PRICING OF CONTINGENT CLAIMS IN LARGE MARKETS

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Abstract. We consider the problem of pricing in large markets in a framework, where the large market limits the small ones with finitely many traded assets. We show that this framework allows accommodating utility-based pricing in stochastic utility settings and arbitrage-free pricing. Adopting a stochastic integration theory with respect to a sequence of semimartingales, we introduce the notion of utility-based prices for the large post-limit market and establish their existence, uniqueness, and relation to arbitrage-free prices. These results rely on a theorem of independent interest on utility maximization with a random endowment in a large market that we state and prove first. Further, we provide approximation results for the utility-based and arbitrage-free prices in the large market by those in small markets. In particular, our framework allows for pricing the asymptotically replicable claims, where we also show a consistency of the pricing methodologies and provide positive examples.

1. Introduction

The size and complexity of financial markets led to the appearance of modes via an infinite number of traded securities. Starting from the usual "small" models and supposing that the number of traded stocks is a finite but random number taking values in the set of natural numbers, one directly

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arrives at the assumption of the availability of countably many tradable assets on the market. Models of this type are considered in [BN98] and [DDGP05], and [Mos18], among others. Further, in the context of fixed income derivatives, it is natural to model interest rates with uncountably many traded instruments, see, e.g., [CT04], [RT06], [DDP05], and [ET05].

The mathematical foundations of the large markets go back to [KK94], where a large market was introduced as a sequence of models with finitely many traded assets. Later, relying on ideas of stochastic integration with respect to infinite-dimensional stochastic processes, characterizations of the large market themselves (post-limit) has appeared. The theory for such markets was further developed through a series of works including [KK98], [CKT16], [KS96b], [KS96a], [Kle00], [Kle03], [Kle06], [KLPO14], and [DDP05]. This topic still attracts researchers, see the recent works [ARS17], [RS18] and [Kar19], among others.

The results on large markets include those where the authors do not model the limiting market, that is, pre-limit results starting from the innovational [KK94], and this approach is still common. On the other hand, there are works where the limiting market is investigated, that is, post-limit results, see [CT04], [DDGP05], and [Kar19], among others. Investigating the post-limit models requires stochastic integration with respect to infinite-dimensional stochastic processes that is less developed than stochastic integration with respect to finite-dimensional semimartingales. Further, *completeness* is a common assumption in traditional interest rate modeling. Thus, pricing in the large (post-limits) market models in the context of fixed-income derivatives often inherits certain replicability assumptions see, e.g., [CT04, Assumption 5.1].

This paper focuses on two *pricing approaches in (fully) incomplete large markets*, without any apriori replicability assumptions, where modeling and establishing results in the large (post-limit) market itself is a significant part of our analysis. In particular, *in stochastic utility settings, we develop the utility-based pricing in the large (post-limit) market and show its consistency with the arbitrage-free pricing.* For this, first, we established a *utility maximization with random endowment theorem for the large market*, a result, which is interesting by itself. Further, we provided an approximation result by the utility-based prices in small markets, again with consistency to the arbitrage-free pricing asymptotics. Finally, we applied our results to the asymptotically replicable claims, where their pricing in the large market has a particularly nice structure. We note that in settings of exponential utility, the problem of utility-based pricing in the large (pre-limit) market has been considered in [ARS17].

On the technical level, as there are fewer stochastic analytic tools for studying stochastic integration with respect to an infinite-dimensional semimartingale, we had to deal with more obstacles. In particular, we do not use the Optional Decomposition Theorem that was crucial for optimal investment with a random endowment in a small market, see [HK04]. We note that the Optional Decomposition Theorem for the large market has been recently developed in [Kar19], under the continuity of the underlying stock price processes. We could not use it as our formulation deals with semimartingales, which might admit jumps. Further, our approach allows to *include the closures of the domains of the key optimization problems*. This is crucial for the proofs and allows for more complete characterizations of the underlying problems. Further, our formulation allows to *circumvent both the non-replicability and asymptotic replicability assumptions* often imposed in the literature, even for small markets (under appropriate adjustments). An application of such a formulation is *pricing of asymptotically replicable claims as a particular case* of our results. The latter *includes asymptotically complete markets*, where every claim is replicable or asymptotically replicable.

The remainder of this paper is organized as follows: in Section 2, we introduce the model. In Section 3, we establish the utility maximization results with random endowment in a large market. In Section 4, we introduce the notions of the utility-based price in a large market and prove its existence, and provide a condition for its uniqueness; in Section 5, we prove the convergence of the utility-based prices in small markets to the ones in the large market. In Section 6, we show an application of our settings to asymptotically replicable claims, where asymptotically complete market come as a particular case.

2. Model

We consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions, $\mathcal{F}_0$ is trivial. We suppose that there is one fixed market that consists of a riskless asses $S^0 \equiv 1$ and a sequence of risky asset $S = (S^n)_{n\in\mathbb{N}}$, where each S^n, $n \in \mathbb{N}$, is a semi-martingale that specifies the price of the nth risky asset. We also suppose that there is a non-traded contingent claim with payment process $(F^i)_{i=1}^N$. If $(q_i)_{i=1}^N = q$ is the finite sequence that specifies the number of such claims, the cumulative payoff is given by

$$qF := (qF_t)_{t\in[0,T]} = \left(\sum_{i=1}^N q_i F_t^i\right)_{t\in[0,T]}.$$

Both processes S and F are given exogenously.

The notion of the trading strategy in the large market is given as follows. For $n \in \mathbb{N}$, an *n-elementary strategy* is a (usual) $\mathbb{R}^n$-valued predictable and $(S^k)_{k=1,\dots,n}$-integrable process. An *elementary strategy* is a strategy, which is n-elementary for some $n \in \mathbb{N}$. Further, an n-elementary strategy H is *x-admissible* for a given $x \geq 0$, if

$$H \cdot S = \int_0^{\cdot} \sum_{k=1}^n H_t^k dS_t^k \geq -x, \quad \mathbb{P}\text{-a.s.}$$

Let $\mathcal{H}^n$ denote the set of n-elementary strategies, which are x-admissible for some $x \geq 0$, and by $\mathcal{H}$ the set of admissible elementary strategies.

To pass to the limit as $n \to \infty$, we will follow [DDP06], and we recall that $\mathbb{R}^{\mathbb{N}}$ is the space of real-valued sequences. An *unbounded functional* on $\mathbb{R}^{\mathbb{N}}$ is a linear functional $\bar{H}$, whose domain, $Dom(\bar{H})$ is a subspace of $\mathbb{R}^{\mathbb{N}}$. A *simple integrand* is a fined sum of bounded predictable processes of the form $\sum\limits_{k=1}^n h^k e^k$, where $(e^k)_{k\in\mathbb{N}}$ is a canonical basis for $\mathbb{R}^{\mathbb{N}}$ and h^k's are one-dimensional bounded and predictable processes.

A process H with values in the set of unbounded functionals on $\mathbb{R}^{\mathbb{N}}$ is *predictable* if there is a sequence of simple integrands (as defined in the previous paragraph) $(H^n)_{n\in\mathbb{N}}$, such that $H = \lim\limits_{n\to\infty} H^n$, pointwise, in the sense that for every $x \in Dom(H)$, the sequence $(H^n(x))_{n\in\mathbb{N}}$ converges to $H(x)$ as $n \to \infty$.

A predictable process H with values in the set of unbounded functionals on $\mathbb{R}^{\mathbb{N}}$ is *integrable* with respect to S if there is a sequence $(H^n)_{n\in\mathbb{N}}$ of simple integrands, such that $(H^n)_{n\in\mathbb{N}}$ converges to H pointwise and the sequence of semimartingales $(H^n \cdot S)_{n\in\mathbb{N}}$ converges to a semimartingale Y in the semimartingale topology. In this case, we set

$$H \cdot S := Y.$$

To put the concept of a stochastic integral as above in the context of optimal investment, we further need to specify the context of admissibility. Thus, for $x \geq 0$, we say that a predictable process with values in the set of unbounded functionals in an *x-admissible generalized strategy* if H is integrable with respect to S and there is an approximating sequence of x-admissible elementary strategies, $(H^n \cdot S)_{n\in\mathbb{N}}$, that converges to $H \cdot S$ in the semimartingale topology.

We suppose that an economic agent can trade in such a market. A *portfolio* is defined as a triple (x, H, q), where a constant $x \geq 0$ is an initial capital, and H is an x-admissible generalized strategy, and q is the number of shares of the non-traded contingent claim. The wealth process $V = (V_t)_{t\in[0,T]}$ generated by the portfolio (x, H, q) is given by

$$V_t = x + H \cdot S_t + qF_t, \quad t \in [0, T].$$

A collection of nonnegative wealth processes generated by x-admissible generalized strategies is denoted by $\mathcal{X}(x)$, that is

$$\mathcal{X}(x) := \{X \geq 0 : \ X_t = x + H \cdot S_t, \quad t \in [0, T]\}, \quad x \geq 0.$$

To rule out suicide strategies in the context of non-traded contingent claim on a large market, we need to introduce the notions of maximality and acceptability, extending the ones in [DS97]. In the context of a large market, we also refer to [Kar19, p. 31] for the definition of maximal processes (at T). Thus, we call that a nonnegative process in $X \in \mathcal{X}(x)$ is *maximal* if its terminal value cannot be dominated by any other process in $\mathcal{X}(x)$, that is if $\tilde{X} \in \mathcal{X}(x)$ is such that $\tilde{X}_T \geq X$, $\mathbb{P}$-a.s., then we have $\tilde{X}_T = X_T$, $\mathbb{P}$-a.s. An *acceptable* process is the one of the form $X = X' - X''$, where X' and X'' are (nonnegative) wealth processes in $\bigcup\limits_{x\geq 0} \mathcal{X}(x)$ and X'' is maximal.

In the large market, we denote the family of acceptable processes starting from $x \in \mathbb{R}$ by $\tilde{\mathcal{X}}(x)$. Similarly, we denote the families of acceptable processes in small markets by $\tilde{\mathcal{X}}^n(x)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. Here the definition of acceptability is consistent with the one in the literature for models with finitely many traded assets, see, e.g., [DS97] and [HK04].

Primal problem

For a nontradable contingent claim(s) $f = F_T$ with payments f^i, $i \in \{1, \dots, N\}$, at T, and with $q = (q_i)_{i=1}^N$, for every $(x, q) \in \mathbb{R}^{N+1}$, we set

$$\begin{aligned} \mathcal{X}^n(x,q) &:= \left\{X \in \tilde{\mathcal{X}}^n(x) : \ X_T + qf \geq 0\right\}, \quad n \in \mathbb{N}, \\ \mathcal{X}(x,q) &:= \left\{X \in \tilde{\mathcal{X}}(x) : \ X_T + qf \geq 0\right\}. \end{aligned} \tag{2.1}$$

One can see that $\mathcal{X}^n(x,q) \subseteq \mathcal{X}^{n+1}(x,q) \subseteq \mathcal{X}(x,q)$ and these sets can be empty for some (x,q)'s. Therefore, we set

$$\begin{aligned} \mathcal{K}^n &:= \left\{(x,q) \in \mathbb{R}^{N+1} : \mathcal{X}^n(x,q) \neq \emptyset\right\}, \quad n \in \mathbb{N}, \\ \mathcal{K} &:= \left\{(x,q) \in \mathbb{R}^{N+1} : \mathcal{X}(x,q) \neq \emptyset\right\}. \end{aligned} \tag{2.2}$$

The preferences of an economic agent are given by a utility stochastic field

$$U = U(\omega, x) : \Omega \times [0, \infty) \to \mathbb{R} \cup \{-\infty\}.$$

We suppose that U satisfies the following assumption.

Assumption 2.1. For every $\omega \in \Omega$, the function $x \to U(\omega, x)$ is strictly concave, strictly increasing, continuously differentiable on $(0, \infty)$, and satisfies the Inada conditions:

$$\lim_{x \downarrow 0} U'(\omega, x) = \infty \quad \textit{and} \quad \lim_{x \uparrow \infty} U'(\omega, x) = 0, \tag{2.3}$$

where U' denotes the partial derivative with respect to the second argument. At $x = 0$, we suppose by continuity $U(\omega, 0) = \lim_{x \downarrow 0} U(\omega, x)$, which may be $-\infty$.

Controlling the investment, the goal of an agent is to maximize the expected utility. The value functions are given by

$$u^n(x,q) = \sup_{X \in \mathcal{X}^n(x,q)} \mathbb{E}\left[U\left(X_T + qF_T\right)\right], \quad (x,q) \in \mathcal{K}^n, \quad n \in \mathbb{N}, \tag{2.4}$$

$$u(x,q) = \sup_{X\in\mathcal{X}(x,q)} \mathbb{E}\left[U\left(X_T + qF_T\right)\right], \quad (x,q)\in\mathcal{K}. \tag{2.5}$$

Here and below, we will use the following convention

$$if \quad \mathbb{E}\left[U^-(X_T+qf)\right] = \infty, \quad we\ set \quad \mathbb{E}\left[U(X_T+qf)\right] := -\infty.$$

If will be convenient to extend the definitions of u^n's and u to $\mathbb{R}^{N+1}$, by setting

$$u^n(x,q) := -\infty, \quad (x,q)\in\mathbb{R}^{N+1}\backslash\mathcal{K}^n, \quad n\in\mathbb{N}, \quad and \quad u(x,q) := -\infty,$$
$$(x,q)\in\mathbb{R}^{N+1}\backslash\mathcal{K}.$$

To ensure that the utility maximization problems (2.4) and (2.5) are non-degenerate, we need to impose no-arbitrage conditions. With $\mathcal{Z}^n$ denoting the set of the densities of the equivalent local martingale measures for S^n, we suppose that

$$\bigcap_{n\geq 1}\mathcal{Z}^n =: \mathcal{Z} \neq \emptyset. \tag{no-Arb}$$

For the contingent claim, we will suppose that

Assumption 2.2. Every component of f is bounded.

