REPRESENTATION OF INDIFFERENCE PRICES ON A FINITE PROBABILITY SPACE

JASON FREITAS, JOSHUA HUANG, AND OLEKSII MOSTOVYI

Abstract. On a finite probability space, we consider the problem of indifference pricing of contingent claims, where the preferences of an economic agent are modeled by an Inada utility stochastic field, the interior of whose effective domain is (a, ∞), for some $a \in \mathbb{R} \cup \{-\infty\}$. This allows for including utilities on both $\mathbb{R}$ and $\mathbb{R}_+$. We consider arbitrary contingent claims and show that, for replicable ones, the indifference price equals the initial value of the replicating strategy and thus depends neither on the agent's initial wealth, for which the indifference pricing problem is well-posed, nor the utility stochastic field. This, in particular, shows the consistency of the indifference and arbitrage-free pricing methodologies for complete models. For non-replicable claims, we show that the indifference price is equal to the expectation of the discounted payoff under the dual-optimal measure, which is equivalent to the reference probability measure. In particular, we demonstrate that the indifference price is unique for every choice of a smooth Inada utility stochastic field and initial wealth in (a, ∞). Our proofs rely on the change of numéraire technique and a reformulation of the indifference pricing problem. The advantages of the settings of this paper and the approach allow for bypassing the technicalities issues related to choosing the notion of admissibility and for including a wide range of utilities, including stochastic ones. We augment the results with examples.

1. Introduction

There are multiple approaches to contingent claims pricing. From statistical to theoretical, different approaches usually allow for assigning a unique number - price, or an interval - the set of prices. In this paper, we focus on indifference pricing, which has been investigated in the literature under multiple names, including utility-based pricing, reservation pricing, private valuation, fair pricing, Davis pricing, and so on, and also under varying definitions of such prices leading to different mathematical problems. All of these notions, however, rely on the preferences of an economic agent that are typically represented via a utility function. We refer to [HH09] for an overview.

Date: September 6, 2023.

2020 *Mathematics Subject Classification.* 60J74, 60G42, 93E20, 91G10, 91G20.

Key words and phrases. Indifference pricing, incomplete markets, duality, numéraire.

This paper is part of an REU project conducted in Summer 2023 at the University of Connecticut. The work was supported by the National Science Foundation under grants No. DMS-1950543 and DMS-1848339.

Our results include representations for the indifference prices in terms of the dual-optimal state price densities and measures in both undiscounted and discounted formulations. We also show that indifference prices are arbitrage-free prices in both complete and incomplete markets. Our results are obtained for a wide class of utility stochastic fields, which naturally generalize the notion of a utility function. In the settings of infinite probability spaces, it is well-known that indifference prices might be non-unique. We refer to [HKS05] for the discussion. Here, we show that, on a finite probability space, the indifference prices are unique.

The main novelty compared to the existing literature is that our results apply to *stochastic Inada utilities* (in the sense of Assumption 2.2) on both $\mathbb{R}_+$ and $\mathbb{R}$. Our formulation and notations, which are consistent with the ones in [Roc70], particularly, extending the utility stochastic field and the primal value function to $\mathbb{R}$ by $-\infty$ outside of their effective domains, allow bypassing the highly technical notion of admissibility, which goes back to [HK79], and which is investigated in [BČ11] and [BS12], among others.

The closest existing results, to the best of our knowledge, are the following ones. Indifference pricing with stochastic preferences on $\mathbb{R}_+$ is considered in [MS23], whereas indifference pricing for *deterministic* utilities on $\mathbb{R}$ is considered in [MZ04], [SZ05], [Mon08], [HH09], and [RS18], among others.

An important role in our analysis is played by the dual problem, which we state and characterize below. In particular, the dual-optimal state price density can be used to establish indifference prices of Arrow-Debreu securities. Contrary to general formulations of indifference pricing (as pointed out in [HH09, p. 45]), indifference pricing in the sense below exhibits linearity; that is, an indifference price of a linear combination of contingent claims is the linear combination of indifference prices of these claims with the same weights. Besides duality, the proofs rely on a change of numéraire technique, which allows for the reduction of the indifference pricing problem to the one with zero interest rate by means of a new stochastic utility.

The remainder of this paper is organized as follows. In Section 2, we present the model and main results. Section 3 contains proof of the main results in the discounted case. Section 4 gives the proof in the general case by reducing the indifference pricing problem to the one in Section 3 via a change of numéraire technique. In Section 5, we show the consistency of the indifference and arbitrage-free pricing methodology for replicable contingent claims, and we conclude the paper with positive examples in Section 6.

2. Model and main results

This section aims to introduce the mathematical settings needed to investigate properties of indifference pricing pricing in a market with multiple stocks. We consider an

$\mathbb{R}^{d+1}$-valued process $(S_t)_{\{t\in\{0,\dots,T\}\}} = \left(S_t^0,\dots,S_t^d\right)_{\{t\in\{0,\dots,T\}\}}$, on a finite filtered probability space $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{\{t\in\{0,\dots,T\}\}},\mathbb{P}\right)$, where $\Omega=\{\omega_1,\dots,\omega_N\}$, and N is finite. Without loss of generality we assume that $\mathbb{P}[\omega_n]>0$, for every $n\in\{1,\dots,N\}$, and that $\mathcal{F}_0$ is trivial, and that $\mathcal{F}_T=\mathcal{F}$ is the power set of Ω.

We suppose that S^0 is the price process of a riskless security is a strictly positive process, that is $S_n^0(\omega)>0$ for every $n\in\{0,\dots,T\}$ and $\omega\in\Omega$. We additionally assume that $S_0^0=1$. Further, we suppose that $S=(S_n)_{\{n=0,\dots,T\}}$ is an $((d+1)$-dimensional vector-valued) $(\mathcal{F})_{\{t\in\{0,\dots,T\}\}}$-adapted process, i.e., each S_n is $\mathcal{F}_n$-measurable, and, for every $i\in\{1,\dots,d\}$, S^i describes the evolution of the i-th risky asset. We take $\left(S^1,\dots,S^d\right)$ to describe the undiscounted price process of d stocks.

Let $H=(H_n)_{\{n=1,\dots,T\}}$ represent a d-dimensional trading strategy corresponding to the number of shares of stocks held at any time n, that is, H_n^i denotes the number of shares of stock $i\in\{1,\dots,d\}$ at time $n\in\{0,\dots,T\}$. We restrict the strategies to the ones that are predictable and *self-financing* in the following senses: since our position in the stock market at time n must be chosen at time $n-1$, we say that H is predictable, namely, that H_n is $\mathcal{F}_{n-1}$-measurable for each $n\in\{1,\dots,T\}$. Formally, let H^0 be the predictable process specifying the number of shares of the riskless asset. Then, denoting

$$\overline{H}:=\left(H^0,H\right)\quad \textit{and}\quad \overline{S}:=(1,S),$$

we call a strategy $\overline{H}$ to be self-financing if

$$\overline{H}_n\cdot\overline{S}_n=\overline{H}_{n+1}\cdot\overline{S}_n,\quad n\in\{1,\dots,T-1\}. \tag{1}$$

where $\cdot$ represents the scalar product in $\mathbb{R}^{d+1}$. Condition (1) implies that the accumulated changes in the value of the wealth process (which is defined in (2) below) result from the fluctuations of the stock prices and not from an external influx or withdrawal of money. This notion goes back to [HK79].

Let $\mathcal{H}$ be the set of predictable $(d+1)$-dimensional processes that correspond to self-financing strategies in the sense of (1). A *portfolio* is defined as a pair $(x,\overline{H})$, where x is the initial wealth and $\overline{H}\in\mathcal{H}$ is a (self-financing) trading strategy satisfying (1). The *wealth process* X associated with a portfolio (x,H) is given by

$$X_n:=x+\sum_{j=1}^{n}\overline{H}_j\cdot\left(\overline{S}_j-\overline{S}_{j-1}\right),\quad n\in\{1,\dots,T\}. \tag{2}$$

Let us denote by $\mathcal{X}(x)$, $x\in\mathbb{R}$, the set of sefl-financing wealth processes starting from an initial wealth x, that is

$$\mathcal{X}(x):=\left\{X:\ X_n=x+\sum_{k=1}^{n}\overline{H}_k\cdot\left(\overline{S}_k-\overline{S}_{k-1}\right),\quad n\in\{1,\dots,T\}\right\},\quad x\in\mathbb{R}. \tag{3}$$

Remark 2.1. Due to the special nature of our formulation, we can bypass the notion of admissibility in the formulation of the value function in (4) and Definition 2.6 of indifference prices below. In general probability spaces, the notion of admissibility is delicate, and the appropriate sets of admissible processes are investigated in [DS97], [BČ11], and [BS12], among others.

