# Indifference pricing of variable annuities with maturity guarantees and periodic fees in a two-regime market

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**Abstract**

We study indifference pricing of a variable annuity with a maturity guarantee and periodic fees in an incomplete market with one risky asset and two observable Markov regimes. The pricing agent has power utility from consumption at finitely many deterministic dates. A statewise Kallsen equation determines the optimal portfolio, and opportunity functions satisfy backward equations with nonlinear consumption-date matching. Their marginal-value process yields an explicit, strictly positive true-martingale dual density. Using transition-counting-process compensators, we prove that the selected measure transforms regime intensities by opportunity-function ratios. The return-noise factor of the density is independent of the consumption clock; its regime factor generally is not. Datewise superhedging bounds establish the dual-martingale representation of indifference prices for the maturity top-up and fee stream. We also prove an account-value identity linking the net rider price to the full annuity value. An extension allows an exogenous settlement time on the deterministic date set. Analytical comparison results show how consumption dates contract opportunity ratios and establish strict clock dependence for regime-contingent claims. Common clock scaling, equal opportunity growth rates and replication yield distinct price-invariance results. Black–Scholes and Merton jump-diffusion specifications give explicit transformed characteristics.

**Keywords:** variable annuity; maturity guarantee; periodic fees; indifference pricing; regime switching; semimartingale characteristics; utility maximization.

## 1 Introduction

A variable-annuity maturity guarantee protects the policyholder against a shortfall in the investment account, while periodic account fees provide income to the insurer. Valuing the guarantee and its fee income consistently is central to assessing the insurer's net liability. When investment opportunities depend on an observable but untraded economic regime, the traded fund does not span all relevant risks. Preferences provide an economic criterion for selecting a pricing measure in this incomplete market.

We develop this criterion for an agent with power utility from consumption at finitely many deterministic dates. The model has one risky asset and two Markov regimes. The resulting indifference price values the maturity top-up and each fee payment at its contractual date. A self-financing account identity links these components: the full annuity value equals the initial account premium plus the net maturity-guarantee rider price. The same construction covers exogenous random settlement on the deterministic date set. American-style exercise and optimal surrender are not considered; in particular, settlement times are not chosen by the policyholder through an optimal stopping problem. Mortality and lapse risks are also excluded to isolate financial regime

risk; the random-settlement extension uses the existing market filtration and does not introduce a mortality or lapse model.

The consumption calendar has a specific pricing role. The factor of the dual density driven by Brownian and return-jump martingales is independent of the calendar, whereas the factor driven by regime transitions generally depends on it. We prove the transformed transition intensities directly from counting-process compensators, including at fixed consumption dates. Analytical comparisons establish strict clock dependence for a regime-contingent claim and identify common clock scaling, equal opportunity growth rates and replication as sources of price invariance. These results distinguish changes in the selected measure from changes in the price of a particular annuity.

The actuarial literature on variable-annuity pricing, valuation and risk management is surveyed by Feng et al. (2022). Related work examines financially optimal surrender of variable annuity guarantees (Bernard et al., 2014) and emphasizes accounting for periodic fee income when hedging the insurer's net liability (Bernard and Kwak, 2016). Building on the maturity-guarantee and periodic-fee structure of Ai et al. (2024), our contribution combines an explicit preference-based measure selection with a finite-date indifference-pricing theorem for the resulting cash flows. The top-up is bounded, and each fee is dominated by a deterministic multiple of the traded fund. These bounds yield dated cash-and-stock superhedges and a direct proof of the pricing representation, including potentially unbounded fee income.

Utility-based valuation of unhedgeable claims has roots in Hodges and Neuberger (1989); Young and Zariphopoulou (2002) develop equivalent-utility pricing for dynamic insurance risks. A distinct approach is the risk-minimization framework for unit-linked life insurance in Møller (1998). Regime-switching equity models and their applications to maturity guarantees are studied by Hardy (2001), while Elliott et al. (2005) select an option-pricing measure through a regime-switching Esscher transform. Portfolio optimization with Markov-modulated investment opportunities is treated by Bäuerle and Rieder (2004). Tiong (2000) uses Esscher transforms to value point-to-point, cliquet and lookback equity-indexed annuities. Lin et al. (2009) apply an Esscher transform to price equity-linked and variable-annuity guarantees in a regime-switching diffusion model. Ng and Li (2011) develop a multivariate regime-switching framework for variable-annuity guarantees, selecting a risk-neutral measure through a multivariate conditional Esscher transform. Our focus is the marginal no-trade price selected by a finite consumption clock in an observable-regime market, including its effect on transition intensities and dated guarantee and fee cash flows.

Garces et al. (2026) study utility-indifference valuation of bounded mortality-linked securities via backward stochastic differential equations. The valuation criterion in the present paper is *marginal no-trade valuation*: a quote at which zero holdings of a fixed claim are optimal for the agent. Following Karatzas and Kardaras (2021, p. 157), we call this *indifference pricing*; other names include Davis pricing, fair pricing and marginal utility-based pricing. The no-trade definition and dual-martingale representation are closely related to Mostovyi and Siorpaes (2024, Definition 4.1 and Theorem 4.2). We establish the corresponding representation for dated cash flows and utility from finitely many consumption dates, together with a joint first-order expansion of the endowed value function.

The portfolio construction builds on the characteristic-based equation and martingale verification of Kallsen (2000), the opportunity-process approach of Nutz (2012), and the intermediate-consumption duality framework of Mostovyi (2015). Opportunity coefficients solve linear backward equations between consumption dates and nonlinear matching conditions at those dates. An explicit wealth construction establishes admissibility, and square-moment estimates establish the true-martingale dual density and the binding budget. Black–Scholes and Merton jump-diffusion specifications provide explicit transformed characteristics.

Section 2 specifies the market, consumption problem and annuity. Section 3 constructs and

verifies the dual optimizer. Section 4 proves the transition-compensator transformation. Section 5 establishes the indifference prices and account-value identity. Section 6 gives clock comparisons and price-invariance results, and Section 7 presents the diffusion and jump-diffusion specifications. Section 8 gives numerical illustrations. Appendix A proves the finite-date pricing theorem.

# 2 Market, consumption and the annuity contract

## 2.1 Noise structure and characteristics

Let the filtered probability space satisfy the usual conditions and let $\alpha_0 = i_0 \in \{1, 2\}$ be fixed. The observable chain has generator

$$Q = \begin{pmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{pmatrix}, \qquad q_{12}, q_{21} > 0.$$

The filtration is generated, and completed, by the chain, a Brownian motion, and independent Poisson random measures used to construct the return jumps. The primitive chain and return noises are independent. Conditional on the chain path, the return process has independent increments with characteristics

$$(b_{\alpha_{t-}}\,\mathrm{d}t,\ c_{\alpha_{t-}}\,\mathrm{d}t,\ F_{\alpha_{t-}}(\,\mathrm{d}z)\,\mathrm{d}t)$$

relative to a fixed bounded truncation function $h$, equal to $z$ near zero. Here $c_i \geq 0$, $F_i$ is a Lévy measure supported on $(-1, \infty)$, and return jumps and regime transitions do not occur simultaneously. Neither jumps at deterministic times. We use the usual augmentation of this natural noise filtration; in particular $\mathcal{F}_{t_k} = \mathcal{F}_{t_k-}$ up to null sets. No extra information source is revealed at the scheduled consumption or fee dates. The regime-dependent characteristics define a switching Lévy model.

The assets are $B_t = e^{rt}$, $B_0 = 1$, and $S_t = S_0\mathcal{E}(L)_t$, where

$$L = b_{\alpha_-} \cdot t + \sqrt{c_{\alpha_-}} \cdot W + h * (\mu^L - \nu^L) + (z - h(z)) * \mu^L, \qquad \nu^L(\,\mathrm{d}t, \mathrm{d}z) = F_{\alpha_{t-}}(\,\mathrm{d}z)\,\mathrm{d}t. \quad (1)$$

Thus $S$ stays positive and has no jump when only the regime changes. Set $R = L - rt$; then $\widetilde{S} = S/B = S_0\mathcal{E}(R)$ has return drift $b_i - r$.

## 2.2 The deterministic consumption calendar

The pricing agent's utility clock is

$$\kappa_t = \sum_{k=1}^{n} a_k \mathbf{1}_{\{t_k \leq t\}}, \qquad 0 < t_1 < \cdots < t_n = T < \infty, \quad a_k > 0. \quad (2)$$

Consuming a rate $\mathfrak{c}_k$ costs $a_k\mathfrak{c}_k$ in bank-account units and yields $a_k U(\mathfrak{c}_k)$. The account premium $P$ and the pricing agent's initial wealth $x$ are different quantities. The utility clock, the fee fractions and the maturity amount are fixed before computing an indifference price.

## 2.3 A maturity guarantee financed by periodic fees

Let $\mathcal{T} = \{t_1, \ldots, t_n = T\}$ be the deterministic date set. A zero fee fraction excludes a grid date from the fee calendar. The policyholder pays an account premium $P > 0$ into a segregated account

invested in the fund $S$. The policy account $\mathcal{V}$ is separate from the pricing agent's wealth $X$ and starts at $\mathcal{V}_0 = P$. With $t_0 = 0$, its pre-fee value and fee update are

$$\boxed{\begin{aligned} \mathcal{V}_{t_k-} &= \mathcal{V}_{t_{k-1}} \frac{S_{t_k}}{S_{t_{k-1}}}, \\ \mathcal{C}_k &= \zeta_k(\alpha_{t_k}, \mathcal{V}_{t_k-})\mathcal{V}_{t_k-}, \qquad 0 \le \zeta_k(i,v) \le \bar{\zeta} < 1, \\ \mathcal{V}_{t_k} &= \mathcal{V}_{t_k-} - \mathcal{C}_k. \end{aligned}} \tag{3}$$

Between dates the account follows the fund proportionally. The fee functions are fixed measurable contract specifications, independent of the pricing agent's controls. Constant proportional fees and threshold-dependent fees are included. All fees in this model are rider income to the insurer. Any management expenses would have to be deducted from that income explicitly.

Let $G > 0$ be a deterministic guaranteed maturity amount, for example a return-of-premium or deterministic roll-up amount. The fee at $T$, if any, is deducted before the maturity benefit is evaluated. The guarantee top-up and the total policyholder benefit are

$$\boxed{\mathcal{H}_T = (G - \mathcal{V}_T)^+, \qquad H_T^{\text{full}} = \mathcal{V}_T + \mathcal{H}_T = \max\{G, \mathcal{V}_T\}.} \tag{4}$$

The account value is passed through to the policyholder. The maturity-guarantee rider pays the shortfall $\mathcal{H}_T$ in addition to the investment account. The insurer's *net rider liability* is this top-up less the periodic fees received. Its dated cash flows are

$$\mathcal{L}_k = -\mathcal{C}_k \ (k < n), \qquad \mathcal{L}_n = \mathcal{H}_T - \mathcal{C}_n. \tag{5}$$

An additional up-front rider charge, if quoted, is paid externally and does not change $P$ in this specification. Deducting it from the account instead would change the account recursion and hence the claim.

If $\nu_0 = P/S_0$, the number of fund units in the account decreases at fee dates and is never above $\nu_0$. Consequently

$$0 \le \mathcal{V}_{t_k-} \le \nu_0 S_{t_k}, \qquad 0 \le \mathcal{C}_k \le \bar{\zeta}\nu_0 S_{t_k}, \qquad 0 \le \mathcal{H}_T \le G. \tag{6}$$

These bounds will verify the pricing theorem for the actual contract, including the potentially unbounded fee income.

### 2.4 The pricing agent: admissibility and utility

We work in bank-account units. A predictable fraction $\pi$ and nonnegative $\mathcal{F}_{t_k}$-measurable consumption rates $\mathfrak{c}_k$ are admissible from $x > 0$ if their stochastic integrals exist and

$$X_t = x + \int_0^t X_{s-}\pi_s \, \mathrm{d}R_s - \sum_{t_k \le t} a_k \mathfrak{c}_k \ge 0. \tag{7}$$

Admissible risky dollar exposure is $\vartheta_t = X_{t-}\pi_t$. We require $\vartheta$ to be $R$-integrable, $1 + \pi_t \Delta R_t \ge 0$ at return jumps, and zero risky exposure while financial wealth is zero. In the baseline problem this makes zero wealth absorbing. With the signed cash flows introduced later, trading may resume after an incoming contractual payment restores positive wealth. Equivalently, the discounted stock holding is $\vartheta_t/\widetilde{S}_{t-}$, with the usual associativity of stochastic integration. These conventions and nonnegative wealth define $\mathcal{A}(x)$. At a fixed consumption date there is almost surely no market or

regime jump, so $X_{t_k} = X_{t_k-} - a_k \mathfrak{c}_k$. Consumption can be chosen from the information at that date; in this filtration it contains no new fixed-time noise.

Let $p > 0$, $p \neq 1$, $\rho = 1 - p$, and $U(x) = x^\rho/\rho$. For undiscounted consumption $C_k = B_{t_k}\mathfrak{c}_k$ and impatience $\delta$, define

$$w_k = e^{-[\delta - r(1-p)]t_k} > 0.$$

The optimization problem is

$$u(x) = \sup_{(\pi,\mathfrak{c})\in\mathcal{A}(x)} \mathbb{E}\sum_{k=1}^{n} a_k w_k U(\mathfrak{c}_k). \tag{8}$$

Set $U(0) = 0$ for $p < 1$ and $U(0) = -\infty$ for $p > 1$. There is no extra terminal utility or endowment. It is optimal to consume all remaining wealth at $T$.

