

# ON PERTURBATIONS OF PREFERENCES AND INDIFFERENCE PRICE INVARIANCE

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ABSTRACT. We investigate indifference pricing under perturbations of preferences in small and large markets. We establish stability results for small perturbations of preferences, where the latter can be stochastic. We obtain a sharp condition in terms of the associated concave and convex envelopes and provide counterexamples demonstrating that, in general, stability fails. Next, we investigate a class of models where the indifference price does not depend on the preferences or the initial wealth. Here, under the existence of an equivalent separating measure, in the settings of deterministic preferences, we show that the class of indifference price invariant models is the class of models where the dual domain is stochastically dominant of the second order. We also provide a counterexample showing that, in general, this result does not hold over stochastic preferences; where instead, we show that the indifference price invariant models are complete models (in both small and large markets). In the process, we establish a theorem of independent interest on the stability of the optimal investment problem under perturbations of preferences. Our results are new in both small and large markets, and thus, in particular, we introduce large stochastically dominant models, give examples of such models, and characterize them as the indifference price invariant ones over deterministic preferences.

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## 1. INTRODUCTION

There are many methodologies for pricing financial instruments, such as contingent claims, options, and other derivatives. From statistical to stochastic analytic, they try to assign a unique value or an interval to a given contingent claim. One of the most far-reaching ideas in financial mathematics is the presumption of pricing by no-arbitrage, which, in the case of complete markets<sup>1</sup>, states that there is a single price for any financial asset and where any other price results in arbitrage leading to the infinite supply or demand. The development of this methodology resulted in the award of the Nobel prize in economic sciences to Robert C. Merton and Myron S. Scholes in 1997.

When financial markets are incomplete, wealth processes associated with dynamically rebalanced portfolios of traded assets cannot span the payoffs of all bounded contingent claims. The no-arbitrage methodology in incomplete markets leads to the concepts of upper and lower hedging prices that define an open interval of arbitrage-free prices for non-replicable contingent claims. Thus, uniqueness of no-arbitrage prices does not hold for a wide class of non-replicable contingent claims. In this case, one can still seek quantities at which the demand and supply match and call them prices, but the underlying theory still needs to be fully established, even though this is an active research area.

One of the mechanisms based on supply and demand that might allow regaining the uniqueness of prices is related to equilibrium theory. However, in continuous-time incomplete market settings, this problem turns out to be too difficult (at least at the current stage of its development) to be handled in full universality. Even the basic questions of fundamental importance, such as the existence and uniqueness of equilibria, are not answered yet in the general settings of incomplete continuous-time markets. The question of effective computations of the equilibria prices in general settings is even less studied.

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<sup>1</sup>A financial market is called complete if every bounded contingent claim (bounded random variable) can be replicated by a wealth processes associated with a dynamic trading strategy.

The notion of indifference pricing has naturally arisen in mathematical economics as a compromise between arbitrage-free pricing (that lacks uniqueness in the incomplete models) and equilibria pricing (whose existence is not established in the great generality). Thus, indifference pricing often (but not always) allows assigning a unique value (price) to a given contingent claim. This pricing methodology depends on the preferences of a rational economic agent that are typically specified via the agent's utility function.

Mathematically, the concepts of existence and uniqueness are closely associated with the well-posedness of a problem. Formally, and following Hadamard, one needs to include the notion of stability, which is of central importance to a wide range of problems, including the ones in mathematical finance. This paper focuses on the stability of indifference pricing under small perturbations of preferences in small and large markets, where by a small/large market, we mean the one with finitely/infinitely, yet countably many traded assets. We provide *sharp conditions* for stability given in terms of the *concave and convex envelopes* of the primal and dual stochastic utilities. Furthermore, we provide a counterexample showing the minimality of the conditions.

While deterministic utility functions are more common in the literature, we state Theorem 2.4, Proposition 2.5, and Proposition 2.9 below, which constitute part of our main results, for stochastic utilities. Such a formulation is made, in particular, to possibly include unbounded contingent claims via a change of numéraire technique, see the discussion in Remark 2.6. Mathematically, stochastic utilities allow for more flexibility yet are technically harder to work with. They arise naturally in many situations, for example, the ones related to stochastic interest rates, aforementioned change of numéraire, and in optimal consumption of multiple goods, see [Mer90], and so on.

Upon establishing the stability of indifference pricing results, we next ask a natural question: in what financial models changes in preferences (or initial wealth) do not affect indifference pricing? Here, for deterministic preferences, we prove that indifference prices invariant models are exactly those

whose dual domains allow for a maximal element in the sense of the second-order stochastic dominance. We show that this holds both in small and large markets. For the large markets, we introduce the notion of *large stochastically dominant models* and provide positive examples of such models, see Examples 4.6 and 4.7. We also provide a counterexample, see Example 4.4, showing that the second-order stochastic dominance does not imply indifference price invariance over *stochastic* utilities. For stochastic utilities, we show that the class of the indifference price invariant models is the class of complete models.

Our results apply to both small and large (post-limit) markets, and they are new, to the best of our knowledge, in both cases. *Abstract theorems*, stated in the measure-theoretic form, are key for handling both cases in one formulation.

Modeling large post-limit markets is based on stochastic integration theory with respect to infinite-dimensional semimartingales that is less developed than its finite-dimensional counterpart. We note that semimartingales constitute the most general class of stochastic processes that allow for no-arbitrage conditions in models without transaction costs. In particular, the Optional Decomposition Theorem for stochastic integration in infinite dimensions is not proven yet, to the best of our knowledge, for large markets (however, [Kar19] has a version of the Optional Decomposition Theorem for continuous stock prices processes), and this theorem is at the core of the proofs of many characterizations of indifference prices in small markets, see, e.g., [HK04].

Our results include a theorem of independent interest on the stability of optimal investment without random endowment with respect to perturbations of preferences. This theorem is stated in stochastic utility settings, which allow to accommodate a wide range of contingent claims via a change of numéraire technique (see a discussion on Remark 2.6).

The remainder of this paper is organized as follows. In section 2, we introduce the model and state the main results. They include the convergence of the value functions and optimizers, a convergence of the indifference prices result, and a characterization of indifference price invariant models as stochastically dominant ones (in the deterministic utility case) and as

complete models (over stochastic utilities). The proofs are given in section 3, and they rely on the abstract versions of the main results also presented in this section, including their proofs. The final section 4 contains positive examples of large indifference price invariant models and two counterexamples, which show that without an integrability condition, in general, the value functions might blow up, making the indifference pricing an ill-posed problem. Section 4 also contains one more counterexample showing that indifference price invariance over stochastic preferences is not equivalent to stochastic dominance of the dual domain and a positive example of the convergence of indifference prices under perturbations of the relative risk aversion in exponential Levy models.

## 2. MODEL AND MAIN RESULTS

We consider a complete stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ , where the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  satisfies the usual conditions,  $\mathcal{F}_0$  is trivial. We assume that there is one fixed market consisting of a riskless asset  $S^0 \equiv 1$  and a sequence of risky asset  $S = (S^n)_{n \in \mathbb{N}}$ , where each  $S^n$ ,  $n \in \mathbb{N}$ , is a semimartingale that gives the price of the  $n$ th risky asset. The process  $S$  is given exogenously.

To specify trading strategies in the large market, we proceed as follows. For a given  $n \in \mathbb{N}$ , an *n-elementary strategy* is defined to be an  $\mathbb{R}^n$ -valued predictable and  $(S^k)_{k=1, \dots, n}$ -integrable process in the sense of vector-valued stochastic integration as in [JS03]. We say that an *n-elementary strategy*  $H$  is *x-admissible* for a given  $x \geq 0$ , if

$$H \cdot S = \int_0^\cdot \sum_{k=1}^n H_t^k dS_t^k \geq -x, \quad \mathbb{P}\text{-a.s.}$$

$\mathcal{H}^n$  denotes the set of *n*-elementary strategies, which are *x*-admissible for some  $x \geq 0$ . An *elementary strategy* is a strategy that is *n*-elementary for some  $n \in \mathbb{N}$ . By  $\mathcal{H}$  we designate the set of admissible elementary strategies.

To pass to the limit as  $n \rightarrow \infty$ , we follow [DDGP05], and recall that  $\mathbb{R}^{\mathbb{N}}$  is the space of real-valued sequences. An *unbounded functional* on  $\mathbb{R}^{\mathbb{N}}$  is a linear functional  $H'$ , whose domain,  $Dom(H')$  is a subspace of  $\mathbb{R}^{\mathbb{N}}$ . With  $\delta^k$  denoting the Dirac delta at point  $k$ , a *simple integrand* is defined as a finite

sum of bounded predictable processes of the form  $\sum_{k=1}^n h^k \delta^k$ , where each  $h^k$  is a one-dimensional bounded and predictable process,  $k \in \mathbb{N}$ .

A process  $H$  with values in the set of unbounded functionals on  $\mathbb{R}^{\mathbb{N}}$  is *predictable* if there is a sequence of simple integrands  $(H^n)_{n \in \mathbb{N}}$ , such that  $H = \lim_{n \rightarrow \infty} H^n$ , pointwise, in the sense that for every  $\tilde{x} \in \text{Dom}(H)$ , the sequence  $(H^n(\tilde{x}))_{n \in \mathbb{N}}$  converges to  $H(\tilde{x})$ .

A predictable process  $H$  with values in the set of unbounded functionals on  $\mathbb{R}^{\mathbb{N}}$  is *integrable* with respect to  $S$  if there exists a sequence  $(H^n)_{n \in \mathbb{N}}$  of simple integrands, such that  $(H^n)_{n \in \mathbb{N}}$  converges to  $H$  pointwise and the sequence of semimartingales  $(H^n \cdot S)_{n \in \mathbb{N}}$ , which converges to a semimartingale  $X$  in the Emery topology. In this case, we set  $H \cdot S := X$ .

To specify problems (2.3) below, we also need the context of admissibility. Thus, for  $x \geq 0$ , we say that a predictable process with values in the set of unbounded functionals in an  *$x$ -admissible generalized strategy* if  $H$  is integrable with respect to  $S$  and there is a sequence of  $x$ -admissible elementary strategies,  $(H^n \cdot S)_{n \in \mathbb{N}}$ , that converges to  $H \cdot S$  in the Emery topology.

We suppose that an agent can trade in such a market. A *portfolio* is defined as a pair  $(x, H)$ , where a constant  $x \geq 0$  is an initial wealth and  $H$  is an  $x$ -admissible generalized strategy. The wealth process  $X = (X_t)_{t \in [0, T]}$  generated by the portfolio  $(x, H)$  is given by

$$X_t = x + H \cdot S_t, \quad t \in [0, T].$$

A collection of nonnegative wealth processes generated by  $x$ -admissible generalized strategies is denoted by  $\mathcal{X}(x)$ , that is

$$(2.1) \quad \mathcal{X}(x) := \{X \geq 0 : X_t = x + H \cdot S_t, \quad t \in [0, T]\}, \quad x \geq 0.$$

Next, we consider a family of utility functions satisfying the following assumption. Here and below  $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ .

**Definition 2.1.** A *stochastic utility field* is a mapping  $U = U(\omega, x) : \Omega \times [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$  such that for every  $\omega \in \Omega$ ,  $U(\omega, \cdot)$  is an Inada utility function that is strictly increasing, strictly concave, continuously differentiable on  $(0, \infty)$  function that satisfies the Inada conditions at 0 and  $\infty$ ; at  $x = 0$ , we suppose by continuity that  $U(\omega, 0) = \lim_{x \downarrow 0} U(\omega, x)$ , this value may be  $-\infty$ ; for every  $x \geq 0$ ,  $U(\cdot, x)$  is measurable.

We denote the convex conjugate of  $U$  by

$$(2.2) \quad V(\omega, y) := \sup_{x>0} (U(\omega, x) - xy), \quad (\omega, x) \in \Omega \times [0, \infty).$$

**Assumption 2.2.**  $U_n, n \in \mathbb{N}^*$ , is a sequence of stochastic utility fields such that  $U_n(\omega, \cdot) \rightarrow U_\infty(\omega, \cdot)$  pointwise on  $[0, \infty)$ , for any  $\omega \in \Omega$ .

