PRICING OF CONTINGENT CLAIMS IN LARGE MARKETS

OLEKSII MOSTOVYI AND PIETRO SIORPAES

Abstract. We consider the problem of pricing in a large market, which arises as a limit of small markets within which there are finitely many traded assets. We show that this framework allows accommodating both marginal utility-based prices (for stochastic utilities) and arbitrage-free prices. Adopting a stochastic integration theory with respect to a sequence of semimartingales, we introduce the notion of marginal utility-based prices for the large (post-limit) market and establish their existence, uniqueness, and relation to arbitrage-free prices. These results rely on a theorem of independent interest on utility maximization with a random endowment in a large market that we state and prove first. Further, we provide approximation results for the marginal utility-based and arbitrage-free prices in the large market by those in small markets. In particular, our framework allows for pricing asymptotically replicable claims, where we also show a consistency of the pricing methodologies and provide positive examples.

1. Introduction

The size and complexity of financial markets have led to the appearance of models with an infinite number of traded securities. Starting from the usual "small" models and supposing that the number of traded stocks is a finite but random number taking values in the set of natural numbers,

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one directly arrives at the assumption of the availability of countably many tradable assets in the market. Models of this type are considered in [BN98], [DDGP05], and [Mos18], among others. Further, in the context of fixed income derivatives, it is natural to model interest rates with uncountably many traded instruments, see, e.g., [CT04], [RT06], [DDP05], and [ET05].

The mathematical foundations of the large markets go back to [KK94], where a large market was introduced as a sequence of models with finitely many traded assets. The theory was further developed [KK98], [Kle00], [KS96a], [KS96b],[Kle03], [Kle06], and [KLPO14]. Recently, the questions of indifference pricing in such markets have been considered in [ARS17] and [RS18]. Relying on ideas of stochastic integration with respect to infinite-dimensional stochastic processes, characterizations of the large market themselves (post-limit) have appeared. The theory for such markets was developed through a series of works including [CT04], [ET05], [DDGP05], [DDP05], [RT06], [KLPO14], [DDP05], and more recently in [CKT16] and [Kar23]. Investigating the post-limit models requires stochastic integration with respect to infinite-dimensional stochastic processes that is less developed than stochastic integration with respect to finite-dimensional semimartingales. Further, *completeness* is a common assumption in traditional interest rate modeling. Thus, pricing in the large (post-limits) market models in the context of fixed-income derivatives often inherits certain replicability assumptions see, e.g., [CT04, Assumption 5.1].

This paper focuses on two *pricing approaches in (fully) incomplete large markets*, without any apriori replicability assumptions, where modeling and establishing results in the large (post-limit) market itself is a significant part of our analysis. In particular, *in stochastic utility settings, we develop the marginal utility-based pricing in the large (post-limit) market and show its consistency with the arbitrage-free pricing.* For this, first, we establish a *utility maximization with random endowment theorem for the large market*, a result, which is interesting by itself. Further, we provide an approximation result by the marginal utility-based prices in small markets, again with consistency to the arbitrage-free pricing asymptotics. Finally, we apply our results to the asymptotically replicable claims, where their pricing in the large market has a particularly nice structure. We note that in settings of exponential utility, the problem of utility-based pricing in the large (pre-limit) market has been considered in [ARS17].

On the technical level, as there are fewer stochastic analytic tools for studying stochastic integration with respect to an infinite-dimensional semimartingale, we deal with more obstacles. In particular, we do not use the Optional Decomposition Theorem, which was crucial for optimal investment with a random endowment in a small market, see [HK04]. We note that the Optional Decomposition Theorem for the large market has been recently developed in [Kar23] under the continuity of the underlying stock price processes. We could not use it as our formulation deals with semimartingales, which might admit jumps. Our approach allows us to include the closures of the domains of the key optimization problems (crucially for the proofs and to obtain more complete characterizations of the underlying problems), and to circumvent both the non-replicability and asymptotic replicability assumptions often imposed in the literature, even for small markets. An application of such a formulation is the pricing of asymptotically replicable claims as a particular case of our results. The latter includes asymptotically complete markets, where every claim is replicable or asymptotically replicable.

The remainder of this paper is organized as follows: in Section 2, we introduce the model. In Section 3, we establish the utility maximization results with random endowment in a large market. In Section 4, we introduce the notions of the marginal utility-based price in a large market and prove its existence, and provide a condition for its uniqueness; in Section 5, we prove the convergence of the marginal utility-based prices in small markets to the ones in the large market. In Section 6, we show an application of our settings to asymptotically replicable claims, where asymptotically complete market come as a particular case.

2. Model

We consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions, $\mathcal{F}_0$ is trivial. We suppose that the large market consists of a riskless asses $S^0 \equiv 1$ and a sequence of risky assets $S = (S^n)_{n\in\mathbb{N}}$, where each S^n, $n \in \mathbb{N}$, is a semimartingale that specifies the price of the nth risky asset. We also suppose that there is a non-traded contingent claim with payment process $(F^i)_{i=1}^N$. If $(q_i)_{i=1}^N = q$ is the finite sequence that specifies the number of such claims, the cumulative payoff is given by

$$qF := (qF_t)_{t\in[0,T]} = \left(\sum_{i=1}^N q_i F_t^i\right)_{t\in[0,T]}.$$

Both processes S and F are given exogenously.

The notion of the trading strategy in the large market is given as follows. For $n \in \mathbb{N}$, an *n-elementary strategy* is a $\mathbb{R}^n$-valued predictable and $(S^k)_{k=1,\dots,n}$-integrable process. An *elementary strategy* is a strategy, which is n-elementary for some $n \in \mathbb{N}$. Further, an n-elementary strategy H is *x-admissible* for a given $x \geq 0$, if

$$H \cdot S = \int_0^\cdot \sum_{k=1}^n H_t^k dS_t^k \geq -x, \quad \mathbb{P}\text{-a.s.}$$

Let $\mathcal{H}^n$ denote the set of n-elementary strategies, which are x-admissible for some $x \geq 0$, and by $\mathcal{H}$ the set of admissible elementary strategies. By n-small market, we call the one where one can trade in $S^0, \dots, S^n$ (and hold q shares of F for some $q \in \mathbb{R}^N$). By a small market, we mean the one, which is n-small for some $n \in \mathbb{N}$.

To pass to the limit as $n \to \infty$, we will follow [DDP06], and we recall that $\mathbb{R}^{\mathbb{N}}$ is the space of real-valued sequences. An *unbounded functional* on $\mathbb{R}^{\mathbb{N}}$ is a linear functional $\bar{H}$, whose domain, $Dom(\bar{H})$ is a subspace of $\mathbb{R}^{\mathbb{N}}$. A *simple integrand* is a finite sum of bounded predictable processes of the form $\sum\limits_{k=1}^n h^k e^k$, where e^k is the Dirac delta at point k, and h^k is a one-dimensional bounded and predictable processes, $k \in \mathbb{N}$.

A process H with values in the set of unbounded functionals on $\mathbb{R}^{\mathbb{N}}$ is *predictable* if there is a sequence of simple integrands (as defined in the previous paragraph) $(H^n)_{n\in\mathbb{N}}$, such that $H = \lim\limits_{n\to\infty} H^n$, pointwise, in the sense that for every $x \in Dom(H)$, the sequence $(H^n(x))_{n\in\mathbb{N}}$ converges to $H(x)$ as $n \to \infty$.

A predictable process H with values in the set of unbounded functionals on $\mathbb{R}^{\mathbb{N}}$ is *integrable* with respect to S if there is a sequence $(H^n)_{n\in\mathbb{N}}$ of simple integrands, such that $(H^n)_{n\in\mathbb{N}}$ converges to H pointwise and the sequence of semimartingales $(H^n \cdot S)_{n\in\mathbb{N}}$ converges to a semimartingale Y

in the semimartingale topology. In this case, we set

$$H \cdot S := Y.$$

To put the concept of a stochastic integral as above in the context of optimal investment, we further need to specify the context of admissibility. Thus, for $x \geq 0$, we say that a predictable process with values in the set of unbounded functionals is an *x-admissible generalized strategy* if H is integrable with respect to S and there is an approximating sequence of x-admissible elementary strategies, $(H^n \cdot S)_{n\in\mathbb{N}}$, that converges to $H \cdot S$ in the semimartingale topology. A predictable process with values in the set of unbounded functionals is an *admissible generalized strategy* if it is x-admissible for some $x \geq 0$.

Primal problem

For a nontradable contingent claim(s) $f = F_T$ with payments f^i, $i \in \{1, \dots, N\}$, at T, that is, F has a single payment of f at T, and with $q = (q_i)_{i=1}^N$, for every $(x, q) \in \mathbb{R}^{N+1}$, we set

(2.1)

$$\begin{aligned} \mathcal{X}^n(x,q) &:= \{X = x + H \cdot S : \ H \in \mathcal{H}^n \textit{ and } X_T + qf \geq 0\}, \quad n \in \mathbb{N}; \\ \mathcal{X}(x,q) &:= \{X = x + H \cdot S : \ H \textit{ is an admissible generalized strategy} \\ &\qquad \textit{and } X_T + qf \geq 0\}. \end{aligned}$$

One can see that, under (noArb), for every $x > 0$, $\mathcal{X}(x, 0)$ consists of the wealth processes associated with x-admissible generalized strategies. Thus, these wealth processes are nonnegative. Likewise, for every $n \in \mathbb{N}$, $\mathcal{X}^n(x, 0)$ consists of the wealth processes associated with x-admissible n-elementary strategies, and so these wealth processes are nonnegative too. Moreover, $\mathcal{X}^n(x, q) \subseteq \mathcal{X}^{n+1}(x, q) \subseteq \mathcal{X}(x, q)$ and these sets can be empty for some (x, q)'s. Therefore, we set

$$\begin{aligned} \mathcal{K}^n &:= \left\{(x,q) \in \mathbb{R}^{N+1} : \mathcal{X}^n(x,q) \neq \emptyset\right\}, \quad n \in \mathbb{N}, \\ \mathcal{K} &:= \left\{(x,q) \in \mathbb{R}^{N+1} : \mathcal{X}(x,q) \neq \emptyset\right\}. \end{aligned} \tag{2.2}$$

For the contingent claim, we will suppose that

Assumption 2.1. Every component of f is bounded.

Remark 2.2 (On the boundedness Assumption 2.1). In many papers, see e.g., [HK04] and [MS20], it is assumed that the contingent claim is bounded by some wealth process in some (typically small) market. Mathematically this amounts to supposing that $|f| \leq C\tilde{X}$, for some maximal $\tilde{X} \in \mathcal{X}^n(1,0)$, for some $n \in \mathbb{N}$. This in particular, allows for pricing contingent claims unbounded from above and below. In this remark, we show that our settings with stochastic utility are closely related to the ones with an unbounded contingent claim.

Let us suppose that one starts from a utility $\tilde{U}$ satisfying Assumption 2.3 (possibly deterministic as in [HK04], but this assumption does not have to be imposed), and a contingent claim $\tilde{f}$ satisfying

$$|\tilde{f}| \leq C\tilde{X}, \; for\ some\ positive\ maximal\ \tilde{X} \in \mathcal{X}^n(1,0), \; and\ C > 0. \tag{2.3}$$

Then, if $\tilde{X}_T > 0$, for a given n-elementary strategy H, there exists an $S^{n,\tilde{X}} := \left(\frac{S^0}{\tilde{X}}, \dots, \frac{S^n}{\tilde{X}}\right)$-integrable predictable process H', such that

$$x + H \cdot S_T + q\tilde{f} = \tilde{X}_T\left(x + H' \cdot S^{\tilde{X}}_T + q\frac{\tilde{f}}{\tilde{X}_T}\right).$$

Next, setting

$$f := \frac{\tilde{f}}{\tilde{X}_T} \quad and \quad U(\omega, x) := \tilde{U}(\omega, \tilde{X}_T(\omega)x), \quad (\omega, x) \in \Omega \times [0,\infty),$$

one can see from (2.3) that $|f| \leq C$, and U satisfies Assumption 2.3. Next, under the local boundedness of the components of $(S^1, \dots, S^n)$, [DS95, Theorem 13] implies that $\left(\frac{S^0}{\tilde{X}}, \dots, \frac{S^n}{\tilde{X}}\right)$ admits an equivalent local martingale measure. This outlines a change of numéraire approach in the context of small markets. For large markets, the change of numéraire calculus and the proper no-arbitrage conditions in the spirit of (noArb), yet under a new numéraire, are yet to be developed.

It follows from Assumption 2.1 that

$$(x, 0) \in int\mathcal{K}^1, \quad for\ every\ x > 0. \tag{2.4}$$

One can show this as follows: let us fix $x > 0$ and consider a ball in $\mathbb{R}^N$ of radius ε; then, for a sufficiently small $\varepsilon > 0$, and every q in this ball, $x + qf \geq 0$, $\mathbb{P}$-a.s., so a portfolio with x units of the riskless asset S^0 and

q shares of the contingent claim (and no risky assets) is admissible. This argument holds for every $x > 0$.

The preferences of an economic agent are given by a utility stochastic field

$$U = U(\omega, x) : \Omega \times [0, \infty) \to \mathbb{R} \cup \{-\infty\}.$$

We suppose that U satisfies the following assumption.

Assumption 2.3. For every $\omega \in \Omega$, the function $x \to U(\omega, x)$ is strictly concave, strictly increasing, continuously differentiable on $(0, \infty)$, and satisfies the Inada conditions:

$$\lim_{x \downarrow 0} U'(\omega, x) = \infty \quad \textit{and} \quad \lim_{x \uparrow \infty} U'(\omega, x) = 0, \tag{2.5}$$

where U' denotes the partial derivative with respect to the second argument. At $x = 0$, we suppose by continuity $U(\omega, 0) = \lim_{x \downarrow 0} U(\omega, x)$, which may be $-\infty$. For every $x > 0$, $U(\cdot, x)$ is $\mathcal{F}$-measurable.

Controlling the investment, the goal of an agent is to maximize the expected utility. The value functions are given by

$$u^n(x, q) = \sup_{X \in \mathcal{X}^n(x,q)} \mathbb{E}\left[U\left(X_T + qF_T\right)\right], \quad (x, q) \in \mathcal{K}^n, \quad n \in \mathbb{N}, \tag{uMax-n}$$

$$u(x, q) = \sup_{X \in \mathcal{X}(x,q)} \mathbb{E}\left[U\left(X_T + qF_T\right)\right], \quad (x, q) \in \mathcal{K}. \tag{uMax-large}$$

Here and below, we will use the following convention

$$\textit{if} \quad \mathbb{E}\left[U^-(X_T + qf)\right] = \infty, \quad \textit{we set} \quad \mathbb{E}\left[U(X_T + qf)\right] := -\infty.$$

If will be convenient to extend the definitions of u^n's and u to $\mathbb{R}^{N+1}$, by setting

$$\begin{aligned} u^n(x, q) &:= -\infty, \quad (x, q) \in \mathbb{R}^{N+1} \backslash \mathcal{K}^n, \quad n \in \mathbb{N}, \quad \textit{and} \\ u(x, q) &:= -\infty, \quad (x, q) \in \mathbb{R}^{N+1} \backslash \mathcal{K}. \end{aligned}$$

To ensure that the utility maximization problems (uMax-n) and (uMax-large) are non-degenerate, we need to impose no-arbitrage conditions. With

$$\begin{aligned} \mathcal{Z}^n = \{&\textit{martingale } Z > 0 : ZX \textit{ is a local martingale} \\ &\qquad\qquad \textit{for every } X \in \mathcal{X}^n(1, 0)\}, \quad n \in \mathbb{N}, \end{aligned} \tag{$\mathcal{Z}^n$}$$

and

$$\mathcal{Z} := \bigcap_{n\geq 1} \mathcal{Z}^n, \tag{$\mathcal{Z}$}$$

we suppose that

$$\mathcal{Z} \neq \emptyset. \tag{noArb}$$

Remark 2.1. Condition (noArb) is closely related, yet stronger than the existence of an equivalent separating measure in the large market, that is, a probability measure $\mathbb{Q}$, such that $\mathbb{E}_{\mathbb{Q}}[X_T] \leq 1$ for every $X \in \mathcal{X}(1,0)$. Unlike small markets, where the existence of an equivalent separating measure implies the existence of an equivalent σ-martingale measure by the results in [DS98] (in particular, [DS98, Proposition 4.7 and Theorem 1.1]), a counterexample in [CKT16, Section 6] demonstrates that in large markets, the existence of an equivalent separating measure does not imply the existence of a σ-martingale measure. Further, by [CKT16, Theorem 3.3], the existence of an equivalent separating measure in the large market is equivalent to *no asymptotic free lunch with vanishing risk* (NAFLVR) condition; see [CKT16] for details.