### 2.5 Kallsen equation and sufficient assumptions

For a fraction $\gamma$ with $1 + \gamma z > 0$ $F_i$-almost everywhere, put

$$g_i(\gamma) = (b_i - r)\gamma - \frac{p}{2}c_i\gamma^2 + \int \left\{ \frac{(1+\gamma z)^{1-p} - 1}{1-p} - \gamma h(z) \right\} F_i(\,\mathrm{d}z). \tag{9}$$

The function $g_i$ is the local portfolio objective. Its stationary equation is

$$(b_i - r) - pc_i\gamma_i + \int \{z(1+\gamma_i z)^{-p} - h(z)\} F_i(\,\mathrm{d}z) = 0. \tag{10}$$

Where differentiation is justified, $g_i''(\gamma) = -pc_i - p\int z^2(1+\gamma z)^{-p-1}F_i(\,\mathrm{d}z) \leq 0$. The admissible deterministic fraction domain is convex: it consists of fractions for which $1 + \gamma z > 0$ $F_i$-almost everywhere and the displayed objective is well defined. Concavity and a valid stationary equation therefore identify a global maximizer on that domain.

In addition to the stationary equation, Kallsen (2000, Theorem 3.2) requires absolute integrability of its jump term. In the present one-asset notation, this condition is

$$\int \left| z(1+\gamma_i z)^{-p} - h(z) \right| F_i(\,\mathrm{d}z) < \infty.$$

It ensures that the jump contribution in (10) is finite. The following assumption includes this condition together with finiteness of the local objective and the moment bounds used in our martingale verification.

**Assumption 2.1** (Interior solution and moments)**.** For each $i = 1, 2$, there is a finite $\gamma_i$ satisfying (10), with $1 + \gamma_i z > 0$ $F_i$-almost everywhere. The integrals in (9) and (10) are absolutely convergent, and the latter is a valid first-order condition (in particular differentiation at $\gamma_i$ is justified). Write

$$f_i(z) = (1+\gamma_i z)^{-p}, \qquad v_i(z) = (1+\gamma_i z)^{1-p}.$$

We assume

$$\int z^2 F_i(\,\mathrm{d}z) < \infty, \quad \int (f_i - 1)^2 \,\mathrm{d}F_i < \infty, \quad \int (v_i - 1)^2 \,\mathrm{d}F_i < \infty, \quad \int z^2 f_i(z) F_i(\,\mathrm{d}z) < \infty. \tag{11}$$

The moment bounds in Assumption 2.1 are sufficient for the martingale estimates below. Near zero the integrands are $O(z^2)$. The nontrivial restrictions concern large jumps and points where $1+\gamma_i z$ approaches zero. Strict concavity gives uniqueness of $\gamma_i$ if $c_i > 0$ or $F_i$ is nonzero. The results below concern interior solutions.

Define

$$\eta_i = (1-p)g_i(\gamma_i), \qquad K = Q + \operatorname{diag}(\eta_1, \eta_2). \tag{12}$$

For $p > 1$, maximizing utility still selects the maximizer of $g_i$; after dividing by $1-p < 0$, the equivalent multiplicative criterion is minimized.

# 3 Optimal consumption and the dual measure

## 3.1 Opportunity functions and fixed-date matching

The opportunity coefficient summarizes a concrete optimization problem. Suppose the agent starts at time $t < T$ with wealth $x > 0$ in regime $i$, after any consumption scheduled at $t$. Let $\mathcal{A}_t(x)$ denote the admissible future investment and consumption plans from that wealth. The continuation value is the greatest expected utility attainable from the remaining consumption dates:

$$u_i(t,x) = \sup_{(\pi,\mathfrak{c})\in\mathcal{A}_t(x)} \mathbb{E}_{t,i}\left[\sum_{k:\, t_k>t} a_k w_k U(\mathfrak{c}_k)\right].$$

Here $\mathbb{E}_{t,i}$ denotes expectation for the market restarted at time $t$ in regime $i$. Power utility and the scaling of admissible wealth plans give the representation

$$u_i(t,x) = A_i(t)\frac{x^{1-p}}{1-p}.$$

The construction and verification below determine $A_i(t)$.

Its marginal interpretation explains the regime adjustment in the dual density. At a fixed time and wealth,

$$\partial_x u_i(t,x) = A_i(t)x^{-p}, \qquad \frac{\partial_x u_j(t,x)}{\partial_x u_i(t,x)} = \frac{A_j(t)}{A_i(t)}.$$

The derivative measures the increase in optimized expected utility produced by a small additional amount of wealth. For example, a ratio of 1.2 means that additional wealth has twenty percent greater marginal utility in regime $j$ than in regime $i$, at the same time and wealth. A regime transition leaves wealth unchanged in this model, but changes this marginal value by the displayed ratio. This is the transition multiplier $h_{ij}$ defined in (20); its appearance in the dual density is established in Theorem 3.4.

At a consumption date, write $A_i(t_k-)$ for the pre-consumption coefficient, with consumption at $t_k$ still available, and $A_i(t_k)$ for the post-consumption coefficient. Start with

$$A_i(T) = 0, \qquad A_i(T-) = a_n^p w_n, \quad i = 1,2. \tag{13}$$

Only the pre-terminal coefficient will be used to construct the terminal density.

**Theorem 3.1** (Backward construction and matching)**.** *Under Assumption 2.1, the opportunity coefficients are obtained uniquely by the following backward recursion. Between consecutive dates,*

$$A'(t) + KA(t) = 0, \qquad A(t) = e^{K(t_k-t)}A(t_k-), \quad t_{k-1} \le t < t_k, \tag{14}$$

*where* $A = (A_1, A_2)^\top$ *and* $t_0 = 0$. *At* $k < n$, *put* $d_k = a_k w_k^{1/p}$ *and impose*

$$\boxed{A_i(t_k-) = \left[d_k + A_i(t_k)^{1/p}\right]^p.} \tag{15}$$

*The optimal fraction of pre-consumption wealth spent at that date and the optimal rate are*

$$m_{ki} = \frac{d_k}{d_k + A_i(t_k)^{1/p}}, \tag{16}$$

$$\mathfrak{c}_k^* = \frac{w_k^{1/p}}{d_k + A_{\alpha_{t_k}}(t_k)^{1/p}} X^*_{t_k-}, \qquad X^*_{t_k} = (1 - m_{k,\alpha_{t_k}}) X^*_{t_k-}. \tag{17}$$

*At* $T$, $m_{ni} = 1$ *and* $\mathfrak{c}_n^* = X^*_{T-}/a_n$. *The coefficients* $A_i(t)$ *for* $t < T$ *and their pre-date limits through* $T-$ *are positive, uniformly bounded and bounded away from zero.*

*Proof.* The generator applied to $A_i(t)x^\rho/\rho$ on an interval without consumption yields

$$\frac{x^\rho}{\rho}\{A_i'(t) + (QA(t))_i\} + x^\rho A_i(t) g_i(\pi).$$

There is no cross-jump term between returns and regimes. Maximization therefore gives (14), because $A_i > 0$ and $g_i$ is concave with stationary point $\gamma_i$. The global verification below identifies this candidate with the value function.

At a consumption date, with $v$ denoting pre-date wealth and $z$ post-date wealth, the decision is

$$\sup_{0 \le z \le v} \left\{a_k w_k U((v-z)/a_k) + A_i(t_k) U(z)\right\}.$$

For $k < n$, both terms are strictly concave and the optimum is interior. The first-order equation is $w_k((v-z)/a_k)^{-p} = A_i(t_k) z^{-p}$. Solving it gives (17); substituting gives (15). At $T$ there is no continuation, so all wealth is consumed. This gives (13).

The matrix $K$ has nonnegative off-diagonal entries, and

$$(e^{Ks} v)_i = \mathbb{E}_i\left[e^{\int_0^s \eta_{\alpha_u}\,\mathrm{d}u} v_{\alpha_s}\right]$$

for deterministic column vectors $v$. This follows from the finite-state backward equation, or directly from conditioning on the first transition. It is positive for strictly positive $v$. Bounded coefficients, a finite horizon and finitely many positive matching maps give all stated bounds and uniqueness by recursion. □

*Remark* 3.2 (Marginal value across a date). For $k < n$ the optimal split satisfies the identity

$$A_i(t_k-) v^{-p} = A_i(t_k)((1 - m_{ki})v)^{-p} = w_k(\mathfrak{c}_k^*)^{-p}. \tag{18}$$

Thus $A$ jumps downward at consumption while the negative power of remaining wealth jumps upward by exactly the inverse factor. The marginal value does not jump. At $T$, its terminal value is defined by the left-hand side of (18), which equals $w_n(\mathfrak{c}_n^*)^{-p}$.

### 3.2 Density construction and martingale verification

Construct $X^*$ explicitly from $X_0^* = x$, the fraction $\pi_t^* = \gamma_{\alpha_{t-}}$ between dates, and the reductions (17). Each inter-date segment is a positive stochastic exponential; each nonterminal reduction has a factor strictly between zero and one. Hence $X^* > 0$ before $T$ and $X_T^* = 0$. The resulting plan is admissible.

Define the marginal-value process by

$$D_t = A_{\alpha_t}(t)(X_t^*)^{-p} \quad (0 \le t < T), \qquad D_T = A_{\alpha_T}(T-)(X_{T-}^*)^{-p}, \qquad Z_t = D_t/D_0. \tag{19}$$

Define the predictable opportunity-function ratios by

$$h_{ij}(t) = \frac{A_j(t-)}{A_i(t-)}, \qquad 0 < t \le T, \quad i, j \in \{1, 2\}, \quad i \ne j. \tag{20}$$

Between consumption dates this equals $A_j(t)/A_i(t)$. At each consumption date, including $T$, the left limits specify the predictable value. The values at the finitely many dates do not affect the Lebesgue or transition-count integrals almost surely.

The transition counting processes and their compensated versions are

$$N_t^{ij} = \sum_{0<s\le t} \mathbf{1}_{\{\alpha_{s-}=i,\alpha_s=j\}}, \qquad M_t^{ij} = N_t^{ij} - \int_0^t \mathbf{1}_{\{\alpha_{s-}=i\}} q_{ij}\, \mathrm{d}s. \tag{21}$$

**Lemma 3.3** (Physical compensators)**.** *The two processes $M^{12}$ and $M^{21}$ are square-integrable martingales on $[0, T]$. Also,*

$$\langle M^{ij}\rangle_t = \int_0^t \mathbf{1}_{\{\alpha_{s-}=i\}} q_{ij}\, \mathrm{d}s, \qquad [M^{12}, M^{21}] = 0.$$

*These statements hold in the full market filtration used here.*

*Proof.* Conditional on the current state $i$, the exponential holding-time law and the Markov property give transition rate $q_{ij}$. Integration against bounded simple predictable processes, followed by monotone approximation, gives

$$\mathbb{E}\int \psi_s\, \mathrm{d}N_s^{ij} = \mathbb{E}\int \psi_s \mathbf{1}_{\{\alpha_{s-}=i\}} q_{ij}\, \mathrm{d}s$$

for nonnegative predictable $\psi$. This characterizes the compensator. Independence of primitive return sources ensures that their past does not alter these conditional rates. Return observations are functions of these sources and the observed chain, so the result also holds in the market filtration. Bounded rates on a finite horizon give integrable counts and square-integrable compensated counts. Their quadratic variations are $N^{ij}$; distinct transition types cannot occur at the same instant. □

**Theorem 3.4** (Explicit normalized dual density)**.** *The normalized density in (19) is the strictly positive square-integrable martingale*

$$\boxed{Z = \mathcal{E}(\mathcal{N}), \quad \mathcal{N} = -p\gamma_{\alpha_-}\sqrt{c_{\alpha_-}} \cdot W + (f_{\alpha_-}(z) - 1) * (\mu^L - \nu^L) + \sum_{i\ne j}(h_{ij} - 1) \cdot M^{ij}.} \tag{22}$$

*Here $h_{ij}$ is the predictable opportunity-function ratio defined in (20). The density has $Z_0 = 1$, $\mathbb{E}Z_t = 1$, and no jump at any $t_k$. In particular $\mathrm{d}\widehat{\mathbb{P}}/\,\mathrm{d}\mathbb{P} = Z_T$ defines an equivalent probability measure on $\mathcal{F}_T$.*

*Proof.* Write $\mathfrak{d}_i$ for the drift rate of $(X^*)^{-p}$ divided by its left limit, between dates. Itô's formula gives

$$\mathfrak{d}_i = -p\gamma_i(b_i - r) + \tfrac{1}{2}p(p+1)c_i\gamma_i^2 + \int \{f_i(z) - 1 + p\gamma_i h(z)\}F_i(\,\mathrm{d}z).$$

Multiplying (10) by $\gamma_i$ shows

$$\mathfrak{d}_i = \eta_i. \tag{23}$$

Indeed the difference $\mathfrak{d}_i - \eta_i$ equals $-\gamma_i$ times the left side of (10); the jump simplification uses $(1+\gamma_i z)f_i(z) = v_i(z)$. Applying the product rule to $A_\alpha(X^*)^{-p}$ now gives drift

$$\frac{A_i'(t) + (QA(t))_i}{A_i(t)} + \mathfrak{d}_i = -\eta_i + \eta_i = 0.$$

Its continuous martingale coefficient is $-p\gamma_i\sqrt{c_i}$; its multiplier at a return jump $z$ is $f_i(z)$, and at a regime transition $i \to j$ it is $h_{ij}(t)$. There are no simultaneous jumps. At a fixed consumption date (18) cancels the two deterministic-time factors exactly, including the separately defined terminal value. This proves (22) as a local-martingale identity.