Next, we specify the preferences of an economic agent as follows. We note that the motivation behind Assumption 2.2 is outlined in remarks 2.3 and 2.4.

Assumption 2.2. A stochastic utility field is a mapping $U = U(\omega, x) : \Omega\times\mathbb{R} \to \mathbb{R}\cup\{-\infty\}$ such that for every $\omega \in \Omega$, $U(\omega, \cdot)$ is an Inada utility function, that is strictly increasing, strictly concave, continuously differentiable on $dom(U)$[1] function that satisfies the Inada conditions at $a(\omega) := \inf\{x : U(\omega, x) \in dom(U)\}$, $\omega \in \Omega$, and ∞:

$$\lim_{x\downarrow a(\omega)} U'(\omega, x) = \infty \quad \textit{and} \quad \lim_{x\uparrow\infty} U'(\omega, x) = 0, \quad \omega \in \Omega,$$

where, here and below, U' denotes the partial derivative of U with respect to the second argument. If $a(\omega) > -\infty$, at $x = a$, we suppose by continuity that $U(\omega, a(\omega)) = \lim\limits_{x\downarrow a(\omega)} U(\omega, x)$, $\omega \in \Omega$, this value may be $-\infty$, and finally we assume that

$$U(\omega, x) = -\infty, \quad x \in (-\infty, a).$$

We also suppose that a does not depend on ω to include the two main cases of utility defined on the positive real line and the whole real line, as pointed out in Remark 2.3 below, and in the spirit of [Sch01, Assumption 1.2].

Remark 2.3. Working with utilities satisfying Assumption 2.2 allows to include in one formulation standard prominent choices of deterministic utilities on $(0, \infty)$, such as

$$\begin{aligned} &U(x) = \ln x, \quad x \in (0, \infty), \\ &U(x) = \frac{x^p}{p}, \quad x \in (0, \infty), \quad p \in (-\infty, 0) \cup (0, 1), \end{aligned}$$

and the exponential utilities on $\mathbb{R}$, given by

$$U(x) = -e^{-\gamma x}, \quad x \in \mathbb{R}, \quad \gamma > 0.$$

Remark 2.4. The main reason why we defined U to be a stochastic utility is to show the flexibility needed to handle a change of numéraire. It is used below in Section 4, in particular, to simplify the proofs for the undiscounted case, that is, when the riskless security is a general strictly positive process (possibly stochastic) and not necessarily a constant.

[1]By $dom(U)$, we denote the interior of $\{U > -\infty\}$.

2.1. **Primal Problem.** Let us define

$$u(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right], \quad x \in \mathbb{R}. \tag{4}$$

We note that for some values of x, $\mathbb{E}\left[U(X_T)\right]$ might equal to $-\infty$ for *every* $X \in \mathcal{X}(x)$. For such x's, we set $u(x) := -\infty$, in a way that is consistent with notations in convex and variational analysis, see, e.g., [Roc70, p. 23].

Remark 2.5. We stress that for some wealth processes $X \in \bigcup\limits_{x\in\mathbb{R}} \mathcal{X}(x)$, where $\mathcal{X}(x)$'s are defined in (3), and for a utility stochastic field satisfying Assumption 2.2, we may have that

$$\mathbb{P}\left[U(X_T) = -\infty\right] > 0,$$

in which case, we also have

$$\mathbb{E}\left[U(X_T)\right] = -\infty. \tag{5}$$

Wealth processes, for which (5) holds, are suboptimal for the utility maximization problem (4). Further, in the context of a finite probability space, having (5) for some wealth processes allows for bypassing the technical issues related to the notion of admissibility in many papers on indifference pricing and optimal investment with random endowment, e.g., in [HK04], [Sio16], and [Sio15], among others.

2.2. **Indifference Pricing.** *By a contingent claim, we mean any random variable on the probability space.* Typical examples include standard call and put options on one of the traded assets. The following definition goes back to [Dav97][2]. Right now, several notions are used in the literature for the same concept, including fair price, indifference price, Davis price, and marginal-utility-based price.

Definition 2.6. For a given contingent claim f, an *indifference price* of f corresponding to the initial wealth x and a utility stochastic field U is a constant $\Pi = \Pi(f, x, U)$, such that

$$\mathbb{E}\left[U\left(X_T + qf\right)\right] \leq u(x), \quad q \in \mathbb{R}, \quad X \in \mathcal{X}(x - q\Pi), \tag{6}$$

where $u(x)$ is given by (4) (corresponding to the stochastic utility field U). As f and U will be fixed below, we will denote the indifference price by $\Pi(x)$.

Remark 2.7. We stress that, in general, the indifference price depends on the initial wealth x (and the preferences of an economic agent that is given via the utility stochastic field U).

[2] Definition 2.6 actually relaxes the differentiability assumption in [Dav97, Definition 1].

2.3. **Absence of arbitrage and the dual problem.** A probability measure $\mathbb{Q}$ is called an equivalent martingale measure for $\frac{\overline{S}}{S^0}$, if $\mathbb{Q} \sim \mathbb{P}$ and $\frac{\overline{S}}{S^0} = \left(1, \frac{S^1_n}{S^0_n}, \dots, \frac{S^d_n}{S^0_n}\right)_{\{n\in\{0,\dots,T\}\}}$ is a $(d+1)$-dimensional martingale under $\mathbb{Q}$. We denote the set of equivalent martingale measures for $\frac{\overline{S}}{S^0}$ by $\mathcal{M}$. In order for (4) to be non-degenerate for every $x > a$, we need to impose the no-arbitrage condition on $\frac{\overline{S}}{S^0}$. We refer to [Shr04, p. 41] for more details.

Assumption 2.8. We suppose that

$$\mathcal{M} \neq \emptyset.$$

To specify the properties of (4) and its optimizers, we need to define the dual problem. Let us denote the convex conjugate of U by

$$V(\omega, y) := \sup_{x\in\mathbb{R}} \left(U(\omega, x) - xy\right), \quad (\omega, x) \in \Omega \times \mathbb{R}. \tag{7}$$

Let us define the dual value function as

$$v(y) := \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[V\left(\frac{y}{S^0_T}\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \quad y > 0. \tag{8}$$

Following [Shr04, Chapter 3], let us consider the

$$\Theta := \left\{\frac{d\mathbb{Q}}{d\mathbb{P}}\frac{1}{S^0_T}\right\}, \quad \mathbb{Q} \in \mathcal{M},$$

that is, the set of the state price densities at time T. Via Θ, we can reformulate the dual problem (8) in an even more concise form as

$$v(y) = \inf_{\zeta\in\Theta} \mathbb{E}\left[V(y\zeta)\right], \quad y > 0. \tag{9}$$

The following theorem is the main result of this paper. The main novelty compared to the existing literature is that the following theorem includes *stochastic utilities* on both $(0, \infty)$ and $\mathbb{R}$, as in Assumption 2.2.

Theorem 2.9. *Let us consider a financial market $S = (S_t)_{\{t\in\{0,\dots,T\}\}}$ defined over the finite filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{\{t\in\{0,\dots,T\}\}}, \mathbb{P}\right)$. Assume that Assumptions 2.2 and 2.8 hold, and let $x \in dom(u)$[3] be fixed.*

Then, for every contingent claim f, the set of indifference prices at x is a singleton, and, with $y = u'(x)$, we have

$$\Pi(x) = \mathbb{E}\left[\frac{\widehat{Y}_T(y)}{y}\frac{f}{S^0_T}\right] = \mathbb{E}_{\widehat{\mathbb{Q}}(y)}\left[\frac{f}{S^0_T}\right] = \mathbb{E}\left[\widehat{\zeta}(y) f\right], \tag{10}$$

where

$$\frac{d\widehat{\mathbb{Q}}(y)}{d\mathbb{P}} = \frac{\widehat{Y}_T(y)}{y},$$

[3]By $dom(u)$, we denote the interior of $\{u > -\infty\}$.

is the density of the dual-optimal equivalent martingale measure in the sense of (8) *and*

$$\widehat{\zeta}(y) = \frac{\widehat{Y}_T(y)}{y}\frac{1}{S^0_T}$$

is the optimal state price density, that is, the optimizer to (9).