We consider the following problems

$$(2.3) \quad u_n(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U_n(X_T)], \quad (x, n) \in (0, \infty) \times \mathbb{N}^*,$$

where we use the convention

$$(2.4) \quad \mathbb{E}[U_n(X_T)] := -\infty \quad \text{if} \quad \mathbb{E}[U_n^-(X_T)] = \infty,$$

and where  $U_n^-$  is the negative part of  $U_n$ .

For (2.3) to be non-degenerate, we need to impose a no-arbitrage type condition. With

$$\mathcal{M}^n := \{\mathbb{Q} \sim \mathbb{P} : H \cdot S \text{ is a } \mathbb{Q}\text{-local martingale for every } H \in \mathcal{H}^n\}, \quad n \in \mathbb{N},$$

we suppose that

$$(2.5) \quad \mathcal{M} := \bigcap_{n \in \mathbb{N}} \mathcal{M}^n \neq \emptyset.$$

This condition is closely related, yet (a little) stronger than the existence of an equivalent separating measure in the large market, see [CKT16] for more details and equivalent characterizations. We set the dual domain as follows

$$(2.6) \quad \mathcal{Y}(y) := \{Y \geq 0 : Y_0 = y \quad \text{and} \quad XY = (X_t Y_t)_{t \in [0, T]}\} \\ \text{is a supermartingale for every } X \in \mathcal{X}(1)\}, \quad y \geq 0.$$

We will denote  $\mathcal{Y}(1)$  by  $\mathcal{Y}$  for brevity. This construction of the dual domain is well-known for small markets and goes back to [KS99] and even to [KLSX91]. In large markets, and somewhat surprisingly, at least compared to [DDGP05], the same set  $\mathcal{Y}$ , whose elements are known as supermartingale deflators, works. We specify the dual problem as

$$(2.7) \quad v_n(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V_n(Y_T)], \quad (y, n) \in (0, \infty) \times \mathbb{N}^*,$$

where we use the convention

$$(2.8) \quad \mathbb{E}[V_n(Y_T)] := \infty \quad \text{if} \quad \mathbb{E}[V_n^+(Y_T)] = \infty,$$

and  $V_n^+$  denotes the positive part of  $V_n$ .

We also set the lower concave envelopes of  $(U_k)_{k \geq n}$ ,  $n \in \mathbb{N}$ , and the upper convex envelopes of  $(V_k)_{k \geq n}$ ,  $n \in \mathbb{N}$ , as

$$(2.9) \quad \bar{U}_n(z) := \inf_{k \geq n} U_n(z) \quad \text{and} \quad \bar{V}_n(z) := \sup_{k \geq n} V_n(z), \quad z > 0, \quad n \in \mathbb{N},$$

and impose the following assumption.

**Assumption 2.3.** Assume that there exists  $n_0 \in \mathbb{N}$ , such that for every  $n \geq n_0$ , we have

$$(2.10) \quad \bar{u}_n(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E} [\bar{U}_n(X_T)] > -\infty, \quad x > 0 \quad \text{and}$$

$$(2.11) \quad \bar{v}_n(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E} [\bar{V}_n(Y_T)] < \infty, \quad y > 0.$$

**2.1. Convergence of the value functions and optimizers.** We note that, under the conditions of Theorem 2.4, the existence and uniqueness of optimizers to (2.3) and (2.7), for every  $z > 0$  and  $n \geq n_0$ , follow from (abstract) [Mos15, Theorem 3.2]. We denote the optimizers by  $\hat{X}^n(z)$  and  $\hat{Y}^n(z)$ , respectively. [Mos15, Theorem 3.2] also provides the differentiability of the value functions in (2.3) and (2.7), for every  $z > 0$  and  $n \geq n_0$ . We use this in Proposition 2.5 below.

**Theorem 2.4.** *We suppose that  $S$  is an  $\mathbb{R}^N$ -valued semimartingale that satisfies (2.5). Then, under Assumptions 2.2 and 2.3, we have:*

- (i) *for every  $n \geq n_0$ , the value functions defined in (2.3) and (2.7) are finite-valued, and*

$$(2.12) \quad \lim_{n \rightarrow \infty} u_n = u_\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} v_n = v_\infty,$$

*pointwise and uniformly on compact subsets of  $(0, \infty)$ ;*

- (ii) *for every sequence  $(x_n)_{n \in \mathbb{N}}$  of strictly positive numbers converging to some  $x > 0$ , the optimizers to (2.3) and (2.7) converge in  $\mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ :*

$$(2.13) \quad \hat{X}_T^\infty(x) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} \hat{X}_T^n(x_n) \quad \text{and} \quad \hat{Y}_T^\infty(x) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} \hat{Y}_T^n(x_n).$$

**2.2. Convergence of the indifference prices.** For a bounded contingent claim  $f \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ , we set

$$(2.14) \quad \mathcal{X}(x, q) := \{X = x + H \cdot S : X_T + qf \geq 0, \mathbb{P}\text{-a.s.}\}, \quad (x, q) \in \mathbb{R}^2,$$



where  $H$  can be an  $x'$ -admissible generalized strategy for some  $x' > 0$ . Here  $x'$  can be different from  $x$ , and  $x$  is allowed to take negative values. We note that under (2.5), and in view of the boundedness of  $f$ , the acceptability requirement typically imposed in small markets as in [DS97] on similarly defined elements of  $\mathcal{X}(x, q)$  is satisfied by every  $X \in \mathcal{X}(x, q)$ ,  $(x, q) \in \{(\tilde{x}, \tilde{q}) \in \mathbb{R}^2 : \mathcal{X}(\tilde{x}, \tilde{q}) \neq \emptyset\}$ .

We recall that for  $f \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ , an indifference price of  $f$  corresponding to the initial wealth  $x$  and a utility stochastic field  $U$  is a constant  $\Pi = \Pi(f, x, U)$ , such that

$$(2.15) \quad \mathbb{E}[U(X_T + qf)] \leq u(x), \quad q \in \mathbb{R}, \quad X \in \mathcal{X}(x - q\Pi, q),$$

where  $u(x)$  is given by (2.3) (corresponding to the stochastic utility field  $U$ ). For characterizations of indifference pricing, we refer to [MZ04], [MS05], [HKS05], [KS06], [Mon08], [BEK09], and [HH09], where this topic goes back to [Dav97].

**Proposition 2.5.** *Let the assumptions of Theorem 2.4 hold and  $\lim_{n \rightarrow \infty} x_n = x_\infty > 0$ ,  $x_n > 0$ . Let  $y_n := u'_n(x_n)$ ,  $n \geq n_0$ , where  $n_0$  is given by Assumption 2.3, and suppose that  $\hat{Y}^n(y_n)$ ,  $n \geq n_0$ , are  $\mathbb{P}$ -martingales. Then, for every bounded contingent claim  $f$ , the indifference prices  $\{\Pi(f, x_n, U_n)\}$ , are singletons,  $n \geq n_0$ , and we have*

$$(2.16) \quad \lim_{n \rightarrow \infty} \Pi(f, x_n, U_n) = \Pi(f, x_\infty, U_\infty),$$

and we have the representation

$$(2.17) \quad \Pi(f, x_n, U_n) = \mathbb{E} \left[ \frac{\hat{Y}_T^n(y_n)}{y_n} f \right], \quad n \geq n_0,$$

*Remark 2.6* (On relaxation on the boundedness condition on  $f$  in Proposition 2.5). The stochastic utility settings considered in the paper, in particular, might allow relaxation of the boundedness condition on the contingent claim  $f$  in Proposition 2.5. While, in our view, the complete analysis of this question is a topic of a separate investigation, we consider the following example.

We suppose that  $f = \max(S_T - K, 0)$ , which is a call option on a stock price and is generally *unbounded*. In this case, in a wide class of models, for example, in the Black-Scholes-Merton stock price model, if the

strike  $K$  is deterministic, option  $f$  is replicable. This implies that it admits a unique arbitrage-free price, and its indifference price equals the arbitrage-free price (see, e.g., the argument in [MS22, Lemma 6.2]).

However, if  $K \geq 0$  is random, then, in general, the contingent claim  $f$  is not replicable. If  $S$  is strictly positive, as in the Black-Scholes-Merton model, one can introduce the following auxiliary stochastic utility. Let

$$(2.18) \quad \tilde{U}(\omega, x) := U(\omega, S_T(\omega)x), \quad x \geq 0,$$

then we have

$$\tilde{V}(\omega, y) = V\left(\omega, \frac{y}{S_T(\omega)}\right), \quad y \geq 0,$$

is the (pointwise in  $\omega$ ) convex conjugate of  $\tilde{U}(\omega, x)$ . Furthermore, we have

$$U(X_T + qf) = \tilde{U}\left(\frac{X_T}{S_T} + q\frac{f}{S_T}\right),$$

and by change of numéraire, formulas, we can restate the primal and dual optimization problems as

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)] = \sup_{X \in \mathcal{X}(x)} \mathbb{E}\left[\tilde{U}\left(\frac{X_T}{S_T}\right)\right] = \sup_{X \in \tilde{\mathcal{X}}(x)} \mathbb{E}\left[\tilde{U}(X_T)\right], \quad x > 0,$$

where  $\tilde{\mathcal{X}}(x) = \{\frac{X}{S} : X \in \mathcal{X}(x)\}$ ,  $x > 0$ . For the dual optimization problem without random endowment, we get

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)] = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[\tilde{V}(Y_T S_T)\right] = \inf_{Y \in \tilde{\mathcal{Y}}(y)} \mathbb{E}\left[\tilde{V}(Y_T)\right], \quad y > 0,$$

where  $\tilde{\mathcal{Y}}(y) = \{YS : Y \in \mathcal{Y}(y)\}$ ,  $y > 0$ . The change of numéraire allows now to specify the utility maximization problem (via (2.18)) with a *bounded* contingent claim  $\tilde{f} := \frac{f}{S_T}$ , as follows: for every  $X \in \mathcal{X}(x, q)$ , the left-hand side in (2.15) can be rewritten as

$$(2.19) \quad \mathbb{E}[U(X_T + qf)] = \mathbb{E}\left[\tilde{U}\left(\frac{X_T + qf}{S_T}\right)\right] = \mathbb{E}\left[\tilde{U}\left(X_T^S + q\tilde{f}\right)\right],$$

where  $X^S = \frac{X}{S}$ , and in a small market and strictly positive  $S$ ,  $X^S$  is a wealth process under the numéraire  $S$ .

The heuristic argument in this remark needs more precise conditions and technical estimates to be made rigorous in general settings. In particular, the acceptability (in the sense of [DS97]) of  $X$  above for an unbounded  $f$  under a change of numéraire (especially in large markets) has to be handled adequately. Nevertheless, this argument shows that *starting from a*

deterministic utility and an unbounded contingent claim  $f$ , by a change of numéraire and passing to a stochastic utility, one can obtain a problem with a bounded contingent claim  $\tilde{f}$  as in (2.19). Here, superreplicability of  $|f|$  by a wealth process is a common condition in the literature that, by a change of numéraire, can lead to the settings of this paper considered in Theorem 2.4 and Proposition 2.5 with a bounded contingent claim and stochastic utility, as for  $\tilde{f} = \frac{f}{S_T}$  in this remark above. This, in particular, allows us to potentially incorporate call options into the modeling framework and is the reason for stating Theorem 2.4 and Proposition 2.5 in stochastic utility settings.

**2.3. Indifference price invariant models over deterministic utilities.**

In this section, working with deterministic utilities, we investigate the indifference price invariant models, that is, the models in which the indifference price does not depend on the choice of a utility function. The main result of this section is that the class of indifference price invariant models is exactly the class of stochastically dominant models, that is, the models, where the dual domain admits the maximal element in the sense of the second-order stochastic dominance.

We recall that for two random variables  $\zeta_1 \geq 0$  and  $\zeta_2 \geq 0$ , we say that  $\zeta_1$  dominates  $\zeta_2$  in the sense of second-order stochastic dominance if

$$(2.20) \quad \int_0^z \mathbb{P}[\zeta_1 \geq t] dt \geq \int_0^z \mathbb{P}[\zeta_2 \geq t] dt, \quad z \geq 0.$$

In this case, we write  $\zeta_1 \succeq_2 \zeta_2$ .