Orthogonality gives the predictable bracket rate

$$\frac{\mathrm{d}\langle\mathcal{N}\rangle_t}{\mathrm{d}t} = p^2\gamma_i^2 c_i + \int (f_i - 1)^2\,\mathrm{d}F_i + \sum_{j\neq i} q_{ij}(h_{ij}(t) - 1)^2 \le C, \qquad i = \alpha_{t-},$$

with deterministic $C < \infty$. The integrals defining $\mathcal{N}$ are therefore locally square integrable. Choose a localizing sequence $\sigma_m$ for $Z = 1 + Z_- \cdot \mathcal{N}$ for which the stopped stochastic integrals are square integrable. The stopped isometry, or Itô's formula for $Z^2$, and the bracket bound imply

$$\mathbb{E}[Z^2_{t\wedge\sigma_m}] \le 1 + C\int_0^t \mathbb{E}[Z^2_{s\wedge\sigma_m}]\,\mathrm{d}s \le e^{Ct}, \qquad 0 \le t \le T.$$

This uniform $L^2$ bound makes the stopped values uniformly integrable. Passing to the limit in conditional martingale identities proves that $Z$ is a true martingale on the closed horizon $[0,T]$, with $\mathbb{E}Z_t = 1$ and $\mathbb{E}Z_t^2 \le e^{Ct}$. Doob's inequality further gives $\mathbb{E}\sup_{t\le T} Z_t^2 \le 4e^{CT}$. All jumps satisfy $1 + \Delta\mathcal{N} > 0$. Since $[\mathcal{N}]_T < \infty$ almost surely, the stochastic-exponential product formula shows strict positivity throughout the horizon, including $T$. This proves equivalence of the terminal measures. □

**Corollary 3.5** (Clock-independent return-noise factor)**.** *The normalized dual density has the factorization*

$$\boxed{Z_t = Z_t^{\mathrm{ret}} Z_t^{\mathrm{reg}}, \qquad Z^{\mathrm{ret}} = \mathcal{E}(\mathcal{N}^{\mathrm{ret}}), \quad Z^{\mathrm{reg}} = \mathcal{E}(\mathcal{N}^{\mathrm{reg}}),} \tag{24}$$

*where*

$$\mathcal{N}^{\mathrm{ret}} = -p\gamma_{\alpha_-}\sqrt{c_{\alpha_-}} \cdot W + (f_{\alpha_-}(z) - 1) * (\mu^L - \nu^L),$$
$$\mathcal{N}^{\mathrm{reg}} = \sum_{i\neq j}(A_j/A_i - 1) \cdot M^{ij}.$$

*The superscripts* ret *and* reg *denote asset returns and regime transitions, respectively. For fixed market parameters, risk aversion and terminal horizon, $Z^{\mathrm{ret}}$ does not depend on the consumption dates, their masses, or the impatience parameter. In general $Z^{\mathrm{reg}}$ and $Z$ do depend on them.*

*Proof.* The return and regime local martingales have zero quadratic covariation: the chain martingales are purely discontinuous and there are no simultaneous return/regime jumps. The stochastic-exponential product formula gives (24). Each factor is a strictly positive square-integrable martingale by the bounded bracket-rate argument in Theorem 3.4. The coefficients of $\mathcal{N}^{\mathrm{ret}}$ involve only the statewise solution of (10), the market characteristics and $p$. In contrast, $\mathcal{N}^{\mathrm{reg}}$ contains the opportunity ratios obtained from the clock-dependent matching recursion. The factors can be statistically dependent because the return-noise coefficients depend on the regime. □

### 3.3 Return characteristics and the martingale stock

The Brownian motion under $\widehat{\mathbb{P}}$ is

$$W_t^{\widehat{\mathbb{P}}} = W_t + p\int_0^t \gamma_{\alpha_{s-}}\sqrt{c_{\alpha_{s-}}}\,\mathrm{d}s, \tag{25}$$

and the return-jump compensator is $f_{\alpha_{t-}}(z)F_{\alpha_{t-}}(\,\mathrm{d}z)\,\mathrm{d}t$. The characteristic triplet of $L$ relative to the original bounded truncation is

$$\widehat{b}_i = b_i - pc_i\gamma_i + \int h(z)(f_i(z)-1)F_i(\,\mathrm{d}z), \qquad \widehat{c}_i = c_i, \qquad \widehat{F}_i = f_iF_i. \tag{26}$$

These are the characteristics of the return process $L$. The regime compensators are proved separately in Section 4; the return formulas follow from the return-noise terms of (22). Relative to the bounded truncation, the excess total mean return vanishes:

$$\widehat{b}_i - r + \int (z-h(z))\widehat{F}_i(\,\mathrm{d}z) = (b_i - r) - pc_i\gamma_i + \int (zf_i(z)-h(z))F_i(\,\mathrm{d}z) = 0. \tag{27}$$

All integrals are legitimate under Assumption 2.1; near zero the drift integrands are $O(z^2)$, and away from zero the stated square-moment bounds give the required first moments. In the fully compensated representation,

$$R = \sqrt{c_{\alpha_-}}\cdot W^{\widehat{\mathbb{P}}} + z*(\mu^L - \widehat{\nu}^L), \qquad \mathrm{d}\langle R\rangle_t = \widehat{a}_{\alpha_{t-}}\,\mathrm{d}t, \quad \widehat{a}_i = c_i + \int z^2 f_i(z)F_i(\,\mathrm{d}z). \tag{28}$$

Thus $R$ is square integrable under $\widehat{\mathbb{P}}$, and $\bar{a} = \max_i \widehat{a}_i < \infty$. Apply the stopped isometry and Gronwall argument from Theorem 3.4 to $\widetilde{S} = S_0\mathcal{E}(R)$, now under $\widehat{\mathbb{P}}$. It gives

$$\mathbb{E}^{\widehat{\mathbb{P}}}[\widetilde{S}_t^2] \le S_0^2 e^{\bar{a}t}, \qquad \mathbb{E}^{\widehat{\mathbb{P}}}\sup_{t\le T}\widetilde{S}_t^2 \le 4S_0^2 e^{\bar{a}T}. \tag{29}$$

Thus the discounted fund is a square-integrable martingale.

**Lemma 3.6** (Admissible trading gains)**.** *Under $\widehat{\mathbb{P}}$, every nonnegative self-financing discounted process $Y = y_0 + \psi\cdot\widetilde{S}$, with a predictable semimartingale-integrable holding $\psi$, is a local martingale and therefore a supermartingale. The same assertion applies after adding a nonnegative tradable reserve to a trading-gain process, whenever the resulting sum is nonnegative and self-financing.*

*Proof.* This is the bounded-below stochastic-integral consequence of the Ansel–Stricker criterion (Ansel and Stricker, 1994). The integrator $\widetilde{S}$ is a local martingale and $\psi\cdot\widetilde{S} = Y - y_0 \ge -y_0$; these are the hypotheses needed to obtain a local martingale. Positivity then gives the supermartingale property by conditional Fatou. For the second assertion, sum the two predictable holdings and apply the same argument to the nonnegative combined portfolio. □

### 3.4 Budget domain and the Kallsen verification step

Following the intermediate-consumption duality framework of Mostovyi (2015), define the normalized consumption polar

$$\mathcal{D} = \left\{ \xi^{\mathrm{pol}} = (\xi_k^{\mathrm{pol}})_{k=1}^n \geq 0 : \mathbb{E} \sum_k a_k \mathfrak{c}_k \xi_k^{\mathrm{pol}} \leq 1 \text{ for every } (\pi, \mathfrak{c}) \in \mathcal{A}(1) \right\}.$$

Nonnegative deflators belong to this polar when they satisfy the consumption budget inequality. The budget argument below establishes this property for $Z$. Dual optimizers are identified on the consumption dates, up to $a_k\mathbb{P}$-null sets.

Put $\beta = (1-p)/p$. The conjugate is $V(y) = y^{-\beta}/\beta$ for $y > 0$, extended to $y = 0$ by its one-sided limit. Define

$$v(y) = \inf_{\xi^{\mathrm{pol}} \in \mathcal{D}} \mathbb{E} \sum_k a_k w_k^{1/p} V(y \xi_k^{\mathrm{pol}}), \qquad y > 0. \tag{30}$$

Fenchel's inequality gives $w_k U(\mathfrak{c}) \leq w_k^{1/p} V(y\xi_k^{\mathrm{pol}}) + y\mathfrak{c}\xi_k^{\mathrm{pol}}$.

**Theorem 3.7** (Admissibility, budgets and primal–dual optimality)**.** *Under Assumption 2.1, the plan constructed in Section 3 is optimal, with*

$$u(x) = \frac{A_{i_0}(0)}{1-p} x^{1-p}, \qquad y = u'(x) = A_{i_0}(0) x^{-p}. \tag{31}$$

*The vector $(Z_{t_k})_k$ belongs to $\mathcal{D}$, $y(Z_{t_k})_k$ is a dual optimizer at scale $y$, and*

$$w_k (\mathfrak{c}_k^*)^{-p} = y Z_{t_k}, \qquad \mathbb{E} \sum_k a_k \mathfrak{c}_k^* Z_{t_k} = x, \qquad u(x) = v(y) + xy. \tag{32}$$

*Optimal consumption is unique on the clock. The normalized density and the optimal fractions do not depend on initial wealth. In original units the consumption is $C_k^* = B_{t_k} \mathfrak{c}_k^*$ and the marginal state-price process is $yZ_t/B_t$.*

*Proof. Step 1: budget inequality for all feasible plans.* Under $\widehat{\mathbb{P}}$, discounted returns have zero total drift by (27). For any admissible plan,

$$\Gamma_t = X_t + \sum_{t_k \leq t} a_k \mathfrak{c}_k = x + \int_0^t X_{s-} \pi_s \, \mathrm{d}R_s$$

is a nonnegative self-financing gain process. Equivalent measures preserve semimartingale integrability, and Lemma 3.6 makes $\Gamma$ a $\widehat{\mathbb{P}}$ local martingale and supermartingale. In particular $\mathbb{E}^{\widehat{\mathbb{P}}} \sum a_k \mathfrak{c}_k \leq x$. Since each $\mathfrak{c}_k$ is $\mathcal{F}_{t_k}$-measurable, the density martingale converts this to $\mathbb{E} \sum a_k \mathfrak{c}_k Z_{t_k} \leq x$. This proves membership in $\mathcal{D}$. It also proves that $ZX + \sum_{t_k \leq t} a_k \mathfrak{c}_k Z_{t_k}$ is a nonnegative $\mathbb{P}$ local martingale and a supermartingale by the product rule; the accumulated payments are deflated at their own payment dates.

*Step 2: budget equality for the constructed plan.* Admissibility was proved by the explicit stochastic-exponential construction. By (28), the return bracket rate is bounded and $\gamma_i$ is bounded. Between dates, Itô's formula for $(X^*)^2$, localization and Gronwall's inequality apply. At a consumption date $\Delta(X^*)^2 \leq 0$, so the reductions do not increase the estimate. Consequently, with $C_X = \max_i \gamma_i^2 \widehat{a}_i$,

$$\mathbb{E}^{\widehat{\mathbb{P}}}[(X_t^*)^2] \leq x^2 e^{C_X t}, \qquad \mathbb{E}^{\widehat{\mathbb{P}}} \int_0^T (X_{s-}^* \gamma_{\alpha_{s-}})^2 \, \mathrm{d}\langle R \rangle_s < \infty.$$

The latter bound proves square-integrability of the gain integral directly, so $\Gamma^*$ is a true martingale. Since $X_T^* = 0$, its expectation gives the equality in (32). The deflated $\mathbb{P}$ gain process is also a true martingale, either by Bayes' formula and conditional budget equality or by its nonnegative local-martingale property and equality of its endpoint expectations.

*Step 3: marginal equality and finiteness.* Equation (18) and the definition of $D_T$ give $w_k(\mathfrak{c}_k^*)^{-p} = D_{t_k} = D_0 Z_{t_k}$. Here $D_0 = y$. Multiplying by $a_k \mathfrak{c}_k^*$ and summing shows

$$\mathbb{E}\sum_k a_k w_k (\mathfrak{c}_k^*)^{1-p} = yx = A_{i_0}(0) x^{1-p} < \infty. \tag{33}$$

In particular both the primal and conjugate objectives at the candidate are finite for either sign of $1-p$. Equality in Fenchel's inequality gives the stated formula for $u(x)$ if optimality is established.

*Step 4: verification and dual minimization.* For any feasible $\mathfrak{c}$, the supporting-line inequality for concave $U$ gives

$$\mathbb{E}\sum_k a_k w_k U(\mathfrak{c}_k) \le \mathbb{E}\sum_k a_k w_k U(\mathfrak{c}_k^*) + y\mathbb{E}\sum_k a_k Z_{t_k}(\mathfrak{c}_k - \mathfrak{c}_k^*) \le \mathbb{E}\sum_k a_k w_k U(\mathfrak{c}_k^*).$$

All right-hand terms are integrable by the budgets and (33). For $p > 1$ the left-hand objective is allowed to be $-\infty$; for $p < 1$ the inequality bounds it above by an integrable quantity.

For any $\xi^{\mathrm{pol}} \in \mathcal{D}$, apply Fenchel's inequality to $\mathfrak{c}^*$ and $y\xi^{\mathrm{pol}}$, and use its polar budget. It yields $\mathbb{E}\sum a_k w_k^{1/p} V(y\xi_k^{\mathrm{pol}}) \ge u(x) - xy$. At $\xi_k^{\mathrm{pol}} = Z_{t_k}$ there is equality at every date and in the budget, proving the dual assertion. For $p > 1$ the negative conjugate is bounded below pointwise by the integrable expression $w_k U(\mathfrak{c}_k^*) - y\mathfrak{c}_k^* \xi_k^{\mathrm{pol}}$, which makes its expectation well defined. The attainable consumption set is convex because nonnegative wealth strategies can be added and scaled. The polar $\mathcal{D}$ is convex as well. Strict concavity of $U$ and strict convexity of $V$ therefore give uniqueness on consumption dates. Homogeneity proves independence from $x$ and gives (31). The same construction and conditional budget argument, restarted at any deterministic time $t < T$ and regime $i$, identify the continuation value as $A_i(t)x^{1-p}/(1-p)$, with the pre-date coefficient when consumption at $t$ is still to occur. This identifies the backward coefficients as opportunity functions. □

*Remark* 3.8 (What is clock-dependent). The local fraction $\gamma_i$ solves (10) and is unaffected by the deterministic clock here, because price and regime jumps never coincide and investment coefficients have no additional state variable. But $A$ depends on all $t_k, a_k, w_k$, so $A_j/A_i$, $\widehat{Q}$ and $Z$ generally depend on the clock and on impatience. A simultaneous regime/price jump would also put opportunity ratios into the portfolio equation.

