{E}\biggl[\int_0^\infty \widehat{c}_t(x) \cdot S_t \frac{\widehat{Y}_t(y)}{y} \,\mathrm{d}\kappa_t\biggr] = x, \tag{2.13}$$

where $y = u'(x)$.

(v) *The functions u and v are Legendre conjugate, i.e.*

$$v(y) = \sup_{x>0}(u(x) - xy), \quad y > 0, \qquad u(x) = \inf_{y>0}(v(y) + xy), \quad x > 0. \tag{2.14}$$

(vi) *The dual value function v can be represented as*

$$v(y) = \inf_{Z\in\mathcal{Z}} \mathbb{E}\biggl[\int_0^\infty V(t, \omega, yZ_t(\omega)) \,\mathrm{d}\kappa_t(\omega)\biggr], \qquad y > 0. \tag{2.15}$$

Remark 2.3. (*On sufficient conditions for the validity of (2.9).*) Condition (2.9) holds if there exists one primal element $c \in \mathcal{A}$ and one dual element $Y \in \mathcal{Y}$ such that

$$\mathbb{E}\biggl[\int_0^\infty U(t, \omega, zc_t(\omega)) \,\mathrm{d}\kappa_t\biggr] > -\infty, \qquad \mathbb{E}\biggl[\int_0^\infty V^*(t, \omega, zY_t(\omega)) \,\mathrm{d}\kappa_t\biggr] < \infty, \qquad z > 0.$$

In particular, for every $x > 0$, as an m-dimensional optional process with constant values $(x/\bar{A}m, \ldots, x/\bar{A}m)$ belonging to $\mathcal{A}(x)$, a sufficient condition in (2.9) for the finiteness of u is

$$\mathbb{E}\biggl[\int_0^\infty U\biggl(t, \omega, \frac{x}{\bar{A}m}, \ldots, \frac{x}{\bar{A}m}\biggr) \,\mathrm{d}\kappa_t\biggr] > -\infty, \qquad x > 0,$$

which typically holds if U is nonrandom. Likewise, as $\mathcal{Z} \neq \varnothing$ (by (NUPBR) and [3, Proposition 2.1]), the finiteness of v holds if, for one equivalent local martingale deflator Z, we have

$$\mathbb{E}\biggl[\int_0^\infty V^*(t, \omega, yZ_t(\omega)) \,\mathrm{d}\kappa_t\biggr] < \infty, \qquad y > 0.$$

3. Examples

3.1. Complete market solution and dual characterization

If the model is complete, the dual characterization of the optimal consumption policies has a particularly nice form since $\mathcal{Z}$ contains a unique element Z. The solutions corresponding to different y in the dual problem (2.8) are $yZ,\ y > 0$. Therefore, in (2.11) and (2.12), we have $\widehat{Y}(y) = yZ,\ y > 0$.

3.2. Special case: additive utility

An important example of U^* corresponds to U having an additive form with respect to its spatial components, i.e. when

$$U(t, \omega, c_1, \dots, c_m) = U^1(t, \omega, c_1) + \dots + U^m(t, \omega, c_m), \qquad (t, \omega) \in [0, \infty) \times \Omega,$$

where for every $k = 1, \dots, m$, U^k is a utility process in the sense of [19, Assumption 2.1], and a utility process in the sense of Assumption 2.1 with $m = 1$. In this case, for every $(t, \omega) \in [0, \infty) \times \Omega$, $U^*(t, \omega, \cdot)$ is the *infimal convolution* of the $U^k(t, \omega, \cdot)$; see the definition in, for example, [21, p. 34]. Let $V^i(t, \omega, \cdot)$ denote the convex conjugate of $U^i(t, \omega, \cdot),\ i = 1, \dots, m$. Then the convex conjugate of $U^*(t, \omega, \cdot)$ is

$$V^*(t, \omega, \cdot) = V^1(t, \omega, \cdot) + \dots + V^m(t, \omega, \cdot).$$