The proof is divided into two steps: first, we prove Theorem 2.9 in the discounted case in Section 3, and then we reduce the general case to the discounted one by a change of numéraire in Section 4.

3. Proof of Theorem 2.9 in the discounted case

This section aims to prove Theorem 2.9 under a simplifying assumption that $S^0 \equiv 1$, that is

$$S^0_n(\omega) = 1, \quad n \in \{0, \dots, T\}, \quad \omega \in \Omega. \tag{11}$$

We note that many results in mathematical finance are proven under this simlifying assumption, see [DS97] and [HK04] among others.

The following theorem summarizes the properties of the primal and dual value functions under (11). We note that for deterministic utilities, these results are presented in [Sch01, Theorem 2.3]. Settings with a stochastic utility defined on the positive real line are considered in [Mos15, Theorems 2.3 and 2.4] and, in large markets, in [Mos18, Theorem 2.2]. The proof is omitted for brevity.

Theorem 3.1. *Let the financial market* $S = (S_t)_{\{t\in\{0,\dots,T\}\}}$ *defined over the finite filtered probability space* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{\{t\in\{0,\dots,T\}\}}, \mathbb{P}\right)$. *Let us suppose that* (11) *and Assumptions 2.2 and 2.8 hold.*

Then, we have

(i) the value functions u *and* v*, defined in* (4) *and* (8)*, respectively, are biconjugate in the sense that*

$$u(x) = \inf_{y>0}\left(v(y) + xy\right), \quad x \in dom(u),$$
$$v(y) = \sup_{x\in dom(u)}\left(u(x) - xy\right), \quad y > 0.$$

u shares the qualitative properties of $U(\omega,\cdot)$, $\omega \in \Omega$ listed in Assumption 2.2.

(ii) For every $x \in dom(u)$, the optimizer $\widehat{X}(x)$ to (4) *exists and is unique; for every $y > 0$, the optimizer to the dual problem $\widehat{Y}(y)$ exists and is unique, $\frac{\widehat{Y}_T(y)}{y}$ is a density of an equivalent martingale measure for S.*

(iii) For every $x \in dom(u)$, with $y = u'(x)$, we have

$$\widehat{Y}_T(y)(\omega) = U'\left(\omega, \widehat{X}_T(x)(\omega)\right), \quad \omega \in \Omega.$$

Equivalently, for every $y > 0$*, for* $x = -v'(y)$*, we have*

$$\widehat{X}_T(x)(\omega) = I\left(\omega, \widehat{Y}_T(y)(\omega)\right), \quad \omega \in \Omega,$$

where $I(\omega, \cdot) = (U')^{-1}(\omega, \cdot) = -V'(\omega, \cdot)$*,* $\omega \in \Omega$*, that is* I *a pointwise (in* ω*) inverse of* U'*.*

(iv) For every $x \in dom(u)$*, for* $y = u'(x)$*, we have*

$$\mathbb{E}\left[\widehat{X}_T(x)\widehat{Y}_T(y)\right] = xy.$$

The following theorem is the version of the main result of this paper, Theorem 2.9, under the additional condition (11).

Theorem 3.2. *Let us suppose that that the assumptions of Theorem 3.1 and* (11) *hold and* $x \in dom(u)$ *is fixed. Then, for every contingent claim* f*, the set of indifference prices at* x *is a singleton, and, with* $y = u'(x)$*, we have*

$$\Pi(x) = \mathbb{E}\left[\frac{\widehat{Y}_T(y)}{y} f\right] = \mathbb{E}_{\widehat{\mathbb{Q}}(y)}[f], \tag{12}$$

where

$$\frac{d\widehat{\mathbb{Q}}(y)}{d\mathbb{P}} = \frac{\widehat{Y}_T(y)}{y}.$$

The proof of Theorem 2.9 is given via several steps. First, we establish the convergence of the dual optimizers in the sense of the next lemma.

Lemma 3.3. *Let us suppose that conditions of Theorem 3.2 hold and consider a sequence of strictly positive numbers* $(y_k)_{k\in\mathbb{N}}$*, that converges to* $y > 0$*. Then, the optimizers to* (8) *converge is the sense that*

$$\lim_{k\to\infty} \widehat{Y}_T(y_k)(\omega) = \widehat{Y}_T(y)(\omega), \quad \omega \in \Omega.$$

Proof. Let us assume by contradition that

$$\left\{\omega \in \Omega : \limsup_{k\to\infty} \left|\widehat{Y}_T(y_k)(\omega) - \widehat{Y}_T(y)(\omega)\right| > 0\right\} \neq \emptyset. \tag{13}$$

It follows from item (ii) of Theorem 3.1 that the dual minimizers $\widehat{Y}_T(y_k)$, $k \in \mathbb{N}$, and $\widehat{Y}_T(y)$ are strictly positive numbers that satisfy

$$\mathbb{E}\left[\widehat{Y}_T(y_k) + \widehat{Y}_T(y)\right] = y_k + y, \quad k \in \mathbb{N}.$$

Therefore, using Markov's inequality, we obtain

$$\mathbb{P}\left[\widehat{Y}_T(y_k) + \widehat{Y}_T(y) \geq M\right] \leq \frac{y_k + y}{M}, \quad k \in \mathbb{N}, \quad M > 0.$$

Consequently, we get

$$\sup_{k\in\mathbb{N}} \mathbb{P}\left[\widehat{Y}_T(y_k) + \widehat{Y}_T(y) \geq M\right] \leq \sup_{k\in\mathbb{N}} \frac{y_k + y}{M}, \quad M > 0. \tag{14}$$

Since $(y_k)_{k\in\mathbb{N}}$ converges to y, it follows from (13) and (14) that there exists $M > 1$, such that

$$\limsup_{k\to\infty} \mathbb{P}\left[\left|\widehat{Y}_T(y_k) - \widehat{Y}_T(y)\right| \geq \frac{1}{M} \quad \textit{and} \quad \widehat{Y}_T(y_k) + \widehat{Y}_T(y) \leq M\right] \geq \frac{1}{M}. \tag{15}$$

Next, from the convexity of $V(\omega, \cdot)$, $\omega \in \Omega$, we get

$$V\left(\omega, \frac{\widehat{Y}_T(y_k)(\omega) + \widehat{Y}_T(y)(\omega)}{2}\right) \leq \frac{1}{2}\left(V\left(\omega, \widehat{Y}_T(y_k)(\omega)\right) + V\left(\omega, \widehat{Y}_T(y)(\omega)\right)\right), \quad \omega \in \Omega, \quad k \in \mathbb{N}. \tag{16}$$

Further, from (15) and the strict convexity of $V(\omega, \cdot)$, $\omega \in \Omega$, we deduce that there exists $\varepsilon > 0$, such that

$$\limsup_{k\to\infty} \mathbb{P}\left[V\left(\omega, \frac{\widehat{Y}_T(y_k)(\omega) + \widehat{Y}_T(y)(\omega)}{2}\right) \leq \frac{1}{2}\left(V\left(\omega, \widehat{Y}_T(y_k)(\omega)\right) + V\left(\omega, \widehat{Y}_T(y)(\omega)\right)\right) - \varepsilon\right] > \varepsilon. \tag{17}$$