## 4 The transformed regime generator: a compensator proof

**Theorem 4.1** (Change of transition compensators)**.** *Under $\widehat{\mathbb{P}}$, the predictable compensator of $N^{ij}$ is*

$$\widehat{\Lambda}_t^{ij} = \int_0^t \mathbf{1}_{\{\alpha_{s-}=i\}} q_{ij} \frac{A_j(s)}{A_i(s)}\,\mathrm{d}s. \tag{34}$$

*Consequently $\alpha$ is a time-inhomogeneous Markov chain with generator, for almost every $t$,*

$$\boxed{\widehat{Q}(t) = \begin{pmatrix} -q_{12}z(t) & q_{12}z(t) \\ q_{21}/z(t) & -q_{21}/z(t) \end{pmatrix}, \qquad z(t) = \frac{A_2(t)}{A_1(t)}.} \tag{35}$$

*There is no atom in either compensator at a fixed consumption date. Although $z$ can have different left and right limits there, the chain has no deterministic-time transition.*

*Proof.* We give both the bracket computation and its compensation interpretation. At an $i \to j$ transition, wealth has no jump, and therefore

$$\Delta Z_t = Z_{t-}(h_{ij}(t) - 1), \qquad \Delta M_t^{ij} = 1.$$

Return-noise martingales have zero covariation with $M^{ij}$. Lemma 3.3 and local square integrability thus give

$$\mathrm{d}\langle M^{ij}, Z\rangle_t = Z_{t-}(h_{ij}(t) - 1)\mathbf{1}_{\{\alpha_{t-}=i\}} q_{ij}\,\mathrm{d}t. \tag{36}$$

The predictable Girsanov formula states that

$$M_t^{ij} - \int_0^t Z_{s-}^{-1}\,\mathrm{d}\langle M^{ij}, Z\rangle_s = N_t^{ij} - \widehat{\Lambda}_t^{ij}$$

is a $\widehat{\mathbb{P}}$ local martingale. All intensities on the right are bounded, so it is a true martingale.

For a direct verification, take a bounded nonnegative predictable $\psi$. The density martingale and optional sampling at transition times imply

$$\begin{aligned}
\mathbb{E}^{\widehat{\mathbb{P}}} \int_0^T \psi_s\,\mathrm{d}N_s^{ij} &= \mathbb{E}\int_0^T \psi_s Z_s\,\mathrm{d}N_s^{ij} \\
&= \mathbb{E}\int_0^T \psi_s Z_{s-} h_{ij}(s)\mathbf{1}_{\{\alpha_{s-}=i\}} q_{ij}\,\mathrm{d}s \\
&= \mathbb{E}^{\widehat{\mathbb{P}}}\int_0^T \psi_s h_{ij}(s)\mathbf{1}_{\{\alpha_{s-}=i\}} q_{ij}\,\mathrm{d}s.
\end{aligned}$$

The middle equality uses the physical compensator, since $\psi Z_- h_{ij}$ is predictable. Localization or monotone approximation justifies unbounded density factors. The last equality uses $\mathbb{E}[Z_T \mid \mathcal{F}_{s-}] = Z_{s-}$ for Lebesgue-almost every $s$. This proves (34) from the defining property of a compensator.

The bounded intensities depend only on time and the current state. Uniqueness for the associated finite-state martingale problem gives the Markov assertion. Finally $\mathbb{P}(\Delta\alpha_{t_k} \neq 0) = 0$ by the continuous physical compensators; equivalence transfers this property to $\widehat{\mathbb{P}}$. The identity (18) also shows directly that $\Delta Z_{t_k} = 0$. The finite set $\{t_1, \dots, t_n\}$ has zero mass under the compensators. □

*Remark* 4.2 (Indicators versus transition counts). The state indicator $I_t^1 = \mathbf{1}_{\{\alpha_t=1\}}$ satisfies $I_t^1 - I_0^1 = N_t^{21} - N_t^{12}$. Its drift is $\int_0^t (q_{21} I_{s-}^2 - q_{12} I_{s-}^1)\,\mathrm{d}s$ under $\mathbb{P}$, with $q$ replaced by $\widehat{q}$ under $\widehat{\mathbb{P}}$. Its martingale part is the difference of the two compensated transition counts.

*Remark* 4.3 (References for compensator changes). General compensator and change-of-measure theory is developed in Brémaud (1981), Jacod and Shiryaev (2003), and Protter (2005, Chapter III). Donnelly (2011) applies the predictable Girsanov formula to Markov-chain transition counts. The direct argument above proves the required compensator transformation in the present model.

# 5 Indifference prices of the guarantee and periodic fees

## 5.1 The pricing definition and its dual-martingale origin

Consider a fixed finite stream $\mathcal{L}_k \in \mathcal{F}_{t_k}$ and let $\mathcal{F}_k = \mathcal{L}_k / B_{t_k}$. Positive cash flows are liabilities of the seller. For a signed position $\theta$, define

$$J(v, \theta) = \sup \mathbb{E} \sum_{k=1}^{n} a_k w_k U(\mathfrak{c}_k),$$

where the feasible plans obey

$$X_t = v + \int_0^t X_{s-}\pi_s \, \mathrm{d}R_s - \sum_{t_k \le t} a_k \mathfrak{c}_k - \theta \sum_{t_k \le t} \mathcal{F}_k \ge 0. \tag{37}$$

The same integrability and zero-exposure conventions as in (7) apply. The dated net outflow $a_k\mathfrak{c}_k + \theta\mathcal{F}_k$ is settled in a single budget at $t_k$, after observation of the scheduled information, and wealth must be nonnegative after settlement. For the annuity this amount uses the already specified fee-before-benefit convention. The cash flows are fixed per contract: scaling the position by $\theta$ does not alter the fee rule, policy account or settlement rule. A scalar $b$ is an *indifference price* for one unit of this stream from initial wealth $x$ if

$$\boxed{J(x + \theta b, \theta) \le J(x, 0) = u(x) \quad \text{for every feasible signed position } \theta.} \tag{38}$$

Thus zero demand for the claim is optimal at the quote. Infeasible points are assigned value $-\infty$. This definition concerns marginal valuation at zero claim holdings.

Mostovyi and Siorpaes (2024, Theorem 4.2) establish a dual-martingale representation in their stochastic-utility, large-market setting: under their standing conditions, including bounded claim components, if $\mathbb{E}[\widehat{Y}_T(y)] = y$, then the marginal utility-based price is unique and

$$\mathfrak{p}(f) = \mathbb{E}\left[\frac{\widehat{Y}_T(y)}{y} f\right]. \tag{39}$$

Their Definition 4.1 gives the corresponding no-trade criterion. Here $yZ$ is the relevant dual optimizer, and $\mathbb{E}Z_T = 1$ has already been proved in Theorem 3.4. The result below adapts this representation to utility from finitely many consumption dates and to the dated annuity cash flows. The proof finances each cash flow at its payment date.

## 5.2 A finite-date representation with checkable superhedges

**Assumption 5.1** (Datewise superreplicating wealth processes)**.** For each payment date $t_k$, there exist a finite deterministic initial capital $w_{k,\mathrm{res}} \ge 0$ and a predictable $\widetilde{S}$-integrable holding $\psi^k$ such that the discounted self-financing wealth process

$$\mathcal{W}^k_t = w_{k,\mathrm{res}} + \int_0^{t\wedge t_k} \psi^k_s \, \mathrm{d}\widetilde{S}_s, \qquad 0 \le t \le T,$$

is nonnegative and satisfies

$$\mathcal{W}^k_{t_k} \ge |\mathcal{F}_k| \quad \mathbb{P}\text{-a.s.} \tag{40}$$

The process $(Z_{t\wedge t_k}\mathcal{W}^k_{t\wedge t_k})_{0\le t\le T}$ is a uniformly integrable $\mathbb{P}$ martingale. The reserve is liquidated at $t_k$ and its proceeds are held in discounted cash thereafter.

In particular, $\mathbb{E}[Z_{t_k}\mathcal{W}^k_{t_k}] = w_{k,\mathrm{res}}$. Since $\mathcal{W}^k$ is constant after $t_k$ and $Z$ is a true martingale, $Z\mathcal{W}^k$ is then a uniformly integrable $\mathbb{P}$ martingale on $[0, T]$.

A sufficient example is *cash-and-stock domination*: for deterministic finite constants $\ell_{k,0}, \ell_{k,1} \ge 0$,

$$|\mathcal{F}_k| \le \ell_{k,0} + \ell_{k,1}\widetilde{S}_{t_k}. \tag{41}$$

Take $\mathcal{W}^k_t = \ell_{k,0} + \ell_{k,1}\widetilde{S}_{t\wedge t_k}$. The true $\widehat{\mathbb{P}}$-martingale property of $\widetilde{S}$ gives the required uniform integrability of $Z\mathcal{W}^k$. This example permits unbounded dated payoffs and verifies Assumption 5.1 for the annuity by (6).

**Theorem 5.2** (Finite-date indifference-price representation)**.** *Assume the market and consumption model of Section 2, Assumptions 2.1 and 5.1. The stream has the unique indifference price*

$$\boxed{\mathfrak{p}(\mathcal{L}) = \mathbb{E}^{\widehat{\mathbb{P}}} \sum_{k=1}^{n} e^{-rt_k} \mathcal{L}_k = \mathbb{E} \sum_{k=1}^{n} Z_{t_k} e^{-rt_k} \mathcal{L}_k.} \tag{42}$$

*In particular, for a terminal payoff $\mathcal{H}$ the representation is*

$$\mathfrak{p}(\mathcal{H}) = \mathbb{E}[Z_T e^{-rT} \mathcal{H}]. \tag{43}$$

*Writing $y = u'(x) > 0$, the endowed value is locally finite and has the joint expansion*

$$J(x + d, \theta) = u(x) + y\{d - \theta \mathfrak{p}(\mathcal{L})\} + o(|d| + |\theta|). \tag{44}$$

*The price is independent of $x$ in this power-utility model, but it generally depends on risk aversion and the consumption clock.*

*Proof.* The dated superhedges permit explicit feasible perturbations for both signs of the position. Their true-martingale budgets cancel the cost of these hedges in the first-order expansion. Appendix A gives the complete budget, admissibility and dominated-convergence argument and proves uniqueness in (38). □

*Remark* 5.3 (Wealth domination and the Mostovyi–Siorpaes approach). Assumption 5.1 requires attainable superhedges and exact martingale budgets, rather than bounded payoffs. Mostovyi and Siorpaes (2024, Remark 2.2) relate bounded claims under stochastic utility to wealth-dominated claims through a numeraire change in finite markets. Here the dated reserves provide a direct finite-date proof. Relaxing the uniform-integrability requirement for general superreplicable payoffs is a further question.

### 5.3 The maturity-guarantee rider, fee income and full annuity value

**Corollary 5.4** (Explicit annuity indifference prices)**.** *For the contract (3)–(5), the guarantee price, the value of fee income and the net rider price are respectively*

$$\boxed{\begin{aligned} \mathcal{P}_0^{\mathrm{guar}} &= \mathbb{E}^{\widehat{\mathbb{P}}}[e^{-rT}(G - \mathcal{V}_T)^+], \\ \mathcal{P}_0^{\mathrm{fee}} &= \mathbb{E}^{\widehat{\mathbb{P}}} \sum_{k=1}^{n} e^{-rt_k} \mathcal{C}_k, \\ \mathcal{P}_0^{\mathrm{rider}} &= \mathcal{P}_0^{\mathrm{guar}} - \mathcal{P}_0^{\mathrm{fee}}. \end{aligned}} \tag{45}$$

*The nonnegative quantity $\mathcal{P}_0^{\mathrm{fee}}$ prices the stream of fee amounts as outflows; receiving those fees contributes $-\mathcal{P}_0^{\mathrm{fee}}$ to the insurer's liability price. The top-up and the signed net rider stream are priced by the same linear functional in (42). The net rider quote may have either sign and is interpreted through the no-trade criterion (38). Moreover,*

$$\mathbb{E}^{\widehat{\mathbb{P}}} \left[ e^{-rT} \mathcal{V}_T + \sum_{k=1}^{n} e^{-rt_k} \mathcal{C}_k \right] = P, \tag{46}$$

*and the full maturity benefit has indifference value*

$$\boxed{\mathcal{P}_0^{\mathrm{full}} = \mathbb{E}^{\widehat{\mathbb{P}}}[e^{-rT} \max\{G, \mathcal{V}_T\}] = P + \mathcal{P}_0^{\mathrm{rider}}.} \tag{47}$$

*Proof.* Equation (6) implies $|\mathcal{L}_k|/B_{t_k} \le \bar{\zeta}\nu_0\widetilde{S}_{t_k}$ for $k < n$, and $|\mathcal{L}_n|/B_T \le Ge^{-rT} + \bar{\zeta}\nu_0\widetilde{S}_T$. The individual fee streams, the bounded top-up and the full benefit also have cash-and-stock bounds of the form (41). Theorem 5.2 therefore proves the representations.

Define the predictable holding explicitly by

$$\nu_s = \sum_{k=1}^{n} \frac{\mathcal{V}_{t_{k-1}}}{S_{t_{k-1}}} \mathbf{1}_{\{t_{k-1}<s\le t_k\}}.$$

The coefficient is known at $t_{k-1}$, so the holding on $(t_{k-1}, t_k]$ is predictable and bounded by $\nu_0$. At $t_k$, the stock has no jump; selling the fee units therefore finances exactly $\mathcal{C}_k$ and gives the post-fee account. Discounted account bookkeeping gives

$$\frac{\mathcal{V}_t}{B_t} + \sum_{t_k\le t} \frac{\mathcal{C}_k}{B_{t_k}} = P + \int_0^t \nu_s \, \mathrm{d}\widetilde{S}_s.$$

Let $M_t^{\mathrm{acct}}$ denote the process on the left. The integral is square integrable, since

$$\mathbb{E}^{\widehat{\mathbb{P}}} \int_0^T \nu_s^2 \, \mathrm{d}\langle \widetilde{S}\rangle_s \le \nu_0^2 \mathbb{E}^{\widehat{\mathbb{P}}} \langle \widetilde{S}\rangle_T < \infty$$

by (29). Hence $M^{\mathrm{acct}}$ is a square-integrable martingale on the closed horizon. Taking expectations gives (46); adding the top-up gives (47). The account bound also gives $e^{-rT}\max\{G, \mathcal{V}_T\} \le Ge^{-rT} + \nu_0\widetilde{S}_T$, so the full benefit meets the pricing theorem's hypotheses separately. □

For a fixed parametric fee schedule $\zeta(\varpi)$, a zero-up-front-charge schedule would satisfy $\mathcal{P}_0^{\mathrm{rider}}(\varpi) = 0$. Existence and uniqueness of such a parameter are separate contract-design questions.