This result was established, for example, in [21, Theorem 16.4, p. 145]. In this case, the optimal $\widehat{c}(x) = (\widehat{c}^1(x), \dots, \widehat{c}^m(x))$ has a more explicit characterization via $I_i(t, \omega, \cdot) \triangleq (U^i_x)^{-1}(t, \omega, \cdot)$, the pointwise inverse of the partial derivative of $U^i(t, \omega, \cdot)$ with respect to the third argument, as (2.11) can be solved for $\widehat{c}^i(x),\ i = 1, \dots, m$, as follows:

$$\widehat{c}^i_t(x)(\omega) = I_i(t, \omega, \widehat{Y}_t(y)(\omega) S^i_t(\omega)), \qquad (\mathrm{d}\kappa \times \mathbb{P})\text{-a.e., } i = 1, \dots, m. \tag{3.1}$$

Using (2.12), we can restate (3.1) as

$$\widehat{c}^i_t(x)(\omega) = I_i(t, \omega, U^*_x(t, \omega, \widehat{c}^*_t(x)(\omega)) S^i_t(\omega)), \qquad (\mathrm{d}\kappa \times \mathbb{P})\text{-a.e., } i = 1, \dots, m,$$

where $\widehat{c}^*(x)$ is the optimizer to the auxiliary problem (4.2) corresponding to the initial wealth $x > 0$.

Remark 3.1. In the following three examples we consider some incomplete models that admit closed-form solutions for one consumption good and show how these results apply to multiple consumption good settings.

3.3. Example of a closed-form solution in an incomplete model with additive logarithmic utility

Let us suppose that d traded discounted assets are modeled with Itô processes of the form

$$\mathrm{d}\widetilde{S}^i_t = \widetilde{S}^i_t b^i_t\,\mathrm{d}t + \widetilde{S}^i_t \sum_{j=1}^{n} \sigma^{ij}_t\,\mathrm{d}W^j_t, \quad i = 1, \dots, d, \qquad \widetilde{S}_0 \in \mathbb{R}^d, \tag{3.2}$$

where W is an $\mathbb{R}^n$-valued standard Brownian motion and b^i, σ^{ij}, $i = 1, \dots, d$, $j = 1, \dots, n$, are predictable processes such that the unique strong solution to (3.2) exists; see, for example, [11]. We suppose that there are m consumption goods and that the value function of a rational economic agent is

$$\sup_{c\in\mathcal{A}(x)} \mathbb{E}\bigg[\int_0^T \mathrm{e}^{-\nu t}\log(c)\,\mathrm{d}t\bigg], \qquad x>0,$$

using the same convention as the one specified after (2.5), where an impatience rate ν and a time horizon T are positive constants. Note that in this case, $\kappa_t = (1-\mathrm{e}^{-\nu t})/\nu$, $t\in[0,T]$, i.e. κ is deterministic. We also suppose that there exists an $\mathbb{R}^d$-valued process γ such that

$$b_t - \sigma_t\sigma_t^{\top}\gamma_t = 0, \qquad (\mathrm{d}\kappa\times\mathbb{P})\text{-a.e.}$$

Let $\mathcal{E}$ denote the Doleans–Dade exponent. Then, using [6, Theorem 3.1 and Example 4.2] and Theorem 2.1, we obtain

$$\widehat{c}_t^*(x) = \frac{x\nu}{1-\mathrm{e}^{-\nu T}}\mathcal{E}\bigg(\int_0^{\cdot}\gamma_s^{\top}\,\mathrm{d}\widetilde{S}_s\bigg)_t, \quad x>0, \qquad \widehat{c}_t^i(x) = \frac{\widehat{c}_t^*(x)}{S_t^iM}, \quad i=1,\dots,m,\ x>0,$$

$$\widehat{Y}_t(y) = \frac{y}{\mathcal{E}(\int_0^{\cdot}\gamma_s^{\top}\,\mathrm{d}\widetilde{S}_s)_t}, \qquad y>0,\ t\in[0,T].$$