From (16) and (17), we conclude that

$$\limsup_{k\to\infty} \mathbb{E}\left[V\left(\frac{\widehat{Y}_T(y_k) + \widehat{Y}_T(y)}{2}\right)\right] \leq \frac{1}{2}\left(\limsup_{k\to\infty} \mathbb{E}\left[V\left(\widehat{Y}_T(y_k)\right)\right] + \mathbb{E}\left[V\left(\widehat{Y}_T(y)\right)\right]\right) - \varepsilon^2,$$

which, by the optimality of $\widehat{Y}(y_k)$, $k \in \mathbb{N}$, and $\widehat{Y}(y)$ for (8), implies that

$$\limsup_{k\to\infty} \mathbb{E}\left[V\left(\frac{\widehat{Y}_T(y_k) + \widehat{Y}_T(y)}{2}\right)\right] \leq \frac{1}{2}\left(\limsup_{k\to\infty} v(y_k) + v(y)\right) - \varepsilon^2. \tag{18}$$

In the right-hand side of (18), by item (i) of Theorem 3.1 (which, in particular, implies the continuity of v on $(0, \infty)$), and we get

$$\frac{1}{2}\left(\limsup_{k\to\infty} v(y_k) + v(y)\right) - \varepsilon^2 = v(y) - \varepsilon^2. \tag{19}$$

For the left-hand side of (18), we first observe that $\frac{\widehat{Y}_T(y_k)+\widehat{Y}_T(y)}{y_k+y}$ is a density of an equivalent martingale measure for S and thus $\frac{\widehat{Y}_T(y_k)+\widehat{Y}_T(y)}{2}$ is admissible for the dual problem (8) at $\frac{y_k+y}{2}$, for every $k \in \mathbb{N}$. Therefore, we have

$$\mathbb{E}\left[V\left(\frac{\widehat{Y}_T(y_k) + \widehat{Y}_T(y)}{2}\right)\right] \geq v\left(\frac{y_k + y}{2}\right), \quad k \in \mathbb{N}.$$

As a consequence, using the continuity of v at y again, we obtain

$$\limsup_{k\to\infty} \mathbb{E}\left[V\left(\frac{\widehat{Y}_T(y_k) + \widehat{Y}_T(y)}{2}\right)\right] \geq \limsup_{k\to\infty} v\left(\frac{y_k + y}{2}\right) = v(y). \tag{20}$$

Combining (18), (19), and (20), we conclude that

$$v(y) \leq \limsup_{k\to\infty} \mathbb{E}\left[V\left(\frac{\widehat{Y}_T(y_k) + \widehat{Y}_T(y)}{2}\right)\right] \leq \frac{1}{2}\left(\limsup_{k\to\infty} v(y_k) + v(y)\right) - \varepsilon^2 = v(y) - \varepsilon^2,$$

which implies that

$$v(y) \leq v(y) - \varepsilon^2,$$

and which is impossible, as ε is strictly positive. We obtained a contradiction, and thus (13) does not hold. This completes the proof of the lemma. □

A a consequence of Lemma 3.3, we obtain the following corollary.

Corollary 3.4. *Let us suppose that conditions of Theorem 3.2 hold and consider a sequence* $(x_k)_{k\in\mathbb{N}} \subset dom(u)$ *that converges to* $x \in dom(u)$. *Then, the maximizers to* (4) *converge in the sense that*

$$\lim_{k\to\infty} \widehat{X}_T(x_k)(\omega) = \widehat{X}_T(x)(\omega), \quad \omega \in \Omega.$$

Proof. Let us denote $y_k := u'(x_k)$, $k \in \mathbb{N}$, and $y := u'(x)$. Then, by the properties of u established item (i) of Theorem 3.1 (monotonicity and continuous differentiability), we deduce that $y_k > 0$, $k \in \mathbb{N}$, $y > 0$, and, as $(x_k)_{k\in\mathbb{N}}$ converges to x, (by item (i) of Theorem 3.1, again) we have

$$\lim_{k\to\infty} y_k = y.$$

Therefore, by Lemma 3.3, we assert that the associated maximizers to (4) converge in the sense that

$$\lim_{k\to\infty} \widehat{Y}_T(y_k)(\omega) = \widehat{Y}_T(y)(\omega), \quad \omega \in \Omega. \tag{21}$$

As for every $\omega \in \Omega$, $I(\omega, \cdot)$ is continuous in the second variable, we deduce from (21) that

$$\lim_{k\to\infty} I\left(\omega, \widehat{Y}_T(y_k)(\omega)\right) = I\left(\omega, \widehat{Y}_T(y)(\omega)\right), \quad \omega \in \Omega. \tag{22}$$

Consequently, from item (iii) of Theorem 3.1 and (22), we deduce that

$$\lim_{k\to\infty} \widehat{X}_T(x_k)(\omega) = \lim_{k\to\infty} I\left(\omega, \widehat{Y}_T(y_k)(\omega)\right) = I\left(\omega, \widehat{Y}_T(y)(\omega)\right) = \widehat{X}_T(x)(\omega), \quad \omega \in \Omega,$$

which completes the proof of this corollary. □

Proof of Theorem 3.2. For $y = u'(x)$, let $\widehat{Y}(y)$ be the minimizer to (8) at y. Let us recall that $\Pi(x)$ is given by (12), that is

$$\Pi(x) = \mathbb{E}\left[\frac{\widehat{Y}_T(y)}{y} f\right].$$

For a given $q \in \mathbb{R}$, let us consider an $X \in \mathcal{X}\left(x - q\Pi(x)\right)$. Let us observe that

$$\mathbb{E}\left[X_T \widehat{Y}_T(y)\right] = \left(x - q\Pi(x)\right) y.$$

Therefore, from the definition of the conjugate stochastic field V, we obtain

$$\begin{aligned}\mathbb{E}\left[U\left(X_T+qf\right)\right] &\leq \mathbb{E}\left[V\left(\widehat{Y}_T(y)\right)+\widehat{Y}_T(y)\left(X_T+qf\right)\right]\\ &= v(y)+\mathbb{E}\left[\widehat{Y}_T(y)X_T\right]+q\mathbb{E}\left[\widehat{Y}_T(y)f\right]\\ &= v(y)+(x-q\Pi(x))y+qy\Pi(x)\\ &= u(x),\end{aligned} \tag{23}$$

where in the last equality, we have used item (i) of Theorem 3.1, which asserts that

$$u(x)=v(y)+xy.$$

We note that, in (23), $\mathbb{E}\left[U\left(X_T+qf\right)\right]$ might be equal to $-\infty$, in which case, the argument in (23) still holds. As $q\in\mathbb{R}$ and $X\in\mathcal{X}(x-q\Pi(x))$ are arbitrary, we deduce from (23) that and Definition 2.6 that $\Pi(x)$ is an indifference price for f.

Let us now consider a constant $\pi\neq\Pi(x)$ and show that this π is not an indifference price at x, in the sense of Definition 2.6. First, let us suppose that $\pi<\Pi(x)$. Let $c:=\|f\|_\infty$, let us consider a sequence of *strictly positive numbers* decreasing to zero, q_k, $k\in\mathbb{N}$, and such that $x-q_k(c+\pi)\in dom(u)$ for every $k\in\mathbb{N}$. Note that, as $dom(u)$ denotes the interior of $\{u>-\infty\}$ and thus is an open subset of $\mathbb{R}$, for $x\in dom(u)$, such a sequence q_k, $k\in\mathbb{N}$, as above, exists. Next, let us define

$$X^k:=\widehat{X}\left(x-q_k(c+\pi)\right)+q_kc,\quad k\in\mathbb{N},$$

where $\widehat{X}(x-q_k(c+\pi))$ is the optimizer to (4) at $(x-q_k(c+\pi))$, $k\in\mathbb{N}$. Then, the sequence X^k, $k\in\mathbb{N}$, satisfies

$$X^k_0=x-q_k\pi\quad and\quad X^k\in\mathcal{X}\left(x-q_k\pi\right),\quad k\in\mathbb{N}.$$

Moreover, by the construction of this sequence, we have

$$X^k_T(\omega)+q_kf(\omega)\geq\widehat{X}_T\left(x-q_k(c+\pi)\right)(\omega),\quad \omega\in\Omega,\quad k\in\mathbb{N},$$

and therefore, with $U'(\xi)$ denoting $U'(\omega,\xi(\omega))$, $\omega\in\Omega$, for a given random variable ξ, from the ω-by-ω concavity of $U(\omega,\cdot)$, we deduce that

$$\begin{aligned}\mathbb{E}\left[U\left(X^k_T+q_kf\right)\right] &\geq \mathbb{E}\left[U\left(\widehat{X}_T(x-q_k(c+\pi))\right)\right]+q_k\mathbb{E}\left[(c+f)\,U'\left(X^k_T+q_kf\right)\right]\\ &= u(x-q_k(c+\pi))+q_k\mathbb{E}\left[(c+f)\,U'\left(X^k_T+q_kf\right)\right],\quad k\in\mathbb{N}.\end{aligned}$$