We consider the class of deterministic Inada utility functions  $U$ , whose convex conjugate  $V$  satisfies

$$(2.21) \quad v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)] < \infty, \quad y > 0.$$

We denote the set of such deterministic Inada utility functions by  $\mathcal{FD}$  for finite dual. We note that, under (2.5),  $\mathcal{FD} \neq \emptyset$ , as, for every  $p > 1$ , it includes  $U(x) = \frac{x^{1-p}}{1-p}$ ,  $x > 0$ .

**Proposition 2.7.** *We suppose that  $S$  is an  $\mathbb{R}^N$ -valued semimartingale that satisfies (2.5). Then the following conditions are equivalent:*

- (i) *there exists  $\hat{Y} \in \mathcal{Y}$  such that  $\hat{Y}_T \succeq_2 Y_T$  for every  $Y \in \mathcal{Y}$ ,*

(ii) the model is indifference price invariant, that is, for every triple of  $U \in \mathcal{FD}$ ,  $x > 0$ , and a bounded contingent claim  $f$ , the set of indifference prices  $\{\Pi(f, x, U)\}$  is a singleton and depends neither on  $x > 0$  nor on  $U \in \mathcal{FD}$ .

Furthermore, in the case when these equivalent assertions hold, we have the representation

$$\Pi(f, x, U) = \mathbb{E}_{\hat{\mathbb{Q}}}[f], \quad f \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}), \quad x > 0, \quad U \in \mathcal{FD},$$

where  $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \hat{Y}_T$ , for  $\hat{Y}$  from the item (i), and, in particular,  $\mathbb{E}[\hat{Y}_T] = 1$ .

*Remark 2.8.* In small markets, the conditions above could be proven to be equivalent to stochastic dominance of the dual domain of the *infinite order*. We refer to [MSZ21] for the definition of stochastic dominance of infinite order and further details.

Furthermore, relying on the results in [SST09, Proposition 3.9, p. 60], one can show that, under the assumptions of Proposition 2.7, each of the assertions is equivalent to the *weak completeness* of the model (again, see [SST09] for the precise definition) and to

$$\mathbb{E}[Y_T | \sigma(\hat{Y}_T)] \leq \hat{Y}_T, \quad \mathbb{P}\text{-a.s.}, \quad \text{for every } Y \in \mathcal{Y},$$

where  $\hat{Y}$  is the dominating element in  $\mathcal{Y}$  as in item (i) of Proposition 2.7.

Also, in small markets, stochastic dominance of the second order (under appropriate conditions) is proven in [KS06] to be equivalent to the existence of the *risk-tolerance wealth process*, that is, a maximal wealth process  $R$ , such that

$$R_T = -\frac{U'(\hat{X}_T(x))}{U''(\hat{X}_T(x))},$$

for an initial wealth  $x > 0$  and a given utility function  $U$  satisfying some technical conditions from [KS06], where  $\hat{X}(x)$  is the associated optimizer to (2.3). For the set of equivalent characterizations of the assertions of Proposition 2.7 in small markets, we refer to [KS06], where we add that the existence of the risk-tolerance wealth process can be characterized as the existence of a solution to a backward stochastic differential equation with jumps, and we refer to [CE12] for this subject.

**2.4. Indifference price invariant models over stochastic utilities.**

In this section working with stochastic utilities, we show that indifference price invariance, under the existence of an equivalent separating measure, is equivalent to the *completeness* of the model. The natural extension of  $\mathcal{FD}$  class is one given by the class of Inada stochastic utility fields  $U$  (in the sense of Definition 2.1), such that the associated primal and dual value functions are finite-valued, that is

$$(2.22) \quad \begin{aligned} u(x) &= \inf_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)] > -\infty, \quad x > 0, \quad \text{and} \\ v(y) &= \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)] < \infty, \quad y > 0, \end{aligned}$$

where  $V$  is given by (2.2). We denote the set of such stochastic utility fields by  $\mathcal{FV}$  for finite value functions. We note that, under (2.5),  $\mathcal{FV} \neq \emptyset$ , as, for every  $p > 1$ , it includes (deterministic)  $U(x) = \frac{x^{1-p}}{1-p}$ ,  $x > 0$ . Conditions (2.5) and (2.22) are known to guarantee the standard assertions of the utility maximization theory in stochastic Inada utility settings.

**Proposition 2.9.** *Let the conditions of Proposition 2.7 hold. Then, the following are equivalent:*

- (i) *the set  $\mathcal{M}$  is a singleton,*
- (ii) *the model is indifference price invariant over stochastic utility fields, that is, for every triple of  $U \in \mathcal{FV}$ ,  $x > 0$ , and a bounded contingent claim  $f$ , the set of indifference prices  $\{\Pi(f, x, U)\}$  is a singleton and depends neither on  $x > 0$  nor on  $U \in \mathcal{FV}$ ,*
- (iii) *every bounded contingent claim  $f$  is replicable, that is there exists  $x' \geq 0$  and  $X \in \bigcup_{x>0} \mathcal{X}(x)$ , such that*

$$X_T - x' = f, \quad \mathbb{P}\text{-a.s.},$$

*that is, the model is complete.*

*Furthermore, in the case when these equivalent assertions hold, we have the representation*

$$(2.23) \quad \Pi(f, x, U) = \mathbb{E}_{\mathbb{Q}}[f], \quad f \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}), \quad x > 0, \quad U \in \mathcal{FV},$$

*where  $\mathbb{Q}$  is the unique element of  $\mathcal{M}$ .*

## 3. ABSTRACT VERSION OF THE MAIN RESULTS AND PROOFS

3.1. **Preliminaries.** We set

(3.1)

$$\mathcal{C}(x) := \{\xi \in \mathbb{L}_+^0(\Omega, \mathcal{F}, \mathbb{P}) : 0 \leq \xi \leq X_T \text{ for some } X \in \mathcal{X}(x)\}, \quad x \geq 0,$$

(3.2)

$$\mathcal{D}(y) := \{\eta \in \mathbb{L}_+^0(\Omega, \mathcal{F}, \mathbb{P}) : 0 \leq \eta \leq Y_T \text{ for some } Y \in \mathcal{Y}(y)\}, \quad y \geq 0.$$

$\mathcal{C}(1)$  and  $\mathcal{D}(1)$  will be denoted by  $\mathcal{C}$  and  $\mathcal{D}$ , respectively, and we have  $\mathcal{C}(z) = z\mathcal{C}$ ,  $\mathcal{D}(z) = z\mathcal{D}$ ,  $z \geq 0$ . We show below that sets  $\mathcal{C}$  and  $\mathcal{D}$  satisfy the following proposition, which is very similar to [KS99, Proposition 3.1], and this also holds for the large market.

**Proposition 3.1.** *We suppose that  $S$  is an  $\mathbb{R}^N$ -valued semimartingale that satisfies (2.5). Then, the sets  $\mathcal{C}$  and  $\mathcal{D}$  defined in (3.1) and (3.2) satisfy the following conditions:*

- (1)  $\mathcal{C}$  and  $\mathcal{D}$  are subsets of  $\mathbb{L}_+^0(\Omega, \mathcal{F}, \mathbb{P})$  that are convex, solid, and closed in  $\mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ ,
- (2)  $\mathcal{C}$  and  $\mathcal{D}$  satisfy the bipolar relations, that is

$$\begin{aligned} \xi \in \mathcal{C} & \text{ if and only if } \mathbb{E}[\xi\eta] \leq 1 \text{ for every } \eta \in \mathcal{D}, \\ \eta \in \mathcal{D} & \text{ if and only if } \mathbb{E}[\xi\eta] \leq 1 \text{ for every } \xi \in \mathcal{C}, \end{aligned}$$

- (3)  $\mathcal{C}$  is bounded in  $\mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$  and contains the constant function 1.

With  $U_n$ ,  $n \in \mathbb{N}^*$ , as above, we can set the abstract versions of primal and dual optimization problems as

$$(3.3) \quad u_n(x) := \sup_{\xi \in \mathcal{C}(x)} \mathbb{E}[U(\xi)] \quad (x, n) \in (0, \infty) \times \mathbb{N}^*,$$

$$(3.4) \quad v_n(y) := \inf_{\eta \in \mathcal{D}(y)} \mathbb{E}[V_n(\eta)], \quad (y, n) \in (0, \infty) \times \mathbb{N}^*,$$

and we have conventions similar to (2.4) and (2.8). Next, we can state the abstract version of Assumption 2.3 as follows.

**Assumption 3.2.** We suppose that there exists  $n_0 \in \mathbb{N}$ , such that

$$\begin{aligned}\bar{u}_n(x) &:= \sup_{\xi \in \mathcal{C}(x)} \mathbb{E} [\bar{U}_n(\xi)] > -\infty, \quad x > 0 \quad \text{and} \\ \bar{v}_n(y) &:= \inf_{\eta \in \mathcal{D}(y)} \mathbb{E} [\bar{V}_n(\eta)] < \infty, \quad y > 0.\end{aligned}$$

**3.2. Abstract version of Theorem 2.4.** We denote the optimizers to (3.3) and (3.4), provided that they exist, by  $\hat{\xi}^n(z)$ , and  $\hat{\eta}^n(z)$ ,  $(z, n) \in (0, \infty) \times \mathbb{N}^*$ . Under the conditions of Theorem 3.3 below, the existence and uniqueness of optimizers to (3.3) and (3.4) for every  $z > 0$  and  $n \geq n_0$  follows from [Mos15, Theorem 3.2].

**Theorem 3.3.** *We consider sets  $\mathcal{C}$  and  $\mathcal{D}$  satisfying the assertions of Proposition 3.1<sup>2</sup>. Then, under Assumptions 2.2 and 3.2, we have:*

(i) *for every  $n \geq n_0$ , the value functions defined in (3.3) and (3.4) are finite-valued and*

$$(3.5) \quad \lim_{n \rightarrow \infty} u_n = u_\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} v_n = v_\infty$$

*pointwise and uniformly on compact subsets of  $(0, \infty)$ ;*

(ii) *for every sequence  $(x_n)_{n \in \mathbb{N}}$  of strictly positive numbers converging to some  $x > 0$ , the optimizers to (3.3) and (3.4) converge in  $\mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ :*

$$(3.6) \quad \hat{\xi}^\infty(x) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} \hat{\xi}^n(x_n) \quad \text{and} \quad \hat{\eta}^\infty(x) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} \hat{\eta}^n(x_n).$$

### 3.3. Convergence of the value functions.

**Lemma 3.4.** *Under the conditions of Theorem 3.3, we have*

$$(3.7) \quad \lim_{n \rightarrow \infty} u_n(z) = u_\infty(z) \quad \text{and} \quad \lim_{n \rightarrow \infty} v_n(z) = v_\infty(z), \quad z > 0.$$

The proof of Lemma 3.4 is given via the following lemmas.

**Lemma 3.5.** *Under the conditions of Theorem 3.3, we have*

$$(3.8) \quad \limsup_{n \rightarrow \infty} v_n(y) \leq v_\infty(y), \quad y > 0.$$

*Proof.* We fix  $z > 0$  and  $\delta \in (0, z)$ , and consider  $\eta := \hat{\eta}^\infty(z - \delta) \in \mathcal{D}(z - \delta)$ , the minimizer to (3.4) at  $(y, n) = (z - \delta, \infty)$ . We consider  $\eta^\delta \in \mathcal{D}(\delta)$ , such that

$$\mathbb{E}[\bar{V}_n(\eta^\delta)] < \infty, \quad n \geq n_0,$$

---

<sup>2</sup>And not necessarily defined by (3.1) and (3.2) above.

where  $n_0$  is given by Assumption 3.2. We note that the existence of such  $\eta^\delta$  follows from Assumption 3.2.