3.4. Example of a closed-form solution and dual characterization in an incomplete additive case

We fix a filtered probability space $(\Omega,\mathcal{F},\mathbb{P})$, where $(\mathcal{F}_t)_{t\ge 0}$ is the augmentation of the filtration generated by a two-dimensional Brownian motion (W^1, W^2). We suppose that there are two traded securities: a risk-free asset B such that

$$B_t = \mathrm{e}^{rt}, \qquad t>0,$$

where r is a nonnegative constant, and a risky stock $\widetilde{S}$ with the dynamics

$$\mathrm{d}\widetilde{S}_t = \widetilde{S}_t\mu_t\,\mathrm{d}t + \widetilde{S}_t\sigma_t\,\mathrm{d}W_t^1, \quad t\ge 0, \qquad \widetilde{S}_0\in\mathbb{R}_+,$$

where processes μ and σ are such that $\theta_t = (\mu_t - r)/\sigma_t$, $t\ge 0$, the market price of risk process, follows from the Ornstein–Uhlenbeck process

$$\mathrm{d}\theta_t = -\lambda_\theta(\theta_t-\bar\theta)\,\mathrm{d}t + \sigma_\theta(\rho\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^2), \quad t\ge 0, \qquad \theta_0\in\mathbb{R}_+,$$

where λ_θ, σ_θ, and $\bar\theta$ are positive constants, and $\rho\in(-1,1)$. We also assume that κ corresponds to the expected utility maximization from terminal wealth, i.e. $\kappa = \mathbf{1}_{[T,\infty]}$, $T\in\mathbb{R}_+$, that there are m consumption goods, where S^i, $i=1,\dots,m$, are *deterministic*, and

$$U(T,\omega,c) = \frac{c_1^p}{p} + \cdots + \frac{c_m^p}{p}, \qquad c\in\mathbb{R}_+^m,\ \omega\in\Omega,$$

where $p<0$. Set

$$q \triangleq \frac{p}{1-p}, \qquad A\triangleq\sum_{i=1}^m (S_T^i)^{-q}, \qquad B\triangleq A^{1-p}.$$

Then, by direct computations, we obtain

$$U^*(T,\omega,x) = \frac{x^p}{p}B, \qquad x>0.$$

Using the argument in [15], one can express the optimal trading strategy $\widehat{H}(x)$ in a closed form in terms of a solution to a system of (nonlinear) ordinary differential equations (see [15, p. 147]), where $\widehat{H}_t(x)$ is the number of shares of the risky asset in the portfolio at time t, $t \in [0, T]$. With $\widehat{X}(x)$ such that

$$\mathrm{d}\widehat{X}_t(x) = \widehat{H}_t(x)\,\mathrm{d}\widetilde{S}_t + (\widehat{X}_t(x) - \widehat{H}_t(x)\widetilde{S}_t)r\,\mathrm{d}t, \qquad \widehat{X}_0(x) = x,$$

and using Theorem 2.1, we obtain

$$\widehat{c}^{\,*}_T(x) = \widehat{X}_T(x), \quad x > 0, \qquad \widehat{Y}_T(y) = \frac{y}{\mathbb{E}[(\widehat{c}^{\,*}_T(1))^p]}(\widehat{c}^{\,*}_T(1))^{p-1}, \quad y > 0,$$

$$\widehat{c}^{\,i}_T(x) = \frac{\widehat{c}^{\,*}_T(x)}{A}(S^i_T)^{-(1+q)}, \qquad x > 0.$$

3.5. Example of a closed-form solution and dual characterization in an incomplete nonadditive case

Here we suppose that $\kappa = \mathbf{1}_{[T,\infty]}$, where $T \in \mathbb{R}_+$, and let

$$U(t, \omega, c_1, c_2) = -\frac{c_1^{p_1}}{p_1}\frac{c_2^{p_2}}{p_2}, \qquad p_1 < 0,\ p_2 < 0,$$

i.e. there are two consumption goods. We see that $U(t, \omega, \cdot)$ is jointly concave since the Hessian of $-U(t, \omega, \cdot)$ is positive definite on $\mathbb{R}^2_{++}$. We also extend $U(t, \omega, \cdot)$ to the boundary of $\mathbb{R}^2_+$ by $-\infty$. Then, with $p \triangleq p_1 + p_2 < 0$, U^* is

$$U^*(t, \omega, x) = \frac{x^p}{p}\frac{(-p_1)^{p_1-1}(-p_2)^{p_2-1}}{(-p)^{p-1}}(S^1_t)^{-p_1}(S^2_t)^{-p_2}, \qquad x > 0.$$