Consequently, we get

$$\liminf_{k\to\infty}\frac{\mathbb{E}\left[U\left(X^k_T+qf\right)\right]-u(x-q_k(c+\pi))}{q_k}\geq\liminf_{k\to\infty}\mathbb{E}\left[(c+f)\,U'\left(X^k_T+q_kf\right)\right]. \tag{24}$$

Next, let us observe that

$$\begin{aligned}\liminf_{k\to\infty} \mathbb{E}\left[(c+f)\,U'\left(X^k_T + q_k f\right)\right] &= \mathbb{E}\left[(c+f)\,U'\left(\widehat{X}_T(x)\right)\right] \\ &= \mathbb{E}\left[(c+f)\,\widehat{Y}_T(y)\right] = cy + \Pi(x)y,\end{aligned} \tag{25}$$

where $\Pi(x)$ is given by (12). In (25), in the first equality, we used Corollary 3.4 (as well as the finiteness of the underlying probability space) and, in the second, item (*iii*) of Theorem 3.1. Combining (24) and (25), we get

$$\liminf_{k\to\infty} \frac{\mathbb{E}\left[U\left(X^k_T + qf\right)\right] - u(x - q_k(c+\pi))}{q_k} \geq (c + \Pi(x))y. \tag{26}$$

Since by the chain rule and item (*i*) of Theorem 3.1, we have

$$\liminf_{k\to\infty} \frac{u(x - q_k(c+\pi)) - u(x)}{q_k} = -u'(x)(c+\pi) = -y(c+\pi). \tag{27}$$

Therefore, from (26) and (27), we deduce that

$$\begin{aligned}\liminf_{k\to\infty} \frac{\mathbb{E}\left[U\left(X^k_T + qf\right)\right] - u(x)}{q_k} &= \liminf_{k\to\infty} \frac{\mathbb{E}\left[U\left(X^k_T + qf\right)\right] - u(x - q_k(c+\pi))}{q_k} \\ &\quad + \liminf_{k\to\infty} \frac{u(x - q_k(c+\pi)) - u(x)}{q_k} \\ &\geq (c + \Pi(x))y - y(c+\pi) = (\Pi(x) - \pi)\,y > 0.\end{aligned}$$

We obtained that

$$\liminf_{k\to\infty} \frac{\mathbb{E}\left[U\left(X^k_T + qf\right)\right] - u(x)}{q_k} = (\Pi(x) - \pi)\,y > 0.$$

Comparing this with Definition 2.6 and since q_k, $k \in \mathbb{N}$, is a sequence of *strictly positive numbers*, we conclude that π is not an indifference price for f at x.

To show that any constant $\pi > \Pi(x)$ is not an indifference price for f at x, we repeat the argument above for the contingent claim $-f$. Thus, the unique indifference price for f at x is given by (12).

□

4. Reformulation of the Indifference Pricing Problem and Proof of Theorem 2.9 in the general case

4.1. **Preliminary notations.** In this section, we reformulate problems (4) and (8) in an equivalent way in terms of the discounted stock price process. That is, we reformulate (4) and (8) as (32) and (34), respectively. Let us denote the set of self-financing wealth processes measured in the units of S^0 and starting from x by[4] $\tilde{\mathcal{X}}(x)$, i.e., with $\tilde{S} :=$

[4]We recall that $S^0_0 = 1$.

$\left(1, \frac{S^1_n}{S^0_n}, \dots, \frac{S^d_n}{S^0_n}\right)_{\{n\in\{0,\dots,T\}\}}$, and for predictable process $\tilde{H}$ that satisfies the self-financing condition

$$\tilde{H}_n \cdot \tilde{S}_n = \tilde{H}_{n+1} \cdot \tilde{S}_n, \quad n \in \{1, \dots, T-1\}, \tag{28}$$

where we recall that $\cdot$ represents the scalar product on $\mathbb{R}^{d+1}$, we set

$$\begin{aligned}\tilde{\mathcal{X}}(x) := \{X_n =& x + \sum_{k=1}^{n} \tilde{H}_k \cdot \left(\tilde{S}_k - \tilde{S}_{k-1}\right),\ n \in \{1, \dots, T\} : \\ &\tilde{H} \textit{ is predictable and satisfies } (28)\Big\}, \quad x \in \mathbb{R}.\end{aligned} \tag{29}$$

4.2. **A change of numéraire lemma.** The following result is key in the reformulation. It is based on the change of the numéraire argument. The proof is contained in [BHMP22, Lemma 3.1, p. 654].

Lemma 4.1. *Let S^0 be a strictly positive process, such that $S^0_0 = 1$. Then, the sets of self-financing wealth processes $\mathcal{X}$ and $\tilde{\mathcal{X}}$, defined in* (3) *and* (29)*, respectively, satisfy*

$$\tilde{\mathcal{X}}(x) = \frac{\mathcal{X}(x)}{S^0} = \left\{\frac{X}{S^0} = \left(\frac{X_n}{S^0_n}\right)_{n\in\{0,\dots,T\}} : X \in \mathcal{X}(x)\right\}, \quad x \in \mathbb{R}.$$

4.3. **Reformulation of problems** (4) **and** (8)**.** Lemma 4.1 allows to reformulate (4) in terms of $\tilde{\mathcal{X}}$ instead of $\mathcal{X}$, that is, in the following form

$$u(x) := \sup_{X\in\tilde{\mathcal{X}}(x)} \mathbb{E}\left[U\left(X_T S^0_T\right)\right], \quad x \in \mathbb{R}. \tag{30}$$

Formulation (30) begs for a change of notation

$$\tilde{U}(\omega, x) := U\left(\omega, x S^0_T(\omega)\right), \quad (\omega, x) \in \Omega \times \mathbb{R}, \tag{31}$$

where, under the strict positivity of S^0_T, $\tilde{U}$ satisfies Assumption 2.2 if and only if U does. Thus, we can reformulate (30) as

$$u(x) := \sup_{X\in\tilde{\mathcal{X}}(x)} \mathbb{E}\left[\tilde{U}(X_T)\right], \quad x \in \mathbb{R}, \tag{32}$$

where, we stress that the value functions in (4), (30), and (32) are equal to each other, and the relationship between the optimizers can be recovered via Lemma 4.1.

Next, to write the dual problem to (32), we introduce the pointwise (in ω) conjugate of $\tilde{U}$ as

$$\tilde{V}(\omega, y) := \sup_{x\in\mathbb{R}} \left(\tilde{U}(\omega, x) - xy\right), \quad (\omega, y) \in \Omega \times (0, \infty), \tag{33}$$

and one can see that V, defined in (7), and $\tilde{V}$, defined in (33), are related via

$$\tilde{V}(\omega, y) = V\left(\omega, \frac{y}{S^0_T(\omega)}\right), \quad (\omega, y) \in \Omega \times (0, \infty).$$

Now, we can restate the dual problem (8) as

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[\tilde{V}\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \quad y > 0. \tag{34}$$

4.4. **A characterization of indifference prices under a change of numéraire.** Let us introduce

$$\tilde{f} := \frac{f}{S^0_T}, \tag{35}$$

which denotes the payoff of the contingent claim under the numéraire S^0. Then, by the strict positivity of S^0_T and (31), for every $X \in \bigcup\limits_{x\in\mathbb{R}} \mathcal{X}(x)$, with $\tilde{X} = \frac{X}{S^0} = \left(\frac{X_t}{S^0_t}\right)_{t\in[0,T]}$, in Definition 2.6, we have

$$U\left(X_T + qf\right) = U\left(S^0_T \frac{(X_T + qf)}{S^0_T}\right) = \tilde{U}\left(\tilde{X}_T + q\tilde{f}\right), \tag{36}$$

where, by Lemma 4.1, we have that $\tilde{X}_T$ is a terminal value of a self-financing wealth process starting from X_0 under the numéraire S^0, i.e., in the market, where the traded assets have the dynamics $\left(1, \frac{S^1_n}{S^0_n}, \dots, \frac{S^d_n}{S^0_n}\right)_{\{n\in\{0,\dots,T\}\}}$. With the notations of Section 4, we can state the following equivalent characterization of indifference prices under the numéraire S^0, in terms of $\tilde{\mathcal{X}}$, $\tilde{U}$, and $\tilde{f}$ defined in (29), (31), and (35), respectively. The proof of Lemma 4.2 immediately follows from the respective definitions of $\tilde{\mathcal{X}}$, $\tilde{U}$, and $\tilde{f}$, and (36). It is, therefore, skipped.