### 5.4 Exogenous random settlement on the deterministic date set

The fixed-maturity contract is the main model. For the variant below, define the potential account and fee sequence by (3) on the whole grid, and fix a stopping time $\tau \in \mathcal{T}$ in the existing market filtration. Its contractual rule is fixed independently of the pricing agent's trading, consumption and position size. The stopping time is prescribed exogenously, rather than optimized over exercise or surrender strategies. Let $G_k \ge 0$ be deterministic, with $\max_k G_k < \infty$. The actual contract follows the potential account only until $\tau$, charges the fee on that date, and then pays the post-fee account plus its top-up. No fees or benefits are paid afterward. The deterministic horizon $T$ and the agent's consumption clock remain unchanged after the contract settles.

**Corollary 5.5** (Indifference prices at random settlement)**.** *Under the same market, consumption-clock and fee assumptions as in Corollary 5.4, the dated net rider cash flows*

$$\mathcal{L}_k^\tau = \mathbf{1}_{\{\tau=t_k\}}(G_k - \mathcal{V}_{t_k})^+ - \mathbf{1}_{\{\tau\ge t_k\}}\mathcal{C}_k \tag{48}$$

*have the unique indifference price*

$$\boxed{\mathcal{P}_0^{\mathrm{rider},\tau} = \mathbb{E}^{\widehat{\mathbb{P}}}\left[e^{-r\tau}(G_\tau - \mathcal{V}_\tau)^+ - \sum_{t_k\le\tau} e^{-rt_k}\mathcal{C}_k\right],} \tag{49}$$

*where $G_\tau = G_k$ on $\{\tau = t_k\}$. The full benefit paid once at $\tau$ has indifference value*

$$\mathcal{P}_0^{\mathrm{full},\tau} = \mathbb{E}^{\widehat{\mathbb{P}}}[e^{-r\tau}\max\{G_\tau, \mathcal{V}_\tau\}] = P + \mathcal{P}_0^{\mathrm{rider},\tau}. \tag{50}$$

*Proof.* The events $\{\tau = t_k\}$ and $\{\tau \geq t_k\}$ are $\mathcal{F}_{t_k}$-measurable; in fact the latter event is known at the preceding grid date. Thus every cash flow is observable at its payment date. On $\{\tau = t_k\}$, the fee uses $\mathcal{V}_{t_k-}$ and the top-up uses $\mathcal{V}_{t_k}$, so the settlement fee is charged exactly once before the benefit. For every $k$,

$$\frac{|\mathcal{L}_k^\tau|}{B_{t_k}} \leq G_k e^{-rt_k} + \bar{\zeta}\nu_0 \widetilde{S}_{t_k}.$$

Theorem 5.2 applies directly to this adapted finite stream and proves (49); it allows dependence of $\tau$ on the market. The stopped account trades with the predictable holding $\nu_s \mathbf{1}_{\{s \leq \tau\}}$. The indicator is left-continuous and adapted, hence predictable. Since $\tau \leq T$ and $M^{\mathrm{acct}}$ is square integrable, optional sampling gives

$$\mathbb{E}^{\widehat{\mathbb{P}}}\left[e^{-r\tau}\mathcal{V}_\tau + \sum_{t_k \leq \tau} e^{-rt_k}\mathcal{C}_k\right] = P.$$

Adding the guarantee top-up proves the last equality in (50). Its interpretation as an indifference price also follows from Theorem 5.2: at each date its payment is bounded after discounting by $G_k e^{-rt_k} + \nu_0 \widetilde{S}_{t_k}$. All stopped terms are square integrable by (29) and the finite grid. □

The exogenous settlement rule is evaluated under $\widehat{\mathbb{P}}$; its distribution generally changes with the measure if it depends on regimes or fund returns. The corollary assumes that the agent's utility horizon remains $T$ and the filtration is the market filtration specified in Section 2.

# 6 Clock dependence and price invariance

All results in this section assume the market and filtration of Section 2 and Assumption 2.1. The physical market, initial regime, risk aversion $p$ and horizon $T$ are fixed, unless explicitly stated otherwise. Each clock has finitely many deterministic dates, positive masses and positive deterministic utility weights, with state-independent terminal weights as in (13). A comparison of pricing clocks holds the contractual cash flows fixed. In particular, adding a utility date does not add a fee or change the annuity account. Prices of dated streams are compared only when the payment dates belong to both consumption grids; a terminal claim always satisfies this requirement.

## 6.1 Opportunity ratios and contraction at a consumption date

Write $z(t) = A_2(t)/A_1(t)$ and $\Delta_\eta = \eta_2 - \eta_1$. The ratios are taken before or after a date as indicated, and $z(T-) = 1$.

**Proposition 6.1** (Ratio dynamics and fixed-date contraction)**.** *On every open interval without consumption,*

$$-z'(t) = \Phi(z(t)), \qquad \Phi(z) = q_{21} + (\Delta_\eta + q_{12} - q_{21})z - q_{12}z^2. \tag{51}$$

*At a nonterminal consumption date with $d_k = a_k w_k^{1/p} > 0$,*

$$z(t_k-) = \left(\frac{d_k + A_1(t_k)^{1/p} z(t_k)^{1/p}}{d_k + A_1(t_k)^{1/p}}\right)^p. \tag{52}$$

*If $z(t_k) > 1$, then $1 < z(t_k-) < z(t_k)$; if $z(t_k) < 1$, then $z(t_k) < z(t_k-) < 1$. Equality with one is preserved. With post-date coefficients fixed, $z(t_k-)$ is strictly decreasing in $d_k$ when $z(t_k) > 1$, and strictly increasing when $z(t_k) < 1$. Its limits as $d_k \downarrow 0$ and $d_k \uparrow \infty$ are respectively $z(t_k)$ and one.*

*For every $0 \le t < T$, on either side of any nonterminal date,*

$$\operatorname{sgn}(z(t) - 1) = \operatorname{sgn}(\Delta_\eta). \tag{53}$$

*Thus, if $\Delta_\eta > 0$, $\widehat{q}_{12} > q_{12}$ and $\widehat{q}_{21} < q_{21}$ away from fixed dates; both inequalities reverse if $\Delta_\eta < 0$.*

*Proof.* Differentiate $A_2/A_1$ using (14); the two equations give (51). Dividing the two matching conditions gives (52). Put $u = A_1(t_k)^{1/p} > 0$ and $\upsilon = z(t_k)^{1/p}$. The expression inside the power is $(d_k + u\upsilon)/(d_k + u)$, strictly between one and $\upsilon$ when $\upsilon \neq 1$. Its derivative with respect to $d_k$ is $u(1-\upsilon)/(d_k+u)^2$. Since $p > 0$, all contraction, monotonicity and limit assertions follow.

For the sign assertion, use backward elapsed time on any interval: the ratio solves $\dot{z} = \Phi(z)$, with $\Phi(1) = \Delta_\eta$. The positive solution exists on the interval by the opportunity construction. If $\Delta_\eta > 0$, a solution starting at one immediately enters $(1, \infty)$, and a solution in that interval cannot cross one: at a first such crossing its derivative would have to be nonpositive, whereas it equals $\Delta_\eta > 0$. The argument for $\Delta_\eta < 0$ is the same with inequalities reversed. If $\Delta_\eta = 0$, uniqueness for the scalar locally Lipschitz equation preserves the constant solution one. Starting at $T-$ and applying the contraction at each of the finitely many dates proves (53). Formula (35) gives the intensity assertions. □

The contraction is a comparison of the pre- and post-consumption opportunity ratios at a fixed date. It creates neither a density jump nor a chain transition at that date. Moreover, because its size also depends on the level $A_1(t_k)$, it does not by itself establish an ordering between arbitrary pairs of consumption calendars.

### 6.2 A strict comparison and a priced witness of clock dependence

Consider a terminal-only clock, with terminal coefficient $A_i^0(T-) = a^p w$, and a second clock with the same terminal parameters and one additional consumption date $s \in (0, T)$ with mass-weight parameter $d_s > 0$. Write $A^+$, $z^+$ and $\widehat{\mathbb{P}}^+$ for the second clock, and use superscript 0 for the terminal-only clock. In particular the two clocks have the same $A$ and $z$ on $(s, T)$; at $s$, the second clock applies the matching map to the terminal-only continuation coefficients.

**Theorem 6.2** (Adding a consumption date changes a regime price)**.** *If $\Delta_\eta > 0$, then*

$$1 < z^+(t) < z^0(t), \qquad \widehat{q}_{12}^{+}(t) < \widehat{q}_{12}^{0}(t), \qquad \widehat{q}_{21}^{+}(t) > \widehat{q}_{21}^{0}(t), \quad 0 \le t < s. \tag{54}$$

*For $\Delta_\eta < 0$, $z^0(t) < z^+(t) < 1$ and both intensity inequalities reverse. For either initial regime and every $u \in (0, T]$,*

$$\operatorname{sgn}\left(\widehat{\mathbb{P}}^+(\alpha_u = 2) - \widehat{\mathbb{P}}^0(\alpha_u = 2)\right) = -\operatorname{sgn}(\Delta_\eta), \qquad \Delta_\eta \neq 0. \tag{55}$$

*In particular the bounded terminal claim $\mathcal{H} = \mathbf{1}_{\{\alpha_T = 2\}}$ has strictly different indifference prices under the two clocks. This gives a strictly clock-dependent price despite the common return-noise factor.*

*Proof.* The strict matching contraction gives $z^+(s-) < z^0(s)$ when $\Delta_\eta > 0$. On $[0, s)$ the two ratios solve the same scalar equation (51). More explicitly, put $\Delta z(t) = z^0(t) - z^+(t)$. Subtracting the equations and solving the resulting linear equation gives

$$\Delta z(t) = (z^0(s) - z^+(s-)) \exp\left( \int_t^s [\Delta_\eta + q_{12} - q_{21} - q_{12}(z^0(v) + z^+(v))] \, \mathrm{d}v \right).$$

The integral is finite by positivity and boundedness of the opportunity coefficients. The exponential is strictly positive, so the order is strict even at $t = 0$. Proposition 6.1 gives the lower bound one. This proves (54); the case $\Delta_\eta < 0$ follows identically with reversed inequalities.

Let $\lambda^\ell(t) = \widehat{q}^\ell_{12}(t)$, $\mu^\ell(t) = \widehat{q}^\ell_{21}(t)$ and $m^\ell(t) = \widehat{\mathbb{P}}^\ell(\alpha_t = 2)$ for $\ell \in \{0, +\}$. The compensator result gives the forward equation

$$(m^\ell)' = \lambda^\ell(1 - m^\ell) - \mu^\ell m^\ell, \qquad m^\ell(0) = \mathbf{1}_{\{i_0=2\}}.$$

These absolutely continuous solutions satisfy $0 < m^\ell(t) < 1$ for $t > 0$, since both transition intensities are positive and bounded, with positive lower bounds on this finite horizon. The difference $D = m^+ - m^0$ therefore solves, almost everywhere,

$$D' = -(\lambda^+ + \mu^+)D + (\lambda^+ - \lambda^0)(1 - m^0) - (\mu^+ - \mu^0)m^0, \qquad D(0) = 0.$$

Denote the forcing term on the right by $\Xi(t)$. The variation-of-constants formula is

$$D(u) = \int_0^u \exp\Big(-\int_v^u [\lambda^+(a) + \mu^+(a)]\,\mathrm{d}a\Big)\,\Xi(v)\,\mathrm{d}v.$$

For $\Delta_\eta > 0$, $\Xi$ is strictly negative on $(0, s)$ and zero on $(s, T)$. Every $u > 0$ has $\min\{u, s\} > 0$, so the integral is strictly negative, including after $s$. For $\Delta_\eta < 0$ it is strictly positive. Values of the intensities at $s$ do not affect this integral equation; there is no point mass or probability jump at that date. Finally Theorem 5.2 applies to the bounded terminal claim for both clocks, and its prices are $e^{-rT}m^\ell(T)$. □

The strict ordering concerns the regime digital. For the maturity guarantee and net rider, the clock effect also involves the fund path and fee income; its sign is contract dependent.

### 6.3 Invariance of the measure and of individual prices

**Proposition 6.3** (Common scaling and equal opportunity growth)**.** *The following invariance statements hold.*

1. *On a fixed consumption grid, replace $d_k = a_k w_k^{1/p}$ by $\widetilde{d}_k = \lambda d_k$ for all $k$, with a common $\lambda > 0$, leaving the market and $p$ unchanged. Then $\widetilde{A}_i = \lambda^p A_i$ on every interval and on both sides of every date. Hence the normalized density and all prices of fixed streams covered by Theorem 5.2 are unchanged. In particular this applies to common scaling of all clock masses, or to common scaling of all utility weights. The latter is merely a normalization of the utility objective; changing impatience generally changes relative weights and is not such a common scaling.*
2. *If $\eta_1 = \eta_2$, then $A_1 = A_2$ for every finite clock with horizon $T$, $\widehat{Q} = Q$, and $Z = Z^{\mathrm{ret}}$. For fixed market, $p$ and $T$, the selected measure is independent of the clock, including its dates and impatience weights. This assertion alone does not give invariance across different risk aversions, because the return-noise factor may depend on $p$. All covered fixed cash-flow prices are therefore clock independent.*
3. *Conversely, independence of the selected measure from every finite clock with horizon $T$, or independence of prices of all bounded terminal claims from every such clock, implies $\eta_1 = \eta_2$.*

*Proof.* For the first assertion, the terminal pre-date value is $d^p_n$, so it scales by $\lambda^p$. Linear propagation preserves this scaling, and

$$(\lambda d_k + (\lambda^p A_i)^{1/p})^p = \lambda^p (d_k + A_i^{1/p})^p$$

preserves it at every matching step. All opportunity ratios, and thus both stochastic-logarithm coefficients of the normalized density, are unchanged. Scaling weights by a common factor $c > 0$ corresponds to $\lambda = c^{1/p}$. Scaling masses by $c$ corresponds to $\lambda = c$.