Dual problem

We begin by setting in small markets

$$\mathcal{L}^n := -(\mathcal{K}^n)^o, \quad n \in \mathbb{N},$$

that is, the respective polars of $-\mathcal{K}^n$, $n \in \mathbb{N}$, in $\mathbb{R}^{N+1}$. We refer to [Roc70] for the definition and properties of a polar of a set. Naturally, we extend this definition to the large markets by setting

$$\mathcal{L} := -\mathcal{K}^o.$$

We introduce, or rather recall, the classical sets of supermartingale deflators in small markets and define

$$\begin{aligned}\mathcal{Y}^n(y) := \{&Y \geq 0 : \ Y_0 = y, \\ &\textit{and } XY \textit{ is a supermartingale for every } X \in \mathcal{X}^n(1,0)\}, \end{aligned} \tag{2.6}$$
$$y > 0, \quad n \in \mathbb{N}.$$

We set

$$\begin{aligned}\mathcal{Y}^n(y,r) := \{Y \in \mathcal{Y}^n(y) :\ \mathbb{E}\left[Y_T(X_T + qf)\right] \leq xy + qr,\\ \textit{for every } (x,q) \in \mathcal{K}^n, \textit{ and } X \in \mathcal{X}^n(x,q)\},\\ (y,r) \in \mathcal{L}^n.\end{aligned} \tag{2.7}$$

Similarly, in the large market, we define

$$\begin{aligned}\mathcal{Y}(y) = \{Y \geq 0 :\ Y_0 = y \textit{ and } XY \textit{ is a supermartingale}\\ \textit{for every } X \in \mathcal{X}(1,0)\}, \quad y > 0,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathcal{Y}(y,r) := \{Y \in \mathcal{Y}(y) :\ \mathbb{E}\left[Y_T(X_T + qf)\right] \leq xy + qr,\\ \textit{for every } (x,q) \in \mathcal{K}, \textit{ and } X \in \mathcal{X}(x,q)\}, \quad (y,r) \in \mathcal{L}.\end{aligned} \tag{2.9}$$

Let us set

$$V(\omega, y) := \sup_{x>0} \left(U(\omega,x) - xy\right), \quad (\omega, y) \in \Omega \times [0,\infty). \tag{2.10}$$

We note that $-V$ satisfies Assumption 2.3. Now, we can state the dual problems for finite markets and the large market.

$$v^n(y,r) = \inf_{Y \in \mathcal{Y}^n(y,r)} \mathbb{E}\left[V\left(Y_T\right)\right], \quad (y,r) \in \mathcal{L}^n, \quad n \in \mathbb{N}, \tag{vMin-n}$$

$$v(y,r) = \inf_{Y \in \mathcal{Y}(y,r)} \mathbb{E}\left[V\left(Y_T\right)\right], \quad (y,r) \in \mathcal{L}, \tag{vMin-large}$$

where we use the convention

$$\mathbb{E}\left[V\left(Y_T\right)\right] := \infty, \quad \textit{if} \quad \mathbb{E}\left[V^+\left(Y_T\right)\right] = \infty.$$

Further, we will extend the definitions of v^n's and v to $\mathbb{R}$ by setting

$$\begin{aligned}v^n(y,r) := \infty, \quad (y,r) \in \mathbb{R}^{N+1} \backslash \mathcal{L}^n, \quad n \in \mathbb{N}, \quad \textit{and} \quad v(y,r) := \infty,\\ (y,r) \in \mathbb{R}^{N+1} \backslash \mathcal{L}.\end{aligned}$$

Let us set

$$\tilde{w}(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[V(Y_T)\right], \quad y > 0, \tag{2.11}$$

and suppose that

$$u(x,0) > -\infty, \quad x > 0, \quad \textit{and} \quad \tilde{w}(y) < \infty, \quad y > 0. \tag{fin u,v}$$

3. Utility maximization with random endowment in large market

Theorem 3.1. *Let us suppose that Assumpitons 2.3, 2.1,* (noArb)*, and* (fin u, v) *hold. Then, we have*

(i) The functions u and v are finite on int$\mathcal{K}$ and ri$\mathcal{L}$, respectively, u and v satisfy

$$\begin{aligned} u(x,q) &= \inf_{(y,r)\in\mathcal{L}} \left(v(y,r) + xy + qr\right), \quad (x,q) \in \mathcal{K}, \\ v(y,r) &= \sup_{(x,q)\in\mathcal{K}} \left(u(x,q) + xy + qr\right), \quad (y,r) \in \mathcal{L}. \end{aligned} \tag{3.1}$$

(ii) The functions u and $-v$ are concave and upper semi-continuous, $u < \infty$ on $\mathcal{K}$. For every $(x,q) \in \{u > -\infty\}$, there exists a unique maximizer to (uMax-large)*. In turn, $v > -\infty$ on $\mathcal{L}$. For every $(y,r) \in \{v < \infty\}$, there exists a unique solution to* (vMin-large).

(iii) For every $(x,q) \in$ int$\mathcal{K}$, the subdifferential of u at (x,q) is a nonempty subset of ri$\mathcal{L}$, $(y,r) \in \partial u(x,q)$ if and only if the following conditions hold:

$$\widehat{Y}_T(y,r) = U'(\cdot, \widehat{X}_T(x,q) + qf), \quad \mathbb{P}\text{-}a.s., \tag{3.2}$$

$$\mathbb{E}\left[\widehat{Y}_T(\widehat{X}_T + qf)\right] = xy + qr, \tag{3.3}$$

$$|v(y,r)| < \infty. \tag{3.4}$$

Structure of the domains to (uMax-n), (uMax-large), (vMin-n), and (vMin-large)

For every (x,q) and (y,r) in $\mathbb{R}^{N+1}$, let us set

$$\begin{aligned} \mathcal{C}^n(x,q) &:= \left\{g \in \mathbb{L}^0_+ : g \le X_T + qf \ for\ some\ X \in \mathcal{X}^n(x,q)\right\}, \quad n \in \mathbb{N}, \\ \mathcal{C}(x,q) &:= \left\{g \in \mathbb{L}^0_+ : g \le X_T + qf \ for\ some\ X \in \mathcal{X}(x,q)\right\}, \\ \mathcal{D}^n(y,r) &:= \left\{h \in \mathbb{L}^0_+ : h \le Y_T \ for\ some\ Y \in \mathcal{Y}^n(y,r)\right\}, \quad n \in \mathbb{N}, \\ \mathcal{D}(y,r) &:= \left\{h \in \mathbb{L}^0_+ : h \le Y_T \ for\ some\ Y \in \mathcal{Y}(y,r)\right\}. \end{aligned} \tag{3.5}$$

For small markets, we recall [HK04, Proposition 1], whose intricate proof was based on a delicate parametrization of the dual domain, the Optional Decomposition Theorem from [Kra94], and superreplications results for finite-dimensional models from [DS98].

Proposition 3.2. *Let us assume that* (noArb) *and Assumption 2.1 hold. Then, for every $n \in \mathbb{N}$, the families* $(\mathcal{C}^n(x,q))_{(x,q)\in\mathcal{K}^n}$ *and* $(\mathcal{D}^n(y,r))_{(y,r)\mathcal{L}^n}$ *defined in* (3.5) *have the following properties:*

(1) For every $(x,q) \in int\mathcal{K}^n$, the set $\mathcal{C}^n(x,q)$ contains a strictly positive constant. A nonnegative random variable g belong to $\mathcal{C}^n(x,q)$ if and only if

$$\mathbb{E}\left[gh\right] \leq xy + qr, \quad \textit{for every } (y,r) \in \mathcal{L}^n \textit{ and } h \in \mathcal{D}^n(y,r)$$

(2) For every $(y,r) \in ri\mathcal{L}^n$, the set $\mathcal{D}^n(y,r)$ contains a strictly positive random variable. A nonnegative function $h \in \mathcal{D}^n(y,r)$ if and only if

$$\mathbb{E}\left[gh\right] \leq xy + qr, \quad \textit{for every } (x,q) \in \mathcal{K}^n \textit{ and } g \in \mathcal{C}^n(x,q).$$

Here is an analogous proposition, but for the large market.

Proposition 3.3. *Let us assume that* (noArb) *and Assumption 2.1 hold. Then for every the families* $(\mathcal{C}(x,q))_{(x,q)\in\mathcal{K}}$ *and* $(\mathcal{D}(y,r))_{(y,r)\mathcal{L}}$ *defined in* (3.5) *have the following properties:*

(i) For every $(x,q) \in int\mathcal{K}$, the set $\mathcal{C}(x,q)$ contains a strictly positive constant. A nonnegative random variable g belong to $\mathcal{C}(x,q)$ if and only if

$$\mathbb{E}\left[gh\right] \leq xy + qr, \quad \textit{for every } (y,r) \in \mathcal{L} \textit{ and } h \in \mathcal{D}(y,r) \tag{3.6}$$

(ii) For every $(y,r) \in ri\mathcal{L}$, the set $\mathcal{D}(y,r)$ contains a strictly positive random variable. A nonnegative function $h \in \mathcal{D}(y,r)$ if and only if

$$\mathbb{E}\left[gh\right] \leq xy + qr, \quad \textit{for every } (x,q) \in \mathcal{K} \textit{ and } g \in \mathcal{C}(x,q). \tag{3.7}$$

We summarize the characterizations of Assumption 2.1 in the following lemma. Here and below, we will also use the notation $\mathcal{M}$ and $\mathcal{M}^n$'s for the sets of probability measures, whose densities are in $\mathcal{Z}$ and $\mathcal{Z}^n$'s, respectively. Both kinds of notations are so common in the literature that we believe this will cause no confusion.

Lemma 3.4. *Let us assume that* (noArb) *and Assumption 2.1 hold. Then, we have*

(i) $(x,0) \in int\mathcal{K}$ for every $x > 0$,
(ii) for every $q \neq 0$, there exists $x > 0$, such that $(x,q) \in int\mathcal{K}$,

(iii) *(trivially) there exists a nonnegative wealth process of a generalized strategy, such that* $X_T \geq \sum\limits_{i=1}^{N} |f^i|$,

(iv) *(trivially)* $\sup\limits_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[\sum\limits_{i=1}^{N} |f^i|\right] < \infty$.

PROOF OF PROPOSITION 3.3 FOR THE LARGE MARKET ONLY

We begin with the following characterization of the set $\mathcal{K}$.

Lemma 3.5. *Under the conditions of Proposition 3.3,* $\mathcal{K}$ *is closed in* $\mathbb{R}^{N+1}$*, and thus, for every* $(x, q) \in cl\mathcal{K}$*,* $\mathcal{X}(x, q) \neq \emptyset$*.*

Proof. Note that the proof bypasses the Optional Decomposition Theorem, which was the center in the analysis of optimal investment with random endowment finite market, see [HK04].

Let $(x^n, q^n) \in \mathcal{K}$, $n \in \mathbb{N}$, be a sequence convergent to $(x, q) \in cl\mathcal{K}$, where the closure is taken in $\mathbb{R}^{N+1}$. Let us consider $X^n \in \mathcal{X}(x^n, q^n)$, $n \in \mathbb{N}$, and let us denote $Z^n := X^n + C$, $n \in \mathbb{N}$, where C is a sufficiently large positive constant such that $Z^n_T \geq 0$, for every $n \in \mathbb{N}$ (e.g., $C = N \max\limits_{n\in\mathbb{N}} ||q^n||_\infty \max\limits_{i\in\{1,\dots,N\}} ||f_i||_\infty$). One can see that Z^n are nonnegative $\mathbb{Q}$-supermartingales for every $\mathbb{Q} \in \mathcal{M}$ (see also [DDGP05, Section 2]). By passing to convex combinations, which we do not relabel, and to Fatou-convergence under any of such $\mathbb{Q}$'s, we can obtain a process Z, a Fatou-limit of $(Z^n)_{n\in\mathbb{N}}$ on the set of rational numbers and T. By construction, we have

$$Z_T - C + qf = \lim_{n\to\infty}(Z^n_T - C + q^n f) = \lim_{n\to\infty}(X^n_T + q^n f) \geq 0. \tag{3.8}$$

We also have

$$Z_0 \leq \liminf_{n\to\infty} Z^n_0 = x + C.$$

Therefore, as Z is a supermartingale for every $\mathbb{Q} \in \mathcal{M}$, we deduce that

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[Z_T\right] \leq Z_0 \leq x + C.$$

Now, [DDGP05, Theorem 3.1] implies that there exists an admissible generalized strategy H, such that

$$x + C + H \cdot S_T \geq Z_T, \quad \mathbb{P}\text{-a.s.} \tag{3.9}$$

Let us set

$$X := x + H \cdot S.$$

Then, using (3.8) and (3.9), we have that

$$X_0 = x \quad and \quad X_T + qf = x + C + H \cdot S_T + qf - C \geq Z_T + qf - C \geq 0,$$

where, in the first inequality, we have used (3.9), and, in the second,(3.8). We deduce that $X \in \mathcal{X}(x, q)$ and thus $(x, q) \in \mathcal{K}$. We conclude that $\mathcal{K}$ is closed. □

Let us consider the following parametrization of the dual domain.

$$\mathcal{M}(\rho) := \{\mathbb{Q} \in \mathcal{M} : \ \mathbb{E}_{\mathbb{Q}}[f] = \rho\}, \quad \rho \in \mathbb{R}^N. \tag{3.10}$$

Let us set

$$\mathcal{P}' := \{\rho \in \mathbb{R}^N : \ \mathcal{M}(\rho) \neq \emptyset\} \quad and \quad \mathcal{P} := \{\rho \in \mathbb{R}^N : \ (1, \rho) \in ri\mathcal{L}\}. \tag{3.11}$$

For the proofs below, we will impose the following non-replicability assumption, which allows us to handle the most difficult case. On the other hand, the cases when some of the components of f are replicable can be handled by reducing the dimensionality of the problem, and if all components of f are replicable, we can analyze (uMax-large) via the results from the optimal investment without random endowment, see, e.g., [DDGP05] and [Mos18], see also the discussion in Section 6.

Definition 3.1. We say that a bounded random variable g is replicable in the large market if there exists an admissible generalized wealth process[1] X, such that $-X$ is also admissible, and $X_T = g$.

Assumption 3.6. We will suppose that every component of f is non-replicable in the following sense: for every $q \in \mathbb{R}^N$, such that $q \neq 0$, the random variable qf is not replicable in the large market. We note that this condition is equivalent to $\mathcal{L}$ being open in $\mathbb{R}^{N+1}$, by a line-by-line adaptation of the argument from [HK04, Lemma 7].