Lemma 4.2. *Let a utility stochastic field U satisfy Assumption 2.2 and a contingent claim f be fixed. Then $\Pi(x)$ is the indifference price of f (in the sense of Definition 2.6) at $x \in dom(u)$, if and only if, $\Pi(x)$ satisfies*

$$\mathbb{E}\left[\tilde{U}\left(\tilde{X}_T + q\tilde{f}\right)\right] \leq u(x), \quad q \in \mathbb{R}, \quad \tilde{X} \in \tilde{\mathcal{X}}(x - q\Pi(x)), \tag{37}$$

where $\tilde{U}$, $\tilde{f}$, and $\tilde{\mathcal{X}}$ are defined in (31), (35), *and* (29), *respectively.*

4.5. **Proof of the main result.** Now, we can prove the main result of this paper, Theorem 2.9, in the undiscounted case, that is, without assuming (11).

Proof of Theorem 2.9. First, we observe that Lemma 4.2, particularly (37), characterizes indifference prices for f (at different x's) via a discounted model as in Section 3 and indifference pricing of the contingent claim $\tilde{f}$ given by (35), that is, in the market, where the traded securities are $\left(1, \frac{S^1_n}{S^0_n}, \dots, \frac{S^d_n}{S^0_n}\right)_{\{n\in\{0,\dots,T\}\}}$, for which the results of Section 3 apply. Let us recall that an equivalent reformulation in the sense of Lemma 4.2 was achieved via passing to another stochastic utility $\tilde{U}$ given by (31), which allows for an equivalent representation of u via (32). In turn, this leads to a reformulation of the dual problem as in (34) via $\tilde{V}$ defined in (33). Thus, we reformulated the indifference pricing problem of f by pricing $\tilde{f}$ defined in (35) in the (discounted) settings of Theorem 3.2.

Next, applying Theorem 3.2 (for $\tilde{f}$), we deduce from this theorem that, for a given $x \in dom(u)$, the unique indifference price of $\tilde{f}$, $\Pi(x)$, is given by

$$\Pi(x) = \mathbb{E}\left[\frac{\widehat{Y}_T(y)}{y}\tilde{f}\right], \tag{38}$$

where $\widehat{Y}(y)$ is the minimizer to (34). By Lemma 4.2, $\Pi(x)$ is the indifference price for f in the sense of Definition 2.6.

It remains to show that (10) holds. Here, one can see that $\zeta(y) := \frac{\widehat{Y}_T(y)}{y}\frac{1}{S^0_T}$ is the dual-optimal state price density, that is, the optimizer to (9) at $y = u'(x) > 0$, and $\widehat{\mathbb{Q}}(y)$, whose Radon-Nikodym derivative is given by $\frac{d\widehat{\mathbb{Q}}(y)}{d\mathbb{P}} = \frac{\widehat{Y}_T(y)}{y}$, is the dual-optimal martingale measure, that is, $\widehat{\mathbb{Q}}(y)$ is the minimizer to both (8) and (34) at y, and since $\tilde{f} = \frac{f}{S^0_T}$ by (35), we can further restate (38) as

$$\Pi(x) = \mathbb{E}\left[\frac{\widehat{Y}_T(y)}{y}\frac{f}{S^0_T}\right] = \mathbb{E}_{\widehat{\mathbb{Q}}(y)}\left[\frac{f}{S^0_T}\right] = \mathbb{E}\left[\widehat{\zeta}(y)f\right],$$

which is precisely representation (10). This completes the proof of this theorem. □

5. Indifference pricing of replicable claims

In this section, we consider the settings of Section 2 (that is, the undiscounted stock price process) and investigate a particular case when the contingent claim f is replicable, that is, f can be replicated by a self-financing wealth process X^r in the sense that

$$f = X^r_T = X^r_0 + \sum_{j=1}^{n}\overline{H}^r_j \cdot \left(\overline{S}_j - \overline{S}_{j-1}\right), \quad n \in \{1, \dots, T\}, \tag{39}$$

for some $X^r_0 \in \mathbb{R}$ and $\overline{H}^r \in \mathcal{H}$ (that is, $\overline{H}$ is a predictable process associated with a self-financing strategy).

The main result of this section is that the indifference price in the sense of Definition 2.6, for replicable f, is the initial value of the replicating strategy, that is, X^r_0. This number is precisely the arbitrage-free price, and thus, this section demonstrates, in the settings of the present paper, the consistency of the arbitrage-free and indifference pricing methodologies for replicable claims. *We recall that for replicable claims, the arbitrage-free price is the initial value of the replicating wealth process. Thus, the arbitrage-free price for f satisfying* (39) *is X^r_0.*

Lemma 5.1. *Let us suppose that the assumptions of Theorem 2.9 hold and f satisfies* (39) *for some $X^r_0 \in \mathbb{R}$ and $\overline{H}^r \in \mathcal{H}$. Then, for every $x \in dom(u)$, the unique indifference price of f is given by X^r_0, that is*

$$\Pi(x) = X^r_0,$$

which is also the unique arbitrage-free price for f.

Proof. Let us first fix an $x \in dom(u)$ and then arbitrary $q \in \mathbb{R}$ and $X \in \mathcal{X}(x - qX^r_0)$. Then, from (39), we have

$$X_T + qf = X_T + qX^r_T.$$

As $X \in \mathcal{X}(x - qX^r_0)$, we have $X + qX^r \in \mathcal{X}(x)$. Therefore, we deduce that

$$\mathbb{E}\left[U\left(X_T + qf\right)\right] = \mathbb{E}\left[U\left(X_T + qX^r_T\right)\right] \leq \sup_{X' \in \mathcal{X}(x)} \mathbb{E}\left[U\left(X'_T\right)\right] = u(x), \tag{40}$$

where the inequality follows from the definition of u in (4). As $q \in \mathbb{R}$ and $X \in \mathcal{X}(x-qX^r_0)$ are arbitrary, we deduce from (40) and Definition 2.6 that X^r_0 is an indifference price for f.

Next, for every $\pi \neq X^r_0$, we show that π is not an indifference price for f at $x \in dom(u)$. Let us set $q := sign\left(X^r_0 - \pi\right)$, so that

$$x + q\left(X^r_0 - \pi\right) > x.$$

It follows from the definition of u in (4) and the strict monotonicity of U in x for every $\omega \in \Omega$ in Definition 2.2 that u is *strictly* increasing. This implies the *strict* positivity of

$$\varepsilon := \frac{1}{2}\left(u\left(x + q\left(X^r_0 - \pi\right)\right) - u(x)\right). \tag{41}$$

Now, let us consider a process $X^\varepsilon \in \mathcal{X}\left(x + q\left(X^r_0 - \pi\right)\right)$, such that

$$\mathbb{E}\left[U(X^\varepsilon_T)\right] \geq u\left(x + q\left(X^r_0 - \pi\right)\right) - \varepsilon. \tag{42}$$

The existence of such a process $X^\varepsilon \in \mathcal{X}\left(x + q\left(X^r_0 - \pi\right)\right)$ follows from the definition of the value function u in (4). Setting $X := X^\varepsilon - qX^r \in \mathcal{X}(x - q\pi)$ and using (39), we can restate (42) as

$$u\left(x + q\left(X^r_0 - \pi\right)\right) - \varepsilon \leq \mathbb{E}\left[U(X^\varepsilon_T)\right] = \mathbb{E}\left[U(X_T - qX^r_T + qf)\right] = \mathbb{E}\left[U(X_T + qf)\right]. \tag{43}$$

Finally, it follows from (41) and (43) that, for $X = X^\varepsilon - qX^r \in \mathcal{X}(x - q\pi)$ and $q = sign\left(X^r_0 - \pi\right)$ (as above), we have

$$\mathbb{E}\left[U(X_T + qf)\right] \geq u\left(x + q\left(X^r_0 - \pi\right)\right) - \varepsilon > u(x),$$

and thus, comparing with (6) in Definition 2.6, we conclude that π is not an indifference price for f at $x \in dom(u)$. As $\pi \neq X^r_0$ is arbitrary, the argument of this paragraph implies that every $\pi \neq X^r_0$ is not an indifference price for f at $x \in dom(u)$. Finally, the fact X^r_0 is also the unique arbitrage-free price follows from (39) and the definition of arbitrage-free prices. ☐

In complete markets every contingent claim f is replicable, that is (39) holds for some self-financing wealth process X^r. Therefore, the argument of this section, in particular, Lemma 5.1, implies that in complete models on finite probability spaces the arbitrage-free and indifference pricing methodologies coincide for stochastic utilities satisfying Assumption 2.2.