Define

$$G \triangleq \frac{(-p_1)^{p_1-1}(-p_2)^{p_2-1}}{(-p)^{p-1}}(S^1_T)^{-p_1}(S^2_T)^{-p_2}.$$

Then

$$U(T, \omega, x) = \frac{x^p}{p}G(\omega), \qquad x > 0.$$

Suppose that W^1 and W^2 are two Brownian motions with a fixed correlation ρ such that $0 < |\rho| < 1$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the usual augmentation of the filtration generated by W^1 and W^2, and $(\mathcal{G}_t)_{t\geq 0}$ be the usual augmentation of the filtration generated by W^2. We also assume that there is a bond B and a stock $\widetilde{S}$ on the market. Their dynamics are

$$\mathrm{d}\widetilde{S}_t = \widetilde{S}_t(\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W^1_t), \quad \widetilde{S}_0 \in \mathbb{R}, \qquad \mathrm{d}B_t = B_t r_t\,\mathrm{d}t, \quad B_0 = 1,$$

where the drift μ, volatility σ, and spot interest rate r are bounded and progressively measurable processes with respect to $(\mathcal{G}_t)_t \in [0, T]$, and σ is strictly positive.

Suppose that S^1_T and S^2_T are $\mathcal{G}_T$-measurable random variables with moments of all orders. Then G is also a $\mathcal{G}_T$-measurable random variable with moments of all orders (by Hölder's inequality), and the auxiliary value function u^* defined in (4.2) satisfies the settings of [25]. Also, as $u^*(x) \geq (x^p/p)\mathbb{E}[G] > -\infty$ and since $V(T, \omega, \cdot)$ is negative-valued (thus, $v(y) \leq 0$), assumption (2.9) holds.

Set

$$\delta \triangleq \frac{1-p}{1-p+\rho^2 p}, \qquad \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \triangleq \exp\biggl(-\frac{\rho^2 p^2}{2(1-p)^2}\int_0^T \lambda_s^2\,\mathrm{d}s + \frac{\rho p}{1-p}\int_0^T \lambda_s\,\mathrm{d}W_s^2\biggr),$$

$$\lambda_t \triangleq \frac{\mu_t - r_t}{\sigma_t}, \qquad K_t \triangleq \frac{p}{(1-p)}\biggl(\lambda_t + \rho\delta\frac{\beta_t}{\mathbb{E}^{\mathbb{Q}}[\exp(\int_0^T (r_s/\delta)\,\mathrm{d}s) \mid \mathcal{F}_t]}\biggr), \qquad t \in [0,T].$$

Then, using [25, Proposition 3.4] and Theorem 2.1, we deduce that

$$\widehat{c}_T^{\,i} = \frac{\widehat{c}_T^{\,*}(x)p_i}{pS_T^i}, \qquad i=1,2,\ x>0,$$

$$\widehat{c}_T^{\,*}(x) = x\exp\biggl(\int_0^T \Bigl(r + K_s\lambda_s - \frac{1}{2}K_s^2\Bigr)\,\mathrm{d}s + \int_0^T K_s\,\mathrm{d}W_s^1\biggr), \qquad x>0,$$

$$\begin{aligned}\widehat{Y}_T(y) = \frac{y}{\mathbb{E}[(\widehat{c}_T^{\,*}(1))^p]}\exp\biggl(&\int_0^T (p-1)\Bigl(r + K_s\lambda_s - \frac{1}{2}K_s^2\Bigr)\,\mathrm{d}s \\ &+ \int_0^T (p-1)K_s\,\mathrm{d}W_s^1\biggr), \qquad y>0,\end{aligned}$$

are the optimizers to (2.5), (2.8), and (4.2), respectively. From Theorem 2.1, we conclude that, for every $x>0$, $\widehat{c}_T^{\,i}(x)$, $i=1,2$, and $\widehat{Y}_T(u'(x))$ are related via (2.11) and (2.12).