Remark 2.3 (On the boundedness Assumption 2.2). In many papers, see e.g., [HK04] and [MS20], it is assumed that the contingent claim is bounded by some wealth process in some (typically small) market. Mathematically this amounts to supposing that $|f|\leq C\tilde{X}$, for some maximal $\tilde{X}\in\mathcal{X}^n(1,0)$, for some $n\in\mathbb{N}$. This in particular, allows for pricing contingent claims unbounded from above and below. In this remark, we show that our settings with stochastic utility are closely related to the ones with an unbounded contingent claim.

Let us suppose that one starts from a utility $\tilde{U}$ satisfying Assumption 2.1 (possibly deterministic as in [HK04], but this assumption does not have to be imposed), and a contingent claim $\tilde{f}$ satisfying

$$|\tilde{f}|\leq C\tilde{X}, \quad for\ some\ positive\ maximal\ \tilde{X}\in\mathcal{X}^n(1,0), \quad and \quad C>0. \tag{2.6}$$

Then one can write

$$\tilde{U}(x + H\cdot S_T + q\tilde{f}) = \tilde{U}\left(\tilde{X}_T\left(x + H\cdot S_T^{\tilde{X}} + q\frac{\tilde{f}}{\tilde{X}_T}\right)\right),$$

where $S^{\tilde{X}} := \left(\frac{1}{\tilde{X}}, \frac{S}{\tilde{X}}\right)$. Setting

$$f := \frac{\tilde{f}}{\tilde{X}_T} \quad and \quad U(\omega, x) := \tilde{U}(\omega, \tilde{X}_T(\omega)x), \quad (\omega, x) \in \Omega \times [0, \infty),$$

one can see from (2.6) that $|f| \leq C$, and U satisfies Assumption 2.1. Under the local boundedness of the components of S, if $\tilde{X}_T > 0$ and $\tilde{X}$ is maximal, then [DS95, Theorem 13] implies that $\left(\frac{S}{\tilde{X}}, \frac{1}{\tilde{X}}\right)$ admits local martingale measure. Further, the set

$$\tilde{Z}^n := \{\tilde{X}Z: \ Z \in \mathcal{Z}^n \ and \ \tilde{X}Z \ is \ a \ martingale\}$$

is dense in $\tilde{X}\mathcal{Z}^n$ with respect to the topology of convergence in $\mathbb{L}^1$, see [DS97, Theorem 5.2]. And thus, to treat the unbounded $\tilde{f}$, we would have to impose $\bigcap\limits_{n \in \mathbb{N}} \tilde{\mathcal{Z}}^n \neq \emptyset$.

It follows from Assumption 2.2 that

$$(x, 0) \in int\mathcal{K}^1, \quad for \ every \ x > 0. \tag{2.7}$$

Dual problem

We begin by recalling that in small markets

$$\mathcal{L}^n := -(\mathcal{K}^n)^o, \quad n \in \mathbb{N},$$

that is, the polars of $-\mathcal{K}^n$ in $\mathbb{R}^{N+1}$. Naturally, we extend this definition to the large markets by setting

$$\mathcal{L} := -\mathcal{K}^o.$$

We recall that

$$\begin{aligned} \mathcal{Y}^n(y) = \{&Y \geq 0: \ Y_0 = y, \\ &and \ XY \ is \ a \ supermartingale \ for \ every \ X \in \mathcal{X}^n(1,0)\}, \\ & \qquad\qquad\qquad\qquad\qquad y > 0, \quad n \in \mathbb{N}. \end{aligned} \tag{2.8}$$

We set

$$\begin{aligned} \mathcal{Y}^n(y,r) := \{&Y \in \mathcal{Y}^n(y): \ \mathbb{E}\left[Y_T(X_T + qf)\right] \leq xy + qr, \\ &for \ every \ (x,q) \in \mathcal{K}^n, \ and \ X \in \mathcal{X}^n(x,q)\}, \\ & \qquad\qquad\qquad\qquad (y,r) \in \mathcal{L}^n. \end{aligned} \tag{2.9}$$

Similarly, in the large market, we define

$$\begin{aligned}\mathcal{Y}(y) = \{Y \geq 0 :\ Y_0 = y \textit{ and } XY \textit{ is a supermartingale} \\ \textit{for every } X \in \mathcal{X}(1,0)\}, \quad y > 0,\end{aligned} \tag{2.10}$$

(def Y(y,r))

$$\begin{aligned}\mathcal{Y}(y,r) := \{Y \in \mathcal{Y}(y) :\ \mathbb{E}\left[Y_T(X_T + qf)\right] \leq xy + qr, \\ \textit{for every } (x,q) \in \mathcal{K}, \textit{ and } X \in \mathcal{X}(x,q)\}, \quad (y,r) \in \mathcal{L}.\end{aligned}$$

Let us set

$$V(\omega, y) := \sup_{x>0} \left(U(\omega, x) - xy\right), \quad (\omega, y) \in \Omega \times [0, \infty). \tag{2.11}$$

We note that $-V$ satisfies Assumption 2.1. Now, we can state the dual problems for finite markets and the large market.

$$v^n(y,r) = \inf_{Y \in \mathcal{Y}^n(y,r)} \mathbb{E}\left[V\left(Y_T\right)\right], \quad (y,r) \in \mathcal{L}^n, \quad n \in \mathbb{N}, \tag{2.12}$$

$$v(y,r) = \inf_{Y \in \mathcal{Y}(y,r)} \mathbb{E}\left[V\left(Y_T\right)\right], \quad (y,r) \in \mathcal{L}, \tag{2.13}$$

where we use the convention

$$\mathbb{E}\left[V\left(Y_T\right)\right] := \infty, \quad if \quad \mathbb{E}\left[V^+\left(Y_T\right)\right] = \infty.$$

Further, we will extend the definitions of v^n's and v to $\mathbb{R}$ by setting

$$\begin{aligned}v^n(y,r) := \infty, \quad (y,r) \in \mathbb{R}^{N+1} \backslash \mathcal{L}^n, \quad n \in \mathbb{N}, \quad \textit{and} \quad v(y,r) := \infty, \\ (y,r) \in \mathbb{R}^{N+1} \backslash \mathcal{L}.\end{aligned}$$

Let us set

$$\tilde{w}(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[V(Y_T)\right], \quad y > 0, \tag{2.14}$$

and suppose that

(fin uv)
$$u(x,0) > -\infty, \quad x > 0, \quad \textit{and} \quad \tilde{w}(y) < \infty, \quad y > 0.$$

3. Utility maximization with random endowment in large market

Theorem 3.1. *Let us suppose that Assumpitons 2.1, 2.2,* (no-Arb)*, and* (fin uv) *hold. Then, we have*

(i) The functions u and v are finite on int$\mathcal{K}$ and ri$\mathcal{L}$, respectively, u and v satisfy

$$
\begin{aligned}
u(x,q) &= \inf_{(y,r)\in\mathcal{L}} \left(v(y,r) + xy + qr\right), \quad (x,q)\in\mathcal{K},\\
v(y,r) &= \sup_{(x,q)\in\mathcal{K}} \left(u(x,q) + xy + qr\right), \quad (y,r)\in\mathcal{L}.
\end{aligned}
\tag{3.1}
$$

(ii) The functions u and $-v$ are concave and upper semi-continuous, $u < \infty$ on $\mathcal{K}$. For every $(x,q) \in \{u > -\infty\}$, there exists a unique maximizer to (2.5)*. In turn, $v > -\infty$ on $\mathcal{L}$. For every $(y,r) \in \{v < \infty\}$, there exists a unique solution to* (2.13)*.*

(iii) For every $(x,q) \in$ int$\mathcal{K}$, the subdifferential of u at (x,q) is a nonempty subset of ri$\mathcal{L}$, $(y,r) \in \partial u(x,q)$ if and only if the following conditions hold:

$$\hat{Y}_T(y,r) = U'(\omega, \hat{X}_T(x,q) + qf), \quad \mathbb{P}\text{-}a.s., \tag{3.2}$$

$$\mathbb{E}\left[\hat{Y}_T(\hat{X}_T + qf)\right] = xy + qr, \tag{3.3}$$

$$|v(y,r)| < \infty. \tag{3.4}$$

Structure of the domains to (2.4), (2.5), (2.12), and (2.13)

For every (x,q) and (y,r) in $\mathbb{R}^{N+1}$, let us set

(3.5)

$$
\begin{aligned}
\mathcal{C}^n(x,q) &:= \left\{g\in\mathbb{L}^0_+ : g \le X_T + qf \ \textit{for some}\ X\in\mathcal{X}^n(x,q)\right\}, \quad n\in\mathbb{N},\\
\mathcal{C}(x,q) &:= \left\{g\in\mathbb{L}^0_+ : g \le X_T + qf \ \textit{for some}\ X\in\mathcal{X}(x,q)\right\},\\
\mathcal{D}^n(y,r) &:= \left\{h\in\mathbb{L}^0_+ : h \le Y_T \ \textit{for some}\ Y\in\mathcal{Y}^n(y,r)\right\}, \quad n\in\mathbb{N},\\
\mathcal{D}(y,r) &:= \left\{h\in\mathbb{L}^0_+ : h \le Y_T \ \textit{for some}\ Y\in\mathcal{Y}(y,r)\right\}.
\end{aligned}
$$

For small markets, we recall [HK04, Proposition 1], whose intricate proof was based on a delicate parametrization of the dual domain, the Optional Decomposition Theorem from [Kra94], and superreplications results for finite-dimensional models from [DS98].

Proposition 3.2. *Let us assume that* (no-Arb) *and Assumption 2.2. Then for every $n \in \mathbb{N}$, the families $(\mathcal{C}^n(x,q))_{(x,q)\in\mathcal{K}^n}$ and $(\mathcal{D}^n(y,r))_{(y,r)\mathcal{L}^n}$ defined in* (3.5) *have the following properties:*

(1) *For every $(x,q) \in int\mathcal{K}^n$, the set $\mathcal{C}^n(x,q)$ contains a strictly positive constant. A nonnegative random variable g belong to $\mathcal{C}^n(x,q)$ if and only if*

$$\mathbb{E}\left[gh\right] \le xy + qr, \quad for\ every\ (y,r) \in \mathcal{L}^n\ and\ h \in \mathcal{D}^n(y,r)$$

(2) *For every $(y,r) \in ri\mathcal{L}^n$, the set $\mathcal{D}^n(y,r)$ contains a strictly positive random variable. A nonnegative function $h \in \mathcal{D}^n(y,r)$ if and only if*

$$\mathbb{E}\left[gh\right] \le xy + qr, \quad for\ every\ (x,q) \in \mathcal{K}^n\ and\ g \in \mathcal{C}^n(x,q).$$

Here is an analogous proposition, but for the large market.

Proposition 3.3. *Let us assume that* (no-Arb) *and Assumption 2.2. Then for every the families $(\mathcal{C}(x,q))_{(x,q)\in\mathcal{K}}$ and $(\mathcal{D}(y,r))_{(y,r)\mathcal{L}}$ defined in* (3.5) *have the following properties:*

(i) *For every $(x,q) \in int\mathcal{K}$, the set $\mathcal{C}(x,q)$ contains a strictly positive constant. A nonnegative random variable g belong to $\mathcal{C}(x,q)$ if and only if*

$$\mathbb{E}\left[gh\right] \le xy + qr, \quad for\ every\ (y,r) \in \mathcal{L}\ and\ h \in \mathcal{D}(y,r) \tag{3.6}$$

(ii) *For every $(y,r) \in ri\mathcal{L}$, the set $\mathcal{D}(y,r)$ contains a strictly positive random variable. A nonnegative function $h \in \mathcal{D}(y,r)$ if and only if*

$$\mathbb{E}\left[gh\right] \le xy + qr, \quad for\ every\ (x,q) \in \mathcal{K}\ and\ g \in \mathcal{C}(x,q). \tag{3.7}$$

We summarize the characterizations of Assumption 2.2 in the following lemma. Here and below, we will also use the notation $\mathcal{M}$ and $\mathcal{M}^n$'s for the sets of probability measures, whose densities are in $\mathcal{Z}$ and $\mathcal{Z}^n$'s, respectively. Both kinds of notations are so common in the literature that we believe this will cause no confusion.

Lemma 3.4. *Let us assume that* (no-Arb) *and Assumption 2.2. Then, we have*

(i) *$(x,0) \in int\mathcal{K}$ for every $x > 0$,*
(ii) *for every $q \neq 0$, there exists $x > 0$, such that $(x,q) \in int\mathcal{K}$,*

(iii) (trivially) there exists a nonnegative wealth process of a generalized strategy, such that $X_T \geq \sum_{i=1}^{N} |f^i|$,

(iv) (trivially) $\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{N} |f^i|\right] < \infty$.

Proof of proposition 3.3 for the large market only

We begin with the following characterization of the set $\mathcal{K}$.

Lemma 3.5. *Under the conditions of Proposition 3.3,* $\mathcal{K}$ *is closed in* $\mathbb{R}^{N+1}$*, and thus, for every* $(x,q) \in cl\mathcal{K}$, $\mathcal{X}(x,q) \neq \emptyset$.

Proof. Note that the proof bypasses the Optional Decomposition Theorem, which was the center in the analysis of optimal investment with random endowment finite market, see [HK04].