For the second assertion put $\eta_1 = \eta_2 = \eta$. Since $Q\mathbf{1} = 0$, $e^{(Q+\eta I)v}\mathbf{1} = e^{\eta v}\mathbf{1}$. State-independent terminal data and matching preserve equality of the components throughout the backward recursion. The regime factor is one, while the return factor is clock independent by Corollary 3.5. Price invariance follows from the representation theorem. For the converse, if $\Delta_\eta \neq 0$, Theorem 6.2 supplies two clocks and a bounded terminal claim with different prices, and hence different selected measures. □

**Proposition 6.4** (Replication and claim-specific invariance)**.** *Consider any collection of the selected pricing measures, possibly obtained from different clocks or risk aversions. For every clock and risk-aversion parameter in this collection, assume that the hypotheses of Theorem 5.2 (the finite-date indifference-price representation) hold for the same fixed discounted cash-flow stream $(\mathcal{F}_k)$. In particular, its payment dates belong to every consumption grid, the superhedge and martingale requirements in Assumption 5.1 hold, and Assumption 2.1 holds at each risk-aversion parameter. Suppose also that a single predictable trading strategy delivers the stream from deterministic cost $v$, with discounted value*

$$Y_t = v + \int_0^t \psi_s \, \mathrm{d}\widetilde{S}_s - \sum_{t_k \le t} \mathcal{F}_k, \qquad Y_T = 0,$$

*and that $\psi \cdot \widetilde{S}$ is a true martingale under each of these measures. Then the indifference price is $v$ under each measure. A sufficient condition for this martingale requirement is that $\psi$ is bounded by a deterministic constant on $[0, T]$, since (29) holds under each selected measure.*

*In particular, the stream consisting of periodic account fees together with the terminal residual account value has price $P$ for every such comparison, even when the individual fee and guarantee prices vary.*

*Proof.* Under any selected measure the terminal replication identity and the true-martingale property give $\mathbb{E}^{\widehat{\mathbb{P}}} \sum_k \mathcal{F}_k = v$. Theorem 5.2 identifies this expectation with the indifference price. For bounded $\psi$, the gain is square integrable because $\mathbb{E}^{\widehat{\mathbb{P}}} \int_0^T \psi_s^2 \, \mathrm{d}\langle \widetilde{S} \rangle_s < \infty$ by (29). For the account stream use the bounded predictable holding $\nu$ from the proof of (46), liquidating the remaining account at $T$. The account and fee bounds verify the pricing hypotheses for each clock, and the initial cost is $P$. □

Replication with a true-martingale budget gives claim-specific invariance. Equal opportunity growth gives invariance of the entire selected measure. Both mechanisms apply in an incomplete regime-switching market.

# 7 Diffusion and jump-diffusion specifications

## 7.1 Regime-switching Black–Scholes model

Suppose $F_i = 0$, $c_i = \sigma_i^2 > 0$, $b_i = \mu_i$, so

$$\frac{\mathrm{d}S_t}{S_t} = \mu_{\alpha_{t-}} \, \mathrm{d}t + \sigma_{\alpha_{t-}} \, \mathrm{d}W_t.$$

Then all moment assumptions hold and

$$\gamma_i = \frac{\mu_i - r}{p\sigma_i^2}, \qquad \eta_i = \frac{1-p}{2p} \left( \frac{\mu_i - r}{\sigma_i} \right)^2. \tag{56}$$

Let $\theta_i = (\mu_i - r)/\sigma_i$. The complete density is

$$Z = \mathcal{E}\left(-\theta_{\alpha_-} \cdot W + \sum_{i \neq j} (A_j/A_i - 1) \cdot M^{ij}\right).$$

Its Brownian coefficient is independent of $p$, while the regime coefficients need not be. Under $\widehat{\mathbb{P}}$, $\mathrm{d}S_t/S_t = r\,\mathrm{d}t + \sigma_{\alpha_{t-}}\,\mathrm{d}W_t^{\widehat{\mathbb{P}}}$ and the generator is (35).

**Proposition 7.1** (Incompleteness and a special cancellation)**.** *In the full observed filtration with $q_{12}, q_{21} > 0$, this diffusion market is incomplete. If $\eta_1 = \eta_2$, the opportunity coefficients coincide, $\widehat{Q} = Q$, and the normalized density reduces to the Brownian density. The equality of opportunity coefficients is a symmetry of the investment opportunities.*

*Proof.* In the diffusion model, replacing the regime coefficients in the density by any bounded deterministic transition multipliers bounded away from zero, while retaining the Brownian coefficient $-\theta_i$, produces a positive true martingale by the argument of Theorem 3.4. Every such measure makes $\widetilde{S}$ a martingale, and different multipliers give different transition probabilities. Thus there are multiple equivalent martingale measures. More directly, a nonconstant compensated-chain martingale has jumps, whereas discounted wealth generated by continuous traded prices is continuous; the chain risk cannot be spanned. If the two $\eta_i$ coincide, the linear evolution and the state-independent matching map preserve equality of $A_1, A_2$ backward from $T-$. □

**Corollary 7.2** (Preference and clock invariance in the diffusion model)**.** *If the squared market prices of risk coincide, $\theta_1^2 = \theta_2^2$, the normalized pricing density is the same for every $p > 0$, $p \neq 1$, and every deterministic finite consumption clock with horizon $T$:*

$$Z = \mathcal{E}(-\theta_{\alpha_-} \cdot W).$$

*Consequently every fixed cash-flow stream satisfying the pricing theorem for the clocks being compared has the same indifference price for all these preferences and clocks. This includes the fixed annuity contract when its payment dates belong to each consumption grid. The market remains incomplete.*

*Proof.* Equation (56) gives $\eta_1 = \eta_2$ for each $p$, so Proposition 6.3 applies to the clock comparison. The Brownian coefficient is $-p\gamma_i\sigma_i = -\theta_i$, independent of $p$, and the regime factor is one. Hence the entire normalized density, not only its Brownian coefficient, is common to all the comparisons. The representation in Theorem 5.2 proves price invariance. The preceding incompleteness argument still applies. □

### 7.2 Regime-switching Merton jump diffusion

In regime $i$, let the return jump be $Y_i = e^{J_i} - 1$, where $J_i \sim N(\mu_i^J, (\sigma_i^J)^2)$, $\sigma_i^J > 0$, and the jump rate is $\lambda_i > 0$. With $\kappa_i^J = \mathbb{E}(e^{J_i} - 1)$, write

$$\frac{\mathrm{d}S_t}{S_{t-}} = (\mu_i - \lambda_i\kappa_i^J)\,\mathrm{d}t + \sigma_i\,\mathrm{d}W_t + \int_{(-1,\infty)} z\,\mu^L(\mathrm{d}t, \mathrm{d}z), \qquad \alpha_{t-} = i. \tag{57}$$

The return jumps form one regime-modulated marked point process with compensator $\lambda_i f_{Y_i}(z)\,\mathrm{d}z\,\mathrm{d}t$, with

$$f_{Y_i}(z) = \frac{1}{(1+z)\sigma_i^J\sqrt{2\pi}} \exp\left(-\frac{(\log(1+z) - \mu_i^J)^2}{2(\sigma_i^J)^2}\right), \quad z > -1.$$

Because all needed jump moments exist, we can alternatively use the full-compensation convention $h(z) = z$. Here the full-compensation representation is integrable by the lognormal moment bounds. In this convention $b_i = \mu_i$, $c_i = \sigma_i^2$, and $F_i(\mathrm{d}z) = \lambda_i f_{Y_i}(z)\,\mathrm{d}z$.

The stationary equation becomes

$$\mu_i - r - p\sigma_i^2\gamma_i + \lambda_i \mathbb{E}\left[Y_i\{(1+\gamma_i Y_i)^{-p} - 1\}\right] = 0. \tag{58}$$

Fractions preserving strictly positive wealth lie in $[0,1]$, since the jump support is the whole interval $(-1,\infty)$. A negative fraction fails positivity at sufficiently large positive jumps, and a fraction above one fails it near $-1$. We retain the interior regime $0 < \gamma_i < 1$. A convenient sufficient condition is

$$\mu_i - r > 0, \qquad \mu_i - r - p\sigma_i^2 + \lambda_i \mathbb{E}[Y_i\{e^{-pJ_i} - 1\}] < 0.$$

Strict monotonicity of the left side of (58), continuity on $[0,1]$ and these endpoint signs give a unique interior root. In this case $1 + \gamma_i Y_i \geq 1 - \gamma_i > 0$; normal exponential moments verify all of Assumption 2.1. These inequalities specify the interior parameter region used here.

Define $\mathcal{M}_i = \mathbb{E}[(1+\gamma_i Y_i)^{-p}]$. Under $\widehat{\mathbb{P}}$,

$$\begin{aligned} \widehat{\lambda}_i &= \lambda_i \mathcal{M}_i, & \widehat{f}_{Y_i}(z) &= \frac{(1+\gamma_i z)^{-p}}{\mathcal{M}_i} f_{Y_i}(z), \\ \widehat{\kappa}_i^J &= \int z \widehat{f}_{Y_i}(z)\,\mathrm{d}z, & \widehat{d}_i &= r - \widehat{\lambda}_i \widehat{\kappa}_i^J = \mu_i - \lambda_i \kappa_i^J - p\sigma_i^2\gamma_i. \end{aligned} \tag{59}$$

Here $\widehat{d}_i$ is the drift in the *uncompensated* return representation; in the fully compensated representation the return drift is $r$. The volatility remains $\sigma_i$, the jump law is generally no longer lognormal, and the chain generator is (35), usually time dependent. Preference dependence enters both the return-jump tilt and the regime tilt.

### 7.3 Collected characteristics under the pricing measure

The following formulas collect the financial and regime transformations. All coefficients use the pre-transition regime. At a deterministic consumption date the density is continuous, neither transition compensator has an atom, and the rates used immediately after the date are computed from the post-consumption opportunity functions. At $T$ use the pre-terminal limit.

**Two regime-transition intensities.** For $t$ between consumption dates,

$$\boxed{\begin{aligned} \widehat{q}_{12}(t) &= q_{12}\frac{A_2(t)}{A_1(t)}, & \widehat{q}_{21}(t) &= q_{21}\frac{A_1(t)}{A_2(t)}, \\ \widehat{\lambda}_t^{12} &= \mathbf{1}_{\{\alpha_{t-}=1\}}\widehat{q}_{12}(t), & \widehat{\lambda}_t^{21} &= \mathbf{1}_{\{\alpha_{t-}=2\}}\widehat{q}_{21}(t). \end{aligned}} \tag{60}$$

Thus $N^{12} - \int_0^{\cdot} \widehat{\lambda}_s^{12}\,\mathrm{d}s$ and $N^{21} - \int_0^{\cdot} \widehat{\lambda}_s^{21}\,\mathrm{d}s$ are $\widehat{\mathbb{P}}$ martingales. The $\widehat{\lambda}^{ij}$ here are transition-count intensities, distinct from the return-jump intensities $\widehat{\lambda}_i$ in the Merton model below.

**General return characteristics.** Relative to the bounded truncation $h$ in (1), put $f_i(z) = (1+\gamma_i z)^{-p}$. The local characteristics of the undiscounted return $L$ under $\widehat{\mathbb{P}}$ are

$$\boxed{\begin{aligned} \widehat{b}_i &= b_i - pc_i\gamma_i + \int h(z)[f_i(z) - 1]F_i(\mathrm{d}z), \\ \widehat{c}_i &= c_i, \qquad \widehat{F}_i(\mathrm{d}z) = f_i(z)F_i(\mathrm{d}z), \\ \widehat{b}_i + \int [z - h(z)]\widehat{F}_i(\mathrm{d}z) &= r, \qquad i = 1,2. \end{aligned}} \tag{61}$$

In particular the predictable characteristic processes are $\int_0^t \widehat{b}_{\alpha_{s-}}\,\mathrm{d}s$, $\int_0^t c_{\alpha_{s-}}\,\mathrm{d}s$, and $\widehat{\nu}^L(\mathrm{d}t,\mathrm{d}z) = \widehat{F}_{\alpha_{t-}}(\mathrm{d}z)\,\mathrm{d}t$. For the discounted return $R$, replace $\widehat{b}_i$ by $\widehat{b}_i - r$.

For completeness, let $\xi = \log(S/S_0)$ and let $\chi$ be a bounded truncation in the log-jump coordinate. Its local characteristics are

$$\boxed{\begin{aligned} &\widehat{b}_i^{\xi} = r - \tfrac{1}{2}c_i + \int [\chi(\log(1+z)) - z]\widehat{F}_i(\mathrm{d}z), \\ &\widehat{c}_i^{\xi} = c_i, \qquad \widehat{F}_i^{\xi}(D) = \int \mathbf{1}_{\{\log(1+z)\in D\}}\widehat{F}_i(\mathrm{d}z), \quad D \subseteq \mathbb{R}\setminus\{0\}. \end{aligned}} \tag{62}$$

The last line is the push-forward of the return-jump measure, and the total-return identity in (61) yields the log drift. This distinguishes return characteristics from log-price characteristics explicitly.

**Black–Scholes specialization.** With $\theta_i = (\mu_i - r)/\sigma_i$, the pricing-measure dynamics are

$$\boxed{\begin{aligned} W_t^{\widehat{\mathbb{P}}} &= W_t + \int_0^t \theta_{\alpha_{s-}}\,\mathrm{d}s, \\ \frac{\mathrm{d}S_t}{S_t} &= r\,\mathrm{d}t + \sigma_{\alpha_{t-}}\,\mathrm{d}W_t^{\widehat{\mathbb{P}}}, \\ \mathrm{d}\xi_t &= (r - \tfrac{1}{2}\sigma_{\alpha_{t-}}^2)\,\mathrm{d}t + \sigma_{\alpha_{t-}}\,\mathrm{d}W_t^{\widehat{\mathbb{P}}}, \end{aligned}} \tag{63}$$

with regime intensities (60). Although the Brownian tilt is clock independent, the regime law under the pricing measure generally is not.