4. Proofs

We begin with a characterization of the utility process U^* defined in (2.7).

Lemma 4.1. *Let U satisfy Assumption 2.1, and U^* be defined in (2.7). Then U^* is an Inada-type utility process for $m=1$ in the sense of Assumption 2.1.*

Proof. For every $(t,\omega) \in [0,\infty)\times\Omega$, and as $U^*(t,\omega,\cdot)$ is an image function under an appropriate linear transformation of a *concave* function $U(t,\omega,\cdot)$, we can use, for example, [7, Theorem B.2.4.2] to show that $U^*(t,\omega,\cdot)$ is concave. In order to show strict concavity of $U^*(t,\omega,\cdot)$, we proceed as follows. First, for some positive numbers $x_1 \neq x_2$, let $c^i \in \mathbb{R}_+^m$ be such that

$$c^i\cdot S_t(\omega) \le x_i, \qquad U^*(t,\omega,x_i) = U(t,\omega,c^i), \quad i=1,2. \tag{4.1}$$

The existence of these c^i follows from the compactness of the domain of the optimization problem in the definition of $U^*(t,\omega,x)$ (for every $x>0$) and the upper semicontinuity of $U(t,\omega,\cdot)$. Since, in (4.1), c^i, $i=1,2$, necessarily satisfies $c^i\cdot S_t(\omega)\le x_i$ with equality, from the strict monotonicity of $U(t,\omega,\cdot)$ in every spatial component and $x_1\neq x_2$, we deduce that $c^1\neq c^2$. Consequently, from the *strict* concavity of $U(t,\omega,\cdot)$, we obtain

$$\begin{aligned}U^*\bigl(t,\omega,\tfrac{1}{2}(x_1+x_2)\bigr) &= \sup_{c\in\mathbb{R}_+^m:\ c\cdot S_t(\omega)\le(x_1+x_2)/2} U(t,\omega,c) \\ &\ge U\bigl(t,\omega,\tfrac{1}{2}(c^1+c^2)\bigr) \\ &> \tfrac{1}{2}U(t,c^1) + \tfrac{1}{2}U(t,\omega,c^2) \\ &= \tfrac{1}{2}U^*(t,\omega,x_1) + \tfrac{1}{2}U^*(t,\omega,x_2).\end{aligned}$$

Therefore, $U^*(t,\omega,\cdot)$ is strictly concave. As $U^*(t,\omega,\cdot)$ is increasing and strictly concave, it is *strictly* increasing.

For every $(t, \omega) \in [0, \infty) \times \Omega$ and $x > 0$, using the Inada conditions for $U(t, \omega, \cdot)$, one can show that there exists c in the *interior* of the first orthant such that $c \cdot S_t(\omega) = x$ and $U^*(t, \omega, x) = U(t, \omega, c)$. As a result, the differentiability of $U^*(t, \omega, \cdot)$ (in the third argument) follows from the differentiability of $U(t, \omega, \cdot)$ and general properties of the subgradient of the image function; see, for example, [7, Corollary D.4.5.2]. As $U^*(t, \omega, \cdot)$ is concave and differentiable, we deduce that $U^*(t, \omega, \cdot)$ is *continuously* differentiable in the interior of its domain; see [7, Theorem D.6.2.4]. The Inada conditions for $U^*(t, \omega, \cdot)$ follow from the (version of the) Inada conditions for $U(t, \omega, \cdot)$ and [7, Theorem D.4.5.1, p. 192].