Let $(x^n, q^n) \in \mathcal{K}$, $n \in \mathbb{N}$, be a sequence convergent to $(x,q) \in cl\mathcal{K}$, where the closure is taken in $\mathbb{R}^{N+1}$. Let us consider $X^n \in \mathcal{X}(x^n, q^n)$, $n \in \mathbb{N}$, and let us denote $Z^n := X^n + C$, $n \in \mathbb{N}$, where C is a sufficiently large positive constant such that $Z^n_T \geq 0$, for every $n \in \mathbb{N}$ (e.g., $C = N \max_{n\in\mathbb{N}} ||q^n||_\infty \max_{i\in\{1,\dots,N\}} ||f_i||_\infty$). One can see that Z^n are nonnegative $\mathbb{Q}$-supermartingales for every $\mathbb{Q} \in \mathcal{M}$ (see also [DDGP05, p. 2011]). By passing to convex combinations, which we do not relabel, and to Fatou-convergence under any of such $\mathbb{Q}$'s, we can obtain a process Z, a Fatou-limit of $(Z^n)_{n\in\mathbb{N}}$ on the set of rational numbers and T. By construction, we have

$$Z_T - C + qf = \lim_{n\to\infty}(Z^n_T - C + q^n f) = \lim_{n\to\infty}(X^n_T + q^n f) \geq 0. \tag{3.8}$$

We also have

$$Z_0 \leq \liminf_{n\to\infty} Z^n_0 = x + C.$$

Therefore, as Z is a supermartingale for every $\mathbb{Q} \in \mathcal{M}$, we deduce that

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[Z_T\right] \leq Z_0 \leq x + C.$$

Now, [DDGP05, Theorem 3.1] implies that there exists an admissible generalized strategy H, such that

$$x + C + H \cdot S_T \geq Z_T, \quad \mathbb{P}\text{-a.s.} \tag{3.9}$$

Let us set

$$X := x + H \cdot S.$$

Then, using (3.8) and (3.9), we have that

$$X_0 = x \quad \textit{and} \quad X_T + qf = x + C + H \cdot S_T + qf - C \geq Z_T + qf - C \geq 0,$$

where, in the first inequality, we have used (3.9), and, in the second,(3.8). We deduce that $X \in \mathcal{X}(x, q)$ and thus $(x, q) \in \mathcal{K}$. We conclude that $\mathcal{K}$ is closed. □

Let us consider the following parametrization of the dual domain.

$$\mathcal{M}(\rho) := \{\mathbb{Q} \in \mathcal{M} : \ \mathbb{E}_{\mathbb{Q}}[f] = \rho\}, \quad \rho \in \mathbb{R}^N. \tag{3.10}$$

Let us set

$$\mathcal{P}' := \{\rho \in \mathbb{R}^N : \ \mathcal{M}(\rho) \neq \emptyset\} \quad \textit{and} \quad \mathcal{P} := \{\rho \in \mathbb{R}^N : \ (1, \rho) \in ri\mathcal{L}\}. \tag{3.11}$$

For the proofs below, we will impose the following non-replicability assumption, which allows us to handle the most difficult case. On the other hand, the cases when some of the components of f are replicable can be handled by reducing the dimensionality of the problem, and if all components of f are replicable, we can analyze (2.5) via the results from the optimal investment without random endowment, see, e.g., [DDGP05] and [Mos18], see also the discussion in Section 6.

Assumption 3.6. We will suppose that every component of f is non-replicable in the following sense: for every $q \in \mathbb{R}^N$, such that $q \neq 0$, the random variable qf is not replicable in the large market. We note that this condition is equivalent to $\mathcal{L}$ being open in $\mathbb{R}^{N+1}$.

Lemma 3.7. *Under the conditions or Proposition 3.3 and Assumption 3.6, we have*

$$\mathcal{P}' = \mathcal{P}. \tag{3.12}$$

and

$$\bigcup_{\rho \in \mathcal{P}} \mathcal{M}(\rho) = \mathcal{M}.$$

Proof. Let us fix $q \in \mathcal{L}$, such that $q \neq 0$, and consider qf. One can see that Assumption 3.6 implies that there exists $X \in \mathcal{X}(x, q)$, such that

$$\mathbb{P}[X_T + qf > 0] > 0.$$

Then, for $\mathbb{Q} \in \mathcal{M}(\rho)$, using the supermartingale property of X under $\mathbb{Q}$, we have

$$0 < \mathbb{E}_{\mathbb{Q}}\left[X_T + qf\right] \leq x + q\rho.$$

As (x, q) is arbitrary in $\mathcal{K}$, we conclude that $\rho \in \mathcal{P}$. Therefore, we get

$$\mathcal{P}' \subseteq \mathcal{P}. \tag{3.13}$$

On the other hand, for a fixed $q \in \mathbb{R}^N$ and $x := \sup\limits_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[qf\right]$, by [DDGP05, Theorem 3.1], there exists an admissible generalized strategy H, such that

$$qf \leq x + H \cdot S_T.$$

This implies that $x + H \cdot S \in \mathcal{X}(x, -q)$, therefore, $(x, -q) \in \mathcal{K}$. As a result, we have

$$q\rho \leq x, \quad \rho \in \mathcal{P}.$$

We deduce that

$$\sup_{\rho \in \mathcal{P}'} q\rho = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[qf\right] = x \geq \sup_{\rho \in \mathcal{P}} q\rho.$$

As q is arbitrary, we conclude that

$$\mathcal{P}' \supseteq \mathcal{P}. \tag{3.14}$$

Combining (3.13) and (3.14), we deduce that

$$\mathcal{P}' = \mathcal{P},$$

and thus

$$\mathcal{M} = \bigcup_{\rho \in \mathcal{P}'} \mathcal{M}(\rho) = \bigcup_{\rho \in \mathcal{P}} \mathcal{M}(\rho).$$

□

Lemma 3.8. *Under the conditions or Proposition 3.3, for every $(x, q) \in \mathcal{K}$, $g \in \mathcal{C}(x, q)$ if and only if*

$$\mathbb{E}_{\mathbb{Q}}\left[g\right] \leq x + q\rho, \quad for\ every\ \rho \in \mathcal{P} \quad and \quad \mathbb{Q} \in \mathcal{M}(\rho). \tag{3.15}$$

Proof. Let us consider a nonnegative random variable g, such that (3.15) holds. Denote

$$h := g - qf.$$

Then boundedness of f implies that $h \geq -C$ for some constant $C > 0$. Therefore, we have

$$\begin{aligned}\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[h+C\right] &= \sup_{\rho\in\mathcal{P}} \sup_{\mathbb{Q}\in\mathcal{M}(\rho)} \mathbb{E}_{\mathbb{Q}}\left[h+C\right] \\ &= \sup_{\rho\in\mathcal{P}} \sup_{\mathbb{Q}\in\mathcal{M}(\rho)} \mathbb{E}_{\mathbb{Q}}\left[g-qf+C\right] \leq x+C.\end{aligned} \tag{3.16}$$

As $h+C \in \mathbb{L}^0_+$, [DDGP05, Theorem 3.1] implies the existence of the $(x+C)$-admissible generalized strategy H, such that

$$h + C \leq x + C + H \cdot S_T,$$

and thus

$$0 \leq g \leq x + H \cdot S_T + qf.$$

We deduce that $g \in \mathcal{C}(x,q)$.

Conversely, let $g \in \mathcal{C}(x,q)$. One can see that, for every $\rho \in \mathcal{P}$, the density process of $\mathbb{Q} \in \mathcal{M}(\rho)$ belongs to $\mathcal{Y}(1,\rho)$. This implies (3.15).

□

Proof of Proposition 3.3. Let $(x,q) \in \mathrm{int}\mathcal{K}$. Then there exists $\varepsilon > 0$, such that $(x-\varepsilon,q) \in \mathcal{K}$. Now, let us pick $X \in \mathcal{X}(x-\varepsilon,q)$, then $X+\varepsilon \in \mathcal{X}(x,q)$, and

$$X + \varepsilon + qf \geq \varepsilon > 0.$$

Therefore, $\varepsilon \in \mathcal{C}(x,q)$, and thus $\mathcal{C}(x,q)$ contains a positive constant.

If $g \in \mathcal{C}(x,q)$, then (3.6) follows from the construction of the sets $\mathcal{D}(y,r), (y,r) \in \mathcal{L}$. Conversely, let us assume that $g \in \mathbb{L}^0_+$, such that (3.6) holds. As, for every $\rho \in \mathcal{P}$, the density process of $\mathbb{Q} \in \mathcal{M}(\rho)$ belongs to $\mathcal{Y}(1,\rho)$, we deduce that g satisfies (3.15). Now, by Lemma 3.8, $g \in \mathcal{C}(x,q)$.

For the item (ii), first it is enough to prove the assertion for $(y,r) = (1,\rho)$ for some $\rho \in \mathcal{P}$, as $c\mathcal{D}(y,r) = \mathcal{D}(cy,cr)$ for every $c > 0$ and $(y,r) \in \mathcal{L}$. By Lemma 3.7, for every $\rho \in \mathcal{P}$, there exists $\mathbb{Q} \in \mathcal{M}(\rho)$. The density process $\mathbb{Q} \in \mathcal{M}(\rho)$ belongs to $\mathcal{Y}(1,\rho)$. As $\mathbb{Q} \sim \mathbb{P}$, $\frac{d\mathbb{Q}}{d\mathbb{P}} > 0$, $\mathbb{P}$-a.s.

If $h \in \mathcal{D}(1,\rho)$, then (3.7) follows from the definition of the set $\mathcal{Y}(1,\rho)$. Conversely, let us consider $h \in \mathbb{L}^0_+$, such that (3.7) holds. Then, in particular, we have

$$\mathbb{E}\left[gh\right] \leq 1, \quad for\ every\ g \in \mathcal{C}(1,0),$$

where the set $\mathcal{C}(1,0) \neq \emptyset$ by Lemma 3.4. Therefore, by [Mos18, Lemma 3.4], h is a terminal value of an element of $\mathcal{Y}(1)$ and is such that (3.7) holds, i.e., $h \in \mathcal{Y}(1,\rho)$. □

Proving the utility maximization Theorem 3.1 for large markets

The proof of the following lemma is an adaptation of the proof of [Mos17, Lemma 2.6], and it is skipped.

Lemma 3.9. *Under the conditions of Theorem 3.1, we have*

$$u(x,q) > -\infty, \quad (x,q) \in int\mathcal{K} \quad and \quad v(y,r) < \infty, \quad (y,r) \in ri\mathcal{L}. \tag{3.17}$$

Lemma 3.10. *Under the conditions of Theorem 3.1, we have*

$$u(x,q) \leq v(y,r) + xy + qr, \quad for\ every\ (x,q) \in \mathcal{K}, \quad and\ every\ (y,r) \in \mathcal{L}. \tag{3.18}$$

As a consequence, we have

$$u(x,q) < \infty, \quad on\ \mathbb{R}^{N+1}, \quad v(y,r) > -\infty, \quad on\ \mathbb{R}^{N+1}. \tag{3.19}$$

Proof. Let us fix $(x,q) \in \mathcal{K}$ and $(y,r) \in \mathcal{L}$. Then, for every $X \in \mathcal{X}(x,q)$ and $Y \in \mathcal{Y}(y,r)$, we have

$$U(X_T + qf) \leq V(Y_T) + (X_T + qf)Y_T, \quad \mathbb{P}\text{-a.s.},$$

and thus, taking the expectation and recalling (2.10), we obtain

$$\begin{aligned} \mathbb{E}\left[U(X_T + qf)\right] &\leq \mathbb{E}\left[V(Y_T)\right] + \mathbb{E}\left[(X_T + qf)Y_T\right] \\ &\leq \mathbb{E}\left[V(Y_T)\right] + xy + qr. \end{aligned} \tag{3.20}$$

As X and Y are arbitrary elements of $\mathcal{X}(x,q)$ and $\mathcal{Y}(y,r)$, taking the supremum over all $X \in \mathcal{X}(x,q)$ and (then) taking the infimum over all $Y \in \mathcal{Y}(y,r)$, in (3.20), we get

$$u(x,q) \leq v(y,r) + xy + qr, \tag{3.21}$$

which is precisely (3.18). In turn, (3.19) follows from (3.18) and Lemma 3.9. □

Proof of Theorem 3.1. The proof is an adaptation of the closely related proof of [Mos17, Theorem 2.4]. We will only highlight one point: for showing that $\partial u(x,q) \subset ri\mathcal{L}$, for $(x,q) \in int\mathcal{K}$, one can observe that, in the fully non-replicable case (as the one in Assumption 3.6), $0 < U'(\omega, \hat{X}_T + qf)$ belongs to $\mathcal{D}(y',r')$ for every $(y',r') \in \partial u(x,q)$, and then, one can show that $(y',r') \in ri\mathcal{L}$. □

4. Utility-based pricing in the large market

We consider the following definition.

Definition 4.1. Let $f^i \in \mathbb{L}^0$, $i \in \{1, \dots, N\}$, and $x > 0$. A vector $\rho \in \mathbb{R}^N$ is a *marginal utility-based price* for f given the initial capital x, if

$$\mathbb{E}\left[U(X_T + qf)\right] \leq u(x,0), \quad q \in \mathbb{R}^N,\ X \in \mathcal{X}(x - q\rho, q). \tag{4.1}$$

We denote the set of utility-based prices by $\Pi(x)$.

This definition is a natural extension of standard definitions of the UBPs in the literature (see, e.g., [HKS05, Definition 3.1]) to stochastic utility and a large market. Let us observe that given (2.5), our formulations of the utility maximization problem for the large market, (4.1) is equivalent to

$$\left\{\rho :\ u(x - q\rho, q) \leq u(x,0), \quad for\ every \quad q \in \mathbb{R}^N\right\}. \tag{4.2}$$

We note that the initial wealth is important in both formulations (4.1) and (4.2), and thus the utility-based prices depend on the initial wealth, in general. This observation has a clear financial interpretation, and therefore, we will denote the set of utility-based prices by $\Pi(x)$.