Next, we observe that from the monotonicity of every  $V_n^+$ ,  $n \in \mathbb{N}$ , and  $\bar{V}_n^+$ 's in  $n$ , we get

$$V_n^+(\eta + \eta^\delta) \leq V_n^+(\eta^\delta) \leq \bar{V}_n^+(\eta^\delta) \leq \bar{V}_{n_0}^+(\eta^\delta) \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \quad n \geq n_0.$$

Therefore, by the Dominated Convergence Theorem, we have

$$(3.9) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[ V_n^+ \left( \eta + \eta^\delta \right) \right] = \mathbb{E} \left[ V_\infty^+ \left( \eta + \eta^\delta \right) \right].$$

Now, by the monotonicity of  $V_\infty^+$  and the construction of  $\eta$ , we have

$$V_\infty^+ \left( \eta + \eta^\delta \right) \leq V_\infty^+ (\eta) \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Combining this inequality with (3.9), we get

$$(3.10) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[ V_n^+ \left( \eta + \eta^\delta \right) \right] \leq \mathbb{E} \left[ V_\infty^+ (\eta) \right] < \infty.$$

In turn, by Fatou's lemma and the monotonicity of  $V_\infty^-$ , we get

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[ V_n^- \left( \eta + \eta^\delta \right) \right] \geq \mathbb{E} \left[ V_\infty^- \left( \eta + \eta^\delta \right) \right] \geq \mathbb{E} \left[ V_\infty^- (\eta) \right].$$

Combining this with (3.10), we deduce that

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[ V_n \left( \eta + \eta^\delta \right) \right] \leq \mathbb{E} \left[ V_\infty (\eta) \right].$$

Therefore, as  $(\eta + \eta^\delta) \in \mathcal{D}(z)$ , we obtain

$$\limsup_{n \rightarrow \infty} v_n(z) \leq \limsup_{n \rightarrow \infty} \mathbb{E} \left[ V_n \left( \eta + \eta^\delta \right) \right] \leq \mathbb{E} \left[ V_\infty (\eta) \right] = v_\infty(z - \delta).$$

Since  $v_\infty$  is continuous by its convexity and finiteness, and since  $\delta \in (0, z)$  is arbitrary, taking the limit as  $\delta \downarrow 0$ , we deduce that

$$\limsup_{n \rightarrow \infty} v_n(z) \leq v(z).$$

As  $z > 0$  is arbitrary, (3.8) follows.  $\square$

Similarly to Lemma 3.5, we can prove the following lemma. The proof is omitted for brevity.

**Lemma 3.6.** *Under the conditions of Theorem 3.3, we have*

$$\liminf_{n \rightarrow \infty} u_n(x) \geq u_\infty(x), \quad x > 0.$$



*Proof of Lemma 3.4.* For  $n = \infty$ , we fix  $x > 0$  and let  $y = u'_\infty(x)$ . Here, we note that the differentiability of  $u_\infty$  follows from [Mos15, Theorem 3.2]. Then, we have

$$(3.11) \quad u_\infty(x) \leq \liminf_{n \rightarrow \infty} u_n(x) \leq \limsup_{n \rightarrow \infty} v_n(y) + xy \leq v_\infty(y) + xy = u_\infty(x),$$

where, in the first inequality, we used Lemma 3.6; in the second - conjugacy relations; in the third - Lemma 3.5; and in the last equality - [Mos15, Theorem 3.2]. Therefore, all inequalities in (3.11) are, in fact, equalities. Moreover, as  $x > 0$  is arbitrary and, by [Mos15, Theorem 3.2],  $u'_\infty$  satisfies the Inada conditions,  $y$  in (3.11) can take any value in  $(0, \infty)$ , we deduce that (3.7) holds.  $\square$

### 3.4. Convergence of the optimizers.

**Lemma 3.7.** *Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of strictly positive numbers converging to  $x > 0$ . Then, under the conditions of Theorem 3.3, we have*

$$(3.12) \quad \widehat{\xi}^\infty(x) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} \widehat{\xi}^n(x_n) \quad \text{and} \quad \widehat{\eta}^\infty(x) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} \widehat{\eta}^n(x_n).$$

*Proof.* We fix  $n \in \mathbb{N}^*$  and observe that for every  $y > x > 0$ , we have

$$(3.13) \quad \begin{aligned} & U_n\left(\frac{x+y}{2}\right) - \frac{1}{2}(U_n(x) + U_n(y)) \\ &= \frac{1}{2}\left(U_n\left(\frac{x+y}{2}\right) - U_n(x)\right) - \frac{1}{2}\left(U_n(y) - U_n\left(\frac{x+y}{2}\right)\right) \\ &= \frac{1}{2}\left(\int_x^{\frac{x+y}{2}} U'_n(z) dz - \int_{\frac{x+y}{2}}^y U'_n(z) dz\right) \\ &= \frac{1}{2}\int_x^{\frac{x+y}{2}} \left(U'_n(z) - U'_n\left(z + \frac{y-x}{2}\right)\right) dz \\ &= \frac{1}{2}\int_0^\infty \left(U'_n(z) - U'_n\left(z + \frac{y-x}{2}\right)\right) \mathbf{1}_{\{y>x\}}(z) \mathbf{1}_{[x, \frac{x+y}{2}]}(z) dz. \end{aligned}$$

Therefore, for any strictly positive  $x$  and  $y$ , we obtain

$$(3.14) \quad \begin{aligned} \Gamma^n(x, y) &:= U_n\left(\frac{x+y}{2}\right) - \frac{1}{2}(U_n(x) + U(y)) \\ &= \int_0^\infty \frac{1}{2} \left( \left( U'_n(z) - U'_n\left(z + \frac{y-x}{2}\right) \right) \mathbf{1}_{\{y>x\}}(z) \mathbf{1}_{[x, \frac{x+y}{2}]}(z) \right. \\ &\quad \left. + \left( U'_n(z) - U'_n\left(z + \frac{x-y}{2}\right) \right) \mathbf{1}_{\{x>y\}}(z) \mathbf{1}_{[y, \frac{x+y}{2}]}(z) \right) dz, \quad n \in \mathbb{N}^*, \end{aligned}$$

where, the integrand is nonnegative and is strictly positive on  $(x \wedge y, x \vee y]$  for  $x \neq y$ , by the strict monotonicity of  $U'_n(\omega, \cdot)$ ,  $\omega \in \Omega$ .

Next, without loss of generality, we suppose that  $n_0 = 1$  in Assumption 3.2, and denote

$$g^n := \widehat{\xi}^n(x_n), \quad n \geq n_0.$$

Then, for every  $\zeta \in \mathbb{L}_+^0(\Omega, \mathcal{F}, \mathbb{P})$ , such that  $\bar{U}_{n_0}^-(\zeta) \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ , we have  $U_n^-(g^\infty + \zeta) \leq \bar{U}_n^-(g^\infty + \zeta) \leq \bar{U}_{n_0}^-(g^\infty + \zeta) \leq \bar{U}_{n_0}^-(\zeta) \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ ,  $n \geq n_0$ , and thus, by Fatou's lemma, we get

$$(3.15) \quad \liminf_{n \rightarrow \infty} \mathbb{E}[U_n(g^\infty + \zeta)] \geq \mathbb{E}[U_\infty(g^\infty + \zeta)] \geq u_\infty(x).$$

Assume by contradiction that  $g^n$ ,  $n \in \mathbb{N}$ , does not converge in probability to  $g^\infty$ . There, there exists  $\varepsilon > 0$ , such that

$$(3.16) \quad \limsup_{n \rightarrow \infty} \mathbb{P}[|g^n - g^\infty| > \varepsilon] > \varepsilon.$$

We fix a constant  $\delta \in (0, \frac{\varepsilon}{2})$  and let  $\delta_m := \frac{\delta}{2^m}$ ,  $m \in \mathbb{N}$ . We consider  $\zeta^m > 0$ ,  $m \in \mathbb{N}$ , a sequence of elements in  $\mathcal{C}(\delta^m)$ ,  $m \in \mathbb{N}$ , such that

$$\bar{U}_{n_0}^-(\zeta^m) \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \quad m \in \mathbb{N}.$$

The existence of such a sequence follows from Assumption 3.2. We set

$$h^m := g^\infty + \zeta^m \quad \text{and} \quad \tilde{g}^{n,m} := g^n + \zeta^m, \quad m \in \mathbb{N}, n \in \mathbb{N}.$$

As, by the assumption of Theorem 3.3,  $\mathcal{C}$  satisfies item (3) of Proposition 3.1, we have that  $(\tilde{g}^{n,m})_{(n,m) \in \mathbb{N}^2}$  is bounded in  $\mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ , as a subset of  $\mathcal{C}(\bar{x})$ , for an appropriate  $\bar{x} \in (0, \infty)$ . Together with (3.16), this implies that there exist strictly positive constants  $\alpha$ ,  $M$ , and  $\varepsilon'$ , such that

$$(3.17) \quad \limsup_{n \rightarrow \infty} \mathbb{P} \left[ |(g^n + \zeta^m) - (g^\infty + \zeta^m)| > \alpha, \quad g^n + g^\infty + \sum_{k \geq 1} \zeta^k < M \right] > \varepsilon',$$

which does not depend on  $m \in \mathbb{N}$  and implies that

$$(3.18) \quad \limsup_{n \rightarrow \infty} \mathbb{P} [|\tilde{g}^{n,m} - h^m| > \alpha, \quad \tilde{g}^{n,m} + h^m < M] > \varepsilon', \quad m \in \mathbb{N}.$$

From (3.14), we obtain

$$U_n \left( \frac{\tilde{g}^{n,m} + h^m}{2} \right) = \frac{1}{2} (U_n(\tilde{g}^{n,m}) + U_n(h^m)) + \Gamma^n(\tilde{g}^{n,m}, h^m),$$

Therefore, we get

$$\mathbb{E} \left[ U_n \left( \frac{\tilde{g}^{n,m} + h^m}{2} \right) \right] = \frac{1}{2} (u_n(x_n) + \mathbb{E} [U_n(h^m)]) + \mathbb{E} [\Gamma^n(\tilde{g}^{n,m}, h^m)].$$

As a consequence, from (3.15), Lemma 3.4, and [Roc70, Theorem 25.7, p. 248], we obtain

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[ U_n \left( \frac{\tilde{g}^{n,m} + h^m}{2} \right) \right] \geq u_\infty(x) + \limsup_{n \rightarrow \infty} \mathbb{E} [\Gamma^n(\tilde{g}^{n,m}, h^m)].$$

and thus, using Lemma 3.4 again, we get

$$\begin{aligned} u_\infty(x + \delta^m) &= \limsup_{n \rightarrow \infty} u_n \left( \frac{x^n + x}{2} + \delta^m \right) \geq \limsup_{n \rightarrow \infty} \mathbb{E} \left[ U_n \left( \frac{\tilde{g}^{n,m} + h^m}{2} \right) \right] \\ &\geq u_\infty(x) + \limsup_{n \rightarrow \infty} \mathbb{E} [\Gamma^n(\tilde{g}^{n,m}, h^m)]. \end{aligned}$$

We deduce that

$$\limsup_{n \rightarrow \infty} \mathbb{E} [\Gamma^n(\tilde{g}^{n,m}, h^m)] \leq u_\infty(x + \delta^m) - u_\infty(x),$$

and therefore, by taking the limit as  $m \rightarrow \infty$ , we obtain

(3.19)

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} [\Gamma^n(\tilde{g}^{n,m}, h^m)] \leq \limsup_{m \rightarrow \infty} (u_\infty(x + \delta^m) - u_\infty(x)) = 0,$$

where the last equality follows from the convexity and finiteness of  $u_\infty$  on  $(0, \infty)$ , where the latter is ensured by Assumption 3.2.