In an incomplete model on a finite probability space, for non-replicable contingent claims (in the sense that there is no self-financing wealth process such (39) holds), arbitrage-free prices and indifference prices do not coincide, in general. Here, the unique indifference price is given by Theorem 2.9 (under assumptions of this theorem) and is a singleton. On the other hand, the arbitrage-free prices of a contingent claim $\overline{f}$ is an interval given by

$$\left(\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[\frac{\overline{f}}{S^0_T}\right], \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[\frac{\overline{f}}{S^0_T}\right]\right), \tag{44}$$

which follows, e.g., from [DS98, Theorem 5.12] plus a change of numéraire argument from Section 4. Comparing (10) in Theorem 2.9 with (44), we further conclude that, under the conditions of Theorem 2.9, the indifference price is an arbitrage-free price for every contingent claim $\overline{f}$, every utility stochastic field U satisfying Assumption 2.2, and every $x \in dom(u)$, where, again, by $dom(u)$ we denote the interior of $\{u > -\infty\}$.

6. Examples

6.1. **One-period Trinomial Model.** Consider a one-period trinomial model, that is, $T = 1$. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ be our probability space. Let us suppose that there are two traded assets: the riskless asset S^0, whose price is equal to 1 at both times and a risky asset, whose evolution S^1 is represented by Figure 1.

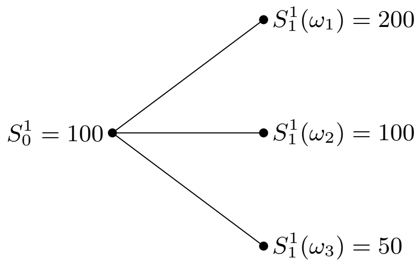

Figure 1. One-period trinomial model.

Here S_0 is the initial price of the stock, and $S_1(\omega)$ is the price of the stock at time $T = 1$ that can take three values. Let us suppose that the probability measure $\mathbb{P}$ is given by

$$p_1 := \mathbb{P}(\omega_1) = 0.2, \quad p_2 := \mathbb{P}(\omega_2) = 0.3, \quad p_3 := \mathbb{P}(\omega_3) = 0.5. \tag{45}$$

and the utility stochastic field is

$$U(\omega, x) = \ln x, \quad (\omega, x) \in \Omega \times (0, \infty). \tag{46}$$

Let us compute indifference prices for the European call and put options with the same strike $K = S^1_0$. In this case, their is the respective payoffs, $C = C(\omega)$ and $P = P(\omega)$, $\omega \in \Omega$, at $T = 1$, are given by

$$\begin{aligned} C_1(\omega_1) &= (S_1(\omega_1) - 100)^+ = 100, & P_1(\omega_1) &= (100 - S_1(\omega_1))^+ = 0, \\ C_1(\omega_2) &= (S_1(\omega_2) - 100)^+ = 0, & P_1(\omega_2) &= (100 - S_1(\omega_2))^+ = 0, \\ C_1(\omega_3) &= (S_1(\omega_3) - 100)^+ = 0, & P_1(\omega_3) &= (100 - S_1(\omega_3))^+ = 50. \end{aligned}$$

We observe that, in this model, neither the call nor put option is replicable in the sense of (39), and so the arbitrage-free prices are intervals given by (44) for both options.

Let us now compute their indifference prices. For this, we observe that the set of equivalent martingale measures $\mathcal{M}$ is given by probability measures $\mathbb{Q}$, such that

$$\begin{aligned} 200\mathbb{Q}(\omega_1) + 100\mathbb{Q}(\omega_2) + 50\mathbb{Q}(\omega_3) = 100, \\ \mathbb{Q}(\omega_1) + \mathbb{Q}(\omega_2) + \mathbb{Q}(\omega_3) = 1, \end{aligned}$$

where, the first equation is the martingale property of S^1 under $\mathbb{Q}$, and the second is the normalization condition, $\mathbb{Q}(\Omega) = 1$. Solving this system of equations and using the positivity of $\mathbb{Q}(\omega)$ for every $\omega \in \Omega$, we can reparametrize the set of equivalent martingale measures $\mathcal{M}$ as follows

$$\mathcal{M} = \left\{ \mathbb{Q} : \ \mathbb{Q}(\omega_1) = t, \quad \mathbb{Q}(\omega_2) = 1 - 3t, \quad \mathbb{Q}(\omega_3) = 2t, \quad t \in \left(0, \frac{1}{3}\right) \right\} . \tag{47}$$

Now, let us compute the dual-optimal measure $\widehat{\mathbb{Q}}$ in the sense of (8). Since

$$V(\omega, y) = -\ln y - 1, \quad (\omega, y) \in \Omega \times (0, \infty), \tag{48}$$

we deduce from (47) and (48) that the dual value function v becomes

$$\begin{aligned} v(y) &= \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}\left[V\left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] \\ &= -\ln(y) - 1 + \inf_{t \in \left(0, \frac{1}{3}\right)} \left(-0.2 \ln\left(\frac{t}{0.2}\right) - 0.3 \ln\left(\frac{1-3t}{0.3}\right) - 0.5 \ln\left(\frac{2t}{0.5}\right) \right), \end{aligned}$$

and one can see from the latter representation that the dual-optimal equivalent martingale measure $\widehat{\mathbb{Q}}$ does not depend on y. In turn, this implies via (10) that the corresponding indifference prices do not depend on the initial wealth $x > 0$, a well-known feature of the logarithmic utility. The expression under the infimum in the latter formula leads to the following smooth function

$$h(t) := -0.2 \ln\left(\frac{t}{0.2}\right) - 0.3 \ln\left(\frac{1-3t}{0.3}\right) - 0.5 \ln\left(\frac{2t}{0.5}\right), \quad t \in \left(0, \tfrac{1}{3}\right),$$

which reaches a unique minimum at $t = \frac{7}{30}$. Therefore, for every $y > 0$, the dual-optimal equivalent martingale measure $\widehat{\mathbb{Q}}$ is given by

$$\widehat{q}_1 := \widehat{\mathbb{Q}}(\omega_1) = \frac{7}{30}, \quad \widehat{q}_2 := \widehat{\mathbb{Q}}(\omega_2) = \frac{9}{30}, \quad \widehat{\mathbb{Q}}(\omega_3) = \frac{14}{30}. \tag{49}$$

The representation formula (10) from Theorem 2.9 in the present setting leads to the following indifference price, at every initial wealth $x > 0$, for the call and put options considered above

$$\begin{aligned} \mathbb{E}_{\widehat{\mathbb{Q}}}\left[C_1\right] &= \sum_{i=1}^{3} C_1(\omega_i)\widehat{q}_i = 100 \cdot \frac{7}{30} + 0 \cdot \frac{9}{30} + 0 \cdot \frac{14}{30} = 23.33, \\ \mathbb{E}_{\widehat{\mathbb{Q}}}\left[P_1\right] &= \sum_{i=1}^{3} P_1(\omega_i)\widehat{q}_i = 0 \cdot \frac{7}{30} + 0 \cdot \frac{9}{30} + 50 \cdot \frac{14}{30} = 23.33. \end{aligned} \tag{50}$$

Remark 6.1. If we suppose that, for the riskless asset, $S^0_0 = 1$, but $S^0_1 \in \left(\frac{1}{2}, 2\right)$, then the dual-optimal equivalent martingale measure $\widehat{\mathbb{Q}}$ is given by (49). However, the indifference prices (50), in this case, need to be divided by S^0_1, in accordance with (10).