Lemma 3.7. *Under the conditions of Proposition 3.3 and Assumption 3.6, we have*

$$\mathcal{P}' = \mathcal{P}. \tag{3.12}$$

[1]That is of the form $X = x + H \cdot S$, where $x \in \mathbb{R}$ and H is an admissible generalized strategy.

and

$$\bigcup_{\rho\in\mathcal{P}}\mathcal{M}(\rho)=\mathcal{M}.$$

Proof. Let us fix $q\in\mathbb{R}^N$, such that $q\neq 0$, and consider qf. One can see that Assumption 3.6 implies that, for every constant x, such that $(x,q)\in\mathcal{K}$, there exists $X\in\mathcal{X}(x,q)$, such that

$$\mathbb{P}[X_T+qf>0]>0.$$

Then, for $\mathbb{Q}\in\mathcal{M}(\rho)$, using the supermartingale property of X under $\mathbb{Q}$, we have

$$0<\mathbb{E}_{\mathbb{Q}}\left[X_T+qf\right]\leq x+q\rho.$$

As (x,q) is arbitrary in $\mathcal{K}$, we conclude that $\rho\in\mathcal{P}$. Therefore, we get

$$\mathcal{P}'\subseteq\mathcal{P}. \tag{3.13}$$

On the other hand, for a fixed $q\in\mathbb{R}^N$ and $x:=\sup\limits_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}_{\mathbb{Q}}\left[qf\right]$, by [DDGP05, Theorem 3.1], there exists an x-admissible generalized strategy H, such that

$$qf\leq x+H\cdot S_T.$$

This implies that $x+H\cdot S\in\mathcal{X}(x,-q)$, therefore, $(x,-q)\in\mathcal{K}$. As a result, we have

$$q\rho\leq x,\quad \rho\in\mathcal{P}.$$

We deduce that

$$\sup_{\rho\in\mathcal{P}'}q\rho=\sup_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}_{\mathbb{Q}}\left[qf\right]=x\geq\sup_{\rho\in\mathcal{P}}q\rho.$$

As q is arbitrary, we conclude that

$$\mathcal{P}'\supseteq\mathcal{P}. \tag{3.14}$$

Combining (3.13) and (3.14), we deduce that

$$\mathcal{P}'=\mathcal{P},$$

and thus

$$\mathcal{M}=\bigcup_{\rho\in\mathcal{P}'}\mathcal{M}(\rho)=\bigcup_{\rho\in\mathcal{P}}\mathcal{M}(\rho).$$

$\square$

Lemma 3.8. *Under the conditions of Proposition 3.3, for every $(x,q)\in\mathcal{K}$, $g\in\mathcal{C}(x,q)$ if and only if*

$$\mathbb{E}_{\mathbb{Q}}\left[g\right]\leq x+q\rho,\quad \textit{for every } \rho\in\mathcal{P}\quad \textit{and}\quad \mathbb{Q}\in\mathcal{M}(\rho). \tag{3.15}$$

Proof. Let us consider a nonnegative random variable g, such that (3.15) holds. Denote

$$h:=g-qf.$$

Then boundedness of f implies that $h\geq -C$ for some constant $C>0$. Therefore, we have

$$\begin{aligned}\sup_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}_{\mathbb{Q}}\left[h+C\right]&=\sup_{\rho\in\mathcal{P}}\sup_{\mathbb{Q}\in\mathcal{M}(\rho)}\mathbb{E}_{\mathbb{Q}}\left[h+C\right]\\&=\sup_{\rho\in\mathcal{P}}\sup_{\mathbb{Q}\in\mathcal{M}(\rho)}\mathbb{E}_{\mathbb{Q}}\left[g-qf+C\right]\leq x+C.\end{aligned}\tag{3.16}$$

As $h+C\in\mathbb{L}^0_+$, [DDGP05, Theorem 3.1] implies the existence of the $(x+C)$-admissible generalized strategy H, such that

$$h+C\leq x+C+H\cdot S_T,$$

and thus

$$0\leq g\leq x+H\cdot S_T+qf.$$

We deduce that $g\in\mathcal{C}(x,q)$.

Conversely, let $g\in\mathcal{C}(x,q)$. One can see that, for every $\rho\in\mathcal{P}$, the density process of $\mathbb{Q}\in\mathcal{M}(\rho)$ belongs to $\mathcal{Y}(1,\rho)$. This implies (3.15).

□

Proof of Proposition 3.3. Let $(x,q)\in\mathrm{int}\mathcal{K}$. Then there exists $\varepsilon>0$, such that $(x-\varepsilon,q)\in\mathcal{K}$. Now, let us pick $X\in\mathcal{X}(x-\varepsilon,q)$, then $X+\varepsilon\in\mathcal{X}(x,q)$, and

$$X_T+\varepsilon+qf\geq\varepsilon>0.$$

Therefore, $\varepsilon\in\mathcal{C}(x,q)$, and thus $\mathcal{C}(x,q)$ contains a positive constant.

If $g\in\mathcal{C}(x,q)$, then (3.6) follows from the construction of the sets $\mathcal{D}(y,r),(y,r)\in\mathcal{L}$. Conversely, let us assume that $g\in\mathbb{L}^0_+$, such that (3.6) holds. As, for every $\rho\in\mathcal{P}$, the density process of $\mathbb{Q}\in\mathcal{M}(\rho)$ belongs to $\mathcal{Y}(1,\rho)$, we deduce that g satisfies (3.15). Now, by Lemma 3.8, $g\in\mathcal{C}(x,q)$.

For the item (ii), first it is enough to prove the assertion for $(y,r)=(1,\rho)$ for some $\rho\in\mathcal{P}$, as $c\mathcal{D}(y,r)=\mathcal{D}(cy,cr)$ for every $c>0$ and $(y,r)\in\mathcal{L}$. By

Lemma 3.7, for every $\rho \in \mathcal{P}$, there exists $\mathbb{Q} \in \mathcal{M}(\rho)$. The density process $\mathbb{Q} \in \mathcal{M}(\rho)$ belongs to $\mathcal{Y}(1,\rho)$. As $\mathbb{Q} \sim \mathbb{P}$, $\frac{d\mathbb{Q}}{d\mathbb{P}} > 0$, $\mathbb{P}$-a.s.

If $h \in \mathcal{D}(1,\rho)$, then (3.7) follows from the definition of the set $\mathcal{Y}(1,\rho)$. Conversely, let us consider $h \in \mathbb{L}^0_+$, such that (3.7) holds. Then, in particular, we have

$$\mathbb{E}\left[gh\right] \leq 1, \quad \textit{for every } g \in \mathcal{C}(1,0),$$

where the set $\mathcal{C}(1,0) \neq \emptyset$ by Lemma 3.4. Therefore, by [Mos18, Lemma 3.4], h is a terminal value of an element of $\mathcal{Y}(1)$ and is such that (3.7) holds, i.e., $h \in \mathcal{Y}(1,\rho)$. □

Proving the utility maximization Theorem 3.1 for large markets

The proof of the following lemma is an adaptation of the proof of [Mos17, Lemma 2.6], and it is skipped.

Lemma 3.9. *Under the conditions of Theorem 3.1, we have*

$$u(x,q) > -\infty, \quad (x,q) \in int\mathcal{K} \quad \textit{and} \quad v(y,r) < \infty, \quad (y,r) \in ri\mathcal{L}. \tag{3.17}$$

Lemma 3.10. *Under the conditions of Theorem 3.1, we have*

$$u(x,q) \leq v(y,r) + xy + qr, \quad \textit{for every } (x,q) \in \mathcal{K}, \quad \textit{and every } (y,r) \in \mathcal{L}. \tag{3.18}$$

As a consequence, we have

$$u(x,q) < \infty, \quad \textit{on } \mathbb{R}^{N+1}, \quad v(y,r) > -\infty, \quad \textit{on } \mathbb{R}^{N+1}. \tag{3.19}$$

Proof. Let us fix $(x,q) \in \mathcal{K}$ and $(y,r) \in \mathcal{L}$. Then, for every $X \in \mathcal{X}(x,q)$ and $Y \in \mathcal{Y}(y,r)$, we have

$$U(X_T + qf) \leq V(Y_T) + (X_T + qf)Y_T, \quad \mathbb{P}\text{-a.s.},$$

and thus, taking the expectation and recalling (2.8), we obtain

$$\begin{aligned} \mathbb{E}\left[U(X_T + qf)\right] &\leq \mathbb{E}\left[V(Y_T)\right] + \mathbb{E}\left[(X_T + qf)Y_T\right] \\ &\leq \mathbb{E}\left[V(Y_T)\right] + xy + qr. \end{aligned} \tag{3.20}$$

As X and Y are arbitrary elements of $\mathcal{X}(x,q)$ and $\mathcal{Y}(y,r)$, taking the supremum over all $X \in \mathcal{X}(x,q)$ and (then) taking the infimum over all $Y \in \mathcal{Y}(y,r)$, in (3.20), we get

$$u(x,q) \leq v(y,r) + xy + qr, \tag{3.21}$$

which is precisely (3.18). In turn, (3.19) follows from (3.18) and Lemma 3.9. □

Proof of Theorem 3.1. The proof is an adaptation of the closely related proof of [Mos17, Theorem 2.4]. We will only highlight one point: for showing that $\partial u(x,q) \subset ri\mathcal{L}$, for $(x,q) \in int\mathcal{K}$, one can observe that, in the fully non-replicable case (as the one in Assumption 3.6), $0 < U'(\omega, \widehat{X}_T + qf)$ belongs to $\mathcal{D}(y', r')$ for every $(y', r') \in \partial u(x,q)$, and then, one can show that $(y', r') \in ri\mathcal{L}$. □

4. Marginal utility-based pricing in the large market

We consider the following definition.

Definition 4.1. Let $f^i \in \mathbb{L}^0$, $i \in \{1, \dots, N\}$, and $x > 0$. A vector $\rho \in \mathbb{R}^N$ is a *marginal utility-based price* for f given the initial capital x, if

$$\mathbb{E}\left[U(X_T + qf)\right] \le u(x, 0), \quad q \in \mathbb{R}^N, \ X \in \mathcal{X}(x - q\rho, q). \tag{4.1}$$

We denote the set of marginal utility-based prices by $\Pi(x)$.

This definition is a natural extension of standard definitions of the UBPs in the literature (see, e.g., [HKS05, Definition 3.1]) to stochastic utility and a large market. Let us observe that given (uMax-large), our formulations of the utility maximization problem for the large market, (4.1) is equivalent to

$$\left\{\rho : \ u(x - q\rho, q) \le u(x, 0), \quad for\ every \quad q \in \mathbb{R}^N\right\}. \tag{4.2}$$

We note that the initial wealth is important in both formulations (4.1) and (4.2), and thus the marginal utility-based prices depend on the initial wealth, in general. This observation has a clear financial interpretation, and therefore, we will denote the set of marginal utility-based prices by $\Pi(x)$.

Further, (4.2) leads to the following natural characterization of the set of marginal utility-based prices.

$$\Pi(x) = \left\{\frac{r}{y} : \ (y, r) \in \partial u(x, 0)\right\}. \tag{4.3}$$

In formulation (uMax-large), given the concavity of u, and in view of Lemma 3.4, we immediately obtain the existence of the marginal utility-based prices for every $x > 0$. If we fix an $x > 0$ first and then compute the $\Pi(x)$, the question of whether $\Pi(x)$ is a singleton or not, becomes important, as the uniqueness of the marginal utility-based prices is a necessary condition for the well-posedness in the sense of Hadamard of the marginal utility-based pricing problem. Such uniqueness is a desirable feature both from the mathematical and financial viewpoints. In the case when $\Pi(x)$ is a singleton, we get the following representation

$$\Pi(x) = \left\{ \frac{u_q(x,0)}{u_x(x,0)} \right\}.$$

Below we provide a sufficient condition for the uniqueness of the marginal utility-based prices starting from x.

Theorem 4.2. *Under the conditions of Theorem 3.1, let $x > 0$ be fixed and consider $y := u_x(x,0)$ and $\widehat{Y}(y)$, the optimizer to* (2.11) *at y. If $\mathbb{E}\left[\widehat{Y}_T(y)\right] = y$, then $\Pi(x)$ is a singleton, and for*

$$\rho := \mathbb{E}\left[\frac{\widehat{Y}_T(y)}{y} f\right] \in \mathcal{P}, \tag{UBP-rep}$$

we have

$$\Pi(x) = \{\rho\}.$$

For the proof of Theorem 4.2, we will need the following lemma.

Lemma 4.3. *Under the conditions of Theorem 3.1, let us consider* (uMax-large) *for $q = 0$, that is, $u(x,0) = \sup\limits_{X \in \mathcal{X}(x,0)} \mathbb{E}[U(X_T)]$, $x > 0$, and let x_n, $n \in \mathbb{N}$ be a sequence of strictly positive numbers converging to $x > 0$. Then, the optimizers to* (uMax-large), *$\widehat{X}_T(x_n,0) \to \widehat{X}_T(x,0)$ in probability.*

Proof. Let us denote

$$g := \widehat{X}_T(x,0), \quad \textit{and} \quad g^n := \widehat{X}_T(x_n,0), \quad n \in \mathbb{N}.$$

Then if g^n does not converge to g in probability, there exists $\varepsilon > 0$, such that

$$\limsup_{n\to\infty} \mathbb{P}\left[|g^n - g| > \varepsilon\right] > \varepsilon.$$

It follows from (noArb) that the set $\{X_T : \ X \in \mathcal{X}(1,0)\}$ is bounded in $\mathbb{L}^1$ under some probability measure, which is equivalent to $\mathbb{P}$. Therefore, $\{X_T : \ X \in \mathcal{X}(1,0)\}$ is bounded in $\mathbb{L}^0$ under $\mathbb{P}$, and, by passing to a smaller ε, if necessary, we get

$$\limsup_{n\to\infty} \mathbb{P}\left[|g^n - g| > \varepsilon, |g^n + g| < \tfrac{1}{\varepsilon}\right] > \varepsilon. \tag{4.4}$$

From the concavity of $U(\omega,\cdot)$, $\omega\in\Omega$, we deduce that

$$U\left(\frac{g^n+g}{2}\right)\geq \tfrac{1}{2}\left(U(g^n)+U(g)\right),$$

whereas (4.4) and the strict concavity of $U(\omega,\cdot)$, $\omega\in\Omega$, imply the existence of a random variable $\eta>0$[2] and a constant $\delta>0$, such that

$$\limsup_{n\to\infty}\mathbb{P}\left[U\left(\frac{g^n+g}{2}\right)\geq \tfrac{1}{2}\left(U(g^n)+U(g)\right)+\eta\right]>\delta.$$

As $u(\cdot,0)$ is concave and finite on $(0,\infty)$, it follows that $u(\cdot,0)$ is continuous on $(0,\infty)$, and with $A_n:=\left\{U\left(\frac{g^n+g}{2}\right)\geq \frac{1}{2}\left(U(g^n)+U(g)\right)+\eta\right\}$, $n\in\mathbb{N}$, we have

$$\limsup_{n\to\infty}\mathbb{E}\left[U\left(\frac{g^n+g}{2}\right)\right]\geq u(x,0)+\limsup_{n\to\infty}\mathbb{E}\left[\eta 1_{A_n}\right]>u(x,0). \tag{4.5}$$

Now, passing to convex combinations $\tilde{g}^n\in conv\left(g^n,g^{n+1},\dots\right)$, $n\in\mathbb{N}$, which converges to some random variable $\tilde{g}$, $\mathbb{P}$-a.s., and invoking [Mos15, Lemma 3.5], which, by the symmetry between the primal and dual value functions, and since $-V$ is the Inada stochastic field, also implies the uniform integrability of $(U^+(\tilde{g}^n))_{n\in\mathbb{N}}$, we get

$$\limsup_{n\to\infty}\mathbb{E}\left[U\left(\frac{\tilde{g}^n+g}{2}\right)\right]\leq\mathbb{E}\left[U\left(\frac{\tilde{g}+g}{2}\right)\right].$$

Therefore, by concavity of U, we get

$$\begin{aligned}\mathbb{E}\left[U\left(\frac{\tilde{g}+g}{2}\right)\right]&\geq\limsup_{n\to\infty}\mathbb{E}\left[U\left(\frac{\tilde{g}^n+g}{2}\right)\right]\\&\geq\limsup_{n\to\infty}\mathbb{E}\left[U\left(\frac{g^n+g}{2}\right)\right]>u(x,0),\end{aligned}\tag{4.6}$$

where, in the last inequality, we have used (4.5). Using [DDGP05, Lemma 3.3], we deduce the existence of $X\in\mathcal{X}(x,0)$, such that

$$X_T\geq\frac{\tilde{g}+g}{2},\quad \mathbb{P}\text{-a.s.}$$

Combining the latter inequality with (4.6), we conclude that

$$\mathbb{E}\left[U\left(X_T\right)\right]>u(x,0),$$

which is a contradiction. □

[2]If U is deterministic, η can be chosen to be a constant.