**Merton specialization.** For $J_i \sim N(\mu_i^J, (\sigma_i^J)^2)$ and $0 < \gamma_i < 1$, define $\omega_i(j) = (1 + \gamma_i(e^j - 1))^{-p}$ and $\mathcal{M}_i = \mathbb{E}[\omega_i(J_i)]$. If $\varphi_i$ is the original normal density, then

$$\boxed{\begin{aligned} \widehat{\lambda}_i &= \lambda_i\mathcal{M}_i, \qquad \widehat{\varphi}_i(j) = \frac{\omega_i(j)}{\mathcal{M}_i}\varphi_i(j), \\ \widehat{\kappa}_i^J &= \frac{\mathbb{E}[(e^{J_i}-1)\omega_i(J_i)]}{\mathcal{M}_i}, \\ \frac{\mathrm{d}S_t}{S_{t-}} &= (r - \widehat{\lambda}_i\widehat{\kappa}_i^J)\,\mathrm{d}t + \sigma_i\,\mathrm{d}W_t^{\widehat{\mathbb{P}}} + \int (e^j - 1)\mu^{\xi}(\mathrm{d}t,\mathrm{d}j), \\ \mathrm{d}\xi_t &= (r - \tfrac{1}{2}\sigma_i^2 - \widehat{\lambda}_i\widehat{\kappa}_i^J)\,\mathrm{d}t + \sigma_i\,\mathrm{d}W_t^{\widehat{\mathbb{P}}} + \int j\,\mu^{\xi}(\mathrm{d}t,\mathrm{d}j), \\ \widehat{\nu}^{\xi}(\mathrm{d}t,\mathrm{d}j) &= \widehat{\lambda}_i\widehat{\varphi}_i(j)\,\mathrm{d}j\,\mathrm{d}t, \qquad \alpha_{t-} = i. \end{aligned}} \tag{64}$$

Here $\mu^{\xi}$ records log-price jumps. These finite-activity equations use uncompensated jump sums. Their mean compensation is already included in the displayed drifts. The Brownian coefficient, jump intensity and jump-mark law are clock independent; the chain rates still use (60).

## 8 Numerical illustrations

We illustrate indifference pricing of a variable annuity with a guaranteed minimum maturity benefit (GMMB). Following Ai et al. (2024), we take $P = 100$, $G = Pe^{0.01T}$ and fees deducted at consumption dates:

$$\mathcal{C}_k = \begin{cases} \left(1 - e^{-0.025(t_k - t_{k-1})}\right)\mathcal{V}_{t_k-}, & \text{if } \mathcal{V}_{t_k-} < 130, \\ \left(1 - e^{-0.02(t_k - t_{k-1})}\right)\mathcal{V}_{t_k-}, & \text{if } \mathcal{V}_{t_k-} \geq 130. \end{cases}$$

Here $\mathcal{V}_{t_k-}$ is the pre-fee account value. We use the market parameters in Table 1, $q_{12} = 0.4$, $q_{21} = 0.5$ and $\delta = 0.02$. Algorithm 1 summarizes the pricing procedure.

With time in years and $T = 15$, the annual schedule is $t_k = k$, $k = 1, \ldots, 15$, and the quarterly schedule is $t_k = k/4$, $k = 1, \ldots, 60$. Each schedule determines consumption, fee deductions and monitoring of the threshold at 130. The comparison therefore changes both the utility clock and the contract; the differing physical-measure benchmarks reflect the latter. Unlike the fixed-claim comparisons in Section 6, these price plots do not isolate a utility-clock effect.

The fee rates are illustrative and are not calibrated to $\mathcal{P}_0^{\text{rider}} = 0$. Fee income exceeds the guarantee value in these examples, giving negative net rider prices.

Table 1: Regime-switching model parameter values.

| Model | Parameters |
|---|---|
| Black–Scholes | $r = 0.05$, $\mu_1 = 0.14$, $\mu_2 = 0.06$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$ |
| Merton | $r = 0.05$, $\mu_1 = 0.14$, $\mu_2 = 0.06$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\mu_1^J = -0.1$, $\mu_2^J = -0.12$, $\sigma_1^J = 0.05$, $\sigma_2^J = 0.07$ |

**Algorithm 1:** Indifference pricing under a regime-switching jump-diffusion model.

**Input:** Market parameters $(r, b_i, c_i, F_i, q_{12}, q_{21})$, risk aversion $p$, impatience $\delta$, consumption dates $\mathcal{T} = \{t_1, \ldots, t_n = T\}$ with weights $a_k$, initial account $P$, guarantee $G$, fee rules $\zeta_k$, number of paths $M$.

**Output:** Indifference prices $\mathcal{P}_0^{\text{guar}}$, $\mathcal{P}_0^{\text{fee}}$, and $\mathcal{P}_0^{\text{rider}}$ in Equation (45).

`// Step 1: Compute Optimal Investment Fractions (`$\gamma_i$`) and Growth Coefficients (`$\eta_i$`)`

**for** $i \in \{1, 2\}$ **do**

  Solve the stationary first-order condition (FOC) in Equation (10) for $\gamma_i$:

    $(b_i - r) - pc_i\gamma_i + \int \{z(1+\gamma_i z)^{-p} - h(z)\} F_i(\mathrm{d}z) = 0$;

  Evaluate the local objective function $g_i(\gamma_i)$ in Equation (9):

    $g_i(\gamma_i) = (b_i - r)\gamma_i - \frac{p}{2}c_i\gamma_i^2 + \int \left\{ \frac{(1+\gamma_i z)^{1-p}-1}{1-p} - \gamma_i h(z) \right\} F_i(\mathrm{d}z)$;

  Compute opportunity growth coefficient $\eta_i$ in Equation (12): $\eta_i \leftarrow (1-p)g_i(\gamma_i)$;

**end**

Construct matrix $K = Q + \mathrm{diag}(\eta_1, \eta_2)$ in Equation (12).

`// Step 2: Backward Recursion for Opportunity Coefficients `$A_i(t)$

Initialize $A_i(T-) = a_n^p w_n$ for $i \in \{1, 2\}$ where $w_n = e^{-[\delta - r(1-p)]T}$;

Set $t_0 = 0$ and propagate from $T$ to $t_{n-1}$: $A(t_{n-1}) = e^{K(T-t_{n-1})}A(T-)$;

**for** $k = n-1$ *down to* $1$ **do**

  $d_k \leftarrow a_k w_k^{1/p}$ where $w_k = e^{-[\delta - r(1-p)]t_k}$;

  Apply matching condition at consumption date $t_k$: $A_i(t_k-) \leftarrow \left[d_k + A_i(t_k)^{1/p}\right]^p$ for $i \in \{1, 2\}$;

  Solve the ODE backward from $t_k$ to $t_{k-1}$: $A(t_{k-1}) = e^{K(t_k - t_{k-1})}A(t_k-)$;

**end**

Construct the piecewise function $A(t)$ on $[0, T)$ and the regime-transition intensity ratio $z(t) = \frac{A_2(t)}{A_1(t)}$.

`// Step 3: Forward Monte Carlo Simulation under `$\widehat{\mathbb{P}}$

**for** $m = 1$ *to* $M$ **do**

  Simulate regime chain $\alpha_t$ under $\widehat{\mathbb{P}}$ using time-dependent intensities:

    $\widehat{q}_{12}(t) = q_{12}z(t), \quad \widehat{q}_{21}(t) = \frac{q_{21}}{z(t)}$;

  Simulate risky asset $S_t$ under dual characteristics $\widehat{b}_i, \widehat{c}_i, \widehat{F}_i$;

  Initialize policy account $\mathcal{V}_0^{(m)} \leftarrow P$;

  **for** $k = 1$ *to* $n$ **do**

    Evolve pre-fee account value to $t_k-$: $\mathcal{V}_{t_k-}^{(m)} = \mathcal{V}_{t_{k-1}}^{(m)} \frac{S_{t_k}}{S_{t_{k-1}}}$;

    Compute periodic fee: $\mathcal{C}_k^{(m)} = \zeta_k(\alpha_{t_k}, \mathcal{V}_{t_k-}^{(m)})\mathcal{V}_{t_k-}^{(m)}$;

    Update account post-fee: $\mathcal{V}_{t_k}^{(m)} = \mathcal{V}_{t_k-}^{(m)} - \mathcal{C}_k^{(m)}$;

  **end**

  Compute maturity-guarantee top-up: $\mathcal{H}_T^{(m)} = \left(G - \mathcal{V}_T^{(m)}\right)^+$;

**end**

`// Step 4: Monte Carlo Price Estimation`

$\mathcal{P}_0^{\text{guar}} \leftarrow \frac{1}{M}\sum_{m=1}^{M} e^{-rT}\mathcal{H}_T^{(m)}$;

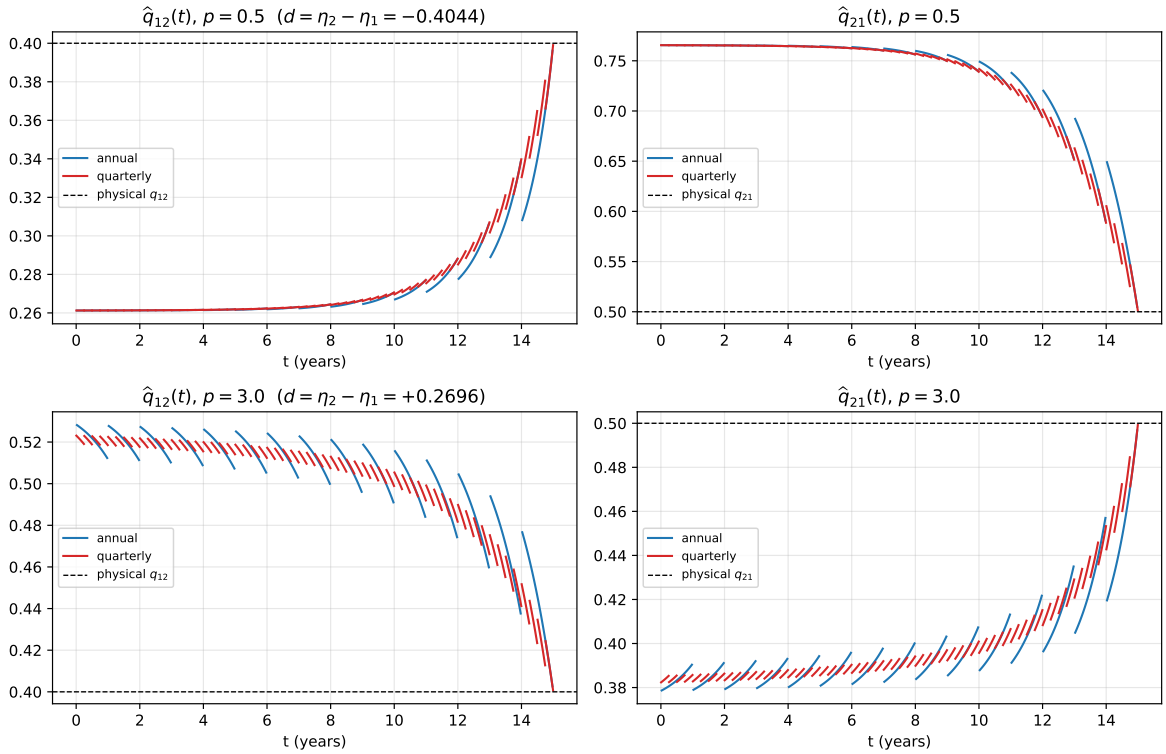
$\mathcal{P}_0^{\text{fee}} \leftarrow \frac{1}{M}\sum_{m=1}^{M}\sum_{k=1}^{n} e^{-rt_k}\mathcal{C}_k^{(m)}$;

$\mathcal{P}_0^{\text{rider}} \leftarrow \mathcal{P}_0^{\text{guar}} - \mathcal{P}_0^{\text{fee}}$;

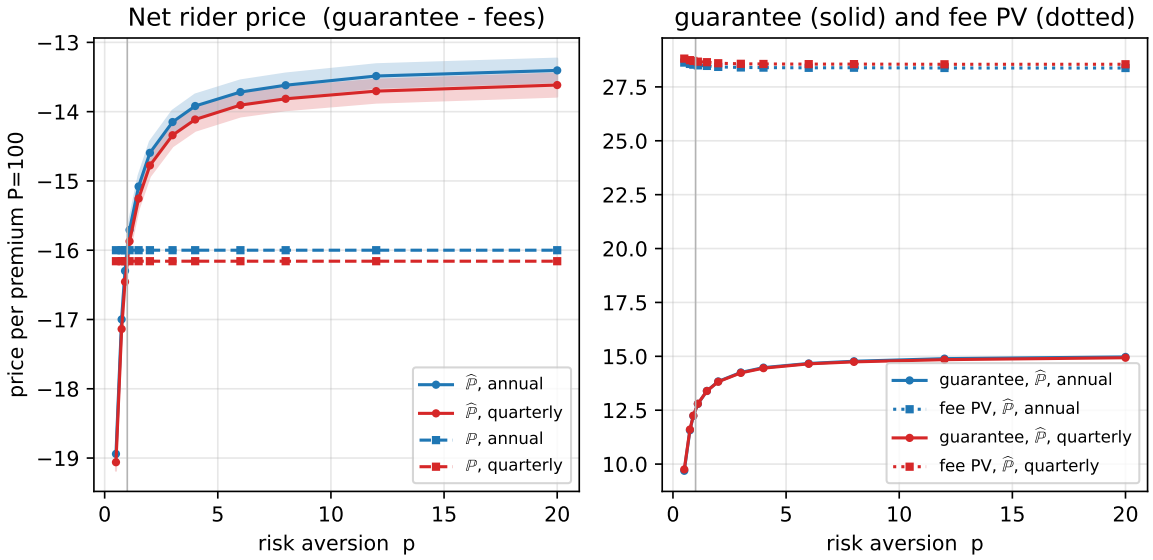
**return** $\mathcal{P}_0^{\text{guar}}$, $\mathcal{P}_0^{\text{fee}}$, $\mathcal{P}_0^{\text{rider}}$

We simulate 100,000 paths using 48 time steps per year over $T = 15$ years. Figures 1 and 2 present the Black–Scholes and Merton results, respectively.