Further, (4.2) leads to the following natural characterization of the set of the utility-based prices.

$$\Pi(x) = \left\{\frac{r}{y} :\ (y,r) \in \partial u(x,0)\right\}. \tag{4.3}$$

In formulation (2.5), given the concavity of u, and in view of Lemma 3.4, we immediately obtain the existence of the utility-based prices for every $x > 0$. If we fix an $x > 0$ first and then compute the $\Pi(x)$, the question of whether $\Pi(x)$ is a singleton or not, becomes important, as the uniqueness of the utility-based prices is a necessary condition for the well-posedness in the sense of Hadamard of the utility-based pricing problem. Such uniqueness is a desirable feature both from the mathematical and financial viewpoints. In the case when $\Pi(x)$ is a singleton, we get the following representation

$$\Pi(x) = \left\{ \frac{u_q(x,0)}{u_x(x,0)} \right\}.$$

Below we provide a sufficient condition for the uniqueness of the utility-based prices starting from x.

Theorem 4.2. *Under the conditions of Theorem 3.1, let $x > 0$ be fixed and consider $y := u_x(x,0)$ and $\hat{Y}(y)$, the optimizer to* (2.14) *at y. If $\mathbb{E}\left[\hat{Y}_T(y)\right] = y$, then $\Pi(x)$ is a singleton, and for*

$$\rho := \mathbb{E}\left[\frac{\hat{Y}_T(y)}{y} f\right] \in \mathcal{P}, \tag{UBP-rep}$$

we have

$$\Pi(x) = \{\rho\}.$$

For the proof of Theorem 4.2, we will need the following lemma.

Lemma 4.3. *Under the conditions of Theorem 3.1, let us consider* (2.5) *for $q = 0$, and let x_n, $n \in \mathbb{N}$ be a sequence of strictly positive numbers converging to $x > 0$. Then, the optimizers to* (2.5), *$\hat{X}_T(x_n, 0) \to \hat{X}_T(x,0)$ in probability.*

Proof. Let us denote

$$g := \hat{X}_T(x,0), \quad \textit{and} \quad g^n := \hat{X}_T(x_n,0), \quad n \in \mathbb{N}.$$

Then if g^n does not converge to g in probability, there exists $\varepsilon > 0$, such that

$$\limsup_{n\to\infty} \mathbb{P}\left[|g^n - g| > \varepsilon\right] > \varepsilon.$$

Since $\{X_T : \ X \in \mathcal{X}(1,0)\}$ is bounded in $\mathbb{L}^0$, by passing to a smaller ε, if necessary, we get

$$\limsup_{n\to\infty} \mathbb{P}\left[|g^n - g| > \varepsilon, |g^n + g| \le \tfrac{1}{\varepsilon}\right] > \varepsilon. \tag{4.4}$$

From the concavity of $U(\omega,\cdot)$, $\omega \in \Omega$, we deduce that

$$U\left(\frac{g^n + g}{2}\right) \ge \tfrac{1}{2}\left(U(g^n) + U(g)\right),$$

whereas the strict concavity of $U(\omega,\cdot)$, $\omega\in\Omega$, implies the existence of a random variable $\eta>0$[1] and a constant $\delta>0$, such that

$$\limsup_{n\to\infty}\mathbb{P}\left[U\left(\frac{g^n+g}{2}\right)\geq \tfrac{1}{2}\left(U(g^n)+U(g)\right)+\eta\right]>\delta.$$

As $u(\cdot,0)$ is concave and finite on $(0,\infty)$, it follows that $u(\cdot,0)$ is continuous on $(0,\infty)$, and with $A_n:=\left\{U\left(\frac{g^n+g}{2}\right)\geq\frac{1}{2}\left(U(g^n)+U(g)\right)+\eta\right\}$, $n\in\mathbb{N}$, we have

$$\limsup_{n\to\infty}\mathbb{E}\left[U\left(\frac{g^n+g}{2}\right)\right]\geq u(x,0)+\limsup_{n\to\infty}\mathbb{E}\left[\eta 1_{A_n}\right]>u(x,0). \tag{4.5}$$

Now, passing to convex combinations $\tilde{g}^n\in conv\left(g^n,g^{n+1},\dots\right)$, $n\in\mathbb{N}$, which converges to some random variable $\tilde{g}$, $\mathbb{P}$-a.s., and invoking [Mos15, Lemma 3.5], which, by the symmetry between the primal and dual value functions, and since $-V$ is the Inada stochastic field, also implies the uniform integrability of $(U^+(\tilde{g}^n))_{n\in\mathbb{N}}$, we get

$$\limsup_{n\to\infty}\mathbb{E}\left[U\left(\frac{\tilde{g}^n+g}{2}\right)\right]\leq\mathbb{E}\left[U\left(\frac{\tilde{g}+g}{2}\right)\right].$$

Therefore, by concavity of U, we get

$$\begin{aligned}\mathbb{E}\left[U\left(\frac{\tilde{g}+g}{2}\right)\right]&\geq\limsup_{n\to\infty}\mathbb{E}\left[U\left(\frac{\tilde{g}^n+g}{2}\right)\right]\\&\geq\limsup_{n\to\infty}\mathbb{E}\left[U\left(\frac{g^n+g}{2}\right)\right]>u(x,0),\end{aligned} \tag{4.6}$$

where, in the last inequality, we have used (4.5). Using [DDGP05, Lemma 3.3], we deduce the existence of $X\in\mathcal{X}(x,0)$, such that

$$X_T\geq\frac{\tilde{g}+g}{2},\quad \mathbb{P}\text{-a.s.}$$

Combining the latter inequality with (4.6), we conclude that

$$\mathbb{E}\left[U\left(X_T\right)\right]>u(x,0),$$

which is a contradiction. $\square$

Proof of Theorem 4.2. Let us fix $q\in\mathbb{R}$, such that $(x-q\rho,q)\in\mathcal{K}$, and consider an arbitrary $X\in\mathcal{X}(x-q\rho,q)$. As every $\frac{\hat{Y}(y)}{y}$ is a density of an element of $\mathcal{M}$, we deduce that (for bounded f), $\hat{Y}(y)X$ is a $\mathbb{P}$-supermartingale.

[1] If U is deterministic, η can be chosen to be a constant.

Therefore, using conjugacy of U and V, we get

$$\begin{aligned}\mathbb{E}\left[U(X_T+qf)\right] &\leq \mathbb{E}\left[V(\hat{Y}_T(y))+\hat{Y}_T(y)(X_T+qf)\right]\\ &= w(y)+\mathbb{E}\left[\hat{Y}_T(y)(X_T+qf)\right]\\ &= u(x,0)-xy+\mathbb{E}\left[\hat{Y}_T(y)(X_T+qf)\right]\\ &\leq u(x,0)-xy+y(x-q\rho+q\rho)=u(x).\end{aligned} \tag{4.7}$$

As q is an arbitrary element of $\mathbb{R}$, such that $(x-q\rho,q)\in\mathcal{K}$ and X is an arbitrary element of $\mathcal{X}(x-q\rho,q)$, we deduce from (4.7) (comparing (4.7) with (4.2)) that ρ is a utility-based price for f.

To show the uniqueness of ρ, let us consider $\pi\neq\rho$. First, we will suppose that $\pi_i<\rho_i$ for some $i\in\{1,\dots,N\}$. For $C_k:=||f_k||_\infty$ and $\vec{c}:=(C_1,\dots,C_N)$, and with e_i being a vector, whose ith component is 1 and all other components are zero, let us consider a sequence of positive numbers $(s_n)_{n\in\mathbb{N}}\to 0$, such that $s_ne_i(\vec{c}+\pi)<x$, $n\in\mathbb{N}$, let us set

$$q_n:=s_ne_i,\quad X^n:=\hat{X}(x-q_n(\vec{c}+\pi),0)+q_n\vec{c},\quad n\in\mathbb{N},$$

Then, we have

$$X^n_0=x-q_n\pi,\quad \textit{and}\quad X^n\in\mathcal{X}(x-q_n\pi,q_n),\quad n\in\mathbb{N}.$$

We deduce that

$$\begin{aligned}u(x-q_n\pi,q_n) &\geq \mathbb{E}\left[U\left(X^n_T+q_nf\right)\right]\\ &\geq \mathbb{E}\left[U\left(\hat{X}_T(x-q_n(\vec{c}+\pi),0)\right)\right]+\mathbb{E}\left[q_n(\vec{c}+f)U'(X^n_T+q_nf)\right]\\ &= u\left(x-q_n(\vec{c}-\pi),0\right)+\mathbb{E}\left[q_n(\vec{c}+f)U'(X^n_T+q_nf)\right].\end{aligned} \tag{4.8}$$

Therefore, we obtain

$$\begin{aligned}\liminf_{n\to\infty}\frac{u(x-q_n\pi,q_n)-u\left(x-q_n(\vec{c}-\pi),0\right)}{s_n} &\geq \liminf_{n\to\infty}\mathbb{E}\left[e_i(\vec{c}+f)U'(X^n_T+q_nf)\right]\\ &\geq \mathbb{E}\left[e_i(\vec{c}+f)U'\left(\hat{X}_T(x,0)\right)\right]\\ &= \mathbb{E}\left[e_i(\vec{c}+f)\hat{Y}_T(y,0)\right]\\ &= e_i(\vec{c}+\rho)y,\end{aligned} \tag{4.9}$$

where, in the second inequality, we have used Fatou's lemma and the assertion of Lemma 4.3. We deduce that

$$\begin{aligned}\liminf_{n\to\infty}\frac{u(x-q_n\pi,q_n)-u(x)}{s_n} =& \liminf_{n\to\infty}\frac{u(x-q_n\pi,q_n)-u\left(x-q_n(\vec{c}+\pi),0\right)}{s_n}\\ &+\liminf_{n\to\infty}\frac{u\left(x-q_n(\vec{c}+\pi),0\right)-u(x)}{s_n}\\ \geq& e_i(\vec{c}+\rho)y-u_x\left(x,0\right)\left(\vec{c}+\pi\right)\\ =& e_i(\rho-\pi)y>0.\end{aligned}$$

As $s_n>0$, $n\in\mathbb{N}$, we deduce that π is not a utility-based price, as this π does not satisfy (4.2). As π_i was an arbitrary number smaller than ρ_i, we deduce that every π with $\pi_i<\rho_i$ is not a utility-based price for f. Denoting $\tilde{f}=-f$, we can apply the argument above to show that every $\tilde{\pi}$, such that $\tilde{\pi}_i<-\rho_i$, is not a utility-based price for $-f$, and thus every π, such that $\pi_i>\rho_i$ is not a utility-based price for f. As $i\in\{1,\dots,N\}$ was arbitrary, we deduce that every π, such that $\pi_i\neq\rho_i$ for some $i\in\{1,\dots,N\}$ is not a utility-based price. That is, under the conditions of this theorem, the utility-based price ρ given by (UBP-rep), is unique.

Finally, to show that $\rho\in\mathcal{P}$, we observe that since $\mathbb{E}[\hat{Y}_T(y)]=y$, we deduce that $\hat{Y}(y)$ is a density process of an element of $\mathcal{M}$. Therefore, $\rho\in\mathcal{P}$, by Lemma 3.7. ☐

5. Utility-based pricing in a large market as a limit of utility-based pricing in small markets

The main result of this section is Theorem 5.7. We begin from auxiliary results.

Lemma 5.1. *Under the conditions of Proposition 3.3, we have*

$$\overline{\bigcup_{n\in\mathbb{N}}\mathcal{K}^n}\subseteq\mathcal{K}, \tag{5.1}$$

where the closure is taken in $\mathbb{R}^{N+1}$.

Proof. Let $(x^n,q^n)\in\mathcal{K}^n$, $n\in\mathbb{N}$, be a sequence convergent to $(x,q)\in\overline{\bigcup_{n\in\mathbb{N}}\mathcal{K}^n}$. Then, for every $n\in\mathbb{N}$, for some $x^n+H^n\cdot S\in\mathcal{X}^n(x^n,q^n)$, we have

$$x^n+H^n\cdot S_T+q^nf\geq 0,\quad \mathbb{P}\text{-a.s.},\quad n\in\mathbb{N}. \tag{5.2}$$

For an appropriate $C \in \mathbb{R}^N$, let us rewrite the latter inequalities as

$$\frac{H^n}{|x^n| + q^n C} \cdot S_T \geq \frac{-x^n - q^n f}{|x^n| + q^n C} \geq -1, \quad \mathbb{P}\text{-a.s.}, \quad n \in \mathbb{N}. \tag{5.3}$$

Therefore, [DDGP05, Lemma 3.3] applies, and $\frac{-x^n - q^n f}{|x^n| + q^n C} \to \frac{-x - qf}{|x| + qC}$, we deduce that there exists a 1-admissible generalized strategy, such that

$$\bar{H} \cdot S_T \geq \frac{-x - qf}{|x| + qC}.$$

Then $H := \bar{H}(|x| + qC)$ is a generalized admissible strategy that satisfies

$$x + H \cdot S_T + qf \geq 0, \quad \mathbb{P}\text{-a.s.}$$

In particular, we deduce that $x + H \cdot S_T \in \mathcal{X}(x, q) \neq \emptyset$ and thus $(x, q) \in \mathcal{K}$. □

We recall that in the finite-dimensional markets, the superreplicaiton price of a contingent claim $\tilde{f} \in \mathbb{L}^0_+$ is characterized as

$$\inf_{x \in \mathbb{R}} \left\{ x + H \cdot S^T \geq \tilde{f},\ \mathbb{P}\text{-a.s.},\ \mathit{for\ some}\ H \in \mathcal{H}^n \right\} = \sup_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{E}_{\mathbb{Q}} \left[\tilde{f} \right]. \tag{5.4}$$

This is the subject of [DS98, Theorem 5.12], for example, among others. This naturally characterizes the superreplication price as

$$\pi_n(\tilde{f}) := \sup_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[\tilde{f}], \quad n \in \mathbb{N}. \tag{5.5}$$

One can see that $(\pi_n(\tilde{f}))_{n \in \mathbb{N}}$ is decreasing. By setting

$$\pi(\tilde{f}) := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[\tilde{f}],$$

we are trying to build an analog of the arbitrage-free price in finite-dimensional markets. Immediately, from the definition of π_n's and π, we get

$$\lim_{n \to \infty} \pi_n(\tilde{f}) = \inf_{n \in \mathbb{N}} \pi_n(\tilde{f}) \geq \pi(\tilde{f}). \tag{5.6}$$

As pointed out in illuminating [DDGP05, Example 3.1], the inequality can be strict. In this case, we might say that the superreplications prices in finite-dimensional markets do not converge to the one in the large market. We show below that in this case, the utility-based prices do not converge, in general, either. We might have situations, when the domains of the optimization problems do not converge in the set-theoretic sense, that is

$$\mathcal{L} \neq \bigcap_{n \in \mathbb{N}} \mathcal{L}^n \quad \mathit{and\ therefore} \quad \mathcal{K} \neq \bigcap_{n \in \mathbb{N}} \mathcal{K}^n.$$

In this case, the model in the large market is not a limit of the finite-dimensional models and thus is not as interesting. Clearly, a similar conclusion holds for the sub replication prices.