Proof of Theorem 4.2. Let us fix $q \in \mathbb{R}^N$, such that $(x - q\rho, q) \in \mathcal{K}$, and consider an arbitrary $X \in \mathcal{X}(x - q\rho, q)$. From the boundedness of f, one can show that $\widehat{Y}(y)X$ is a $\mathbb{P}$-supermartingale. From [Mos18, Theorem 2.2], we have

$$u(x,0) - xy = \tilde{w}(y).$$

Therefore, using conjugacy of U and V, we get

$$\begin{aligned}\mathbb{E}\left[U(X_T + qf)\right] &\le \mathbb{E}\left[V(\widehat{Y}_T(y)) + \widehat{Y}_T(y)(X_T + qf)\right] \\ &= \tilde{w}(y) + \mathbb{E}\left[\widehat{Y}_T(y)(X_T + qf)\right] \\ &= u(x,0) - xy + \mathbb{E}\left[\widehat{Y}_T(y)(X_T + qf)\right] \\ &\le u(x,0) - xy + y(x - q\rho + q\rho) = u(x,0).\end{aligned} \tag{4.7}$$

As q is an arbitrary element of $\mathbb{R}^N$, such that $(x - q\rho, q) \in \mathcal{K}$ and X is an arbitrary element of $\mathcal{X}(x - q\rho, q)$, we deduce from (4.7) (comparing (4.7) with (4.1)) that ρ is a marginal utility-based price for f.

To show the uniqueness of ρ, let us consider $\pi \neq \rho$. First, we will suppose that $\pi_i < \rho_i$ for some $i \in \{1, \dots, N\}$. For $C_k := ||f_k||_\infty$ and $\vec{c} := (C_1, \dots, C_N)$, and with e_i being a vector, whose ith component is 1 and all other components are zero, let us consider a sequence of positive numbers $(s_n)_{n \in \mathbb{N}} \to 0$, such that $s_n e_i(\vec{c} + \pi) < x$, $n \in \mathbb{N}$, let us set

$$q_n := s_n e_i, \quad X^n := \widehat{X}(x - q_n(\vec{c} + \pi), 0) + q_n\vec{c}, \quad n \in \mathbb{N},$$

Then, we have

$$X_0^n = x - q_n\pi, \quad \textit{and} \quad X^n \in \mathcal{X}(x - q_n\pi, q_n), \quad n \in \mathbb{N}.$$

We deduce that

$$\begin{aligned}u(x - q_n\pi, q_n) &\ge \mathbb{E}\left[U\left(X_T^n + q_n f\right)\right] \\ &\ge \mathbb{E}\left[U\left(\widehat{X}_T(x - q_n(\vec{c} + \pi), 0)\right)\right] + \mathbb{E}\left[q_n(\vec{c} + f)U'(X_T^n + q_n f)\right] \\ &= u\left(x - q_n(\vec{c} - \pi), 0\right) + \mathbb{E}\left[q_n(\vec{c} + f)U'(X_T^n + q_n f)\right].\end{aligned} \tag{4.8}$$

Therefore, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \frac{u(x - q_n\pi, q_n) - u\left(x - q_n(\vec{c} - \pi), 0\right)}{s_n} &\geq \liminf_{n\to\infty} \mathbb{E}\left[e_i(\vec{c} + f)U'(X^n_T + q_n f)\right] \\
&\geq \mathbb{E}\left[e_i(\vec{c} + f)U'\left(\widehat{X}_T(x, 0)\right)\right] \\
&= \mathbb{E}\left[e_i(\vec{c} + f)\widehat{Y}_T(y, 0)\right] \\
&= e_i(\vec{c} + \rho)y,
\end{aligned} \tag{4.9}$$

where, in the second inequality, we have used Fatou's lemma and the assertion of Lemma 4.3. We deduce that

$$\begin{aligned}
\liminf_{n\to\infty} \frac{u(x - q_n\pi, q_n) - u(x, 0)}{s_n} =& \liminf_{n\to\infty} \frac{u(x - q_n\pi, q_n) - u\left(x - q_n(\vec{c} + \pi), 0\right)}{s_n} \\
&+ \liminf_{n\to\infty} \frac{u\left(x - q_n(\vec{c} + \pi), 0\right) - u(x, 0)}{s_n} \\
\geq& e_i(\vec{c} + \rho)y - e_i u_x\left(x, 0\right)\left(\vec{c} + \pi\right) \\
=& e_i(\rho - \pi)y > 0.
\end{aligned}$$

As $s_n > 0$, $n \in \mathbb{N}$, we deduce that π is not a marginal utility-based price, as this π does not satisfy (4.2). As π_i was an arbitrary number smaller than ρ_i, we deduce that every π with $\pi_i < \rho_i$ is not a marginal utility-based price for f. Denoting $\tilde{f} = -f$, we can apply the argument above to show that every $\tilde{\pi}$, such that $\tilde{\pi}_i < -\rho_i$, is not a marginal utility-based price for $-f$, and thus every π, such that $\pi_i > \rho_i$ is not a marginal utility-based price for f. As $i \in \{1, \dots, N\}$ was arbitrary, we deduce that every π, such that $\pi_i \neq \rho_i$ for some $i \in \{1, \dots, N\}$ is not a marginal utility-based price. That is, under the conditions of this theorem, the marginal utility-based price ρ given by (UBP-rep), is unique.

Finally, to show that $\rho \in \mathcal{P}$, we observe that since $\mathbb{E}[\widehat{Y}_T(y)] = y$, we deduce that $\widehat{Y}(y)$ is a density process of an element of $\mathcal{M}$. Therefore, $\rho \in \mathcal{P}$, by Lemma 3.7. □

5. Marginal utility-based pricing in a large market as a limit of marginal utility-based pricing in small markets

For the convergence results, we need to strengthen (fin u, v) to ensure that the value functions in small markets to be finite-valued too.

Assumption 5.1. There exists $\tilde{n} \in \mathbb{N}$, such that

$$u^{\tilde{n}}(x,0) > -\infty, \quad x > 0, \quad \text{and} \quad \tilde{w}(y) < \infty, \quad y > 0. \tag{5.1}$$

Remark 5.2. Under Assumptions 2.3, 2.1, (noArb), and 5.1, an application of [Mos17, Lemma 2.6] imply (fin u, v) and

$$u^n(x,q) > -\infty, \ (x,q) \in int\mathcal{K}^n \quad and \quad v^n(y,r) < \infty, \ (y,r) \in ri\mathcal{L}^n, \quad n \geq n'.$$

Under Assumption 5.1, the marginal utility-based prices in small markets, $\Pi^n(x)$, $n \in \mathbb{N}$, can be characterized similarly to Theorem 4.2.

5.1. **The case when dual minimizers are true martingales.** Let us recall the dual problems in small markets without endowment.

$$\tilde{w}^n(y) := \inf_{Y \in \mathcal{Y}^n(y)} \mathbb{E}[V(Y_T)], \quad y > 0, \quad n \in \mathbb{N}, \tag{5.2}$$

It follows from Theorem 6.5 that under Assumptions 2.3, 5.1, and (noArb), that (5.2) admit unique minimizers for every $(y,n) \in (0,\infty) \times \mathbb{N}$.

The following lemma gives convergence of the marginal utility-based prices in small markets to the one in the large market under the assumption that the dual minimizers are true martingales.

Lemma 5.3. *Let us suppose that Assumptions 2.3, 2.1,* (noArb)*, and 5.1 hold. Let us consider a sequence of strictly positive numbers* $(x^n)_{n\in\mathbb{N}}$ *converging to* $x > 0$*. Then* $y^n := u^n_x(x^n, 0)$*,* $n \geq \tilde{n}$*, and* $y := u_x(x,0)$*, are well-defined and we have*

$$\lim_{n\to\infty} y^n = y > 0.$$

Moreover, if both $\widehat{Y}(y)$*, the minimizer to* (2.11)*, and* $\widehat{Y}^n(y^n)$*,* $n \geq \tilde{n}$*, the minimizers to* (5.2) *are martingales, then we have*

$$\lim_{n\to\infty} \rho^n(x) = \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{y^n}\widehat{Y}^n_T(y^n,0)f\right] = \mathbb{E}\left[\frac{1}{y}\widehat{Y}_T(y,0)f\right] = \rho(x), \tag{5.3}$$

that is, the marginal utility-based prices are singletons, have representations as in (5.3)*, and converge.*

Proof. By [Mos18, Lemma 2.4], we have $u^n(\cdot,0) \to u(\cdot,0)$ (in the case of deterministic utility this follows from [DDGP05, Proposition 4.3]). By [Mos18, Theorem 2.2], $u(\cdot,0)$ is continuously differentiable, and, by Theorem 6.5, so are $u^n(\cdot,0)$, $n \in \mathbb{N}$. Therefore, y^n, $n \geq \tilde{n}$, and y are well-defined, and by [Roc70, Theorem 24.5], we have $y^n \to y$.

Next, similarly to Lemma 4.3, one can show that $\widehat{Y}^n_T(y^n) \to \widehat{Y}_T(y)$ in probability. As $\widehat{Y}^n(y^n)$, $n \geq \tilde{n}$, and $\widehat{Y}(y)$ are martingales, and $y^n \to y$, using Scheffe's lemma, we deduce that $\widehat{Y}^n_T(y^n)f \to \widehat{Y}_T(y)f$ in $\mathbb{L}^1(\mathbb{P})$. Finally, using Theorem 4.2, where a similar argument can be used to obtain the representation in (5.3) of the marginal utility-based prices in small markets, we deduce that (5.3) holds. □

The following example illustrates the assertions of Lemma 5.3. It also demonstrates that superreplication prices in small markets do not converge to the one(s) in the large market, in general. Nevertheless, the martingale property of the dual minimizers, as in Lemma 5.3, ensures the convergence of the marginal utility-based prices.

Example 5.4. In a one period-setting, where $\Omega = \{\omega_n\}_{n=0}^{\infty}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}_1$ is discrete σ-algebra on Ω. We suppose that $\mathbb{P}[\omega_n] > 0$, $n \geq 0$, and that the asset prices are given by

$$S^0 \equiv 1, \quad S^n_0 = s^n, \quad S^n_1(\omega_n) = 1, \quad S^n_1(\omega_k) = 0, \quad k \neq n,$$

where s^n are strictly positive numbers such that $\sum\limits_{k=1}^{\infty} s^n < 1$.

One can see that for every $n \in \mathbb{N}$, the market with traded securities $S^0, \dots, S^n$ is incomplete. An example of a non-replicable claim in every such small market is

$$f := 1 - 1_{\{\omega_0\}} = \sum_{k=1}^{\infty} S^k_1. \tag{5.4}$$

The superreplication price for f, $\pi^n = 1$, $n \in \mathbb{N}$. This can also be obtained via (5.13), as

$$\pi^n = \sup_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[1 - 1_{\{\omega_0\}}\right] = 1 - \inf_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{Q}\left[1_{\{\omega_0\}}\right] = 1, \quad n \in \mathbb{N}.$$

Here, we have that the elements of $\mathcal{M}^n$ can be identified with sequences of strictly positive numbers $(q_0, q_1, \dots)$ adding up to 1 and such that $q_j = s^j$, $j = 1, \dots n$. Here, for a given $\mathbb{Q} \in \mathcal{M}^n$, q_k represents $\mathbb{Q}(\omega_k)$, $k \in \{0, 1, \dots\}$.

On the other hand, the large market is complete, as every $1_{\{\omega_n\}}$, $n \in \mathbb{N}$, can be replicated by one share of S^n with the initial cost s^n, and $1_{\{\omega_0\}} =$

$S_1^0 - \sum_{k=1}^{\infty} S_1^k$, and so $1_{\{\omega_0\}}$ can be replicated with the initial cost $1 - \sum_{k=1}^{\infty} s^k$. In particular, f in (5.4) can be replicated with the initial cost $\pi = \sum_{k=1}^{\infty} s^k < 1$.

To recapitulate, we do not have convergence of superreplication prices for f, as

$$\lim_{n\to\infty} \pi^n = 1 > \sum_{k=1}^{\infty} s^k = \pi.$$

Let us fix a *deterministic* utility function U satisfying Assumption 2.3. In the large market, the marginal utility-based price is equal to $\pi = \sum_{k=1}^{\infty} s^k$ by Theorem 4.2, and since the unique martingale measure in this market has the density

$$Z_1 = \frac{1 - \sum_{j=1}^{\infty} s^j}{\mathbb{P}[\omega_0]} 1_{\{\omega_0\}} + \sum_{i=1}^{\infty} \frac{s^i}{\mathbb{P}[\omega_i]} 1_{\{\omega_i\}}, \tag{5.5}$$

therefore $\widehat{Y}_1(y) = yZ_1$ is the minimizer to (2.11) for every $y > 0$.

In the market with $S^0, \dots, S^n$ are traded denoting by h^i the proportion of wealth invested in S^i, $i = 0, \dots, n$, so that the corresponding wealth at time 1 is given by

$$X_1^n(x) = x \left(h^0 + \sum_{i=1}^{n} \frac{h^i}{s^i} 1_{\{\omega_i\}} \right).$$

Let us state an auxiliary optimization problem

$$\begin{aligned} &\max_{h^0,\dots,h^n} \quad (1 - \sum_{i=1}^{n} \mathbb{P}[\omega_i]) U\left(xh^0\right) + \sum_{i=1}^{n} \mathbb{P}[\omega_i] U \left(x \left(h^0 + \frac{h^i}{s^i} \right) \right), \\ &\text{subject to} \quad \sum_{i=0}^{n} h^i = 1. \end{aligned} \tag{5.6}$$

This formulation does not include the admissibility condition. However, the computations below show that the solution to this problem gives a positive wealth process and, thus, the optimizer to the utility maximization problem without random endowment. Introducing the Lagrangian $L(h, \lambda)$, $(h, \lambda) \in \mathbb{R}^{n+2}$, by

$$L(h,\lambda) := \left(1 - \sum_{i=1}^{n} \mathbb{P}[\omega_i]\right) U\left(xh^0\right) + \sum_{i=1}^{n} \mathbb{P}[\omega_i] U \left(x \left(h^0 + \frac{h^i}{s^i} \right) \right) + \lambda \left(1 - \sum_{i=0}^{n} h^i \right),$$

from the optimality conditions, we get

$$\begin{aligned} &x\left(1-\sum_{k=1}^{n}\mathbb{P}[\omega_k]\right)U'\left(xh^0\right)+x\sum_{j=1}^{n}\mathbb{P}[\omega_j]U'\left(x\left(h^0+\frac{h^j}{s^j}\right)\right)-\lambda=0,\\ &\mathbb{P}[\omega_i]U'\left(x\left(h^0+\frac{h^i}{s^i}\right)\right)\frac{x}{s^i}-\lambda=0,\quad i=1,\dots,n, \end{aligned} \tag{5.7}$$

which leads to

$$\begin{aligned} &x\left(1-\sum_{k=1}^{n}\mathbb{P}[\omega_k]\right)U'\left(xh^0\right)-\lambda\left(1-\sum_{j=1}^{n}s^j\right)=0,\\ &\mathbb{P}[\omega_i]U'\left(x\left(h^0+\frac{h^i}{s^i}\right)\right)\frac{x}{s^i}-\lambda=0,\quad i=1,\dots,n, \end{aligned} \tag{5.8}$$

Thus, the optimal $h^0,\dots,h^n$ are given by

$$\begin{aligned} h^0&=\frac{1}{x}(U')^{-1}\left(\frac{\lambda}{x}\frac{\left(1-\sum\limits_{j=1}^{n}s^j\right)}{\left(1-\sum\limits_{k=1}^{n}\mathbb{P}[\omega_k]\right)}\right),\\ h^i&=s^i\frac{1}{x}(U')^{-1}\left(\frac{\lambda}{x}\frac{s^i}{\mathbb{P}[\omega_i]}\right)-h^0s^i, \end{aligned} \tag{5.9}$$

where λ is the unique solution to

$$\frac{\left(1-\sum\limits_{j=1}^{n}s^j\right)}{x}(U')^{-1}\left(\frac{\lambda}{x}\frac{\left(1-\sum\limits_{j=1}^{n}s^j\right)}{\left(1-\sum\limits_{k=1}^{n}\mathbb{P}[\omega_k]\right)}\right)+\sum_{i=1}^{n}\frac{s^i}{x}(U')^{-1}\left(\frac{\lambda}{x}\frac{s^i}{\mathbb{P}[\omega_i]}\right)=1,$$

where the existence and uniqueness of such λ follows from the strict monotonicity of U' and the Inada conditions. Therefore the optimal wealth at time 1 is

$$\widehat{X}_1^n(x)=(U')^{-1}\left(\frac{\lambda}{x}\frac{1-\sum\limits_{j=1}^{n}s^j}{1-\sum\limits_{k=1}^{n}\mathbb{P}[\omega_k]}\right)\left(1-\sum_{l=1}^{n}1_{\{\omega_l\}}\right)+\sum_{i=1}^{n}(U')^{-1}\left(\frac{\lambda}{x}\frac{s^i}{\mathbb{P}[\omega_i]}\right)1_{\{\omega_i\}}. \tag{5.10}$$

From (5.10), one can see that $\widehat{X}^n(x)>0$ for every $\omega\in\Omega$. Therefore, considering (5.6) allowed for finding a candidate solution, which satisfies the admissibility constraint(s), and, thus, is the optimizer to $u^n(x,0)$, $x>0$.