Next, we consider  $\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} [\Gamma^n(\tilde{g}^{n,m}, h^m)]$ . We denote

$$A_n := \left\{ |(g^n + \zeta^m) - (g^\infty + \zeta^m)| > \alpha, g^n + g^\infty + \sum_{k \geq 1} \zeta^k < M \right\}, \quad n \in \mathbb{N}.$$

This allows rewriting (3.17) as

$$(3.20) \quad \limsup_{n \rightarrow \infty} \mathbb{P} [A_n] > \varepsilon',$$

It follows from Assumption 2.2 via [Roc70, Theorem 25.7, p. 248] that  $U'_n(\omega, \cdot)$  converges to  $U'_\infty(\omega, \cdot)$  uniformly on compact subsets of  $(0, \infty)$  (and this was the idea behind introducing  $\zeta^m$ 's), for every  $\omega \in \Omega$ . This allows to deduce that

(3.21)

$$\limsup_{n \rightarrow \infty} \mathbb{E} [1 \wedge \Gamma^n(\tilde{g}^{n,m}, h^m)] = \limsup_{n \rightarrow \infty} \mathbb{E} [1 \wedge \Gamma^\infty(\tilde{g}^{n,m}, h^m)], \quad m \in \mathbb{N}.$$

Denoting by  $\phi := 1 \wedge \frac{\alpha}{2} \inf_{x \in (0, M]} (U'_\infty(x) - U'_\infty(x + \alpha/2))$ , where  $M$  and  $\alpha$  are given in (3.17), we observe that  $\phi$  is a *strictly* positive random variable. We conclude from (3.14), (3.20), and (3.21) that

$$\limsup_{n \rightarrow \infty} \mathbb{E} [1 \wedge \Gamma^n(\tilde{g}^{n,m}, h^m)] \geq \limsup_{n \rightarrow \infty} \mathbb{E} [\phi 1_{A_n}] > 0,$$

where the middle term,  $\limsup_{n \rightarrow \infty} \mathbb{E} [\phi 1_{A_n}]$ , does not depend on  $m \in \mathbb{N}$ , and thus

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} [\Gamma^n(\tilde{g}^{n,m}, h^m)] \geq \limsup_{n \rightarrow \infty} \mathbb{E} [\phi 1_{A_n}] > 0,$$

which contradicts to (3.19). We conclude that  $\mathbb{P}$ - $\lim_{n \rightarrow \infty} \hat{\xi}^n(x_n) = \hat{\xi}^\infty(x)$ . Similarly, we can show that  $\mathbb{P}$ - $\lim_{n \rightarrow \infty} \hat{\eta}^n(x_n) = \hat{\eta}^\infty(x)$ .  $\square$

*Proof of Theorem 3.3.* The assertions of the item (i), (3.5), follow from Lemma 3.4 and [Roc70, Theorem 10.8, p. 90]. The conclusions of the item (ii), (3.6), follow from Lemma 3.7.  $\square$

**3.5. Abstract version of Proposition 2.5.** To give the abstract version of the indifference pricing characterizations, for a fixed  $f \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ , we set

$$(3.22) \quad \mathcal{C}(x, q) := \{ \xi \in \mathbb{L}_+^0(\Omega, \mathcal{F}, \mathbb{P}) : \xi \leq X_T + qf \text{ for some } X \in \mathcal{X}(x, q) \}, \quad (x, q) \in \mathbb{R}^2.$$

We note that the set  $\mathcal{C}(x, q)$  might be empty for certain  $(x, q) \in \mathbb{R}^2$ .

An abstract indifference price of  $f$  corresponding to the initial wealth  $x$  and a stochastic utility function  $U$  is a constant  $\Pi = \Pi(f, x, U)$ , such that

$$(3.23) \quad \mathbb{E} [U(\xi)] \leq u(x), \quad q \in \mathbb{R}, \quad \xi \in \mathcal{C}(x - q\Pi, q), \quad (x, q) \in \mathbb{R}^2,$$

where  $u(x)$  is given by (3.3).

**Proposition 3.8.** *Let the assumptions of Theorem 3.3 hold, and consider a sequence of strictly positive numbers  $x_n$  converging to  $x_\infty > 0$ . Let  $y_n := u'_n(x_n)$ ,  $n \geq n_0$ <sup>3</sup>, and suppose that*

$$(3.24) \quad \mathbb{E} [\hat{\eta}^n(y_n)] = y_n, \quad n \geq n_0.$$

<sup>3</sup>Here, by [Mos15, Theorem 3.2],  $y_n$ 's are well-defined and strictly positive for every  $n \geq n_0$  and  $n_0$  is given by Assumption 3.2.

Then, for every bounded contingent claim  $f$ , the indifference prices  $\{\Pi(f, x_n, U_n)\}$ , are singletons,  $n \geq n_0$ , and we have

$$(3.25) \quad \lim_{n \rightarrow \infty} \Pi(f, x_n, U_n) = \Pi(f, x_\infty, U_\infty),$$

and the following representation holds

$$(3.26) \quad \Pi(f, x_n, U_n) = \mathbb{E} \left[ \frac{\widehat{\eta}^n(y_n)}{y_n} f \right], \quad n \geq n_0.$$

*Proof.* We observe that by Theorem 3.3, item (i), and [Roc70, Theorem 25.7, p. 248], the sequence  $y_n$ ,  $n \geq n_0$ , converges to  $y_\infty$ , whereas by Theorem 3.3, item (ii), the terminal values of the dual minimizers converge in probability. That is

$$\mathbb{P}\text{-} \lim_{n \rightarrow \infty} \widehat{\eta}^n(y_n) = \widehat{\eta}^\infty(y_\infty).$$

Therefore, we obtain

$$(3.27) \quad \mathbb{P}\text{-} \lim_{n \rightarrow \infty} \frac{\widehat{\eta}^n(y_n)}{y_n} = \frac{\widehat{\eta}^\infty(y_\infty)}{y_\infty}.$$

Next, from (3.24), along the lines of [MS22, Theorem 4.2] (see also [HKS05, Theorem 3.1] for the case of the deterministic utilities) that the representation (2.17) holds.

As (3.24) implies that

$$1 = \lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{\widehat{\eta}^n(y_n)}{y_n} \right] = \mathbb{E} \left[ \frac{\widehat{\eta}^\infty(y_\infty)}{y_\infty} \right],$$

and therefore, in view of (3.27) and Scheffe's lemma, we deduce that the sequence  $\frac{\widehat{\eta}^n(y_n)}{y_n}$ ,  $n \geq n_0$ , is uniformly integrable and the convergence in (3.27) also holds in  $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ . Therefore, for every bounded contingent claim  $f$ , the sequence  $\frac{\widehat{\eta}^n(y_n)}{y_n} f$ ,  $n \geq n_0$ , is uniformly integrable, and converges to  $\frac{\widehat{\eta}^\infty(y_\infty)}{y_\infty} f$  in probability  $\mathbb{P}$ , and consequently also in  $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ . Together with earlier established (3.26), this implies (3.25).  $\square$

### 3.6. Abstract version of the stochastic dominance characterization.

We denote by  $\mathcal{FD}$  the class of (deterministic) Inada utility functions  $U$ , whose convex conjugate  $V$  satisfies

$$(3.28) \quad v(y) = \inf_{\eta \in \mathcal{D}(y)} \mathbb{E}[V(\eta)] < \infty, \quad y > 0.$$

**Proposition 3.9.** *We consider the sets  $\mathcal{C}$  and  $\mathcal{D}$  satisfying the assertions of Proposition 3.1 and assume that there exists  $\tilde{\eta} \in \mathcal{D}$  with  $\tilde{\eta} > 0$ ,  $\mathbb{P}$ -a.s., and  $\mathbb{E}[\tilde{\eta}] = 1$ . Then the following conditions are equivalent:*

- (i)  $\mathcal{D}$  admits a maximal element  $\hat{\eta}$  in the sense of the second-order stochastic dominance, (2.20),
- (ii) the model is indifference price invariant over deterministic utilities, that is, for every  $U \in \mathcal{FD}$ ,  $x > 0$ , and every bounded contingent claim  $f$ , the set of indifference prices  $\{\Pi(f, x, U)\}$  is a singleton and depends neither on  $x > 0$  nor on  $U \in \mathcal{FD}$ .

Furthermore, in the case when these equivalent assertions hold, we have the representation

$$\Pi(f, x, U) = \mathbb{E}_{\hat{\mathbb{Q}}} [f], \quad f \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}), \quad x > 0, \quad U \in \mathcal{FD},$$

where  $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \hat{\eta}$  from the item (i), and, in particular,  $\mathbb{E}[\hat{\eta}] = 1$ .

*Proof of Proposition 3.9.* (i)  $\Rightarrow$  (ii). Let  $\hat{\eta}$  denote the maximal element in  $\mathcal{D}$  in the sense of the second-order stochastic dominance. Next, one can show that (i) is equivalent to

$$(3.29) \quad \mathbb{E}[\eta | \sigma(\hat{\eta})] \leq \hat{\eta}, \quad \mathbb{P}\text{-a.s.}, \quad \text{for every } \eta \in \mathcal{D},$$

which can be proven similarly to [SST09, Proposition 3.9, p. 60]. As there exists  $\tilde{\eta} \in \mathcal{D}$ , such that  $\tilde{\eta} > 0$ ,  $\mathbb{P}$ -a.s. and  $\mathbb{E}[\tilde{\eta}] = 1$ , by taking an expectation in (3.29) for  $\eta = \tilde{\eta}$ , we deduce that

$$(3.30) \quad \mathbb{E}[\tilde{\eta}] = 1.$$

In particular, the probability measure  $\hat{\mathbb{Q}}$ , whose density with respect to  $\mathbb{P}$  is  $\hat{\eta}$ , is well-defined. Further, the strict positivity of  $\tilde{\eta}$  and (3.29) imply that  $\hat{\eta} > 0$ ,  $\mathbb{P}$ -a.s.. Next, from the characterization of the second-order stochastic dominance via convex functions (as in (3.31) below), one can see that  $\hat{\eta}$  is the minimizer to (3.28) for every  $U \in \mathcal{FD}$  and every  $y > 0$ . This and (3.30) (along the lines of the proof of [HKS05, Theorem 3.1]) imply that, for every bounded contingent claim  $f$ , the indifference price has the representation

$$\Pi(f, x, U) = \mathbb{E}[\hat{\eta}f] = \mathbb{E}_{\hat{\mathbb{Q}}}[f],$$

which depends neither on  $x > 0$  nor on  $U \in \mathcal{FD}$ . That is (2) holds.

(ii)  $\Rightarrow$  (i). First, it is well known, see, e.g., [KS06], that for random variables  $\zeta^i$ ,  $i = 1, 2$ ,  $\zeta_1 \succeq_2 \zeta_2$  in the sense of (2.20) if and only if

$$(3.31) \quad \mathbb{E} \left[ \tilde{V}(\zeta_1) \right] \leq \mathbb{E} \left[ \tilde{V}(\zeta_2) \right],$$

for every convex decreasing function  $\tilde{V}$  on  $[0, \infty)$ , such that the expectations in (3.31) are well-defined. Using an approximation argument, one can show that (2.20) is equivalent to (3.31) for every convex conjugate of an element of  $\mathcal{FD}$ , that is, for every

$$(3.32) \quad \tilde{V} \in \left\{ V : V(y) = \sup_{x>0} (U(x) - xy), y > 0, U \in \mathcal{FD} \right\}.$$

Next, we assume by contradiction that (ii) holds, but the model is not stochastically dominant. Then, there exist two utility functions  $U_1$  and  $U_2$  in  $\mathcal{FD}$ , and  $x_1, x_2$ , such that the dual minimizers for  $y_i := u'_i(x_i)$ ,  $i = 1, 2$ , are not equal to each other. Here we note that  $u'_i$ ,  $i = 1, 2$ , are well-defined and strictly positive, e.g., by (the abstract) [Mos15, Theorem 3.2]. We denote these minimizers by  $\hat{\eta}^i(y_i) \in \mathcal{D}(y_i)$ ,  $i = 1, 2$ , and set  $\eta^i := \frac{\hat{\eta}^i(y_i)}{y_i}$ ,  $i = 1, 2$ . That is

$$(3.33) \quad \mathbb{P} [\eta^1 \neq \eta^2] > 0.$$

We consider the primal optimizers  $\xi^i$ , such that  $\xi^i = -V'_i(y_i \eta^i)$ ,  $i = 1, 2$ . By the [Mos15, Theorem 3.2], we have that  $\mathbb{E} [\xi^i \eta^i] = x_i$ , and that  $\xi^i$  are maximal in  $\mathcal{C}$ ,  $i = 1, 2$ . We consider a contingent claim

$$f := \min(1, \xi^1, \xi^2) 1_{\{\eta^1 > \eta^2\}}.$$

Then,  $f$  is nonnegative and bounded, and, on  $\{\eta^1 > \eta^2\}$ ,  $f$  is *strictly* positive. Furthermore, as  $|f| \leq \xi^i$ ,  $i = 1, 2$ , following [HKS05, Theorem 3.1], one can show that

$$\Pi(f, x^i, U^i) = \mathbb{E} [\eta^i f], \quad i = 1, 2.$$

Note that we do not suppose that  $\mathbb{E}[\eta^i] = 1$  for either  $i \in \{1, 2\}$ . On the other hand, by (ii), we have

$$\Pi(f, x^1, U^1) = \Pi(f, x^2, U^2),$$

and thus

$$(3.34) \quad 0 = \mathbb{E} [(\eta^1 - \eta^2) f] = \mathbb{E} [(\eta^1 - \eta^2) \min(1, \xi^1, \xi^2) 1_{\{\eta^1 > \eta^2\}}].$$