Example 5.2. This example is an extension of [DDGP05, Example 3.1]. In a one period-settings, where $\Omega = \{\omega_n\}_{n=0}^{\infty}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}_1$ is discrete σ-algebra on Ω. The probability and the asset prices at time 1 are given by

$$\mathbb{P}(\omega_0) = 1 - \alpha, \quad \mathbb{P}(\omega_n) = \alpha 2^{-n}, \quad n \geq 1.$$

$$S_1^n(\omega_0) = 1, \quad S_1^n(\omega_n) = 2^n, S_1^n(\omega_k) = 0, \quad k \notin \{0, n\}.$$

The initial price of all assets is a constant $c > 0$. For every $n \in \mathbb{N}$, in the associated finite-dimensional market, we deduce that the martingale measures could be characterized as follows, with $q_k = \mathbb{Q}(\omega_k)$, $k \geq 0$, we have

$$q_0 \in (0, c) \quad \textit{and} \quad q_k = 2^{-k}(c - q_0), \quad k \in \{1, \dots, n\}. \tag{5.7}$$

In the large market, we obtain that there exists a unique martingale measure, which makes this model asymptotically complete, and this, in particular, connects this example to the one of Section 6 below. This unique martingale measure in the large market is given by

$$q_0 = 2c - 1, \quad \textit{and} \quad q_k = 2^{-k}(c - q_0), \quad k \in \mathbb{N}. \tag{5.8}$$

Let us consider the contingent claim

$$f = 1_{\{\omega_0\}}.$$

In every finite-dimensional market, the super replication price of f is given by

$$\sup_{q_0 \in (0,c)} q_0 = c,$$

and it does not converge to the one in the large market given by $q_0 = 2c - 1$ for every $c < 1$. This is where [DDGP05, Example 3.1] ends. For such f, we show below that, for example, for power and logarithmic utilities, the associated utility-based prices for f also do not converge. The plan is to use Theorem 4.2. Using [Mos15, Theorem 2.4], the problem can be restated as

$$v^n(y) = \inf_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{E}\left[V\left(y \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \quad y > 0, \quad n \in \mathbb{N}.$$

In our case, this reduces to

$$v^n(y) = \inf_{q_0\in(0,c)} \left(V\left(y\frac{q_0}{1-\alpha}\right)(1-\alpha) + \sum_{k=1}^{n} V\left(y\frac{c-q_0}{\alpha}\right)\frac{\alpha}{2^k} \right),$$
$$y>0, \quad n\in\mathbb{N},$$

and, with $\phi_n := 1-\frac{1}{2^n}$, $n\in\mathbb{N}$, we get

$$v^n(y) = \inf_{q_0\in(0,c)} \left(V\left(y\frac{q_0}{1-\alpha}\right)(1-\alpha) + V\left(y\frac{c-q_0}{\alpha}\right)\alpha\phi_n \right),$$
$$y>0, \quad n\in\mathbb{N}.$$

The first-order conditions lead to

$$V'\left(y\frac{q_0}{1-\alpha}\right) = V'\left(y\frac{c-q_0}{\alpha}\right)\phi_n, \quad y>0, \quad n\in\mathbb{N}. \tag{5.9}$$

If $U(x) = \frac{x^p}{p}$, $x>0$, and $p\in(-\infty,0)\cup(0,1)$, $V(y) = -\frac{y^\gamma}{\gamma}$, $y>0$, where $\gamma := \frac{p}{p-1}$. Therefore, we have $V'(y) = -y^{\gamma-1}$, $y>0$, and the first-order conditions lead to

$$\left(y\frac{q_0}{1-\alpha}\right)^{\gamma-1} = \left(y\frac{c-q_0}{\alpha}\right)^{\gamma-1}\phi_n, \quad y>0, \quad n\in\mathbb{N}.$$

Solving for q_0, where by q_0^n, we denote q_0 in the n-th market, we get

$$q_0^n = \frac{c\phi_n^{\frac{1}{\gamma-1}}(1-\alpha)}{\alpha+\phi_n^{\frac{1}{\gamma-1}}}, \quad n\in\mathbb{N}. \tag{5.10}$$

Note that in the power utility case, the dual minimizer does not depend on y, and thus, neither does the optimal dual measure in Theorem 4.2. This holds for every small market and the large market.

Now from (5.10), using Theorem 4.2, we have

$$\rho^n = \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = q_0^n,$$

where ρ^n is the utility-based price in the n-th market, and from (5.10), we get

$$\lim_{n\to\infty}\rho^n = \frac{c(1-\alpha)}{\alpha+1}. \tag{5.11}$$

On the other hand, in the large market, the unique martingale measure $\mathbb{Q}$ is the optimizer to the dual problem for every y, and, using Theorem 4.2, we have

$$\rho = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f] = q_0 = 2c - 1, \tag{5.12}$$

where, in the last equality, we have used (5.8). Comparing (5.11) and (5.12), we conclude that

$$\lim_{n\to\infty} \rho^n \neq \rho,$$

if $c \neq \frac{\alpha+1}{3\alpha+1}$.

If $U(x) = \log x$, $x > 0$, then $V(y) = -\log y - 1$, $y > 0$, and $V'(y) = -\frac{1}{y}$, $y > 0$. The first-order conditions (5.9), lead to

$$\frac{1-\alpha}{q_0} = \frac{\alpha}{c - q_0}\phi_n, \quad n \in \mathbb{N},$$

and thus

$$q_0 = \frac{c(1-\alpha)}{\alpha\phi_n + 1 - \alpha}, \quad n \in \mathbb{N}, \tag{5.13}$$

We deduce that

$$\rho^n = \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = q_0^n = \frac{\alpha}{c - q_0}\phi_n, \quad x > 0, \quad n \in \mathbb{N},$$

where, in the logarithmic-utility case, and similarly to the power-utility case, the optimal dual measure does not depend on x in every small market and the large market. Again, in the large market, there exists a unique martingale measure, and, via (5.8), we have

$$\rho = 2c - 1.$$

Comparing this with (5.13), we conclude that

$$\lim_{n\to\infty} \rho^n = c(1-\alpha) \neq \rho = 2c - 1,$$

if $c \neq \frac{1}{\alpha+1}$.

Thus, for the positive convergence result, we need to strengthen (fin uv) to the following conditions. First, we need the value functions in some small markets to be finite-valued too.

Assumption 5.3. There exists $\tilde{n} \in \mathbb{N}$, such that

$$u^{\tilde{n}}(x, 0) > -\infty, \quad x > 0, \quad \textit{and} \quad \tilde{w}(y) < \infty, \quad y > 0. \tag{5.14}$$

Remark 5.4. Under Assumptions 2.1, 2.2, (no-Arb), and 5.3, an application of [Mos17, Lemma 2.6] imply (fin uv) and

$$u^n(x,q) > -\infty, \quad (x,q) \in int\mathcal{K}^n \quad and \quad v^n(y,r) < \infty, \quad (y,r) \in ri\mathcal{L}^n, \quad n \geq n'.$$

Assumption 3.6 is helpful in seeing how this works in the most difficult, fully non-replicable case first. Further, under Assumption 5.3, the utility-based prices in small markets, $\Pi^n(x)$, $n \in \mathbb{N}$, can be characterized similarly to Theorem 4.2.

For every $(x,q) \in int\mathcal{K}$, let us define the following sets.

$$\begin{aligned}
\mathcal{B}^n(x,q) &:= \{(y,r) \in \mathcal{L}^n : \ xy + qr \leq 1\}, \quad n \in \mathbb{N}, \\
\mathcal{B}(x,q) &:= \{(y,r) \in \mathcal{L} : \ xy + qr \leq 1\}, \\
\tilde{\mathcal{D}}^n(x,q) &:= \bigcup_{(y,r)\in\mathcal{B}^n(x,q)} \mathcal{D}^n(y,r), \quad n \in \mathbb{N}, \\
\tilde{\mathcal{D}}(x,q) &:= \bigcup_{(y,r)\in\mathcal{B}(x,q)} \mathcal{D}(y,r).
\end{aligned}$$

The sets $\tilde{\mathcal{D}}^n(x,q)$ and $\tilde{\mathcal{D}}^n(x,q)$ are polar to $\mathcal{C}^n(x,q)$ and $\mathcal{C}(x,q)$, respectively, and for every $Z \in \mathcal{M}(\rho)$, $\rho \in \mathcal{P}$, there exists $\alpha = \alpha(\rho) := \frac{1}{x+q\rho}$, such that $\alpha Z \in \tilde{\mathcal{D}}(x,q)$ and $(\alpha+\delta)Z \notin \tilde{\mathcal{D}}(x,q)$, for every $\delta > 0$. Let us denote

(5.15)
$$\tilde{\tilde{Z}}^n(x,q) := \bigcup_{\rho\in\mathcal{P}^n} \bigcup_{Y\in\mathcal{M}^n(\rho)} \alpha(\rho)Y, \quad n \in \mathbb{N}, \quad \tilde{\tilde{Z}}(x,q) := \bigcup_{\rho\in\mathcal{P}} \bigcup_{Y\in\mathcal{M}(\rho)} \alpha(\rho)Y.$$

Remark 5.5. A convenient way of thinking about the construction of $\tilde{\tilde{Z}}^n(x,q)$'s and $\tilde{\tilde{Z}}(x,q)$ is via

$$\tilde{\tilde{Z}}^n(x,q) := \bigcup_{Y\in\mathcal{M}^n} \frac{1}{x+q\rho(Z)}Y, \quad n \in \mathbb{N}, \quad \tilde{\tilde{Z}}(x,q) := \bigcup_{Y\in\mathcal{M}} \frac{1}{x+q\rho(Z)}Y,$$

where $\rho(Z) := \mathbb{E}\left[Z_T f\right]$.

We will show below that $\tilde{\tilde{Z}}$'s are the generating set for $\tilde{\mathcal{D}}(x,q)$, in the following sense. If we set

$$\tilde{\mathcal{Z}}^n(x,q) := \overline{conv\left(\tilde{\tilde{\mathcal{Z}}}^n(x,q)\right)}, \quad n \in \mathbb{N}, \quad and \quad \tilde{\mathcal{Z}}(x,q) := \overline{conv\left(\tilde{\tilde{\mathcal{Z}}}^n(x,q)\right)},$$

the closures in $\mathbb{L}^0$ of the convex and solid hull of $\tilde{\tilde{\mathcal{Z}}}^n(x,q)$ and $\tilde{\tilde{\mathcal{Z}}}^n(x,q)$, respectively. We note that $\tilde{\tilde{Z}}(x,q)$ play exactly the same role with random endowment as the set $\mathcal{Z}$ without random endowment, that is, it is the generating set for the dual domain. Further, we can extend (no-Arb), which asserts that $\mathcal{Z} = \bigcap\limits_{n\in\mathbb{N}} \mathcal{Z}^n$, where the monotonicity of $\mathcal{Z}^n$, $n \in \mathbb{N}$, is crucial to $\tilde{\tilde{Z}}$'s, a "tilted" and rescaled version of $\mathcal{Z}$, where the monotonicity will also hold. That is we will have

$$\tilde{\tilde{Z}}(x,q) = \bigcap_{n\in\mathbb{N}} \tilde{\tilde{Z}}^n(x,q).$$

However, showing that $\tilde{Z}(x,q) = \bigcap\limits_{n\in\mathbb{N}} \tilde{Z}^n(x,q)$ requires an extra condition, as a closure of a countable intersection is not equal to a countable intersection of closures, in general. The example above implicitly supports this point. This is despite the fact that for $(x,q) = (1,0)$, it is proven in [DDGP05] that $\tilde{Z}(1,0) = \bigcap\limits_{n\in\mathbb{N}} \tilde{Z}^n(1,0)$, for other $(x,q) \in int\mathcal{K}$, such that $q \neq 0$, more work is needed.

Thus, we impose the following assumption, see how it holds in examples of Section 6. In cases when the optimal Z's for the large market are elements of $\mathcal{Z}$, one can typically have a natural candidate for the approximating sequence as in examples of Section 6.