Consequently, from (5.10), we obtain that the density of the martingale measure for $(S^0, \dots, S^n)$ given by

$$Z_1^n = \frac{1 - \sum\limits_{j=1}^{n} s^j}{1 - \sum\limits_{k=1}^{n} \mathbb{P}[\omega_k]} \left(1 - \sum_{l=1}^{n} 1_{\{\omega_l\}}\right) + \sum_{i=1}^{n} \frac{s^i}{\mathbb{P}[\omega_i]} 1_{\{\omega_i\}}, \tag{5.11}$$

is, up to a multiplicative constant, the (dual) minimizer to (5.2) for every deterministic utility in the market with traded securities $S^0, \dots, S^n$. Therefore, one can see that for every bounded contingent claim $\tilde{f}$, the set of the marginal utility-based prices in such a market is

$$\Pi^n(x) = \left\{\mathbb{E}\left[Z_1^n \tilde{f}\right]\right\}, \quad n \in \mathbb{N}, \quad x > 0.$$

Next, from (5.11), we get

$$\lim_{n\to\infty} Z_1^n = \frac{1 - \sum\limits_{j=1}^{\infty} s^j}{1 - \sum\limits_{k=1}^{\infty} \mathbb{P}[\omega_k]} 1_{\{\omega_0\}} + \sum_{i=1}^{\infty} \frac{s^i}{\mathbb{P}[\omega_i]} 1_{\{\omega_i\}} = Z_1, \tag{5.12}$$

where Z_1 is given by (5.5), which is the density of the unique martingale measure in the large market. As $\mathbb{E}[Z_1^n] = 1$, $n \in \mathbb{N}$, and $\mathbb{E}[Z_1] = 1$, Scheffe's lemma implies that convergence in (5.12) also takes place in $\mathbb{L}^1(\mathbb{P})$. Therefore, for every bounded contingent claim $\tilde{f}$, the corresponding sequence of the marginal utility-based prices in a sequence of small markets converge to the one in the large market. The argument above, with some modifications, can be extended to stochastic utilities satisfying Assumption 2.3 and such that Assumption 5.1 holds.

5.2. **The case of no martingale assumption.** The main result of this subsection is Theorem 5.8, stated without assuming that the marginal utility-based prices in large or small markets are singletons, yet under an additional Assumption 5.7. The key role in the proof is played by the auxiliary minimization problems (5.39). In turn, their domains are given by the polars in $\mathbb{L}^0(\mathbb{P})$ to $\mathcal{C}(x,q)$ defined in (3.5) for $(x,q) \in int\mathcal{K}$. Here a special role is played by the sets $\Theta(x,q)$'s defined in (5.16). Lemma 5.6 establishes properties of $\Theta(x,q)$'s, and Lemma 5.11 shows that they generate the polars to $\mathcal{C}(x,q)$'s. Further, we will need the convergence of the domains of the value functions from (uMax-large); this is established in Lemma 5.15. In turn, this will allow showing the convergence of the value functions in (5.39).

We recall that in the finite-dimensional markets, the superreplicaiton price of a contingent claim $\tilde{f} \in \mathbb{L}^0_+$ is characterized as

$$\inf_{x\in\mathbb{R}} \left\{ x + H\cdot S_T \geq \tilde{f},\ \mathbb{P}\text{-a.s.},\ \mathit{for\ some}\ H \in \mathcal{H}^n \right\} = \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[\tilde{f}\right]. \tag{5.13}$$

This is the subject of [DS98, Theorem 5.12], for example, among others, which characterizes the superreplication price as

$$\pi_n(\tilde{f}) := \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[\tilde{f}], \quad n\in\mathbb{N}. \tag{5.14}$$

One can see that $(\pi_n(\tilde{f}))_{n\in\mathbb{N}}$ is decreasing. By setting

$$\pi(\tilde{f}) := \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[\tilde{f}],$$

we are trying to build an analog of the arbitrage-free price in finite-dimensional markets. Immediately, from the definition of π_n's and π, we get

$$\lim_{n\to\infty} \pi_n(\tilde{f}) = \inf_{n\in\mathbb{N}} \pi_n(\tilde{f}) \geq \pi(\tilde{f}). \tag{5.15}$$

As pointed out in Example 5.4 above, the inequality can be strict. In this case, we might say that the superreplications prices in finite-dimensional markets do not converge to the one in the large market. We might have situations, when the domains of the optimization problems do not converge in the set-theoretic sense, that is

$$\mathcal{L} \neq \bigcap_{n\in\mathbb{N}} \mathcal{L}^n \quad \mathit{and\ therefore} \quad \mathcal{K} \neq \bigcup_{n\in\mathbb{N}} \mathcal{K}^n.$$

In this case, the model in the large market is not a limit of the finite-dimensional models and thus is not as interesting.

Let us recall that $\mathcal{M}(\rho)$ and $\mathcal{P}$ are defined in (3.10) and (3.11), respectively. Analogously, we can specify $\mathcal{M}^n(\rho)$ and $\mathcal{P}^n$ for the market with traded $(S^0,\dots,S^n)$, $n\in\mathbb{N}$, and set

$$\begin{aligned} \Theta^n(x,q) &:= \bigcup_{\rho\in\mathcal{P}^n} \bigcup_{\mathbb{Q}\in\mathcal{M}^n(\rho)} \frac{1}{x+q\rho}\frac{d\mathbb{Q}}{d\mathbb{P}}, \quad n\in\mathbb{N}, \\ \Theta(x,q) &:= \bigcup_{\rho\in\mathcal{P}} \bigcup_{\mathbb{Q}\in\mathcal{M}(\rho)} \frac{1}{x+q\rho}\frac{d\mathbb{Q}}{d\mathbb{P}}. \end{aligned} \tag{5.16}$$

We recall that for $(x,q) \in int\mathcal{K}$, $\rho \in \mathcal{P}$, and $\mathbb{Q} \in \mathcal{M}(\rho)$, we have that $x + q\rho > 0$. The latter inequality also holds for $(x,q) \in int\mathcal{K}^n$, $\rho \in \mathcal{P}^n$, and $\mathbb{Q} \in \mathcal{M}^n(\rho)$.

Remark 5.5. With $\rho(\mathbb{Q}) := \mathbb{E}_{\mathbb{Q}}[f]$, $\mathbb{Q} \in \mathcal{M}^1$, a convenient way of thinking about the construction of $\Theta^n(x,q)$'s and $\Theta(x,q)$ is via

$$\Theta^n(x,q) := \bigcup_{\mathbb{Q}\in\mathcal{M}^n} \frac{1}{x + q\rho(\mathbb{Q})}\frac{d\mathbb{Q}}{d\mathbb{P}}, \quad n \in \mathbb{N}, \quad \Theta(x,q) := \bigcup_{\mathbb{Q}\in\mathcal{M}} \frac{1}{x + q\rho(\mathbb{Q})}\frac{d\mathbb{Q}}{d\mathbb{P}}.$$

We will show below that $\Theta^n(x,q)$, $n \in \mathbb{N}$, and $\Theta(x,q)$ are the generating sets for the respective polars to $\mathcal{C}^n(x,q)$, $n \in \mathbb{N}$, and $\mathcal{C}(x,q)$ in $\mathbb{L}^0(\mathbb{P})$, in the following sense. If we set

$$\tilde{\Theta}^n(x,q) := \overline{conv\,(\Theta^n(x,q))}, \quad n \in \mathbb{N}, \quad and \quad \tilde{\Theta}(x,q) := \overline{conv\,(\Theta(x,q))},$$

the closures in $\mathbb{L}^0$ of the convex and solid hull of $\Theta^n(x,q)$ and $\Theta^n(x,q)$, respectively. We note that $\Theta(x,q)$ play exactly the same role with random endowment as the set $\mathcal{Z}$ without random endowment, that is, it is the generating set for the dual domain. Further, we can extend (noArb), which asserts that $\mathcal{Z} = \bigcap\limits_{n\in\mathbb{N}} \mathcal{Z}^n$, to

$$\Theta(x,q) = \bigcap_{n\in\mathbb{N}} \Theta^n(x,q),$$

which is a "tilted" and rescaled version of $\mathcal{Z}$. Here, the monotonicity of $\mathcal{Z}^n$, $n \in \mathbb{N}$, in n, is crucial, and the monotonicity of Θ^n's, in n, will also hold. However, showing that $\tilde{\Theta}(x,q) = \bigcap\limits_{n\in\mathbb{N}} \tilde{\Theta}^n(x,q)$ requires an extra condition, as a closure of a countable intersection is not equal to a countable intersection of closures, in general. Example 5.4 above implicitly supports this point. This is despite the fact that for $(x,q) = (1,0)$, it follows from [DDGP05, Lemma 3.4] that $\tilde{\Theta}(1,0) = \bigcap\limits_{n\in\mathbb{N}} \tilde{\Theta}^n(1,0)$, for other $(x,q) \in int\mathcal{K}$, such that $q \neq 0$, more work is needed.

Lemma 5.6. *Under the conditions of Proposition 3.3, for every $(x,q) \in int\mathcal{K}$, the set $\Theta(x,q)$ is convex, closed under the countable convex combinations*[3]*, and we have*

$$\sup_{h\in\Theta(x,q)} \mathbb{E}\left[gh\right] = \sup_{h\in\tilde{\Theta}(x,q)} \mathbb{E}\left[gh\right], \quad g\in\mathcal{C}(x,q). \tag{5.17}$$

Proof. Since $(x,q) \in int\mathcal{K}$, there exists $\delta > 0$, such that $(x-\delta, q) \in int\mathcal{K}$. Therefore, for every $\rho \in \mathcal{P}$, we have

$$x + q\rho \geq \delta > 0.$$

Next, we show that $\Theta(x,q)$ is closed under countable convex combinations. Let $\rho^i \in \mathcal{P}$, $\mathbb{Q}^i \in \mathcal{M}(\rho^i)$, and $\alpha^i = \frac{1}{x+q\rho^i}$, $i\in\mathbb{N}$. As $(x,q)\in int\mathcal{K}$, one can see that α^i's are uniformly bounded from above. Then $h^i = \alpha^i\frac{d\mathbb{Q}^i}{d\mathbb{P}} \in \Theta(x,q)$. For convex weights $\lambda^i \in [0,1]$, such that $\sum\limits_{i=1}^{\infty}\lambda^i = 1$, we want to show that

$$h := \sum_{i=1}^{\infty}\lambda^i h^i \in \Theta(x,q).$$

Let us set

$$\alpha := \sum_{j=1}^{\infty}\lambda^j\alpha^j \quad \textit{and} \quad \mu^i := \frac{\lambda^i\alpha^i}{\sum\limits_{j=1}^{\infty}\lambda^j\alpha^j} = \frac{\lambda^i\alpha^i}{\alpha} \in [0,1], \quad i\in\mathbb{N},$$

and a probability measure $\mathbb{Q}$, whose density is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \sum_{i=1}^{\infty}\mu^i\frac{d\mathbb{Q}^i}{d\mathbb{P}} \quad \textit{and} \quad \rho := \sum_{i=1}^{\infty}\mu^i\rho^i. \tag{5.18}$$

Then $\mathbb{Q}\in\mathcal{M}$, as an application of the monotone convergence theorem shows that $\mathcal{M}$ is closed under countable convex combinations (see, e.g., proof of [Mos18, Lemma 3.5]), and moreover, one can see that $\mathbb{Q}\in\mathcal{M}(\rho)$. Then we

[3] That is under combinations of the form $\sum\limits_{k=1}^{\infty}\lambda^k h^k$, where $(h^k)_{k\in\mathbb{N}} \subset \Theta(x,q)$ and λ^k's are nonnegative constants, such that $\sum\limits_{k=1}^{\infty}\lambda^k = 1$.

have

$$\begin{aligned} h &= \sum_{i=1}^{\infty} \lambda^i h^i = \sum_{i=1}^{\infty} \lambda^i \alpha^i \frac{d\mathbb{Q}^i}{d\mathbb{P}} \\ &= \left(\sum_{k=1}^{\infty} \lambda^k \alpha^k\right) \sum_{i=1}^{\infty} \frac{\lambda^i \alpha^i}{\sum\limits_{j=1}^{\infty} \lambda^j \alpha^j} \frac{d\mathbb{Q}^i}{d\mathbb{P}} = \alpha \sum_{i=1}^{\infty} \mu^i \frac{d\mathbb{Q}^i}{d\mathbb{P}} = \alpha \frac{d\mathbb{Q}}{d\mathbb{P}}. \end{aligned} \tag{5.19}$$

Next, let us observe that

$$\begin{aligned} 1 = \sum_{i=1}^{\infty} \lambda^i = \sum_{i=1}^{\infty} \lambda^i \underbrace{\alpha^i \left(x + q\rho^i\right)}_{=1} \\ &= x \sum_{i=1}^{\infty} \lambda^i \alpha^i + q \sum_{j=1}^{\infty} \lambda^j \alpha^j \rho^j \\ &= \left(x + q \sum_{j=1}^{\infty} \frac{\lambda^j \alpha^j}{\sum\limits_{k=1}^{\infty} \lambda^k \alpha^k} \rho^j\right) \left(\sum_{i=1}^{\infty} \lambda^i \alpha^i\right) \\ &= \left(x + q \sum_{j=1}^{\infty} \mu^j \rho^j\right) \left(\sum_{i=1}^{\infty} \lambda^i \alpha^i\right). \end{aligned}$$

That is, we have

$$\left(\sum_{i=1}^{\infty} \lambda^i \alpha^i\right) \left(x + q \sum_{j=1}^{\infty} \mu^j \rho^j\right) = 1,$$

and thus, recalling the definitions of α and ρ, we conclude that

$$\alpha(x + q\rho) = 1. \tag{5.20}$$

To recapitulate, (5.18), (5.19), and (5.20) imply that

$$h = \sum_{i=1}^{\infty} \lambda^i h^i = \alpha \frac{d\mathbb{Q}}{d\mathbb{P}}, \quad \textit{where} \quad \mathbb{Q} \in \mathcal{M}(\rho) \quad \textit{and} \quad \alpha = \frac{1}{x + q\rho},$$

that is $h \in \Theta(x, q)$. Therefore, $\Theta(x, q)$ is closed under countable convex combinations. In turn, (5.17) follows from the respective constructions of $\Theta(x, q)$ and $\tilde{\Theta}(x, q)$, and Fatou's lemma.