As on  $\{\eta^1 > \eta^2\}$ , we have

$$(3.35) \quad (\eta^1 - \eta^2) \min(1, \xi^1, \xi^2) = (\eta^1 - \eta^2) \min(1, -V_1'(y_1\eta^1), -V_2'(y_2\eta^2)) > 0, \quad \mathbb{P}\text{-a.s.},$$

we conclude from (3.34) and (3.35) that

$$\mathbb{P}[\eta^1 > \eta^2] = 0.$$

Similarly, we can show that

$$\mathbb{P}[\eta^1 < \eta^2] = 0.$$

We assert that the elements of  $\mathcal{D}$ ,  $\eta^i$ ,  $i = 1, 2$ , are equal,  $\mathbb{P}$ -a.s., which contradicts (3.33).  $\square$

**3.7. Abstract version of Proposition 2.9.** We denote by  $\mathcal{FV}$  the class of Inada utility stochastic fields  $U$  (in the sense of Definition 2.1), such that

$$(3.36) \quad \begin{aligned} u(x) &= \sup_{\xi \in \mathcal{C}(x)} \mathbb{E}[U(\xi)] > -\infty, \quad x > 0, \quad \text{and} \\ v(y) &= \inf_{\eta \in \mathcal{D}(y)} \mathbb{E}[V(\eta)] < \infty, \quad y > 0, \end{aligned}$$

where  $V$  is given by (2.2).

**Proposition 3.10.** *Under the conditions of Proposition 3.9, the following conditions are equivalent:*

(i)  $\tilde{\eta}$  is the unique maximal element of  $\mathcal{D}$ , in the sense that

$$\tilde{\eta} \geq \eta, \quad \mathbb{P}\text{-a.s.}, \quad \text{for every } \eta \in \mathcal{D},$$

(ii) *the model is indifference price invariant over stochastic utilities, that is, for every  $U \in \mathcal{FV}$ ,  $x > 0$ , and every bounded contingent claim  $f$ , the set of indifference prices  $\{\Pi(f, x, U)\}$  is a singleton and depends neither on  $x > 0$  nor on  $U \in \mathcal{FV}$ .*

Furthermore, in the case when these equivalent assertions hold, we have the representation

$$(3.37) \quad \Pi(f, x, U) = \mathbb{E}_{\hat{\mathbb{Q}}} [f], \quad f \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}), \quad x > 0, \quad U \in \mathcal{FV},$$

where  $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \tilde{\eta}$ .



*Proof.* (i)  $\Rightarrow$  (ii). From (i), it follows that  $\tilde{\eta}$  is the dual minimizer for every (convex conjugate of)  $U \in \mathcal{FV}$ . As  $\mathbb{E}[\tilde{\eta}] = 1$ , similarly to [MS22, Theorem 4.2], one can show that for every  $f \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ ,  $x > 0$ , and  $U \in \mathcal{FV}$ , we have

$$\Pi(f, x, U) = \mathbb{E}[\tilde{\eta}f].$$

(ii)  $\Rightarrow$  (i). Assume by contradiction that (ii) holds, but (i) does not, that is the set of maximal elements in  $\mathcal{D}$  is not a singleton.

We consider a bounded from above deterministic  $\tilde{V}$ , such that  $-\tilde{V}$  is an Inada utility function (that is a deterministic function satisfying Definition 2.1), and  $\tilde{U}(x) = \inf_{y < 0} (\tilde{V}(y) + xy)$ ,  $x > 0$ . Then, one can see that  $\tilde{U} \in \mathcal{FV}$ . Therefore, by [Mos15, Theorem 3.2], there exists a unique optimizer  $\hat{\eta}$  to the dual problem corresponding to  $y = 1$  and the utility function  $\tilde{U}$ . As the set of maximal elements in  $\mathcal{D}$  is not a singleton, there exists  $\eta \in \mathcal{D}$ , such that  $A := \{\eta > \hat{\eta}\}$  satisfies  $\mathbb{P}[A] > 0$ . We set

$$(3.38) \quad \alpha := \frac{1}{2} \frac{\mathbb{E} \left[ \left( \tilde{V}(\hat{\eta}) - \tilde{V}(\eta) \right) 1_A \right]}{\mathbb{E} \left[ \left( \tilde{V}(\eta) - \tilde{V}(\hat{\eta}) \right) 1_{A^c} \right]}.$$

One can see that  $\alpha > 0$  and that  $\mathbb{E} \left[ \left( \tilde{V}(\eta) - \tilde{V}(\hat{\eta}) \right) 1_{A^c} \right] > 0$ , where the inequality is strict, by the maximality of the optimizer  $\hat{\eta}$  in  $\mathcal{D}$ .

With these preparations, we set

$$U(\omega, x) := \tilde{U}(x)1_A + \alpha\tilde{U}(x)1_{A^c}, \quad (\omega, x) \in \Omega \times [0, \infty).$$

Then its convex conjugate is

$$V(\omega, y) := \tilde{V}(y)1_A + \alpha\tilde{V}(y)1_{A^c}, \quad (\omega, y) \in \Omega \times [0, \infty),$$

and, using (3.38) and since  $\frac{1}{2}\mathbb{E} \left[ \left( \tilde{V}(\hat{\eta}) - \tilde{V}(\eta) \right) 1_A \right] > 0$ , we get

$$(3.39) \quad \begin{aligned} \mathbb{E}[V(\hat{\eta})] &= \mathbb{E} \left[ \left( \tilde{V}(\hat{\eta}) - \tilde{V}(\eta) \right) 1_A \right] + \alpha \mathbb{E} \left[ \left( \tilde{V}(\hat{\eta}) - \tilde{V}(\eta) \right) 1_{A^c} \right] + \mathbb{E}[V(\eta)] \\ &= \frac{1}{2} \mathbb{E} \left[ \left( \tilde{V}(\hat{\eta}) - \tilde{V}(\eta) \right) 1_A \right] + \mathbb{E}[V(\eta)] > \mathbb{E}[V(\eta)] > -\infty, \end{aligned}$$

where, the last inequality holds, as  $U$  is bounded from below by a deterministic function. We deduce from (3.39) that  $\hat{\eta}$  is not the maximizer to the dual problem associated with the stochastic utility  $U$  at  $y = 1$ . As, by the construction of  $U$ , it is bounded from below by a deterministic function,

and  $V$  is bounded from above by  $|\tilde{V}(0)|(\alpha + 1) < \infty$ , we have that  $U \in \mathcal{FV}$ . As a result, by [Mos15, Theorem 3.2], there exists a unique optimizer to the dual problem corresponding to the utility  $U$  (whose convex conjugate is  $V$ ) at  $y = 1$ . We denote this optimizer by  $\zeta$ . It follows from (3.39) that  $\mathbb{P}[\hat{\eta} \neq \zeta] > 0$  and from the maximality of the optimizer  $\zeta$  in  $\mathcal{D}$  that

$$(3.40) \quad \mathbb{P}[\zeta > \hat{\eta}] > 0.$$

We consider a bounded contingent claim

$$f = \min \left( 1, -V'(\zeta), -\tilde{V}'(\hat{\eta}) \right) 1_{\{\zeta > \hat{\eta}\}}.$$

As, on  $\{\zeta > \hat{\eta}\}$ ,  $f$  is *strictly* positive,  $\mathbb{P}$ -a.s., in view of (3.40), we get

$$(3.41) \quad 0 < \mathbb{E}[f(\zeta - \hat{\eta})] = \Pi(f, -v'(1), U) - \Pi(f, -\tilde{v}'(1), \tilde{U}),$$

where the last equality can be shown along the lines of [HKS05, Theorem 3.1] (and the representation formula (3.26) established in Proposition 3.8),  $v$  and  $\tilde{v}$  denote the dual value functions corresponding to  $V$  and  $\tilde{V}$ , respectively. As both  $U$  and  $\tilde{U}$  are in  $\mathcal{FV}$ , we conclude from (3.41) that indifference price invariance fails over  $\mathcal{FV}$ , which is a contradiction. Therefore (ii) implies (i).

Finally, under the equivalent assertions (i) and (ii) one can show (3.37) from  $\mathbb{E}[\hat{\eta}] = 1$  and by the optimality of  $\tilde{\eta}$  via Proposition 3.8, representation (3.26).  $\square$

### 3.8. Proof of the main results.

*Proof of Proposition 3.1.* The proof of this proposition in small markets is established in [KS99]. In the large market case, the only part that needs work is closedness in  $\mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$  of the set  $\mathcal{D}$ . This, however, holds, as the proof from [KS99] applies line by line, specifically the proofs of [KS99, Lemmas 4.1 and 4.2] via the argument based on the Fatou convergence for stochastic processes.  $\square$

*Proof of Theorem 2.4.* The connection between Theorem 2.4 and its abstract counterpart, Theorem 3.3, is Proposition 3.1, which is proven above. Therefore, the assertions of Theorem 2.4 follow from Theorem 3.3.  $\square$

*Proof of Proposition 2.5.* The assertions of Proposition 2.5 follows from Proposition 3.8, once we observe that the martingale assumption on  $\widehat{Y}^n(y_n)$ ,  $n \in \mathbb{N}^*$ , implies that the terminal values of this sequence satisfy (3.24).  $\square$

*Proof of Proposition 2.7.* We observe that (2.5) implies the existence of a separating measure  $\mathbb{Q} \sim \mathbb{P}$ , whose Radon-Nikodym derivative,  $\frac{d\mathbb{Q}}{d\mathbb{P}}$ , is strictly positive  $\mathbb{P}$ -a.s. and satisfies  $\mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \right] = 1$ . Now, assertions of Proposition 2.7 follows from Proposition 3.9.  $\square$

*Proof of Proposition 2.9.* The equivalence of (i) and (iii) is a well-known characterization of completeness. We outline the main steps.

(i)  $\Rightarrow$  (iii). This direction follows from the dual representation of super-replicable claims (under  $\mathcal{M} = \{\mathbb{Q}\}$ ) as in [DDGP05, Theorem 3.1] (see also [DS98, 5.12 Theorem] for the case of small markets) and the  $\mathbb{Q}$ -supermartingale property of the wealth processes associated with  $x$ -admissible generalized strategies, for every  $x > 0$ .

(iii)  $\Rightarrow$  (i). We consider an arbitrary event  $A \in \Omega$ , and set  $f := 1_A$ . From the (replicability and) boundedness of  $f$ , one can show that the replicating process for  $f$  is a  $\mathbb{Q}$ -martingale for every  $\mathbb{Q} \in \mathcal{M}$ , where  $\mathcal{M} \neq \emptyset$  by (2.5). Next, one can see that  $\mathbb{Q}[A]$  is the initial value of the replicating strategy for  $f$ , which does not depend on the choice of  $\mathbb{Q} \in \mathcal{M}$ . As  $A$  is arbitrary, we conclude that  $\mathcal{M}$  contains only one probability measure.

The remaining assertions of Proposition 2.9 follow from the (abstract) Proposition 3.10.  $\square$

## 4. EXAMPLES

**4.1. Counterexamples.** The following two examples show the necessity of Assumption 2.3 for the assertions of Theorem 2.4 to hold.

*Example 4.1.* In this example, we construct a model where Assumption 2.3 does not hold, specifically (2.21) fails, and the assertions of Theorem 2.4 also fail to hold.

We consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and let  $B$  be a one-dimensional Brownian motion on this space. Let  $(\mathcal{F}_t)_{t \in [0, T]}$  be the augmentation under  $\mathbb{P}$  of the filtration generated by  $B$ . Also, we suppose that the time horizon

$T = 1$ . With

$$g_n(x) := \left(\frac{1}{2} - \frac{1}{n}\right)x^2, \quad x \in \mathbb{R}, \quad n \in \mathbb{N},$$

$$\alpha_n := \frac{1}{2^n \mathbb{E} [e^{g_n(B_1)}]} = 2^{-n} \sqrt{\frac{2}{n}}, \quad n \in \mathbb{N},$$

we consider a nonnegative random variable defined as

$$(4.1) \quad \phi := 1 + \sum_{n \geq 1} \alpha_n e^{g_n(B_1)} > 1.$$

One can see that

$$(4.2) \quad \mathbb{E}[\phi] = 1 + \sum_{n \geq 1} \frac{1}{2^n} = 2 < \infty,$$

and, for every  $\mu > 1$  and  $k > \frac{\mu}{2(\mu-1)}$ , we obtain

$$\mathbb{E}[\phi^\mu] \geq \mathbb{E} \left[ \left( \sum_{n \geq 1} \alpha_n e^{g_n(B_1)} \right)^\mu \right] \geq \mathbb{E} \left[ \left( \alpha_k e^{g_k(B_1)} \right)^\mu \right] = \infty.$$

To recapitulate, we have

$$(4.3) \quad \mathbb{E}[\phi^\mu] = \infty, \quad \mu > 1.$$

For  $p_\infty \in (0, 1)$ , we consider a power utility  $U_\infty(x) = \frac{x^{1-p_\infty}}{1-p_\infty}$ ,  $x > 0$ . We note that the convex conjugate of  $U_\infty$  is  $V_\infty(y) = \frac{y^{-q_\infty}}{q_\infty}$ ,  $y > 0$ , where  $q_\infty = \frac{1-p_\infty}{p_\infty} > 0$ . With  $c := \mathbb{E} \left[ \phi^{-\frac{1}{q_\infty}} \right]$ , where  $c < \infty$  by (4.1), we consider a martingale

$$Z_t := \frac{1}{c} \mathbb{E} \left[ \phi^{-\frac{1}{q_\infty}} | \mathcal{F}_t \right], \quad t \in [0, 1].$$

As  $\phi^{-\frac{1}{q_\infty}}$  takes values in  $(0, 1)$ ,  $\mathbb{P}$ -a.s.,  $Z_t$  takes values in  $(0, \frac{1}{c})$ ,  $t \in [0, 1]$ .

In particular, by [JS03, Theorem II.8.3, p. 134], a strictly positive bounded and continuous martingale  $Z$  can be written in the form

$$Z = \mathcal{E}(M), \quad \text{where} \quad M = \frac{1}{Z_-} \cdot Z.$$

One can see, e.g., using [Pro04, Theorem III. 29, p. 128], that  $M$  is a continuous local martingale. Therefore, the martingale representation theorem [KS98, Theorem 3.4.15, p. 182] and localization  $M$  can be represented as

$$M = -\lambda \cdot B,$$

for some progressively measurable process  $\lambda$ , such that  $\int_0^\cdot \lambda_s^2 ds$  is locally integrable.

Next, we consider a stock market with two traded securities: one riskless, whose price process is equal to 1 at all times, and one risky stock, whose return is given by

$$R = \int_0^\cdot \lambda_s ds + B.$$

This market admits a unique equivalent local martingale measure, whose density process is  $Z$ .

For a sequence  $p_n \in (0, 1)$ ,  $n \in \mathbb{N}$ , such that  $\lim_{n \rightarrow \infty} p_n = p_\infty$  and  $p_n < p_\infty$  for infinitely many  $n \in \mathbb{N}$ , we consider a family of deterministic utilities given by  $U_n(x) = \frac{x^{1-p_n}}{1-p_n}$ ,  $x > 0$ ,  $n \in \mathbb{N}$ . We define  $q_n := \frac{1-p_n}{p_n}$ ,  $n \in \mathbb{N}$ , and we consider  $V_n(y) = \frac{y^{-q_n}}{q_n}$ ,  $y > 0$ . We observe that  $-q_n \geq -q_\infty$  for infinitely many  $n \in \mathbb{N}$ , and

$$\mu_n := \frac{q_n}{q_\infty} > 1,$$

for infinitely many  $n \in \mathbb{N}$ . Then, for every  $\mu_n > 1$ , as  $\bar{V}_n \geq V_n$  by the definition of  $\bar{V}_n$ 's in (2.9), we have

$$\begin{aligned} \bar{v}_n(y) &= \mathbb{E} [\bar{V}_n(yZ_1)] \geq \mathbb{E} [V_n(yZ_1)] = \mathbb{E} \left[ \frac{(yZ_1)^{-q_n}}{q_n} \right] \\ (4.4) \quad &= \frac{1}{q_n} \mathbb{E} [((yZ_1)^{-q})^{\mu_n}] = \frac{1}{q_n} \left( \frac{c}{y} \right)^{q_n} \mathbb{E} [\phi^{\mu_n}] = \infty, \quad y > 0, \end{aligned}$$

where, in the last equality, we have used (4.3). Therefore, *Assumption 2.21 does not hold for this model*.

Next, for every  $n$ , such that  $\mu_n > 1$ , by similar computations, we get

$$\begin{aligned} v_n(y) &= \mathbb{E} [V_n(yZ_1)] = \mathbb{E} \left[ \frac{(yZ_1)^{-q_n}}{q_n} \right] \\ (4.5) \quad &= \frac{1}{q_n} \mathbb{E} [((yZ_1)^{-q})^{\mu_n}] = \frac{1}{q_n} \left( \frac{c}{y} \right)^{q_n} \mathbb{E} [(\phi)^{\mu_n}] = \infty, \quad y > 0, \end{aligned}$$

On the other hand, for the limiting dual value function, we have

$$(4.6) \quad v_\infty(y) = \mathbb{E} [V_\infty(yZ_1)] = \frac{y^{-q_\infty}}{q_\infty} \mathbb{E} [Z_1^{-q_\infty}] = \frac{1}{q_\infty} \left( \frac{c}{y} \right)^{q_\infty} \mathbb{E} [\phi] < \infty,$$

where, in the last inequality, we have used (4.2). From (4.5) and (4.6), we conclude that the dual value functions *do not converge for every  $y > 0$* . That is, the assertion of Theorem 2.4, item (i), fails.

*Example 4.2.* In this example, we construct a model where Assumption 2.3 does not hold, specifically (2.10) fails, and the assertions of Theorem 2.4 also fail to hold.

We consider a negative-valued deterministic Inada utility function  $U_\infty$ , e.g., of the form  $U_\infty(x) = \frac{x^{1-p}}{1-p}$ ,  $x > 0$ , where  $p > 1$ , and let

$$(4.7) \quad U_n(\omega, x) = U_\infty(x) - \frac{1}{n}\phi(\omega), \quad (\omega, x) \in \Omega \times (0, \infty),$$

where  $\phi$  is a nonnegative random variable with  $\mathbb{E}[\phi] = \infty$  on an infinite probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , which supports any reasonable model of the stock price, e.g, Black-Scholes, binomial, or trinomial model with one stock, the exponential Levy models in Example 4.5, or even a large market as the ones in Examples 4.6 and 4.7, so that  $u_\infty$  is a *finite-valued function*. Observe that (4.7) implies that  $U_n$ ,  $n \in \mathbb{N}$ , converge to  $U_\infty$  for every  $\omega \in \Omega$ , and thus, Assumption 2.2 is satisfied.

On the other hand, for every admissible wealth process  $X$  starting from any  $x > 0$ , as  $U_\infty$  is negative-valued, we get

$$\mathbb{E} [\bar{U}_n^-(X_T)] = \mathbb{E} \left[ \frac{1}{n}\phi - U_\infty(X_T) \right] \geq \frac{1}{n}\mathbb{E}[\phi] = \infty, \quad n \in \mathbb{N}.$$

Therefore, in view of the convention (2.4), we deduce that

$$\bar{u}_n(x) = -\infty, \quad (x, n) \in (0, \infty) \times \mathbb{N},$$

so Assumption 2.3 fails. Specifically, (2.10) does not hold. Similarly, we have

$$u_n(x) = -\infty, \quad (x, n) \in (0, \infty) \times \mathbb{N},$$

and there is no convergence of the value functions  $u_n$ ,  $n \in \mathbb{N}$ , to  $u_\infty$ . Thus, the assertion of Theorem 2.4 in item (i) does not hold.

*Remark 4.3.* The lack of convergence of the utility functions in Example 4.2 cannot occur with *deterministic* utility functions, and Assumption 2.2 for deterministic  $U_n$ 's will lead to (2.10) in Assumption 2.21 being valid. Namely, in this case, we have

$$(4.8) \quad \bar{u}_n(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E} [\bar{U}_n(X_T)] \geq \bar{U}_n(x) > -\infty, \quad (x, n) \in (0, \infty) \times \mathbb{N}.$$

This happens because the wealth process taking a constant value  $x$  is an element of  $\mathcal{X}(x)$  for every  $x > 0$ , and thus, pointwise convergence of  $U_n$ 's in Assumption 2.2 implies (4.8).

*Example 4.4.* Here we show that *the assertions of Proposition 2.7 fail for stochastic utility fields*, in general. We consider the following stochastically dominant in the sense of Proposition 2.7, item (i), market. Let  $(B_t)_{t \in [0, T]}$  and  $(W_t)_{t \in [0, T]}$  be independent Brownian motions on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We suppose that these two Brownian motions generate the filtration. We suppose that there are two traded securities on this market. The price process of the riskless traded asset is equal to 1 at all times, and the evolution of the risky traded asset is given by

$$dS_t = S_t(\mu dt + \sigma dB),$$

where  $\mu$  and  $\sigma > 0$  are constants. Let  $\hat{Y} = \mathcal{E}(-\frac{\mu}{\sigma}B)$  denotes the density process of the minimal martingale measure. One can show that  $\hat{Y}_T$  dominates the terminal value of every element of the dual domain in the sense of (2.20). Let  $Z := \hat{Y}\mathcal{E}(W)$  and  $\tilde{V}$  be a bounded from the above deterministic function that is a convex conjugate, in the sense of (2.2), of a deterministic Inada utility function  $\tilde{U}$ . Let

$$A := \{Z_T > \hat{Y}_T\},$$

and we see that  $\mathbb{P}[A] = \mathbb{P}[\mathcal{E}(W)_T > 1] \in (0, 1)$ . We define

$$(4.9) \quad \alpha := \frac{1}{2} \frac{\mathbb{E} \left[ \left( \tilde{V}(\hat{Y}_T) - \tilde{V}(Z_T) \right) 1_A \right]}{\mathbb{E} \left[ \left( \tilde{V}(Z_T) - \tilde{V}(\hat{Y}_T) \right) 1_{A^c} \right]},$$

and observe that  $\alpha > 0$ . Now, we set the stochastic utility as

$$U(\omega, y) := \tilde{U}(x)1_A + \alpha\tilde{U}(x)1_{A^c}, \quad (\omega, x) \in \Omega \times [0, \infty).$$

Then, the convex conjugate of  $U$  in the sense of (2.2) is

$$V(\omega, y) := \tilde{V}(y)1_A + \alpha\tilde{V}(y)1_{A^c}, \quad (\omega, y) \in \Omega \times [0, \infty).$$

Then, in view of (4.9), we have

$$\mathbb{E}[V(\hat{Y}_T)] = \mathbb{E}[\tilde{V}(\hat{Y}_T)1_A + \alpha\tilde{V}(\hat{Y}_T)1_{A^c}] > \mathbb{E}[\tilde{V}(Z_T)1_A + \alpha\tilde{V}(Z_T)1_{A^c}] = \mathbb{E}[V(Z_T)].$$

Therefore,  $\hat{Y}$  is not the dual minimizer, with the utility stochastic field  $U$ , whose convex conjugate is  $V$ , for  $y = 1$ . As  $V$  is bounded from above and

$U$  is bounded from below by a deterministic function, where we note that  $U \geq \min(1, \alpha)\tilde{U}$ , which is deterministic, using [Mos15, Theorem 2.3], one can show that the optimal solution to this dual problem (with  $V$ ) exists for every  $y > 0$ . For  $y = 1$ , we denote its optimizer by  $\hat{Z}$ . Next, we consider a bounded contingent claim

$$f := 1 \wedge (-V'(\hat{Z}_T))1_{\{\hat{Y}_T > \hat{Z}_T\}}.$$

Then, as  $1 \wedge (-V'(\hat{Z}_T)) > 0$  on  $\{\hat{Y}_T > \hat{Z}_T\}$ , we obtain

$$\mathbb{E} \left[ (\hat{Y}_T - \hat{Z}_T) f \right] > 0,$$

and thus, in particular, we have

$$(4.10) \quad \mathbb{E} \left[ \hat{Y}_T f \right] \neq \mathbb{E} \left[ \hat{Z}_T f \right].$$

On the other hand, one can show that  $\mathbb{E} \left[ \hat{Y}_T f \right] = \Pi(f, -\tilde{v}'(1), \tilde{U})$  and  $\mathbb{E} \left[ \hat{Z}_T f \right] = \Pi(f, -v'(1), U)$ , where  $\tilde{v}$  and  $v$  denote the dual value functions corresponding to  $\tilde{V}$  and  $V$ , respectively. Therefore, (4.10) yields

$$\Pi(f, -\tilde{v}'(1), \tilde{U}) \neq \Pi(f, -v'(1), U).$$

We conclude that this stochastically dominant model, in the sense of Proposition 2.7, item (i), does not possess indifference price invariance over *stochastic* utilities.