Assumption 5.6. For every $(x,q) \in int\mathcal{K}$, there exists a uniformly integrable sequence sequence $Z^n_T : Z^n \in Z \in \tilde{\tilde{\mathcal{Z}}}^n(x,q)$, $n \in \mathbb{N}$, such that

$$\sup_{n\in\mathbb{N}} \inf_{Z\in\tilde{\tilde{\mathcal{Z}}}^n(x,q)} \mathbb{E}\left[V(Z_T)\right] = \lim_{n\to\infty} \mathbb{E}\left[V(zZ^n_T)\right],$$

and for every $q \in \mathbb{R}^N$, there exists a uniformly integrable sequence Z^n_T : $Z^n \in \mathcal{Z}$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sup_{Z\in\mathcal{Z}^n} \mathbb{E}[Z^n_T qf] = \lim_{n\to\infty} \mathbb{E}\left[Z^n_T qf\right].$$

We note that the latter assumption holds if $\{Z_T :\ Z \in \mathcal{Z}^n\}$ is uniformly integrable for some $n \in \mathbb{N}$, which is much stronger. The primary result of this section is the following theorem. We recall that $\Pi(x)$ is given in (4.3), and $\Pi^n(x)$'s are specified entirely similarly for the market with n risky assets.

Theorem 5.7. *Let us suppose that Assumptions 2.1, 2.2,* (no-Arb)*, 5.3, and 5.6 hold. Then, for every sequence of strictly positive numbers* $(x^n)_{n\in\mathbb{N}}$

converging to $x > 0$, we have

$$\Pi^n(x^n) \to \Pi(x),$$

in the sense that, for every $\varepsilon > 0$, there exists $n' \in \mathbb{N}$, such that

$$\Pi^n(x^n) \subset \Pi(x) + \varepsilon B, \quad \textit{for every} \quad n \geq n', \tag{5.16}$$

where B is the Euclidean unit ball of $\mathbb{R}^N$.

Remark 5.8. The assertions of Theorem 5.7 hold without any assumption on whether and of the sets $\Pi^n(x)$ or $\Pi(x)$ are singletons or not.

Corollary 5.9. *Under the conditions of Theorem 5.7, if $\Pi^n(x)$ or $\Pi(x)$ are singletons, that is, if both $\hat{Y}(y)$, the minimizer to* (2.14)*, and $\hat{Y}^n(y^n)$, the minimizers to*

$$\tilde{w}^n(y) := \inf_{Y \in \mathcal{Y}^n(y)} \mathbb{E}[V(Y_T)], \quad y > 0, \quad n \in \mathbb{N}, \tag{5.17}$$

are martingales, for $y^n := u^n_x(x, 0)$, $n \in \mathbb{N}$, and $y := u_x(x, 0)$, then we have

$$\lim_{n\to\infty} \rho^n(x) = \lim_{n\to\infty} \mathbb{E}\left[\hat{Y}^n(y^n, 0) f\right] = \mathbb{E}\left[\hat{Y}(y, 0) f\right] = \rho(x).$$

Lemma 5.10. *Under the conditions of Proposition 3.3, for every $(x, q) \in$ int$\mathcal{K}$, the set $\tilde{\tilde{Z}}(x, q)$ is convex, closed under the countable convex combinations*[2]*, and we have*

$$\sup_{Z \in \tilde{\tilde{Z}}(x,q)} \mathbb{E}\left[g Z_T\right] = \sup_{Z \in \tilde{Z}(x,q)} \mathbb{E}\left[g Z_T\right], \quad g \in \mathcal{C}(x, q). \tag{5.18}$$

Proof. We show that $\tilde{\tilde{Z}}(x, q)$ is closed under countable convex combinations. Let $\rho^i \in \mathcal{P}$, $Y^i \in \mathcal{M}(\rho^i)$, and $\alpha^i = \frac{1}{x + q\rho^i}$, $i \in \mathbb{N}$. Then $Z^i = \alpha^i Y^i \in \tilde{\tilde{\mathcal{Z}}}(x, q)$. For convex weights $\lambda^i \in [0, 1]$, such that $\sum\limits_{i=1}^{\infty} \lambda^i = 1$, we want to show that

$$Z := \sum_{i=1}^{\infty} \lambda^i Z^i \in \tilde{\tilde{\mathcal{Z}}}(x, q).$$

[2]That is under combinations of the form $\sum\limits_{k=1}^{\infty} \lambda^k Z^k$, where $(Z^k)_{k\in\mathbb{N}} \subset \tilde{\tilde{Z}}(x, q)$ and λ^k's are nonnegative constants, such that $\sum\limits_{k=1}^{\infty} \lambda^k = 1$.

Let us set

$$\alpha := \sum_{j=1}^{\infty} \lambda^j \alpha^j \quad \textit{and} \quad \mu^i := \frac{\lambda^i \alpha^i}{\sum_{j=1}^{\infty} \lambda^j \alpha^j} = \frac{\lambda^i \alpha^i}{\alpha} \in [0,1], \quad i \in \mathbb{N},$$

and

$$\rho := \sum_{i=1}^{\infty} \mu^i \rho^i \quad \textit{and} \quad Y := \sum_{i=1}^{\infty} \mu^i Y^i. \tag{5.19}$$

Then $Y \in \mathcal{M}$, as an application of the monotone convergence theorem shows that $\mathcal{M}$ is closed under countable convex combinations (see, e.g., proof of [Mos18, Lemma 3.5]), and moreover, one can see that $Y \in \mathcal{M}(\rho)$. Then we have

(5.20)

$$Z = \sum_{i=1}^{\infty} \lambda^i Z^i = \sum_{i=1}^{\infty} \lambda^i \alpha^i Y^i = \left(\sum_{k=1}^{\infty} \lambda^k \alpha^k \right) \sum_{i=1}^{\infty} \frac{\lambda^i \alpha^i}{\sum_{j=1}^{\infty} \lambda^j \alpha^j} Y^i = \alpha \sum_{i=1}^{\infty} \mu^i Y^i = \alpha Y.$$

Next, let us observe that

$$\begin{aligned} 1 = \sum_{i=1}^{\infty} \lambda^i &= \sum_{i=1}^{\infty} \lambda^i \underbrace{\alpha^i \left(x + q\rho^i \right)}_{=1} \\ &= x \sum_{i=1}^{\infty} \lambda^i \alpha^i + q \sum_{j=1}^{\infty} \lambda^j \alpha^j \rho^j \\ &= \left(x + q \sum_{j=1}^{\infty} \frac{\lambda^j \alpha^j}{\sum_{k=1}^{\infty} \lambda^k \alpha^k} \rho^j \right) \left(\sum_{i=1}^{\infty} \lambda^i \alpha^i \right) \\ &= \left(x + q \sum_{j=1}^{\infty} \mu^j \rho^j \right) \left(\sum_{i=1}^{\infty} \lambda^i \alpha^i \right). \end{aligned}$$

That is, we have

$$\left(\sum_{i=1}^{\infty} \lambda^i \alpha^i \right) \left(x + q \sum_{j=1}^{\infty} \mu^j \rho^j \right) = 1,$$

and thus, recalling the definitions of α and ρ, we conclude that

$$\alpha(x + q\rho) = 1. \tag{5.21}$$

To recapitulate, (5.19), (5.20), and (5.21) imply that

$$Z = \sum_{i=1}^{\infty} \lambda^i Z^i = \alpha Y, \quad \textit{where} \quad Y \in \mathcal{M}(\rho) \quad \textit{and} \quad \alpha = \frac{1}{x + q\rho},$$

that is $Z \in \tilde{\tilde{\mathcal{Z}}}(x,q)$. Therefore, $\tilde{\tilde{\mathcal{Z}}}(x,q)$ is closed under countable convex combinations. In turn, (5.18) follows from the respective constructions of $\tilde{\tilde{Z}}(x,q)$ and $\tilde{Z}(x,q)$, and Fatou's lemma.

□

Lemma 5.11. *Under the conditions of Proposition 3.3, for every $(x,q) \in int\mathcal{K}$, we have*

$$\tilde{\mathcal{Z}}(x,q) = \tilde{\mathcal{D}}(x,q). \tag{5.22}$$

Proof. Let $(x,q) \in int\mathcal{K}$ be fixed and consider an arbitrary $g \in \mathcal{C}(x,q)$. Then, we have

$$g \leq X_T + qf, \quad \mathbb{P}\text{-a.s.},$$

for some $X \in \mathcal{X}(x,q)$. Therefore, for every $\rho \in \mathcal{P}$, and $Y \in \mathcal{M}(\rho)$, by the supermartingale property of XY, we have

$$\mathbb{E}[Y_T g] \leq \mathbb{E}\left[X_T + qf\right] \leq x + q\rho.$$

This implies that for $Z := \frac{1}{x+q\rho} Y \in \tilde{\tilde{\mathcal{Z}}}(x,q)$, we have

$$\mathbb{E}[Z_T g] \leq 1. \tag{5.23}$$

Next, for every $h \in \tilde{\mathcal{Z}}(x,q)$ there exists a sequence of processes $Z^n = \frac{1}{x+q\rho^n} Y^n \in \tilde{\tilde{\mathcal{Z}}}(x,q)$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} Z^n_T \geq h$, and such that (5.23) holds for every Z^n. Fatou's lemma implies that $\mathbb{E}[hg] \leq 1$, and thus by Proposition 3.3, $h \in \tilde{\mathcal{D}}(x,q)$. In turn, as $g \in \mathcal{C}(x,q)$ and $h \in \tilde{\mathcal{Z}}(x,q)$ are arbitrary, this shows that

$$\tilde{\mathcal{Z}}(x,q) \subseteq \tilde{\mathcal{D}}(x,q). \tag{5.24}$$

Conversely, let us consider $g \in \mathbb{L}^0_+$, such that

$$\mathbb{E}\left[g Z_T\right] \leq 1, \quad \textit{for every} \quad Z \in \tilde{\tilde{\mathcal{Z}}}(x,q). \tag{5.25}$$

We want to show that

$$g \leq X_T + qf$$

for some $X \in \mathcal{X}(x,q)$. (5.25) implies that for every $\rho \in \mathcal{P}$ and $Y \in \mathcal{M}(\rho)$, we have

$$\mathbb{E}\left[g\frac{1}{x+q\rho}Y_T\right] \le 1. \tag{5.26}$$

Then, we get

$$\mathbb{E}\left[(g-qf)\frac{1}{x+q\rho}Y_T\right] \le 1 - \frac{q\rho}{x+q\rho} = \frac{x}{x+q\rho}.$$

Therefore, we obtain

$$\mathbb{E}\left[(g-qf)Y_T\right] \le x. \tag{5.27}$$

Let $C \in \mathbb{R}^N$ be such that $g - q(f-C) \in \mathbb{L}^0_+$. Then, from (5.27), we have

$$\mathbb{E}\left[(g-q(f-C))Y_T\right] \le x + qC.$$

The latter inequality holds for every $\rho \in \mathcal{P}$, and $Y \in \mathcal{M}(\rho)$, where the right-hand side does not depend on ρ. Consequently, from Lemma 3.7, we deduce that

$$\sup_{Y\in\mathcal{M}} \mathbb{E}\left[(g-q(f-C))Y_T\right] \le x + qC.$$

Now we apply [DDGP05, Theorem 3.1], which asserts that there exists an $(x+qC)$-admissible generalized strategy H, such that

$$g - q(f-C) \le x + qC + H\cdot S_T,$$

and thus $X := x + H\cdot S \in \mathcal{X}(x,q)$ and X super replicates $g - qf$. In turn, this implies that $g \in \mathcal{C}(x,q)$. Therefore, $\mathcal{C}(x,q) \supseteq (\tilde{\mathcal{Z}}(x,q))^o$. As a result, we obtain

$$\tilde{\mathcal{D}}(x,q) = (\mathcal{C}(x,q))^o \subseteq (\tilde{\mathcal{Z}}(x,q))^{oo} = \tilde{\mathcal{Z}}(x,q), \tag{5.28}$$

where in the last equality, we have used the bipolar theorem of Brannath and Schachermayer, and we note that $\tilde{\mathcal{Z}}(x,q)$ is convex, solid, and closed in $\mathbb{L}^0$ by construction.