□

We impose the following assumption, and see how it holds in examples of Section 6. In cases when the optimal Z's for the large market are elements of $\mathcal{Z}$, one can typically have a natural candidate for the approximating sequence as in examples of Section 6.

Assumption 5.7. For every $(x, q) \in int\mathcal{K}$ and $z > 0$, there exists a uniformly integrable sequence sequence $h^n \in \Theta^n(x, q)$, $n \in \mathbb{N}$, such that

$$\sup_{n\in\mathbb{N}} \inf_{h\in\Theta^n(x,q)} \mathbb{E}\left[V(zh)\right] = \lim_{n\to\infty} \mathbb{E}\left[V(zh^n)\right],$$

and for every $q \in \mathbb{R}^N$, there exists a uniformly integrable sequence $\tilde{h}^n$, $n \in \mathbb{N}$, where each $\tilde{h}^n$ is a terminal value of an element of $\mathcal{Z}^n$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sup_{h\in\mathcal{Z}^n} \mathbb{E}[hqf] = \lim_{n\to\infty} \mathbb{E}\left[\tilde{h}^n qf\right].$$

We note that the latter assumption holds if $\{Z_T : \ Z \in \mathcal{Z}^n\}$ is uniformly integrable for some $n \in \mathbb{N}$, which is much stronger. The primary result of this section is the following theorem. We recall that $\Pi(x)$ is given in (4.3), and $\Pi^n(x)$'s are specified entirely similarly for the market with n risky assets.

Theorem 5.8. *Let us suppose that Assumptions 2.3, 2.1,* (noArb)*, 5.1, and 5.7 hold. Then, for every sequence of strictly positive numbers $(x^n)_{n\in\mathbb{N}}$ converging to $x > 0$, we have*

$$\Pi^n(x^n) \to \Pi(x),$$

in the sense that, for every $\varepsilon > 0$, there exists $n' \in \mathbb{N}$, such that

$$\Pi^n(x^n) \subset \Pi(x) + \varepsilon B, \quad for\ every \quad n \geq n', \tag{5.21}$$

where B is the Euclidean unit ball of $\mathbb{R}^N$.

Remark 5.9. The assertions of Theorem 5.8 hold without any assumption on whether and of the sets $\Pi^n(x)$ or $\Pi(x)$ are singletons or not.

Lemma 5.10. *Under the conditions of Proposition 3.3, we have*

$$\overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n} \subseteq \mathcal{K}, \tag{5.22}$$

where the closure is taken in $\mathbb{R}^{N+1}$.

Proof. Let $(x^n, q^n) \in \mathcal{K}^n$, $n \in \mathbb{N}$, be a sequence convergent to $(x, q) \in \overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n}$. Then, for every $n \in \mathbb{N}$, for some $x^n + H^n \cdot S \in \mathcal{X}^n(x^n, q^n)$, we have

$$x^n + H^n \cdot S_T + q^n f \geq 0, \quad \mathbb{P}\text{-a.s.}, \quad n \in \mathbb{N}. \tag{5.23}$$

For an appropriate $C \in \mathbb{R}^N$, let us rewrite the latter inequalities as

$$\frac{H^n}{|x^n| + q^n C} \cdot S_T \geq \frac{-x^n - q^n f}{|x^n| + q^n C} \geq -1, \quad \mathbb{P}\text{-a.s.}, \quad n \in \mathbb{N}. \tag{5.24}$$

Therefore, [DDGP05, Lemma 3.3] applies, and $\frac{-x^n - q^n f}{|x^n| + q^n C} \to \frac{-x-qf}{|x|+qC}$, we deduce that there exists a 1-admissible generalized strategy, such that

$$\bar{H} \cdot S_T \geq \frac{-x - qf}{|x| + qC}.$$

Then $H := (|x| + qC)\bar{H}$ is a generalized admissible strategy that satisfies

$$x + H \cdot S_T + qf \geq 0, \quad \mathbb{P}\text{-a.s.}$$

In particular, we deduce that $x + H \cdot S_T \in \mathcal{X}(x, q) \neq \emptyset$ and thus $(x, q) \in \mathcal{K}$. $\square$

For every $(x, q) \in int\mathcal{K}$, let us define the following sets.

$$\begin{aligned}
\mathcal{B}^n(x, q) &:= \{(y, r) \in \mathcal{L}^n : \ xy + qr \leq 1\}, \quad n \in \mathbb{N}, \\
\mathcal{B}(x, q) &:= \{(y, r) \in \mathcal{L} : \ xy + qr \leq 1\}, \\
\tilde{\mathcal{D}}^n(x, q) &:= \bigcup_{(y,r) \in \mathcal{B}^n(x,q)} \mathcal{D}^n(y, r), \quad n \in \mathbb{N}, \\
\tilde{\mathcal{D}}(x, q) &:= \bigcup_{(y,r) \in \mathcal{B}(x,q)} \mathcal{D}(y, r).
\end{aligned} \tag{5.25}$$

Using Propositions 3.2 and 3.3, one can show that the sets $\tilde{\mathcal{D}}^n(x, q)$ and $\tilde{\mathcal{D}}(x, q)$ are polar to $\mathcal{C}^n(x, q)$ and $\mathcal{C}(x, q)$, respectively, and for every $\mathbb{Q} \in \mathcal{M}(\rho)$, $\rho \in \mathcal{P}$, there exists $\alpha = \alpha(\rho) = \frac{1}{x + q\rho}$, such that $\alpha \frac{d\mathbb{Q}}{d\mathbb{P}} \in \tilde{\mathcal{D}}(x, q)$ and $(\alpha + \delta)\frac{d\mathbb{Q}}{d\mathbb{P}} \notin \tilde{\mathcal{D}}(x, q)$, for every $\delta > 0$.

Lemma 5.11. *Under the conditions of Proposition 3.3, for every $(x, q) \in int\mathcal{K}$, we have*

$$\tilde{\Theta}(x, q) = \tilde{\mathcal{D}}(x, q). \tag{5.26}$$

Proof. Let $(x, q) \in int\mathcal{K}$ be fixed and consider an arbitrary $g \in \mathcal{C}(x, q)$. Then, we have

$$g \leq X_T + qf, \quad \mathbb{P}\text{-a.s.},$$

for some $X \in \mathcal{X}(x, q)$. Therefore, for every $\rho \in \mathcal{P}$, and $\mathbb{Q} \in \mathcal{M}(\rho)$, by the supermartingale property of X under $\mathbb{Q}$, we have

$$\mathbb{E}_{\mathbb{Q}}[g] \leq \mathbb{E}_{\mathbb{Q}}[X_T + qf] \leq x + q\rho.$$

This implies that for $h := \frac{1}{x+q\rho}\frac{d\mathbb{Q}}{d\mathbb{P}} \in \Theta(x,q)$, we have

$$\mathbb{E}[hg] \leq 1. \tag{5.27}$$

Next, for every $\tilde{h} \in \tilde{\Theta}(x,q)$ there exists a sequence $h^n \in \Theta(x,q)$, $n \in \mathbb{N}$, such that $\lim\limits_{n\to\infty} h^n \geq \tilde{h}$, and such that (5.27) holds for every h^n. The existence of such a sequence follows from the respective definitions of $\tilde{\Theta}(x,q)$ and $\Theta(x,q)$. Fatou's lemma implies that $\mathbb{E}[\tilde{h}g] \leq 1$, and thus by Proposition 3.3, $\tilde{h} \in \tilde{\mathcal{D}}(x,q)$. In turn, as $g \in \mathcal{C}(x,q)$ and $\tilde{h} \in \tilde{\Theta}(x,q)$ are arbitrary, this shows that

$$\tilde{\Theta}(x,q) \subseteq \tilde{\mathcal{D}}(x,q). \tag{5.28}$$

Conversely, let us consider $g \in \mathbb{L}^0_+$, such that

$$\mathbb{E}\left[gZ\right] \leq 1, \quad for\ every \quad Z \in \Theta(x,q). \tag{5.29}$$

We want to show that

$$g \leq X_T + qf$$

for some $X \in \mathcal{X}(x,q)$. (5.29) implies that for every $\rho \in \mathcal{P}$ and $\mathbb{Q} \in \mathcal{M}(\rho)$, we have

$$\mathbb{E}_{\mathbb{Q}}\left[g\frac{1}{x+q\rho}\right] \leq 1. \tag{5.30}$$

Then, we get

$$\mathbb{E}_{\mathbb{Q}}\left[(g-qf)\frac{1}{x+q\rho}\right] \leq 1 - \frac{q\rho}{x+q\rho} = \frac{x}{x+q\rho}.$$

Therefore, we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[g-qf\right] \leq x. \tag{5.31}$$

Let $C \in \mathbb{R}^N$ be such that $g - q(f-C) \in \mathbb{L}^0_+$. Then, from (5.31), we have

$$\mathbb{E}_{\mathbb{Q}}\left[(g-q(f-C))\right] \leq x + qC.$$

The latter inequality holds for every $\rho \in \mathcal{P}$, and $\mathbb{Q} \in \mathcal{M}(\rho)$, where the right-hand side does not depend on ρ. Consequently, from Lemma 3.7, we deduce that

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[(g-q(f-C))\right] \leq x + qC.$$

Now we apply [DDGP05, Theorem 3.1], which asserts that there exists an $(x+qC)$-admissible generalized strategy H, such that

$$g - q(f - C) \leq x + qC + H \cdot S_T,$$

and thus $X := x + H \cdot S \in \mathcal{X}(x,q)$ and X super replicates $g - qf$. In turn, this implies that $g \in \mathcal{C}(x,q)$. Therefore, $\mathcal{C}(x,q) \supseteq (\tilde{\Theta}(x,q))^o$. Also, from the construction of $\tilde{\mathcal{D}}(x,q)$ in (5.25) and Proposition (3.3), it follows that $\tilde{\mathcal{D}}(x,q) = (\mathcal{C}(x,q))^o$. As a result, we obtain

$$\tilde{\mathcal{D}}(x,q) = (\mathcal{C}(x,q))^o \subseteq (\tilde{\Theta}(x,q))^{oo} = \tilde{\Theta}(x,q), \tag{5.32}$$

where in the last equality, we have used the bipolar theorem of Brannath and Schachermayer, and we note that $\tilde{\Theta}(x,q)$ is convex, solid, and closed in $\mathbb{L}^0$ by construction.

Finally, (5.28) and (5.32) imply the assertion of the lemma, (5.26). □

Lemma 5.12. *Under the conditions of Theorem 5.8, let $(x,q) \in int\mathcal{K}$ be fixed, and $\bar{h}^n \in \Theta^n(x,q)$, $n \in \mathbb{N}$, be a uniformly integrable sequence, such that $\lim\limits_{n\to\infty} \bar{h}^n = h$, $\mathbb{P}$-a.s.. Then, $h \in \tilde{\mathcal{D}}(x,q)$.*

Proof. Let us consider an arbitrary $g \in \mathcal{C}(x,q)$. Then there exists $X \in \mathcal{X}(x,q)$, such that

$$g \leq X_T + qf.$$

For a sufficiently large constant $C > 0$, such that $X + C \in \mathcal{X}(x + C, 0)$, let us consider an approximating sequence $\tilde{X}^n \in \mathcal{X}^n(x + C, 0)$, $n \in \mathbb{N}$, and set $X^n := \tilde{X}^n - C$, $n \in \mathbb{N}$. By passing if necessary to subsequences, which we do not relabel, we have

$$\begin{aligned}\mathbb{E}[h(g+C)] &\leq \mathbb{E}[h(X_T + qf + C)] = \mathbb{E}[\liminf_{n\to\infty} \bar{h}^n(X^n_T + qf + C)] \\ &\leq \liminf_{n\to\infty} \mathbb{E}[\bar{h}^n(X^n_T + qf + C)] \leq 1 + C \liminf_{n\to\infty} \mathbb{E}[\bar{h}^n],\end{aligned}$$

where, in the second inequality, we have used Fatou's lemma, and, in the last inequality, we used the definition of the sets $\Theta^n(x,q)$ and the uniform integrability of $\bar{h}^n$, $n \in \mathbb{N}$. Now, one can see that $\mathbb{E}[hg] \leq 1$. Since g was an arbitrary element of $\mathcal{C}(x,q)$, we deduce that $h \in (\mathcal{C}(x,q))^o = \tilde{\mathcal{D}}(x,q)$. □

Remark 5.13. Assumption 5.7 implies the following condition

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[qf] = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[qf] \quad \textit{for every} \quad q \in \mathbb{R}^N. \tag{5.33}$$

This can be proven as follows. Let us fix $q \in \mathbb{R}^N$. By Assumption 5.7, there exists $Z^n \in \mathcal{Z}^n$, $n \in \mathbb{N}$, such that $\mathbb{Q}^n$, $n \in \mathbb{N}$, with $\frac{d\mathbb{Q}^n}{d\mathbb{P}} = Z^n_T$, is a maximizing sequence for (5.33), where Z^n_T, $n \in \mathbb{N}$, is uniformly integrable. By passing to convex combinations, we obtain a sequence, which we still denote $Z^n \in \mathcal{Z}^n$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} Z^n_T = h$, $\mathbb{P}$-a.s., for some nonnegative random variable h. Then we have

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[qf] = \lim_{n\to\infty} \mathbb{E}\left[Z^n_T qf\right] = \mathbb{E}\left[hqf\right]. \tag{5.34}$$

Lemma 5.12 implies that $h \in \tilde{\mathcal{D}}(1,0)$. Using Lemmas 5.6 and 5.11 (note that since qf is bounded, so it is in $\mathcal{C}(x,0)$ for a sufficiently large x, see Proposition 3.3), we deduce that

$$\sup_{Z\in\Theta(1,0)} \mathbb{E}[Z_T qf] = \sup_{h\in\mathcal{D}(1,0)} \mathbb{E}[hqf] \geq \mathbb{E}\left[hqf\right]. \tag{5.35}$$

Combining (5.34) and (5.35), we deduce (5.33).

Remark 5.14. When $N = 1$, (5.33) is equivalent to assuming that

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f] \quad \textit{and} \quad \lim_{n\to\infty} \inf_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f], \tag{5.36}$$

which is convergence of super and subreplication prices to the ones in the large market, respectfully. Without passing to the limit, that is without considering $\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f]$ and $\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f]$ in the limiting market, (5.36) is closely related to [ARS17, Assumption 4.1]. We also note that, for every fixed n, the consistency of the arbitrage-free pricing and utility-based pricing is established in [Sio16].

For the proof of Theorem 5.8, we will need the following lemma. With Assumption 5.7, we can get the opposite inclusion comparing to Lemma 5.10.