For every $(t, \omega) \in [0, \infty) \times \Omega$, and as $U(t, \omega, \cdot)$ is a closed concave function, using, for example, [21, Theorem 9.2, p. 75], we deduce that $U^*(t, \omega, \cdot)$ is also a *closed* concave function. (Note that, in general, the image of a closed convex or concave function under a linear transformation is not necessarily closed; see the discussion in [7, p. 97].) In particular, we have

$$U^*(t, \omega, 0) = \lim_{z \downarrow 0} U^*(t, \omega, z), \qquad (t, \omega) \in [0, \infty) \times \Omega.$$

Finally, for every $x \geq 0$, $U^*(\cdot, \cdot, x)$ is optional as a supremum of countably many optional processes (from the continuity of $U(t, \omega, \cdot)$ in the relative interior of its effective domain, it is enough to take the supremum—in the definition of $U^*(t, \omega, \cdot)$—over the m-dimensional vectors, whose components take only *rational* values). □

Remark 4.1. Lemma 4.1 asserts that U^* satisfies [19, Assumption 2.1].

For every $x > 0$, we denote by $\mathcal{A}^*(x)$ the set of one-dimensional optional processes c^*, for which there exists an $\mathbb{R}^d$-valued predictable $\widetilde{S}$-integrable process H such that

$$X_t \triangleq x + \int_0^t H_u \, \mathrm{d}\widetilde{S}_u - \int_0^t c_u^* \, \mathrm{d}\kappa_u, \qquad t \geq 0,$$

is nonnegative, $\mathbb{P}$-a.s. We also define

$$u^*(x) \triangleq \sup_{c^* \in \mathcal{A}^*(x)} \mathbb{E}\bigg[\int_0^\infty U(t, \omega, c_t^*(\omega)) \, \mathrm{d}\kappa_t(\omega)\bigg], \qquad x > 0, \tag{4.2}$$

with the convention that, analogous to (2.6),

$$\mathbb{E}\bigg[\int_0^\infty U^*(t, \omega, c_t^*(\omega)) \, \mathrm{d}\kappa_t(\omega)\bigg] \triangleq -\infty \quad \text{if } \mathbb{E}\bigg[\int_0^\infty U^{*-}(t, \omega, c_t^*(\omega)) \, \mathrm{d}\kappa_t(\omega)\bigg] = \infty.$$

Proof of Theorem 2.1. Let $x > 0$ be fixed, and $c \in \mathcal{A}(x)$. Then $c_t^* \triangleq c_t \cdot S_t$, $t \geq 0$, is an optional process such that $c^* \in \mathcal{A}^*(x)$. Therefore,

$$u^*(x) \geq u(x) > -\infty, \qquad x > 0. \tag{4.3}$$

Since U^* satisfies the assertions of Lemma 4.1, and using standard techniques in convex analysis, we see that $-V^*$ has the same properties as U^*. Therefore, optimization problems (2.8) and (4.2) satisfy the assumptions of [19, Theorem 3.2]. Consequently, the result of [19, Theorem 3.2] applies, which, in particular, asserts that u^* and v are finite-valued and that for every $x > 0$, the exists a strictly positive optional process $\widehat{c^*}(x)$, the unique maximizer to (4.2).

Consider

$$\sup_{x \in \mathbb{R}_+^m : \, x \cdot S_t(\omega) \leq \widehat{c}_t^*(x)(\omega)} U(t, \omega, x), \qquad (t, \omega) \in [0, \infty) \times \Omega, \tag{4.4}$$

and define a correspondence $\varphi\colon [0,\infty)\times\Omega \twoheadrightarrow \mathbb{R}^m$ as

$$\varphi(t,\omega) \triangleq \{x \in \mathbb{R}^m_+ : x\cdot S_t(\omega) \le \widehat{c}^*_t(x)(\omega)\}.$$

From the *strict* positivity of the S^k, and the positivity and $(\mathrm{d}\kappa\times\mathbb{P})$-a.e. finiteness of $\widehat{c}^*(x)$ (see [19, Theorem 3.2]), we deduce that φ has nonempty compact values $(\mathrm{d}\kappa\times\mathbb{P})$-a.e. (Note that the origin in $\mathbb{R}^m$ is in $\varphi(t,\omega)$ for every $(t,\omega)\in[0,\infty)\times\Omega$.) Consider the lower inverse of φ^l, i.e.