6.2. **Multiperiod trinomial model.** Let us consider a multi-period trinomial model. Let $T \in \mathbb{N}$ be the time horizon, and Ω be the set of sequences of T elements, each of these elements taking one of the three possible values. A generic element of Ω is denoted by $\omega = (\overline{\omega}_{1j_1}, \dots, \overline{\omega}_{Tj_T})$, where each $j_i \in \{1, 2, 3\}$ and $i \in \{1, \dots, T\}$. Let us suppose that there are two traded assets: riskless S^0, whose price equals 1 at all times, and a risky stock, whose price process has the multiplicative increments u, 1, or d, with the one-step *conditional* probabilities under $\mathbb{P}$ being given as in (45). Here, $\overline{\omega}_{i1}$ corresponds to the stock price going up by a multiplicative factor $u = 2$ at the i-th experiment, $\overline{\omega}_{i2}$ - staying the same, and $\overline{\omega}_{i3}$ - going down by a multiplicative factor $d = \frac{1}{2}$. This construction is entirely similar to [Shr04, Section 1.2], however, in trinomial, not binomial settings.

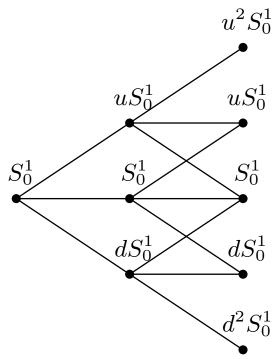

Figure 2. Two-period trinomial model.

Let us consider the same logarithmic utility as in the one-period case given by (46). To obtain the dual minimizer, let Z_n, $n \in \{0, \dots, T\}$, denote the density process of an element of $\mathcal{M}$. As this process is strictly positive, we can represent it as

$$Z_n = \frac{Z_1}{Z_0} \dots \frac{Z_n}{Z_{n-1}}, \quad n \in \{1, \dots, T\}. \tag{51}$$

Therefore, since

$$V(yZ_T) = -\ln(yZ_T) - 1 = -\ln y - 1 - \sum_{n=1}^{T} \ln\left(\frac{Z_n}{Z_{n-1}}\right), \quad y > 0,$$

we deduce that

$$\mathbb{E}\left[V(yZ_T)\right] = -\ln y - 1 - \sum_{n=1}^{T} \mathbb{E}\left[\ln\left(\frac{Z_n}{Z_{n-1}}\right)\right], \quad y > 0,$$

which, in turn, implies that *dual problem* (8)*, in the logarithmic utility case, is reduced to minimizing each of the terms* $\mathbb{E}\left[\ln\left(\frac{Z_n}{Z_{n-1}}\right)\right]$*,* $n \in \{1, \dots, T\}$*, separately.* This is the idea behind representation (51). Therefore, as in the one-period trinomial case and, in particular, (49), each one-step *conditional* probability under the dual-optimal equivalent martingale measure is

$$\widehat{q}_n(\overline{\omega}_{n1}) := \frac{7}{30}, \quad \widehat{q}_n(\overline{\omega}_{n2}) := \frac{9}{30}, \quad \widehat{q}_n(\overline{\omega}_{n3}) := \frac{14}{30}, \quad n \in \{1, \dots, T\},$$

and, thus, the optimal $\widehat{\mathbb{Q}}$ is given by

$$\widehat{\mathbb{Q}}\left(\omega = (\overline{\omega}_{1j_1}, \dots, \overline{\omega}_{Tj_T})\right) = \prod_{n=1}^{T} \widehat{q}_n\left(\overline{\omega}_{nj_n}\right), \quad j_n \in \{1, 2, 3\}, \quad n \in \{1, \dots, T\}.$$

With this $\widehat{\mathbb{Q}}$, contingent claims in this model can be priced similarly to the one-period case via (10).

References

[BČ11] S. Biagini and A. Černý. Admissible strategies in semimartingale portfolio selection. *SIAM J. Control Optim.*, 49(1):42–72, 2011.

[BHMP22] W. Busching, D. Hintz, O. Mostovyi, and A. Pozdnyakov. Fair pricing and hedging under small perturbations of the numéraire on a finite probability space. *Involve*, 15(4):649 – 668, 2022.

[BS12] S. Biagini and M. Sirbu. A note on admissibility when the credit line is infinite. *Stochastics*, 84(2–3):157–169, 2012.

[Dav97] M. H. Davis. Option pricing in incomplete markets. In *Mathematics of derivative securities*, pages 216–226. Cambridge University Press, 1997.

[DS97] F. Delbaen and W. Schachermayer. The Banach space of workable contingent claims in arbitrage theory. *Ann. Inst. H. Poincaré Statist. Probab.*, 33:113–144, 1997.

[DS98] F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.*, 312:215–250, 1998.

[HH09] V. Henderson and D. Hobson. Utility indifference pricing: an overview. In Rene Carmona, editor, *Indifference Pricing: Theory and Applications*. Princeton University Press, 2009.

[HK79] J. M. Harrison and D. M. Kreps. Martingales and arbitrage in multiperiod securities market. *J. Econom. Theory*, 20:381–408, 1979.

[HK04] J. Hugonnier and D. Kramkov. Optimal investment with random endowments in incomplete markets. *Ann. Appl. Probab.*, 14(2):845–864, 2004.

[HKS05] J. Hugonnier, D. Kramkov, and W. Schachermayer. On utility-based pricing of contingent claims in incomplete markets. *Math. Finance*, 15(2):203–212, 2005.

[Mon08] M. Monoyios. Utility indifference pricing with market incompleteness. In Ehrhardt M., editor, *Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing*, pages 67–100. Nova Science Publishers, Hauppauge, New York, 2008.

[Mos15] O. Mostovyi. Necessary and sufficient conditions in the problem of optimal investment with intermediate consumption. *Finance Stoch.*, 19(1):135–159, 2015.

[Mos18] O. Mostovyi. Utility maximization in a large market. *Math. Finance*, 28(1):106–118, 2018.

[MS23] O. Mostovyi and P. Siorpaes. Pricing of contingent claims in large markets. submitted, 50 pages, 2023.

[MZ04] M. Musiela and T. Zariphopoulou. An example of indifference prices under exponential preferences. *Finance Stoch.*, 8:229–239, 2004.

[Roc70] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[RS18] S. Robertson and K. Spiliopoulos. Indifference pricing for contingent claims: Large deviations effects. *Math. Finance*, 28(1):335–371, 2018.

[Sch01] W. Schachermayer. Optimal investment in incomplete financial markets. In H. Geman, D. Madan, St.R. Pliska, and T. Vorst, editors, *Mathematical Finance: Bachelier Congress 2000*, pages 427–462. Springer, 2001.

[Shr04] S. Shreve. *Stochastic Calculus for Finance I The Binomial Asset Pricing Model*. Springer, 2004.

[Sio15] P. Siorpaes. Optimal investment and price dependence in a semi-static market. *Finance Stoch.*, 19(1):161–187, 2015.

[Sio16] P. Siorpaes. Do arbitrage-free prices come from utility maximization? *Math. Finance*, 26(3):602–616, 2016.

[SZ05] R. Sircar and T. Zariphopoulou. Bounds and asymptotic approximations for utility prices when volatility is random. *SIAM J. Control Optim.*, 43(4):1328–1353, 2005.

Jason Freitas, University of Connecticut, Storrs, CT 06269, United States
Email address: jason.freitas@uconn.edu

Joshua Huang, Tufts University, Medford, MA 02155, United States
Email address: joshua.j.huang1@gmail.com

Oleksii Mostovyi, Department of Mathematics, University of Connecticut, Storrs, CT 06269, United States
Email address: oleksii.mostovyi@uconn.edu