Finally, (5.24) and (5.28) imply the assertion of the lemma, (5.22). □

Lemma 5.12. *Under the conditions of Theorem 5.7, let $(x,q) \in int\mathcal{K}$ be fixed, and $Z^n \in \tilde{\tilde{\mathcal{Z}}}^n(x,q)$, $n \in \mathbb{N}$, be a uniformly integrable sequence, such that $\lim\limits_{n\to\infty} Z^n_T = h$, $\mathbb{P}$-a.s. Then $h \in \tilde{\mathcal{D}}(x,q)$.*

Proof. Let us consider an arbitrary $g \in \mathcal{C}(x,q)$. Then there exists $X \in \mathcal{X}(x,q)$, such that

$$g \leq X_T + q'f.$$

Let X^n, $n \in \mathbb{N}$, be an approximating sequence of elements of $\mathcal{X}^n(x)$ for X_T. By passing if necessary to subsequences, which we do not relabel, and for a sufficiently large constant $C > 0$, we have

$$\begin{aligned}\mathbb{E}[h(g+C)] \leq \mathbb{E}[h(X_T + q + C)] &= \mathbb{E}[\liminf_{n\to\infty} Z^n_T(X^n_T + qf + C)] \\ &\leq \liminf_{n\to\infty} \mathbb{E}[Z^n_T(X^n_T + qf + C)] \leq 1 + C \liminf_{n\to\infty} \mathbb{E}[Z^n_T],\end{aligned}$$

where, in the second inequality, we have used Fatou's lemma, and, in the last one, we used the definition of the sets $\tilde{\tilde{Z}}^n(x,q)$ and the uniform integrability of Z^n, $n \in \mathbb{N}$. Now, one can see that $\mathbb{E}[hg] \leq 1$. Since g was an arbitrary element of $\mathcal{C}(x,q)$, we deduce that $h \in (\mathcal{C}(x,q))^o = \tilde{\mathcal{D}}(x,q)$. □

Remark 5.13. Assumption 5.6 implies the following condition

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[qf] = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[qf] \quad \textit{for every} \quad q \in \mathbb{R}^N. \tag{5.29}$$

This can be proven as follows. Let us fix $q \in \mathbb{R}^n$. By Assumption 5.6, there exists $Z^n \in \mathcal{Z}^n$, $n \in \mathbb{N}$, a maximizing sequence for (5.29) that is uniformly integrable. By passing to convex combinations, we obtain a sequence, which we still denote $Z^n \in \mathcal{Z}^n$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} Z^n_T = h$, $\mathbb{P}$-a.s., for some nonnegative random variable h. Then we have

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[qf] = \lim_{n\to\infty} \mathbb{E}\left[Z^n_T qf\right] = \mathbb{E}\left[hqf\right]. \tag{5.30}$$

Lemma 5.12 implies that $h \in \tilde{\mathcal{D}}(1,0)$. Using Lemmas 5.10 and 5.11 (note that since qf is bounded, so it is in $\mathcal{C}(x,0)$ for a sufficiently large x, see Proposition 3.3), we deduce that

$$\sup_{Z\in\tilde{\tilde{Z}}(1,0)} \mathbb{E}[Z_T qf] = \sup_{h\in\mathcal{D}(1,0)} \mathbb{E}[hqf] \geq \mathbb{E}\left[hqf\right]. \tag{5.31}$$

Combining (5.30) and (5.31), we deduce (5.29).

Remark 5.14. When $N = 1$, (5.29) is equivalent to assuming that

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f] \quad \textit{and} \quad \lim_{n\to\infty} \inf_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f], \tag{5.32}$$

which is convergence of super and subreplication prices to the ones in the large market, respectfully. Without passing to the limit, that is without considering $\inf\limits_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f]$ and $\sup\limits_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f]$ in the limiting market, (5.32) is closely related to [ARS17, Assumption 4.1].

For the proof of Theorem 5.7, we will need the following lemma. With Assumption 5.6, we can get the opposite inclusion comparing to Lemma 5.1.

Lemma 5.15. *Let the assumptions of Theorem 5.7 hold. Then, we have*

$$\overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n} = \mathcal{K}. \tag{5.33}$$

Proof. We recall that, by Lemma 3.5, $\mathcal{K}$ is closed. Let $(x,q)\in\mathcal{K}$ and let $((x^k,q^k))_{k\in\mathbb{N}}\subset int\mathcal{K}$ convergent to (x,q). As $(x,0)\in\mathcal{K}^n$ for every $n\in\mathbb{N}$, it is enough to consider $q\neq 0$, and thus, it is enough to consider $q^k\neq 0$, $k\in\mathbb{N}$. Let us fix $k\in\mathbb{N}$. As $(x^k,q^k)\in int\mathcal{K}$, we deduce that there exists $\delta^k>0$, such that

$$(x^k-\delta^k,q^k)\in\mathcal{K}.$$

Definition of $\mathcal{K}$, there exists $X\in\mathcal{X}(x^k-\delta^k,q^k)$, such that

$$X+q^kf\geq 0,\quad \mathbb{P}\text{-a.s.}$$

Via [DDGP05, Theorem 3.1], the latter inequality is equivalent to

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[-q^kf]\leq x^k-\delta^k.$$

Then, using Assumption 5.6, we can rewrite the latter inequality as

$$x^k-\delta^k\geq \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[-q^kf\right] = \lim_{n\to\infty}\sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[-q^kf\right],$$

and thus, we obtain

$$\lim_{n\to\infty}\sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[-q^kf\right]\leq x^k-\delta^k.$$

Let us fix an arbitrary $\varepsilon_n>0$. Then, the latter inequality implies the existence of $n=n(\varepsilon_n)\in\mathbb{N}$, such that

$$\sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[-q^kf\right]\leq x^k-\delta^k+\varepsilon_n \tag{5.34}$$

By the super replication results for finite-dimensional models, see, e.g., [DS98, Theorem 5.12], (5.34) implies the existence of an $(x^k - \delta^k + \varepsilon_n)$-admissible n-elementary strategy H^n, such that

$$x^k - \delta^k + \varepsilon_n + H^n \cdot S_T + q^k f \geq 0, \quad \mathbb{P}\text{-a.s.}$$

Therefore $X := x^k - \delta^k + \varepsilon_n + H^n \cdot S \in \mathcal{X}^n(x^k - \delta^k + \varepsilon_n, q^k)$, and in particular $(x^k - \delta^k + \varepsilon_n, q^k) \in \mathcal{K}^n$. We deduce that

$$(x^k - \delta^k + \varepsilon_n, q^k) \in \bigcup_{n\in\mathbb{N}} \mathcal{K}^n$$

As ε_n was arbitrary, by picking, for example $\varepsilon_n = \delta^k$, we deduce that

$$((x^k, q^k))_{k\in\mathbb{N}} \subset \bigcup_{n\in\mathbb{N}} \mathcal{K}^n,$$

and therefore, since $((x^k, q^k))_{k\in\mathbb{N}} \subset int\mathcal{K}$ is convergent to (x, q), we deduce that $(x, q) \in \overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n}$. Therefore (5.33) holds.

□

Corollary 5.16. *Let the assumptions of Theorem 5.7 hold. Then, we have*

$$\mathcal{L} = \bigcap_{n\in\mathbb{N}} \mathcal{L}^n.$$

Corollary 5.17. *Let the assumptions of Theorem 5.7 hold. Then the sets of closures of arbitrage-free prices in small markets converge to the closure of the set of arbitrage-free prices in the large market.*

Now, for every $(x, q) \in int\mathcal{K}$, let us define

$$\begin{aligned} \tilde{v}^n(z) &:= \inf_{Z\in\tilde{\bar{\mathcal{Z}}}^n(x,q)} \mathbb{E}\left[V(zZ_T)\right], \quad z > 0, \quad n \geq n_0, \\ \tilde{v}(z) &:= \inf_{Z\in\tilde{\bar{\mathcal{Z}}}(x,q)} \mathbb{E}\left[V(zZ_T)\right], \quad z > 0. \end{aligned} \tag{5.35}$$

Lemma 5.18. *Under the conditions of Theorem 5.7, for every $(x, q) \in int\mathcal{K}$, there exists $n_0 \in \mathbb{N}$, such that*

$$\begin{aligned} \tilde{v}^n(z) &:= \inf_{h\in\tilde{\mathcal{D}}^n(x,q)} \mathbb{E}\left[V(zh)\right], \quad z > 0, \quad n \in \mathbb{N}, \\ \tilde{v}(z) &:= \inf_{h\in\tilde{\mathcal{D}}(x,q)} \mathbb{E}\left[V(zh)\right], \quad z > 0. \end{aligned}$$

Proof. First, we observe that $(x, q) \in int\mathcal{K}$ and Lemma 5.15, imply that there exists $n_0 \in \mathbb{N}$, such that $(x, q) \in \mathcal{K}^n$ for every $n \geq n_0$. Entirely similarly to the proofs of Lemmas 5.10 and 5.11, we can obtain similar assertions (convexity, closedness under countable convex combinations, (5.18) and (5.22)) for $\tilde{\mathcal{Z}}^n(x, q)$, $n \geq n_0$, besides $\tilde{\mathcal{Z}}(x, q)$. Now, the assertion of the lemma follows from [Mos15, Theorem 2.4]. □

Proof of Theorem 5.7. Let $(x, q) \in int\mathcal{K}$ be fixed. One can see (e.g., using Lemma 5.18) the monotonicity of $\tilde{v}^n$: as $\mathcal{M}^n$, $n \in \mathbb{N}$, is decreasing, $\tilde{v}^n$, $n \geq n_0$, where n_0 is given by Lemma 5.18, is increasing and

$$\sup_{n\to\infty} \tilde{v}^n(z) = \lim_{n\to\infty} \tilde{v}^n(z) \leq \tilde{v}(z) < \infty, \quad z > 0. \tag{5.36}$$

Let us fix $z > 0$. Assumption 5.6 implies the existence of a uniformly integrable sequence $h^n \in \tilde{\tilde{\mathcal{Z}}}^n(x, q)$, $n \geq n_0$, such that

$$\liminf_{n\to\infty} \tilde{v}^n(z) = \lim_{n\to\infty} \mathbb{E}\left[V(zh^n)\right]. \tag{5.37}$$

By passing to convex combinations $\tilde{h}^n \in conv(h^n, h^{n+1}, \dots)$, $n \in \mathbb{N}$, we can obtain a sequence such that $\tilde{h}^n \in \tilde{\tilde{\mathcal{Z}}}^n(x, q)$, $n \in \mathbb{N}$, and such that $\tilde{h}^n$ converges to some limit denoted by $\tilde{h}$, $\mathbb{P}$-a.s. By Lemma 5.12, we deduce that $\tilde{h} \in \tilde{\mathcal{D}}(x, q)$.

As $\tilde{h} \in \bigcap\limits_{n\in\mathbb{N}} \mathcal{Y}^n(y^n) \subseteq \mathcal{Y}^1(\bar{y})$ for an some $\bar{y} \in (0, \infty)$, and also $(\tilde{h}^n)_{n\geq\tilde{n}} \subseteq \mathcal{Y}^{\tilde{n}}(\bar{y})$, where $\tilde{n}$ is given by Assumption 5.3, via [Mos15, Lemma 3.5], we conclude that the sequence $V^-(z\tilde{h}^n)$, $n \in \mathbb{N}$, is uniformly integrable. Therefore, using the convexity of $V(\cdot, \omega)$, $\omega \in \Omega$, we obtain

$$\tilde{v}(z) \leq \mathbb{E}\left[V(z\tilde{h})\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[V(z\tilde{h}^n)\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[V(zh^n)\right] = \liminf_{n\to\infty} \tilde{v}^n(z), \tag{5.38}$$

where, in the last equality, we have used (5.37). Combining (5.36) and (5.38), we obtain

$$\tilde{v}(z) = \lim_{n\to\infty} \tilde{v}^n(z), \quad z > 0. \tag{5.39}$$

Now, from Theorem 3.1, we deduce that

$$\begin{aligned} \tilde{u}(s) := u(sx, sq) &= \inf_{(y,r)\in\mathcal{L}} \left(v(y, r) + sxy + sqr\right) \\ &= \inf_{z>0} \inf_{(y,r)\in z\mathcal{B}(x,q)} \left(v(y, r) + sxy + sqr\right) \\ &= \inf_{z>0} \left(\tilde{v}(z) + sz\right), \quad s > 0. \end{aligned} \tag{5.40}$$

By construction, both $-\tilde{u}$, v, and $\liminf_{n\to\infty} \tilde{v}^n$ are convex and finite-valued. From (5.39) and (5.40), we get

$$\tilde{u}(s) = \inf_{z>0} \left(\tilde{v}(z) + sz\right) = \inf_{z>0} \left(\lim_{n\to\infty} \tilde{v}^n(z) + sz\right), \quad s > 0. \tag{5.41}$$

A similar construction

$$\tilde{u}^n(s) := u^n(sx, sq) = \inf_{z>0} \left(\tilde{v}^n(z) + sz\right), \quad s > 0, \quad n \geq n_0, \tag{5.42}$$

shows that $\tilde{u}^n$, $n \geq n_0$, is a monotone sequence and

$$-\infty < \tilde{u}^n(s) \leq \tilde{u}(s), \quad s > 0, \quad n \geq n_0.$$

Therefore, we have

$$\tilde{u}^\infty(s) := \lim_{n\to\infty} \tilde{u}^n(s) \leq \tilde{u}(s), \quad s > 0. \tag{5.43}$$

Further, combining (5.43) with (5.41) and (5.42), and using the monotonicity of $\tilde{v}^n$, $n \geq n_0$, we get

$$\inf_{z>0} \left(\sup_{k\geq n_0} \tilde{v}^k(z) + sz\right) = \tilde{u}(s) \geq \inf_{z>0} \left(\tilde{v}^n(z) + sz\right), \quad n \geq n_0. \tag{5.44}$$

By conjugacy of $\tilde{u}^n$ and $\tilde{v}^n$, and from the monotonicity of $\tilde{u}^n$, $n \geq n_0$, we obtain

$$\tilde{v}^n(z) = \sup_{s>0} (\tilde{u}^n(s) - sz) \leq \sup_{s>0} (\tilde{u}^\infty(s) - sz), \quad z > 0, \quad n \geq n_0.$$

Therefore, using (5.39), we obtain

$$\tilde{v}(z) \leq \sup_{s>0} (\tilde{u}^\infty(s) - sz), \quad z > 0. \tag{5.45}$$

One can see that $\tilde{u}^\infty$ is a concave function as a pointwise limit of concave functions, and further that $\tilde{u}^\infty$ is finite-valued. Let $\hat{v}$ denote its convex conjugate, then (5.45) implies that

$$\hat{v}(z) = \sup_{s>0} (\tilde{u}^\infty(s) - sz) \geq \tilde{v}(z), \quad z > 0. \tag{5.46}$$

Therefore, the biconjugation characterization and (5.46), imply that

$$\tilde{u}^\infty(s) = \inf_{z>0} (\hat{v}(z) + zs) \geq \inf_{z>0} (\tilde{v}(z) + zs) = \tilde{u}(s), \quad s > 0, \tag{5.47}$$

where, in the last equality, we have used (5.41). As a result, combining (5.43) and (5.47), we get

$$\tilde{u}^\infty(s) = \tilde{u}(s), \quad s > 0.$$

In particular, recalling (5.40) and (5.42), we conclude that

$$u(x,q) = \lim_{n\to\infty} u^n(x,q).$$

As (x,q) was an arbitrary point in $int\mathcal{K}$, we deduce that

$$u(x,q) = \lim_{n\to\infty} u^n(x,q), \quad (x,q) \in int\mathcal{K},$$

which, via [Roc70, Theorem 24.5], implies (5.16). □

6. Pricing of asymptotically replicable claims

The asymptotically replicable claim is the one that is replicable in the large market, but possibly not in any small market. We give examples of such claims and markets admitting such claims below Intuitively, the arbitrage-free prices for small markets should converge to singletons.