Lemma 5.15. *Let the assumptions of Theorem 5.8 hold. Then, we have*

$$\overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n} = \mathcal{K}. \tag{5.37}$$

Proof. We recall that, by Lemma 3.5, $\mathcal{K}$ is closed. Let $(x,q) \in \mathcal{K}$ and let $((x^k,q^k))_{k\in\mathbb{N}} \subset int\mathcal{K}$ convergent to (x,q). As $(x,0) \in \mathcal{K}^n$ for every $n \in \mathbb{N}$, it is enough to consider $q \neq 0$, and thus, it is enough to consider $q^k \neq 0$,

$k \in \mathbb{N}$. Let us fix $k \in \mathbb{N}$. As $(x^k, q^k) \in int\mathcal{K}$, we deduce that there exists $\delta^k > 0$, such that

$$(x^k - \delta^k, q^k) \in \mathcal{K}.$$

From the definition of $\mathcal{K}$, it follows that, there exists $X \in \mathcal{X}(x^k - \delta^k, q^k)$, such that

$$X + q^k f \geq 0, \quad \mathbb{P}\text{-a.s.}$$

Using the supermartingale property of X under every $\mathbb{Q} \in \mathcal{M}$, we deduce that

$$x^k - \delta^k + \mathbb{E}_{\mathbb{Q}}\left[q^k f\right] \geq \mathbb{E}_{\mathbb{Q}}\left[X_T + q^k f\right] \geq 0, \quad \mathbb{Q} \in \mathcal{M},$$

and thus

$$\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[-q^k f] \leq x^k - \delta^k.$$

Then, using Assumption 5.7, we can rewrite the latter inequality as

$$x^k - \delta^k \geq \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[-q^k f\right] = \lim_{n \to \infty} \sup_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[-q^k f\right],$$

and thus, we obtain

$$\lim_{n \to \infty} \sup_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[-q^k f\right] \leq x^k - \delta^k.$$

Let us fix an arbitrary $\varepsilon_n > 0$. Then, the latter inequality implies the existence of $n = n(\varepsilon_n) \in \mathbb{N}$, such that

$$\sup_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[-q^k f\right] \leq x^k - \delta^k + \varepsilon_n \tag{5.38}$$

By the super replication results for finite-dimensional models, see, e.g., [DS98, Theorem 5.12], (5.38) implies the existence of an $(x^k - \delta^k + \varepsilon_n)$-admissible n-elementary strategy H^n, such that

$$x^k - \delta^k + \varepsilon_n + H^n \cdot S_T + q^k f \geq 0, \quad \mathbb{P}\text{-a.s.}$$

Therefore $X := x^k - \delta^k + \varepsilon_n + H^n \cdot S \in \mathcal{X}^n(x^k - \delta^k + \varepsilon_n, q^k)$, and in particular $(x^k - \delta^k + \varepsilon_n, q^k) \in \mathcal{K}^n$. We deduce that

$$(x^k - \delta^k + \varepsilon_n, q^k) \in \bigcup_{n \in \mathbb{N}} \mathcal{K}^n$$

As ε_n was arbitrary, by picking, for example $\varepsilon_n = \delta^k$, we deduce that

$$((x^k, q^k))_{k \in \mathbb{N}} \subset \bigcup_{n \in \mathbb{N}} \mathcal{K}^n,$$

and therefore, since $((x^k, q^k))_{k\in\mathbb{N}} \subset int\mathcal{K}$ is convergent to (x, q), we deduce that $(x, q) \in \overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n}$. Therefore (5.37) holds.

□

Corollary 5.16. *Let the assumptions of Theorem 5.8 hold. Then, we have*

$$\mathcal{L} = \bigcap_{n\in\mathbb{N}} \mathcal{L}^n.$$

Corollary 5.17. *Let the assumptions of Theorem 5.8 hold. Then the sets of closures of arbitrage-free prices in small markets converge to the closure of the set of arbitrage-free prices in the large market.*

Now, for every $(x, q) \in int\mathcal{K}$, let us define

$$\begin{aligned} \tilde{v}^n(z) &:= \inf_{h\in\Theta^n(x,q)} \mathbb{E}\left[V(zh)\right], \quad z > 0, \quad n \geq n_0, \\ \tilde{v}(z) &:= \inf_{h\in\Theta(x,q)} \mathbb{E}\left[V(zh)\right], \quad z > 0. \end{aligned} \tag{5.39}$$

Lemma 5.18. *Under the conditions of Theorem 5.8, for every $(x, q) \in int\mathcal{K}$, there exists $n_0 \in \mathbb{N}$, such that*

$$\begin{aligned} \tilde{v}^n(z) &= \inf_{h\in\tilde{\mathcal{D}}^n(x,q)} \mathbb{E}\left[V(zh)\right], \quad z > 0, \quad n \geq n_0, \\ \tilde{v}(z) &= \inf_{h\in\tilde{\mathcal{D}}(x,q)} \mathbb{E}\left[V(zh)\right], \quad z > 0. \end{aligned}$$

Proof. First, we observe that $(x, q) \in int\mathcal{K}$ and Lemma 5.15, imply that there exists $n_0 \in \mathbb{N}$, such that $(x, q) \in \mathcal{K}^n$ for every $n \geq n_0$. Entirely similarly to the proofs of Lemmas 5.6 and 5.11, we can obtain similar assertions (convexity, closedness under countable convex combinations, (5.17) and (5.26)) for $\tilde{\Theta}^n(x, q)$, $n \geq n_0$, besides $\tilde{\Theta}(x, q)$. Now, the assertion of the lemma follows from Theorem 6.6. □

Proof of Theorem 5.8. Let $(x, q) \in int\mathcal{K}$ be fixed. One can see (e.g., using Lemma 5.18) the monotonicity of $\tilde{v}^n$: as $\mathcal{M}^n$, $n \in \mathbb{N}$, is decreasing, $\tilde{v}^n$, $n \geq n_0$, where n_0 is given by Lemma 5.18, is increasing and

$$\sup_{n\geq 1} \tilde{v}^n(z) = \lim_{n\to\infty} \tilde{v}^n(z) \leq \tilde{v}(z) < \infty, \quad z > 0. \tag{5.40}$$

Let us fix $z > 0$. Assumption 5.7 implies the existence of a uniformly integrable sequence $h^n \in \Theta^n(x, q)$, $n \geq n_0$, such that

$$\liminf_{n\to\infty} \tilde{v}^n(z) = \lim_{n\to\infty} \mathbb{E}\left[V(zh^n)\right]. \tag{5.41}$$

By passing to convex combinations $\tilde{h}^n \in conv(h^n, h^{n+1}, \dots)$, $n \in \mathbb{N}$, we can obtain a sequence such that $\tilde{h}^n \in \Theta^n(x,q)$, where the convexity of $\Theta^n(x,q)$ can be shown similarly to Lemma 5.6, $n \in \mathbb{N}$, and such that $\tilde{h}^n$ converges to some limit denoted by $\tilde{h}$, $\mathbb{P}$-a.s. By Lemma 5.12, we deduce that $\tilde{h} \in \tilde{\mathcal{D}}(x,q)$.

As $\tilde{h} \in \bigcap_{n\in\mathbb{N}} \mathcal{Y}^n(y^n) \subseteq \mathcal{Y}^1(\bar{y})$ for some $\bar{y} \in (0,\infty)$, and also $(\tilde{h}^n)_{n\geq \tilde{n}} \subseteq \mathcal{Y}^{\tilde{n}}(\bar{y})$, where $\tilde{n}$ is given by Assumption 5.1, via [Mos15, Lemma 3.5], we conclude that the sequence $V^-(z\tilde{h}^n)$, $n \in \mathbb{N}$, is uniformly integrable. Therefore, using the convexity of $V(\cdot,\omega)$, $\omega \in \Omega$, we obtain

(5.42)
$$\tilde{v}(z) \leq \mathbb{E}\left[V(z\tilde{h})\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[V(z\tilde{h}^n)\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[V(zh^n)\right] = \liminf_{n\to\infty} \tilde{v}^n(z),$$

where, in the last equality, we have used (5.41). Combining (5.40) and (5.42), we obtain

$$\tilde{v}(z) = \lim_{n\to\infty} \tilde{v}^n(z), \quad z > 0. \tag{5.43}$$

Let us recall that, by the construction of $\tilde{\mathcal{D}}(x,q)$ in (5.25) and Proposition (3.3), it follows that $\tilde{\mathcal{D}}(x,q)$ and $\mathcal{C}(x,q)$ satisfy the assumptions of Theorem 6.5. Further, by Assumption 5.1, finiteness of the value functions as in Theorem 3.1 holds. Therefore, from Theorem 6.5, we get

$$\begin{aligned} \tilde{u}(s) := u(sx,sq) &= \sup_{g\in\mathcal{C}(x,q)} \mathbb{E}\left[U(sg)\right] \\ &= \inf_{z>0}\left(\inf_{h\in\tilde{\mathcal{D}}(x,q)} \mathbb{E}\left[V(zh)\right] + zs\right) \\ &= \inf_{z>0}\left(\tilde{v}(z) + sz\right), \quad s > 0, \end{aligned} \tag{5.44}$$

where, in the last equality, we used Lemma 5.18. By construction, both $-\tilde{u}$, v, and $\liminf_{n\to\infty} \tilde{v}^n$ are convex and finite-valued. From (5.43) and (5.44), we get

$$\tilde{u}(s) = \inf_{z>0}\left(\tilde{v}(z) + sz\right) = \inf_{z>0}\left(\lim_{n\to\infty} \tilde{v}^n(z) + sz\right), \quad s > 0. \tag{5.45}$$

A similar construction

$$\tilde{u}^n(s) := u^n(sx,sq) = \inf_{z>0}\left(\tilde{v}^n(z) + sz\right), \quad s > 0, \quad n \geq n_0, \tag{5.46}$$

shows that $\tilde{u}^n$, $n \geq n_0$, is a monotone sequence and

$$-\infty < \tilde{u}^n(s) \leq \tilde{u}(s), \quad s > 0, \quad n \geq n_0.$$

Therefore, we have

$$\tilde{u}^{\infty}(s) := \lim_{n\to\infty} \tilde{u}^n(s) \le \tilde{u}(s), \quad s > 0. \tag{5.47}$$

Further, combining (5.47) with (5.45) and (5.46), and using the monotonicity of $\tilde{v}^n$, $n \ge n_0$, we get

$$\inf_{z>0}\left(\sup_{k\ge n_0} \tilde{v}^k(z) + sz\right) = \tilde{u}(s) \ge \inf_{z>0}\left(\tilde{v}^n(z) + sz\right), \quad n \ge n_0. \tag{5.48}$$

By conjugacy of $\tilde{u}^n$ and $\tilde{v}^n$, and from the monotonicity of $\tilde{u}^n$, $n \ge n_0$, we obtain

$$\tilde{v}^n(z) = \sup_{s>0}(\tilde{u}^n(s) - sz) \le \sup_{s>0}(\tilde{u}^{\infty}(s) - sz), \quad z > 0, \quad n \ge n_0.$$

Therefore, using (5.43), we obtain

$$\tilde{v}(z) \le \sup_{s>0}(\tilde{u}^{\infty}(s) - sz), \quad z > 0. \tag{5.49}$$

One can see that $\tilde{u}^{\infty}$ is a concave function as a pointwise limit of concave functions, and further that $\tilde{u}^{\infty}$ is finite-valued. Let $\widehat{v}$ denote its convex conjugate, then (5.49) implies that

$$\widehat{v}(z) = \sup_{s>0}(\tilde{u}^{\infty}(s) - sz) \ge \tilde{v}(z), \quad z > 0. \tag{5.50}$$

Therefore, the biconjugation characterization and (5.50), imply that

$$\tilde{u}^{\infty}(s) = \inf_{z>0}(\widehat{v}(z) + zs) \ge \inf_{z>0}(\tilde{v}(z) + zs) = \tilde{u}(s), \quad s > 0, \tag{5.51}$$

where, in the last equality, we have used (5.45). As a result, combining (5.47) and (5.51), we get

$$\tilde{u}^{\infty}(s) = \tilde{u}(s), \quad s > 0.$$

In particular, recalling (5.44) and (5.46), we conclude that

$$u(x,q) = \lim_{n\to\infty} u^n(x,q).$$

As (x,q) was an arbitrary point in $int\mathcal{K}$, we deduce that

$$u(x,q) = \lim_{n\to\infty} u^n(x,q), \quad (x,q) \in int\mathcal{K},$$

which, via [Roc70, Theorem 24.5], implies (5.21). Here we note that division by y in (4.3) leads to no issues as $\partial u(x,q) \subset ri\mathcal{L}$ by item *(iii)* of Theorem 3.1, and thus, as the subdifferential is closed, the set $\{y : (y,r) \in \partial u(x,q)\}$ is bounded away from 0. $\square$

Remark 5.19. A close look at the proof of Theorem 5.8 suggests that it applies to show convergence of utility-based prices, and not only marginal utility based prices.

6. Pricing of asymptotically replicable claims

The asymptotically replicable claim is the one that is replicable in the large market, but possibly not in any small market. We give examples of such claims and markets admitting such claims below. Intuitively, the arbitrage-free prices for small markets should converge to singletons.

Remark 6.1. If every bounded contingent claim is replicable in the large markets, one can intuitively call such markets asymptotically complete. We also compare the definition to the one in [Kar23, Definition 3.10]. In [ARS17] the asymptotically complete markets are investigated, without passing to the limiting market.

Contingent claims, which are replicable in some finite markets are well-studied in the literature. To develop a theory of arbitrage-free or marginal utility-based pricing for such claims, one does need to analyze the large markets. Below, we focus on the claims, which are not replicable in the small markets, but are replicable in the large one. Examples below show such markets and claims. Under (noArb), the definition of asymptotic replicability can be stated as follows.

Definition 6.1. A component-wise bounded contingent claim f is *asymptotically replicable* if it is not replicable in any small market and it is replicable in the large market; that is, every component of f^i is replicable in the sense of Definition 3.1.

We also denote by AFP^n, $n \in \mathbb{N}$, and AFP the sets of the arbitrage-free prices for f in small and large markets, respectively. The following lemma shows the consistency of various pricing methodologies for asymptotically replicable claims.

Lemma 6.2. *Let us assume the conditions of Theorem 5.8, and suppose that f is asymptotically replicable at $\pi \in \mathbb{R}^N$. Then, we have*

$$\mathcal{P} = \Pi(x) = AFP = \{\pi\}, \quad x > 0, \tag{6.1}$$

and for every $\varepsilon > 0$, there exists $n' \in \mathbb{N}$, such that for every $n \geq n'$, we have

$$\Pi^n \subseteq \mathcal{P}^n \subseteq \mathcal{P} + \varepsilon B, \quad x > 0, \tag{6.2}$$

and

$$AFP^n \subseteq AFP + \varepsilon B, \tag{6.3}$$

where B is the unit ball of $\mathbb{R}^N$.

Proof. If one can replicate f with an initial price $\pi \in \mathbb{R}^N$, then, the utility maximization problem degenerates to the one without f, as follows. One can see that

$$u(x, q) = u(x + q\pi, 0), \quad (x, q) \in \mathbb{R}^{N+1},$$

where we note that, from the boundedness of f and (noArb), no admissibility issues arise by passing from $u(x, q)$ to $u(x+q\pi, 0)$ and back. Then for $\rho = \pi$ and every $x > 0$, we have

$$u(x - q\rho, q) = u(x + q\pi - q\rho, 0) = u(x, 0), \quad q \in \mathbb{R}^N,$$

and thus (4.2) holds, i.e., π is the marginal utility-based price at x, for every $x > 0$, i.e., $\pi \in \Pi(x)$, $x > 0$.

If $\rho^i \neq \pi^i$, for some $i \in \{1, \dots, N\}$, for $q = sign(\pi^i - \rho^i)e^i$, where e^i is the canonical basis vector in $\mathbb{R}^N$, we have

$$u(x - q\rho, q) = u(x + q\pi - q\rho, 0) > u(x, 0),$$

as $u(\cdot, 0)$ is *strictly* increasing (see e.g., [Mos18, Theorem 2.2]), that is ρ is not a marginal utility-based price, for every $x > 0$.

Further, e.g., by observing that $\mathcal{K}$ contains straight lines passing through the origin and using [Roc70, Theorem 14.6], one can see that $\pi \in \mathcal{P}$, and $\mathcal{L} = \{y(1, \pi) : \ y > 0\}$. As $AFP = \{\pi\}$, we conclude (6.1). In turn, analogously to the proof of Theorem 3.1, as $\partial u^n \subseteq ri\mathcal{L}^n$, we obtain that $\Pi^n \subseteq \mathcal{P}^n$ and $AFP^n = \mathcal{P}^n$. Now, (6.2) and (6.3) follow from Corollary 5.16. □

Examples. The following (positive) Examples 6.3 and 6.4 illustrate the results above. Note that the framework of Section 5, is particularly convenient for the characterization of asymptotically replicable claims (and, in particular, asymptotically complete markets). We also note that the examples also allow to consider multi-agent (in fact, countably many agent) setting and, relying on ideas from [BEK05] and [Sio15], to analyze partial equilibria pricing.

Example 6.3. Let $S^n_0 = \exp\left(-\frac{1}{2^n}\right) - \frac{1}{2} \in (0,1)$, $n \in \mathbb{N}$, and let S^n_1 are IID Bernoulli random variable, such that $\mathbb{P}[S^n_1 = 0] = \mathbb{P}[S^n_1 = 1] = \frac{1}{2}$. Let us suppose that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_1 = \sigma(S^n_1, n \in \mathbb{N})$ Let us consider an asymptotically replicable $f = \sum\limits_{k=1}^{\infty} \frac{1}{2^k} S^k_1$, and fix a deterministic Inada utility U, such that Assumption 5.1 holds.