$$\varphi^l(G) \triangleq \{(t,\omega)\in[0,\infty)\times\Omega : \varphi(t,\omega)\cap G \neq \varnothing\}, \qquad G\subset\mathbb{R}^m.$$

Consider also a subset of $\mathbb{R}^m$ of the form $A \triangleq [a_1,b_1]\times\cdots\times[a_m,b_m]$, where a_i and b_i are real numbers. In view of the weak measurability of φ (see [1, Definition 18.1, p. 592]) that we are planning to show, it is enough to consider $b_i\ge 0$, $i=1,\dots,m$. In addition, set $\bar a_i = \max(0,a_i)$. We see that for such a set A, as

$$\varphi^l(A) = \varphi^l([\bar a_1,b_1]\times\cdots\times[\bar a_m,b_m])$$

with $\bar a \triangleq (\bar a_1,\dots,\bar a_m)$, we have

$$\varphi^l(A) = \{(t,\omega) : \bar a\cdot S_t(\omega) \le \widehat{c}^*_t(x)(\omega)\}.$$

As $\widehat{c}^*(x)$ and S^i are optional processes, and since $\varphi^l(\bigcup_{n\in\mathbb{N}} A_n) = \bigcup_{n\in\mathbb{N}}\varphi^l(A_n)$ (see [1, Section 17.1], where the A_n are subsets of $\mathbb{R}^m$), we deduce that $\varphi^l(G)\in\mathcal{O}$ for every open subset G of $\mathbb{R}^m$, i.e. φ is weakly measurable. As U is a Carathéodory function (see [1, Definition 4.50, p. 153]), we conclude using [1, Theorem 18.19, p. 605] to see that there exists an *optional* $\mathbb{R}^m$-valued process $\widehat{c}_t(x)$, $t\in[0,T]$, the maximizer of (4.4) for $(\mathrm{d}\kappa\times\mathbb{P})$-a.e., $(t,\omega)\in[0,\infty)\times\Omega$. The uniqueness of such a maximizer follows from the *strict* concavity of $U(t,\omega,\cdot)$ (for every $(t,\omega)\in[0,\infty)\times\Omega$). (See [1, Theorem 18.19, p. 605] for a maximizer, which is a measurable multifunction, and from the uniqueness of the maximizer it is a single-valued multifunction, for which the concept of measurability coincides with measurability for functions.) As $\widehat{c}^*(x)\in\mathcal{A}^*(x)$, we deduce that $\widehat{c}(x)\in\mathcal{A}(x)$. Combining this with (4.3), we conclude that $\widehat{c}(x)$ is the unique (up to an equivalence class) maximizer to (2.5).

For $x>0$, let $\widehat{c}^i_t(x)$, $i=1,\dots,m$, denote the components of $\widehat{c}_t(x)$. As $\widehat{c}_t(x)(\omega)\cdot S_t(\omega) = \widehat{c}^*_t(\omega)$, $(\mathrm{d}\kappa\times\mathbb{P})$-a.e., (where the argument here is similar to the discussion after (4.1)) relations (2.10) and (2.12)–(2.14) follow from [19, Theorem 3.2], whereas (2.15) results from [19, Theorem 3.3] (equivalently, from [3, Theorem 2.4]). In turn, combining (2.12) with [7, Theorem D.4.5.1], we conclude that

$$\begin{aligned}\widehat{Y}_t(\omega) &= U^*_x(t,\omega,\widehat{c^*_t}(x)(\omega))\\ &= \{s(t,\omega)\in\mathbb{R} : S^i_t(\omega)s(t,\omega)\\ &\qquad\qquad = \partial_{x_i}U(t,\omega,\widehat{c}_t(x)(\omega)),\ i=1,\dots,m\}, \qquad (\mathrm{d}\kappa\times\mathbb{P})\text{-a.e.,}\end{aligned}$$

i.e. (2.11) holds. □

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