Remark 6.1. If every bounded contingent claim is replicable in the large markets, one can intuitively call such markets asymptotically complete. We also compare the definition to the one in [Kar19, Definition 3.10]. In [ARS17] the asymptotically complete markets are investigated, without passing to the limiting market.

Contingent claims, which are replicable in some finite markets are well-studied in the literature. To develop a theory of arbitrage-free or utility-based pricing for such claims, one does need to analyze the large markets. Below, we focus on the claims, which are not replicable in the small markets, but are replicable in the large one. Examples below show such markets and claims.

Under (no-Arb), the definition of asymptotic replicability can be stated as follows. A bounded f is *asymptotically replicable*, if f is not replicable in any small market, and there exist processes in $\bigcup\limits_{x\in\mathbb{R}} \tilde{\mathcal{X}}(x)$ whose terminal values $\mathbb{P}$-a.s. equal to the respective components of f. If the initial values of processes in $\bigcup\limits_{x\in\mathbb{R}} \tilde{\mathcal{X}}(x)$ that replicate f^i's equal to π^i's, then f is *asymptotically replicable at* $\pi \in \mathbb{R}^N$. We also denote by AFP^n, $n \in \mathbb{N}$, and AFP the sets of the arbitrage-free prices for f in small and large markets, respectively.

The following lemma shows the consistency of various pricing methodologies for asymptotically replicable claims.

Lemma 6.2. *Let us assume the conditions of Theorem 5.7, and suppose such that f is asymptotically replicable at $\pi \in \mathbb{R}^N$. Then, we have*

$$\mathcal{P} = \Pi(x) = AFP = \{\pi\}, \quad x > 0, \tag{6.1}$$

and for every $\varepsilon > 0$, there exists $n' \in \mathbb{N}$, such that for every $n \geq n'$, we have

$$\Pi^n \subseteq \mathcal{P}^n \subseteq \mathcal{P} + \varepsilon B, \quad x > 0, \tag{6.2}$$

and

$$AFP^n \subseteq AFP + \varepsilon B, \tag{6.3}$$

where B is the unit ball of $\mathbb{R}^N$.

Proof. If one can replicate f with an initial price $\pi \in \mathbb{R}^N$, then, the utility maximization problem degenerates to the one without f, as follows. One can see that

$$u(x, q) = u(x + q\pi, 0), \quad (x, q) \in \mathbb{R}^{N+1}.$$

Then for $\rho = \pi$ and every $x > 0$, we have

$$u(x - q\rho, q) = u(x + q\pi - q\rho, 0) = u(x, 0), \quad q \in \mathbb{R}^N,$$

and thus (4.2) holds, i.e., π is the utility-based price at x, for every $x > 0$, i.e., $\pi \in \Pi(x)$, $x > 0$.

If $\rho^i \neq \pi^i$, for some $i \in \{1, \ldots, N\}$, for $q = sign(\pi^i - \rho^i)e^i$, where e^i is the canonical basis vector in $\mathbb{R}^N$, we have

$$u(x - q\rho, q) = u(x + q\pi - q\rho, 0) > u(x, 0),$$

as $u(\cdot, 0)$ is *strictly* increasing (see e.g., [Mos18, Theorem 2.2]), that is ρ is not a utility-based price, for every $x > 0$.

Further, e.g., by observing that $\mathcal{K}$ contains straight lines passing through the origin and using [Roc70, Theorem 14.6], one can see that $\pi \in \mathcal{P}$, and $\mathcal{L} = \{y(1, \pi) : \ y > 0\}$. As $AFP = \{\pi\}$, we conclude (6.1). In turn, analogously to the proof of Theorem 3.1, as $\partial u^n \subseteq ri\mathcal{L}^n$, we obtain that $\Pi^n \subseteq \mathcal{P}^n$ and $AFP^n = \mathcal{P}^n$. Now, (6.2) and (6.3) follow from Corollary 5.16. $\square$

Examples. The following (positive) Examples 6.3 and 6.4 illustrate the results above. Note that the framework of Section 5, is particularly convenient for the characterization of asymptotically replicable claims (and, in particular, asymptotically complete markets). We also note that the examples also allow to consider multi-agent (in fact, countably many agent) setting and, relying on ideas from [BEK05] and [Sio15], to analyze partial equilibria pricing.

Example 6.3. Let $S^n_0 = c \in (0,1)$ and let S^n_1 are IID Bernoulli random variable, such that $\mathbb{P}[S^n_1 = 0] = \mathbb{P}[S^n_1 = 1] = \frac{1}{2}$. Let us suppose that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_1 = \sigma(S^n_1, n \in \mathbb{N})$ Let us consider an asymptotically replicable $f = \sum\limits_{k=1}^{\infty} \frac{1}{2^k} S^k_1$.

In the small markets, the super replication price of f is given by

$$\begin{aligned}\sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] &= \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{n} \frac{1}{2^k} S^k_1 + \sum_{j=n+1}^{\infty} \frac{1}{2^j} S^j_1\right] \\ &= c\sum_{k=1}^{n} \frac{1}{2^k} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} = c + \frac{1}{2^n}(1-c), \quad n \in \mathbb{N}.\end{aligned}$$

By similar computations, the sub replication price is given by

$$\inf_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = c\left(1 - \frac{1}{2^n}\right), \quad n \in \mathbb{N}.$$

We see that the set of the arbitrage-free prices in small markets is given by

$$AFP^n = \left(c\left(1-\frac{1}{2^n}\right), c + \frac{1}{2^n}(1-c)\right) \to \{c\} = AFP,$$

where c is also the unique arbitrage-free price in the large market (and the convergence has to be understood in the sense of convergence of sets as in Lemma 6.2), as under the minimal martingale measure in the large market, $\hat{\mathbb{Q}}$, we have $\mathbb{E}_{\hat{\mathbb{Q}}}[f] = c$. In particular, Assumption 5.6 from Section 5 holds.

For every deterministic Inada utility U, such that Assumption 5.3 holds, one can show that $\hat{\mathbb{Q}}$ is the dual minimizer in the large market. By Theorem 4.2, the unique utility-based price (in the large market) is given by

$$\mathbb{E}_{\hat{\mathbb{Q}}}[f] = c. \tag{6.4}$$

In every small market, we set

$$
\begin{aligned}
\zeta^k &:= 2c1_{\{S^k=1\}} + 2(1-c)1_{\{S^k=0\}}, \quad k \in \{1,\dots,n\},\\
\tilde{\zeta}^j(\alpha^j) &:= 2\alpha^j 1_{\{S^j=1\}} + 2(1-\alpha^j)1_{\{S^j=0\}}, \quad \alpha^j \in (0,1), \quad j \in \{n+1,\dots\},\\
\mathcal{A}^n &:= \{(\alpha^j)_{j\in\{n+1,\dots\}} : \ \alpha^j \in (0,1), j \in \{n+1,\dots\}\}.
\end{aligned}
$$

Let V be the convex conjugate of U. Then, we can specify the dual problem as

$$
v^n(y) = \inf_{(\alpha^j)_{j\in\{n+1,\dots\}}\in\mathcal{A}^n} \mathbb{E}\left[V\left(y\prod_{k=1}^n \zeta^k \prod_{j=n+1}^{\infty}\tilde{\zeta}^j(\alpha^j)\right)\right], \quad y>0. \tag{6.5}
$$

One can see that the density of the minimal martingale measure, $\hat{\mathbb{Q}}^n$, is the minimizer to (6.5). The corresponding density is given by

$$
\frac{d\hat{\mathbb{Q}}^n}{d\mathbb{P}} = \prod_{k=1}^n \zeta^k
$$

and it corresponds to $\alpha^j = \frac{1}{2}$ for every $j \in \{n+1,\dots\}$. As in the large market case, this does not depend on y, and thus, on the initial wealth.

Finally, as $\mathbb{E}\left[\frac{d\hat{\mathbb{Q}}^n}{d\mathbb{P}}\right] = 1$, $n \in \mathbb{N}$, by Theorem 4.2, whose proof also applies to small markets, the unique utility-based price in the market with n stocks is given by

$$
\begin{aligned}
\rho^n = \mathbb{E}_{\hat{\mathbb{Q}}^n}[f] &= \mathbb{E}_{\hat{\mathbb{Q}}^n}\left[\sum_{k=1}^n \frac{1}{2^k}S_1^k + \sum_{j=n+1}^{\infty}\frac{1}{2^j}S_1^j\right]\\
&= \sum_{k=1}^n \frac{1}{2^k}c + \sum_{j=n+1}^{\infty}\frac{1}{2^j}\frac{1}{2} = c\left(1-\frac{1}{2^n}\right) + \frac{1}{2^{n+1}}, \quad n\in\mathbb{N}.
\end{aligned}
$$

Therefore, we obtain

$$
\lim_{n\to\infty}\rho^n = c,
$$

which is the unique utility based price for f in the large market by (6.4).

Example 6.4. Let us consider a model, which is not asymptotically complete, where
$\left(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,T]}, \mathbb{P}\right)$, is a complete stochastic basis supporting a countable set of one-dimensional independent Brownian motions W^n, $n \in \mathbb{N}$, $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$ is generated by W^n, $n \in \mathbb{N}$, and some other finite-dimensional

Brownian motion. Let the riskless asset $S^0 \equiv 1$, whereas the dynamics of risky assets is given by

$$dS^n = S^n\left(\mu^n dt + \sigma^n dW_t^n\right), \quad n \in \mathbb{N}, \tag{6.6}$$

where S^0 is (deterministic and) strictly positive, and where the constants μ^n and $\sigma^n > 0$, $n \in \mathbb{N}$, are and such that the market price of risks satisfy

$$\lambda^n := \frac{\mu^n}{\sigma^n}, \quad n \in \mathbb{N},$$

satisfy

$$\sum_{n=1}^{\infty} (\lambda^n)^2 < \infty.$$

Let us consider the minimal martingale measures for the n-stock model, who's density processes are given by

$$Z^n = \prod_{k=1}^{n} \mathcal{E}\left(\lambda^k \cdot W^k\right), \quad n \in \mathbb{N},$$

One can show that the family Z_T^n, $n \in \mathbb{N}$, is uniformly integrable and it converges to a random variable Z_T, which admits a representation

$$Z_T = \exp\left(\sum_{k=1}^{\infty} \lambda^k W_T^k - \tfrac{T}{2} \sum_{k=1}^{\infty} (\lambda^k)^2\right), \tag{6.7}$$

where $\sum\limits_{k=1}^{\infty} \lambda^k W_T^k$ is also a limit of a uniformly integrable sequence of terminal values of martingales $\sum\limits_{k=1}^{n} \lambda^k W_T^k$, $n \in \mathbb{N}$.

One can see that (no-Arb) satisfied. Let us introduce $\mathcal{H}_t^n := \mathcal{F}_t \vee \mathcal{F}_T^{W^1,\dots,W^n}$, $t \in [0,T]$, $n \in \mathbb{N}$. Thus, for such a model and a deterministic Inada utility U, such that Assumptions 5.3 hold, and with V being the conjugate of U, for every $\tilde{Z} \in \mathcal{Z}^n$, $n \in \mathbb{N}$, we have

(6.8)

$$\mathbb{E}[V(y\tilde{Z}_T)] = \mathbb{E}[\mathbb{E}[V(y\tilde{Z}_T)|\mathcal{H}_T^n]] \geq \mathbb{E}[V(y\mathbb{E}[\tilde{Z}_T|\mathcal{H}_T^n])] \geq \mathbb{E}[V(yZ_T^n)], \quad n \in \mathbb{N}.$$

Let

$$f = \sum_{k=1}^{\infty} h^k(S_T^k),$$

where h^k's are smooth functions such that $\sum\limits_{k=1}^{\infty} ||h^k||_\infty < \infty$. Then f is demonstratively asymptotically replicable, and

$$\lim_{n\to\infty} \mathbb{E}[Z^n_T f] = \mathbb{E}[Z_T f]. \tag{6.9}$$

Using (6.8) and (6.9), one can see that Assumption 5.6 holds.

Since additionally $\mathbb{E}\left[Z^n_T\right] = 1$, one can show along the lines of Theorem 4.2 that the set of utility-based prices for f (in the market with the first n risky assets) is given by

$$\Pi^n(x) = \{\mathbb{E}\left[Z^n_T f\right]\}, \quad n \in \mathbb{N}, \quad x > 0.$$

In the large market, Lemma 6.2 (and Theorem 4.2) implies that

$$\Pi(x) = \{\mathbb{E}\left[Z_T f\right]\}, \quad x > 0,$$

is the set of utility-based prices in the large market. In view of (6.9), we have $\lim\limits_{n\to\infty} \Pi^n(x_n) = \Pi(x)$ for every sequence of strictly positive numbers x_n, $n \in \mathbb{N}$, convergent to $x > 0$, where the convergence is in the sense of Lemma 6.2, which in the present settings reduces to convergence of singletons.

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