#### 4.2. Exponential Levy models and perturbations of the relative risk aversion.

*Example 4.5.* The following positive example illustrates the assertions of Proposition 2.5. We consider the settings of [Kal00]. The discounted stock price process is assumed to be a  $d$ -dimensional semimartingale of the form

$$S^i = S_0^i \mathcal{E}(L^i), \quad i = 1, \dots, d,$$

where  $L = (L^i)_{i=1, \dots, d}$ , is an  $\mathbb{R}^d$ -valued Levy process with a characteristic triplet  $(b, c, F)$  relative to some truncation function  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ .

We suppose that the element of the sequence  $p_n$ ,  $n \in \mathbb{N}$ , take values in  $(0, 1) \cup (1, \infty)$ , and that this sequence converges to  $p_\infty \in (0, 1) \cup (1, \infty)$ .



These  $p_n$ 's specify the relative risk aversions of  $U_n(x) = \frac{x^{1-p_n}}{1-p_n}$ ,  $x > 0$ ,  $n \in \mathbb{N}^*$ . Further, we suppose that  $(b, c, F)$  is such that

$$b - p_n c \gamma_n + \int_{\mathbb{R}^d} \left( \frac{x}{(1 + \gamma_n^\top x)^{p_n}} - h(x) \right) F(dx) = 0,$$

admits a unique solution  $\gamma_n$  for every  $n \in \mathbb{N}^*$ , and we have

$$F\left(\{x \in \mathbb{R}^d : 1 + \gamma_n^\top x \leq 0\}\right) = 0,$$

$$\int_{\mathbb{R}^d} \left| \frac{x}{(1 + \gamma_n^\top x)^{p_n}} - h(x) \right| F(dx) < \infty,$$

$$\lim_{n \rightarrow \infty} \gamma_n = \gamma_\infty.$$

With  $L^c$  denoting the continuous martingale part of  $L$ ,  $\mu^L$  denoting the random measure associated with the jumps of  $L$  and with  $\nu^L(dx, dt) = F(dx)dt$  denoting the predictable  $\mathbb{P}$ -compensator of  $\mu^L$ , we consider

$$(4.11) \quad Z^n := \mathcal{E} \left( -p_n \gamma_n^\top L^c + \left( (1 + \gamma_n^\top x)^{-p} - 1 \right) * (\mu^L - \nu^L) \right), \quad n \in \mathbb{N}^*.$$

Then the results in [Kal00, proof of Theorem 3.2, Third step] ensure that  $Z^n$ ,  $n \in \mathbb{N}^*$ , is the sequence of dual minimizers to (2.7) corresponding to  $y = 1$ . By [Kal00, Lemmas 4.2 and 4.4] (see also [Kal00, proof of Theorem 3.2, Fourth step]),  $Z^n$  is a martingale for every  $n \in \mathbb{N}^*$ . Therefore, the assumptions of Proposition 2.5 are satisfied. Therefore, for every bounded contingent claim  $f$ , we have

$$\lim_{n \rightarrow \infty} \Pi(f, x_n, U_n) = \lim_{n \rightarrow \infty} \mathbb{E}[Z_T^n f] = \mathbb{E}[Z_T^\infty f] = \Pi(f, x_\infty, U_\infty),$$

where, in this case,  $x_n$ ,  $n \in \mathbb{N}^*$ , can be any sequence of strictly positive numbers. That is, the convergence of the indifference prices holds under perturbations of relative risk aversion. In view of (4.11), one can also see that the exponential Levy models, in general, are not indifference price invariant (and not stochastically dominant). We note that a characterization of  $Z^n$ 's in (4.11) in terms of a solution to a deterministic problem in  $\mathbb{R}^d$  is obtained in [JKM07].

We also note that risk aversion asymptotics for power utility maximization is investigated in [Nut12]. Also, [Kal00] covers logarithmic and exponential preferences, where indifference prices can be characterized similarly.

**4.3. Examples of large incomplete stochastically dominant markets.** Specific examples of large markets in various settings are considered in [ET05], [DDP05], [DDGP05], [HK17], and [HK20], among others. Below we give examples of large *stochastically dominant* markets in the sense above. Possibly the simplest example can be built from Bernoulli random variables.

*Example 4.6.* [Large incomplete stochastically dominant Bernoulli-driven market]

Let  $s_n, n \in \mathbb{N}$ , be a sequence of positive numbers taking values in  $(1, 2)$  and converging to 2 sufficiently fast, so that

$$(4.12) \quad s_n \in \left( \frac{2}{2 - e^{-\frac{1}{n^2}}}, 2 \right), \quad n \in \mathbb{N}.$$

We consider a sequence of independent Bernoulli random variables  $S_1^n, n \in \mathbb{N}$ , such that  $\mathbb{P}[S_1^n = 0] = \mathbb{P}[S_1^n = s_n] = \frac{1}{2}$ . We further suppose that  $S_0^n = 1, n \in \mathbb{N}$ , and consider a market model, where the evolution of risky assets is modeled by  $S^n$  and the riskless asset is given by  $S^0 \equiv 1$ . We assume that  $\mathcal{F}_0$  is trivial and  $\mathcal{F} = \mathcal{F}_1 = \sigma(S_1^n, n \in \mathbb{N}, \text{ and } \phi)$ , where  $\phi$  is a non-constant-valued random variable independent of  $S_1^n, n \in \mathbb{N}$ . One can see that the model is incomplete, and an example of a nonreplicable contingent claim is  $f = \phi$ .

Next, we define

$$\eta^n := \frac{2}{s_n} 1_{\{S_1^n = s_n\}} + \left( 2 - \frac{2}{s_n} \right) 1_{\{S_1^n = 0\}}, \quad n \in \mathbb{N}.$$

Then

$$(4.13) \quad \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} := \prod_{n=1}^{\infty} \eta^n,$$

is a density of an equivalent separating measure  $\widehat{\mathbb{Q}}$ , where (4.12) ensures that  $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}$  above is well-defined and bounded away from 0 and  $\infty$ ,  $\mathbb{P}$ -a.s.. Here we note that in (4.12), we have

$$s_n > \frac{2}{2 - e^{-\frac{1}{n^2}}} = \max \left( 2e^{-\frac{1}{n^2}}, \frac{2}{2 - e^{-\frac{1}{n^2}}} \right), \quad n \in \mathbb{N}.$$

Using this, one can show that  $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}$  defined in (4.13) is bounded away from 0 and  $\infty$ .

Further, one can show that  $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$  stochastically dominates every element of the dual domain,  $\{Y_T : Y \in \mathcal{Y}\}$ , in the sense of (2.20). In particular, the set of equivalent separating measures is non-empty, and (2.5) holds. Next, for every bounded contingent claim, initial wealth  $x > 0$ , and (a deterministic)  $U \in \mathcal{FD}$  that is, for every  $U$  whose convex conjugate  $V$  satisfies

$$(4.14) \quad \mathbb{E} \left[ V \left( y \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} \right) \right] < \infty, \quad y > 0,$$

for  $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$  given by (4.13), by Proposition 2.7, we have

$$(4.15) \quad \Pi(f, x, U) = \mathbb{E}_{\hat{\mathbb{Q}}}[f].$$

Thus, this model is indifference price invariant over deterministic Inada utility functions (and incomplete). We note that, as  $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$  given by (4.13), is bounded away from 0, the set  $\mathcal{FD}$  in this market equals to the set of all deterministic Inada utility functions  $U$ , since for every such  $U$ , (4.14) will hold for its convex conjugate  $V$  in the sense of (2.2).

The construction in the following example (with some variations) is quite common in the literature. Below, we show that it also gives an incomplete market that satisfies the assertions of Proposition 2.7.

*Example 4.7.* [Large incomplete stochastically dominant Brownian market] We consider a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  supporting a sequence of one-dimensional independent Brownian motions  $W^n$ ,  $n \in \mathbb{N}^*$ ,  $\mathcal{F} = \mathcal{F}_T$  is generated by  $W^n$ ,  $n \in \mathbb{N}^*$ . Let the riskless asset  $S^0 \equiv 1$ , whereas the dynamics of risky assets is given by

$$S^n = \mathcal{E}(\mu^n I + \sigma^n W^n), \quad n \in \mathbb{N},$$

where  $I_t = t$ ,  $t \in [0, T]$ , and the constants  $\mu^n$  and  $\sigma^n > 0$ ,  $n \in \mathbb{N}$ , are and such that

$$(4.16) \quad \sum_{n=1}^{\infty} \left( \frac{\mu^n}{\sigma^n} \right)^2 < \infty.$$

One can see that this model is incomplete. An example of a non-replicable claim is  $1_{\{W_T^\infty > 0\}}$ .

Next, by (4.16),  $\sum_{n=1}^{\infty} \frac{\mu^n}{\sigma^n} W_T^n$  is a limit of a uniformly integrable sequence of terminal values of martingales  $\sum_{n=1}^m \frac{\mu^n}{\sigma^n} W^n$ ,  $m \in \mathbb{N}$ , and thus  $\sum_{n=1}^{\infty} \frac{\mu^n}{\sigma^n} W_T^n$  is well-defined and, by (4.16) and [CE15, Theorem 15.4.2, p. 384], we have

$$\mathbb{E} \left[ \mathcal{E} \left( - \sum_{n=1}^{\infty} \frac{\mu^n}{\sigma^n} W^n \right)_T \right] = 1.$$

Therefore, we can define a probability measure  $\widehat{\mathbb{Q}}$  via its density as follows

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} := \mathcal{E} \left( - \sum_{n=1}^{\infty} \frac{\mu^n}{\sigma^n} W^n \right)_T.$$

Thus,  $\widehat{\mathbb{Q}}$  is a probability measure, which is equivalent to  $\mathbb{P}$  and is separating, that is,  $\widehat{\mathbb{Q}} \in \mathcal{M}$ , and where we recall that  $\mathcal{M}$  is defined in (2.5). In particular, the set of equivalent separating measures is non-empty, and (2.5) holds.

Next, one can show that  $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}$  stochastically dominates every element of the dual domain,  $\{Y_T : Y \in \mathcal{Y}\}$ , in the sense of (2.20). For every deterministic Inada utility  $U \in \mathcal{FD}$ , that is, for every  $U$ , whose convex conjugate  $V$ , in the sense (2.2), satisfies

$$\mathbb{E} \left[ V \left( y \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \right) \right] < \infty, \quad y > 0,$$

and for every bounded contingent claim  $f$  and every  $x > 0$ , by Proposition 2.7, we have

$$(4.17) \quad \Pi(f, x, U) = \mathbb{E}_{\widehat{\mathbb{Q}}} [f], \quad x > 0, \quad U \in \mathcal{FD}.$$

We note that (4.17), in particular, applies to non-replicable contingent claims, including  $1_{\{W_T^\infty > 0\}}$ . In view of (4.17), the model considered in this example is indifference price invariant with respect to the choice of both  $U \in \mathcal{FD}$  and the initial wealth  $x > 0$ .

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