In the small markets, the super replication price of f is given by

$$\begin{aligned}\sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[f\right] &= \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{n} \frac{1}{2^k} S^k_1 + \sum_{j=n+1}^{\infty} \frac{1}{2^j} S^j_1\right] \\ &= \sum_{k=1}^{n} \frac{1}{2^k}\left(\exp\left(-\frac{1}{2^k}\right) - \frac{1}{2}\right) + \sum_{j=n+1}^{\infty} \frac{1}{2^j}, \quad n \in \mathbb{N}.\end{aligned}$$

By similar computations, the sub replication price is given by

$$\inf_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\left[f\right] = \sum_{k=1}^{n} \frac{1}{2^k}\left(\exp\left(-\frac{1}{2^k}\right) - \frac{1}{2}\right), \quad n \in \mathbb{N}.$$

We see that the set of the arbitrage-free prices in small markets is given by

$$AFP^n = \left(\sum_{k=1}^{n} \frac{1}{2^k}\left(\exp\left(-\frac{1}{2^k}\right) - \frac{1}{2}\right), \sum_{k=1}^{n} \frac{1}{2^k}\left(\exp\left(-\frac{1}{2^k}\right) - \frac{1}{2}\right) + \sum_{j=n+1}^{\infty} \frac{1}{2^j}\right), \quad n \in \mathbb{N},$$

which converges to

$$\left\{\sum_{k=1}^{\infty} \frac{1}{2^k}\left(\exp\left(-\frac{1}{2^k}\right) - \frac{1}{2}\right)\right\} = AFP,$$

and where $\sum\limits_{k=1}^{\infty} \frac{1}{2^k}\left(\exp\left(-\frac{1}{2^k}\right) - \frac{1}{2}\right)$ is also the unique arbitrage-free price in the large market (and the convergence has to be understood in the sense of convergence of sets as in Lemma 6.2), as under the minimal martingale measure in the large market, $\widehat{\mathbb{Q}}$, we have $\mathbb{E}_{\widehat{\mathbb{Q}}}\left[f\right] = \sum\limits_{k=1}^{\infty} \frac{1}{2^k}\left(\exp\left(-\frac{1}{2^k}\right) - \frac{1}{2}\right)$. We note that careful choice of S^n_0, $n \in \mathbb{N}$, ensures that $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}$ is strictly positive and finite $\mathbb{P}$-a.s., in fact, it is bounded away from 0 and ∞ (this can be seen via taking a limit in (6.6) below).

One can show that $\widehat{\mathbb{Q}}$ is the dual minimizer in the large market. By Theorem 4.2, the unique marginal utility-based price (in the large market) is given by

$$\mathbb{E}_{\widehat{\mathbb{Q}}}[f] = \sum_{k=1}^{\infty} \frac{1}{2^k}\left(\exp\left(-\frac{1}{2^k}\right) - \frac{1}{2}\right). \tag{6.4}$$

In every small market, we set

$$\begin{aligned}
\zeta^k &:= 2S_0^k 1_{\{S_1^k=1\}} + 2(1-S_0^k)1_{\{S_1^k=0\}}, \quad k \in \{1,\dots,n\},\\
\tilde{\zeta}^j(\alpha^j) &:= 2\alpha^j 1_{\{S^j=1\}} + 2(1-\alpha^j)1_{\{S^j=0\}}, \quad \alpha^j \in (0,1), \quad j \in \{n+1,\dots\},\\
\mathcal{A}^n &:= \{(\alpha^j)_{j\in\{n+1,\dots\}} : \ \alpha^j \in (0,1), j \in \{n+1,\dots\}\}.
\end{aligned}$$

Let V be the convex conjugate of U. Then, we can specify the dual problem as

$$v^n(y) = \inf_{(\alpha^j)_{j\in\{n+1,\dots\}}\in\mathcal{A}^n} \mathbb{E}\left[V\left(y\prod_{k=1}^{n}\zeta^k \prod_{j=n+1}^{\infty}\tilde{\zeta}^j(\alpha^j)\right)\right], \quad y>0. \tag{6.5}$$

One can see that the density of the minimal martingale measure, $\widehat{\mathbb{Q}}^n$, is the minimizer to (6.5). The corresponding density is given by

$$\frac{d\widehat{\mathbb{Q}}^n}{d\mathbb{P}} = \prod_{k=1}^{n}\zeta^k \tag{6.6}$$

and it corresponds to $\alpha^j = \frac{1}{2}$ for every $j \in \{n+1,\dots\}$. As in the large market case, this does not depend on y, and thus, on the initial wealth. From the explicit form of the minimizers in (6.6) and the characterizations of the arbitrage-free prices in large and small markets above, one can see that Assumption 5.7 from Section 5 holds.

Finally, as $\mathbb{E}\left[\frac{d\widehat{\mathbb{Q}}^n}{d\mathbb{P}}\right] = 1$, $n \in \mathbb{N}$, by Theorem 4.2, whose proof also applies to small markets, the unique marginal utility-based price in the market with n stocks is given by

$$\begin{aligned}
\rho^n = \mathbb{E}_{\widehat{\mathbb{Q}}^n}[f] &= \mathbb{E}_{\widehat{\mathbb{Q}}^n}\left[\sum_{k=1}^{n}\frac{1}{2^k}S_1^k + \sum_{j=n+1}^{\infty}\frac{1}{2^j}S_1^j\right]\\
&= \sum_{k=1}^{n}\frac{1}{2^k}\left(\exp\left(-\frac{1}{2^n}\right) - \frac{1}{2}\right) + \sum_{j=n+1}^{\infty}\frac{1}{2^j}\frac{1}{2}, \quad n \in \mathbb{N}.
\end{aligned}$$

Therefore, we obtain

$$\lim_{n\to\infty}\rho^n = \sum_{k=1}^{\infty}\frac{1}{2^k}\left(\exp\left(-\frac{1}{2^k}\right)-\frac{1}{2}\right),$$

which is the unique utility based price for f in the large market by (6.4).

Example 6.4. Let us consider a model, which is not asymptotically complete, where
$\left(\Omega,\mathcal{F},\mathcal{F}_{t\in[0,T]},\mathbb{P}\right)$, is a complete stochastic basis supporting a countable set of one-dimensional independent Brownian motions W^n, $n\in\mathbb{N}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}=\mathcal{F}_T$ is generated by W^n, $n\in\mathbb{N}$, and some other finite-dimensional Brownian motion. Let the riskless asset $S^0\equiv 1$, whereas the dynamics of risky assets is given by

$$dS^n = S^n\left(\mu^n dt+\sigma^n dW^n_t\right),\quad n\in\mathbb{N}, \tag{6.7}$$

where S^0 is (deterministic and) strictly positive, and where the constants μ^n and $\sigma^n>0$, $n\in\mathbb{N}$, are and such that the market price of risks satisfy

$$\lambda^n := \frac{\mu^n}{\sigma^n},\quad n\in\mathbb{N},$$

and

$$\sum_{n=1}^{\infty}(\lambda^n)^2<\infty.$$

Let us consider the minimal martingale measures for the n-stock model, who's density processes are given by

$$Z^n = \prod_{k=1}^{n}\mathcal{E}\left(-\lambda^k\cdot W^k\right),\quad n\in\mathbb{N},$$

One can show that the family Z^n_T, $n\in\mathbb{N}$, is uniformly integrable and it converges to a random variable Z_T, which admits a representation

$$Z_T = \exp\left(-\sum_{k=1}^{\infty}\lambda^k W^k_T - \tfrac{T}{2}\sum_{k=1}^{\infty}(\lambda^k)^2\right), \tag{6.8}$$

where $\sum\limits_{k=1}^{\infty}\lambda^k W^k_T$ is also a limit of a uniformly integrable sequence of terminal values of martingales $\sum\limits_{k=1}^{n}\lambda^k W^k_T$, $n\in\mathbb{N}$.

One can see that (noArb) satisfied. Let us introduce $\mathcal{H}^n_t := \mathcal{F}_t\vee\mathcal{F}_T^{W^1,\dots,W^n}$, $t\in[0,T]$, $n\in\mathbb{N}$. Thus, for such a model and a deterministic Inada utility

U, such that Assumptions 5.1 hold, and with V being the conjugate of U, for every $\tilde{Z} \in \mathcal{Z}^n$, $n \in \mathbb{N}$, we have

(6.9)
$$\mathbb{E}[V(y\tilde{Z}_T)] = \mathbb{E}[\mathbb{E}[V(y\tilde{Z}_T)|\mathcal{H}^n_T]] \geq \mathbb{E}[V(y\mathbb{E}[\tilde{Z}_T|\mathcal{H}^n_T])] \geq \mathbb{E}[V(yZ^n_T)], \quad n \in \mathbb{N}.$$

Let
$$f = \sum_{k=1}^{\infty} h^k(S^k_T),$$
where h^k's are smooth functions such that $\sum\limits_{k=1}^{\infty} ||h^k||_\infty < \infty$. Then f is demonstratively asymptotically replicable, and

$$\lim_{n\to\infty} \mathbb{E}[Z^n_T f] = \mathbb{E}[Z_T f]. \tag{6.10}$$

Using (6.9) and (6.10), one can see that Assumption 5.7 holds.

Since additionally $\mathbb{E}\left[Z^n_T\right] = 1$, one can show along the lines of Theorem 4.2 that the set of marginal utility-based prices for f (in the market with the first n risky assets) is given by
$$\Pi^n(x) = \{\mathbb{E}\left[Z^n_T f\right]\}, \quad n \in \mathbb{N}, \quad x > 0.$$
In the large market, Lemma 6.2 (and Theorem 4.2) implies that
$$\Pi(x) = \{\mathbb{E}\left[Z_T f\right]\}, \quad x > 0,$$
is the set of marginal utility-based prices in the large market. In view of (6.10), we have $\lim\limits_{n\to\infty} \Pi^n(x_n) = \Pi(x)$ for every sequence of strictly positive numbers x_n, $n \in \mathbb{N}$, convergent to $x > 0$, where the convergence is in the sense of Lemma 6.2, which in the present settings reduces to convergence of singletons.

Appendix

Below we state [Mos15, Theorem 3.2 and Theorem 3.3], which are used above. Their proofs are contained in [Mos15]. Let μ be a finite and positive measure on a measurable space $(\Omega, \mathcal{F})$. Denote by $\mathbf{L}^0 = \mathbf{L}^0(\Omega, \mathcal{F}, \mu)$ the vector space of (equivalence classes of) real-valued measurable functions on $(\Omega, \mathcal{F}, \mu)$ topologized by convergence in measure μ. Let $\mathbf{L}^0_+$ denote its positive orthant, i.e.,
$$\mathbf{L}^0_+ = \left\{\xi \in \mathbf{L}^0(\Omega, \mathcal{F}, \mu) : \ \xi \geq 0\right\}.$$

For any ξ and η in $\mathbf{L}^0$ we write

$$\langle \xi, \eta \rangle := \int_\Omega \xi \eta d\mu.$$

If ξ and η are both nonnegative, the integral is well-defined in $[0, \infty]$. Let $\mathcal{C}, \mathcal{D}$ be subsets of $\mathbf{L}^0_+$ that satisfy the conditions below.

(1)

$$\begin{aligned} \xi \in \mathcal{C} \quad &\Leftrightarrow \quad \langle \xi, \eta \rangle \leq 1 \text{ for all } \eta \in \mathcal{D}, \\ \eta \in \mathcal{D} \quad &\Leftrightarrow \quad \langle \xi, \eta \rangle \leq 1 \text{ for all } \xi \in \mathcal{C}. \end{aligned} \tag{6.11}$$

(2) $\mathcal{C}$ and $\mathcal{D}$ contain at least one strictly positive element:

$$\text{there are } \xi^* \in \mathcal{C},\ \eta^* \in \mathcal{D} \text{ such that } \min(\xi^*,\ \eta^*) > 0\ \mu\ a.e. \tag{6.12}$$

Now we can state the optimization problems:

$$u(x) = \sup_{\xi \in \mathcal{C}} \int_\Omega U(x\xi) d\mu, \qquad x > 0, \tag{6.13}$$

$$v(y) = \inf_{\eta \in \mathcal{D}} \int_\Omega V(y\eta) d\mu, \qquad y > 0. \tag{6.14}$$

Theorem 6.5. [Mos15, Theorem 3.3] *Assume that $\mathcal{C}$ and $\mathcal{D}$ satisfy conditions (6.11) and (6.12). Let Assumption 2.3 hold and suppose*

$$v(y) < \infty \text{ for all } y > 0 \quad \text{and} \quad u(x) > -\infty \text{ for all } x > 0. \tag{6.15}$$

Then we have:

(1) $u(x) < \infty$ for all $x > 0$, $v(y) > -\infty$ for all $y > 0$. The functions u and v satisfy the biconjugacy relations, i.e.,

$$\begin{aligned} v(y) &= \sup_{x>0} \left(u(x) - xy \right), \quad y > 0, \\ u(x) &= \inf_{y>0} \left(v(y) + xy \right), \quad x > 0. \end{aligned} \tag{6.16}$$

The functions u and $-v$ are continuously differentiable on $(0, \infty)$, strictly increasing, strictly concave, and satisfy the Inada conditions:

$$\begin{aligned} u'(0) &:= \lim_{x \downarrow 0} u'(x) = \infty, &\qquad -v'(0) &:= \lim_{y \downarrow 0} -v'(y) = \infty, \\ u'(\infty) &:= \lim_{x \to \infty} u'(x) = 0, &\qquad -v'(\infty) &:= \lim_{y \to \infty} -v'(y) = 0. \end{aligned}$$

(2) *For every $x > 0$ the optimal solution $\hat{\xi}(x)$ to (6.13) exists and is unique. For every $y > 0$ the optimal solution $\hat{\eta}(y)$ to (6.14) exists and is unique. If $y = u'(x)$, we have the dual relations*

$$\hat{\eta}(y) = U'\left(\hat{\xi}(x)\right) \quad \mu \text{ a.e.}$$

and

$$\langle \hat{\xi}(x), \hat{\eta}(y) \rangle = xy.$$

Let $\tilde{\mathcal{D}}$ be a subset of $\mathcal{D}$ such that

(i) $\tilde{\mathcal{D}}$ is closed under the countable convex combinations,
(ii) for every $\xi \in \mathcal{C}$ we have

$$\sup_{\eta \in \mathcal{D}} \langle \xi, \eta \rangle = \sup_{\eta \in \tilde{\mathcal{D}}} \langle \xi, \eta \rangle. \tag{6.17}$$

Likewise, define $\tilde{\mathcal{C}}$ to be a subset of $\mathcal{C}$ such that

(iii) $\tilde{\mathcal{C}}$ is closed under the countable convex combinations,
(iv) for every $\eta \in \mathcal{D}$ we have

$$\sup_{\xi \in \mathcal{C}} \langle \xi, \eta \rangle = \sup_{\xi \in \tilde{\mathcal{C}}} \langle \xi, \eta \rangle.$$

Theorem 6.6. [Mos15, Theorem 3.3] *Under the conditions of Theorem 6.5, we have*

$$\begin{aligned} v(y) &= \inf_{\eta \in \tilde{\mathcal{D}}} \int_\Omega V(y\eta)\, d\mu, \qquad y > 0. \\ u(x) &= \sup_{\xi \in \tilde{\mathcal{C}}} \int_\Omega U(x\xi)\, d\mu, \qquad x > 0. \end{aligned}$$

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Oleksii Mostovyi, University of Connecticut, Department of Mathematics, Storrs, CT 06269, United States

Email address: `oleksii.mostovyi@uconn.edu`

Pietro Siorpaes, Imperial College London, Department of Mathematics, London SW7 2AZ, United Kingdom

Email address: `p.siorpaes@imperial.ac.uk`