In the Black–Scholes model, $\eta_2 - \eta_1$ has the sign of $p - 1$ for the parameters in Table 1. Proposition 6.1 therefore gives $\widehat{q}_{12} > q_{12}$ and $\widehat{q}_{21} < q_{21}$ for $p > 1$, with both inequalities reversed for $p < 1$, as illustrated in Figure 1(a). The shift toward the high-volatility regime for $p > 1$ is

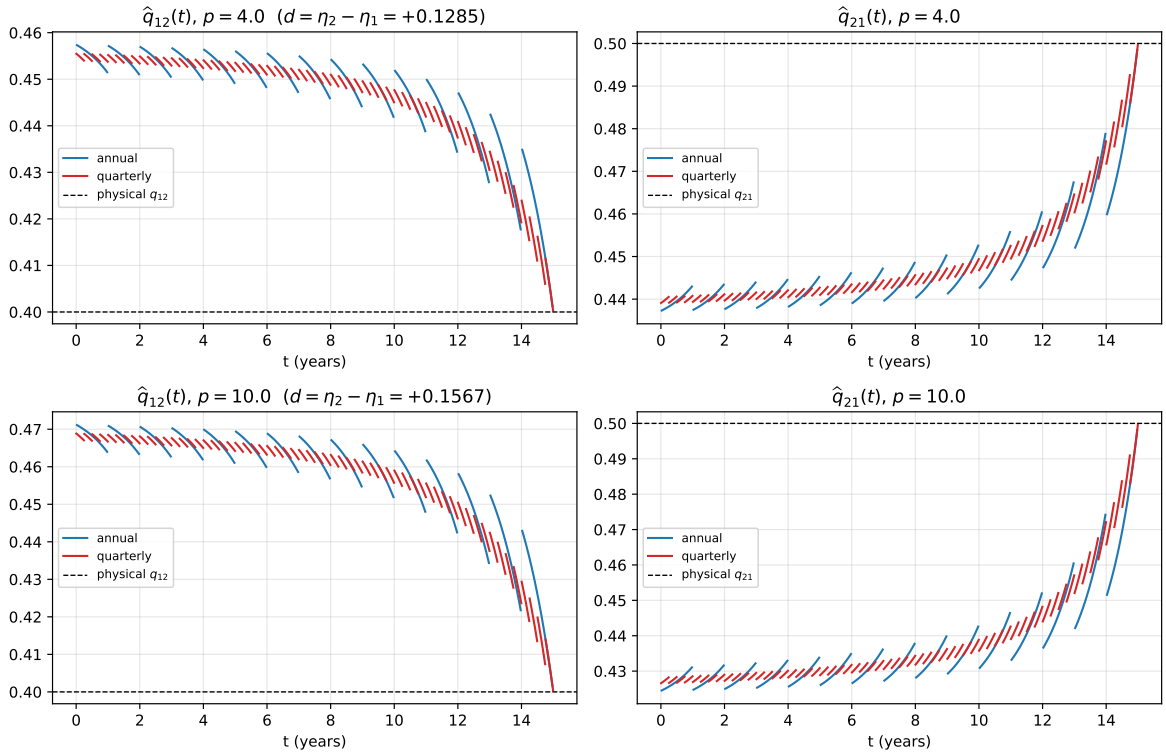



(a) Transformed regime-switching intensities under different consumption clocks. The symbol $d$ in the panel titles denotes $\Delta_\eta = \eta_2 - \eta_1$.

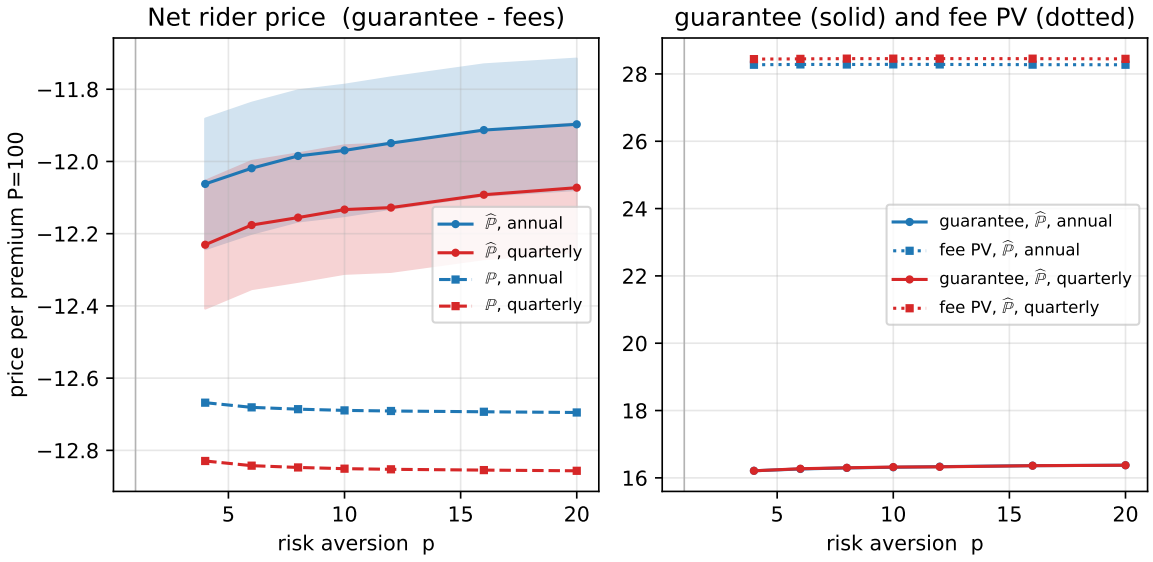



(b) Marginal indifference prices as functions of the risk-aversion parameter $p$ under joint changes of the consumption and fee schedules. Shaded bands are the reported 95% Monte Carlo confidence intervals. In the left panel, $\mathbb{P}$ denotes the physical measure, whereas $\widehat{\mathbb{P}}$ denotes the equivalent utility-based pricing measure.

Figure 1: Joint consumption- and fee-schedule comparisons under the Black–Scholes model.




(a) Transformed regime-switching intensities under different consumption clocks. The symbol $d$ in the panel titles denotes $\Delta_\eta = \eta_2 - \eta_1$.




(b) Marginal indifference prices as functions of the risk-aversion parameter $p$ under joint changes of the consumption and fee schedules. Shaded bands are the reported 95% Monte Carlo confidence intervals. In the left panel, $\mathbb{P}$ denotes the physical measure, whereas $\widehat{\mathbb{P}}$ denotes the equivalent utility-based pricing measure.

Figure 2: Joint consumption- and fee-schedule comparisons under the Merton jump-diffusion model.

consistent with the increasing guarantee-price curves in Figure 1(b); it does not establish general price monotonicity.

The diffusion example allows substantial leverage: $\gamma_1 = 9/p$, giving $\gamma_1 = 18$ at $p = 0.5$ and $\gamma_1 = 3$ at $p = 3$. In the Merton model, jump support $(-1, \infty)$ excludes fractions above one under strict wealth positivity. We retain interior solutions $0 < \gamma_i < 1$, excluding smaller $p$ under the chosen parameters. Figure 2(a) shows the intensities for $p = 4$ and $p = 10$, while Figure 2(b) shows a modest guarantee-price increase over $p \in [4, 20]$.

Clock dependence is explicit in the transition intensities, computed from the deterministic backward equations. The rider-price bands overlap substantially in the Merton example and partly in the diffusion example. Separate confidence bands do not test price differences, and the figures establish neither statistical significance nor a pure utility-clock price effect for a fixed contract.

Table 2: Present value estimates and standard deviations of terminal account values and fees under different risk-aversion parameters in the Black–Scholes model.

| | Annual | | Quarterly | |
|---|---|---|---|---|
| $p$ | Estimate | SE | Estimate | SE |
| 0.5 | 100.0113 | 0.2036 | 100.0167 | 0.2023 |
| 0.75 | 100.0789 | 0.2381 | 100.0729 | 0.2362 |
| 0.9 | 100.1049 | 0.2520 | 100.1129 | 0.2496 |
| 1.1 | 100.1060 | 0.2637 | 100.0983 | 0.2611 |
| 1.5 | 100.1942 | 0.2806 | 100.2329 | 0.2784 |
| 2 | 100.0064 | 0.2864 | 100.0050 | 0.2828 |
| 3 | 99.8532 | 0.2922 | 99.8692 | 0.2889 |
| 4 | 99.8734 | 0.2969 | 99.8780 | 0.2936 |
| 6 | 99.9572 | 0.3024 | 99.9139 | 0.2981 |
| 8 | 100.0028 | 0.3048 | 99.9793 | 0.3001 |
| 12 | 99.9435 | 0.3065 | 99.9620 | 0.3021 |
| 20 | 99.9502 | 0.3081 | 100.0191 | 0.3053 |

Table 3: Present value estimates and standard deviations of terminal account values and fees under different risk-aversion parameters in the Merton jump-diffusion model.

| | Annual | | Quarterly | |
|---|---|---|---|---|
| $p$ | Estimate | SE | Estimate | SE |
| 4 | 99.8845 | 0.3287 | 99.8942 | 0.3252 |
| 6 | 100.0220 | 0.3317 | 100.0015 | 0.3280 |
| 8 | 100.0387 | 0.3317 | 100.0834 | 0.3294 |
| 10 | 100.0872 | 0.3335 | 100.0742 | 0.3297 |
| 12 | 100.1102 | 0.3336 | 100.1017 | 0.3299 |
| 16 | 100.0755 | 0.3334 | 100.1388 | 0.3306 |
| 20 | 100.0325 | 0.3332 | 100.0949 | 0.3304 |

Equation (46) provides a numerical consistency check: the simulated mean of the discounted terminal account value plus fees should be close to the initial premium, $P = 100$, up to Monte Carlo

and discretization errors. Tables 2 and 3 report the corresponding simulated means and standard deviations. As shown in the tables, the simulated means are consistently close to the initial premium of 100, supporting the numerical consistency of the simulation results. Moreover, the relatively small standard deviations indicate limited variability in the simulated discounted values.

# 9 Conclusions

The finite consumption clock selects an explicit indifference pricing measure for a maturity guarantee funded by periodic account fees. The regime generator under this measure depends on opportunity-function ratios and generally on the consumption calendar. The return-noise factor of the density remains clock independent. The annuity cash flows satisfy directly verifiable datewise superhedging bounds, so their indifference prices and the account-value consistency identity follow from a proved finite-date representation.

Random settlement is specified by an exogenous stopping time on the finite date set in the existing filtration. More general superreplicable cash flows and additional insurance risks require further analysis. The analytical comparisons show strict clock dependence for a regime-contingent claim, while identifying common clock scaling, equal opportunity growth rates and replication as distinct sources of invariance. They do not imply a universal ordering of guarantee or net rider prices across consumption calendars. Valuation of American-style contracts under the preference-selected measure, with endogenous surrender as in Bernard et al. (2014), is a possible topic for a separate paper.

# A Proof of the finite-date indifference-price representation

We prove Theorem 5.2. Put $\mathfrak{p} = \sum_k \widehat{\mathbb{E}^{\mathbb{P}}} \mathcal{F}_k$ and $y = u'(x)$. For the stopped superreplicating wealth processes in Assumption 5.1, write

$$w_{k,\mathrm{res}} = \mathcal{W}_0^k, \qquad w_{\mathrm{res}} = \sum_k w_{k,\mathrm{res}}, \qquad \mathcal{W}_t = \sum_k \mathcal{W}_t^k.$$

The uniform-integrability requirement in Assumption 5.1 gives $\mathbb{E}[Z_{t_k} \mathcal{W}_{t_k}^k] = w_{k,\mathrm{res}}$. Each reserve is held in cash after its payment date, so $\mathcal{W}_T = \sum_k \mathcal{W}_{t_k}^k$ and

$$\widehat{\mathbb{E}^{\mathbb{P}}} \mathcal{W}_T = w_{\mathrm{res}} < \infty, \qquad \mathbb{E}[Z_T \mathcal{W}_T] = \sum_k \mathbb{E}[Z_{t_k} \mathcal{W}_{t_k}^k] = w_{\mathrm{res}}. \tag{65}$$

These equalities also show that $\mathfrak{p}$ is finite.

**Budget inequality and the supporting price.** For a feasible plan at $(v, \theta)$, define

$$\Gamma_t = X_t + \sum_{t_k \le t} a_k \mathfrak{c}_k + \theta \sum_{t_k \le t} \mathcal{F}_k = v + \int_0^t X_{s-} \pi_s \, \mathrm{d}R_s.$$

Each $\mathcal{W}^k$ is held in discounted cash after $t_k$, so

$$\mathcal{W}_t \ge \sum_{t_k \le t} \mathcal{W}_{t_k}^k \ge \sum_{t_k \le t} |\mathcal{F}_k|.$$

Therefore $\Gamma + |\theta|\mathcal{W}$ is a nonnegative self-financing discounted portfolio, with the combined predictable stock holding. Lemma 3.6 makes this sum a local martingale and a supermartingale. Its terminal budget, using (65), yields

$$\mathbb{E}\sum_k a_k Z_{t_k}\mathfrak{c}_k + \theta\mathfrak{p} \le v. \tag{66}$$

Consequently the baseline first-order identities and the concavity of $U$ give

$$J(v,\theta) \le u(x) + y(v - x - \theta\mathfrak{p}). \tag{67}$$

Indeed, apply the pointwise supporting-line inequality at $\mathfrak{c}_k^*$ and sum over dates, then use (66) and the baseline binding budget. Integrability of the bound follows from those budgets. For $p > 1$, a competing utility value may be $-\infty$; the integrable upper bound remains valid. Setting $v = x + \theta\mathfrak{p}$ proves the indifference-pricing no-trade inequality for every feasible position.

**A feasible perturbation that pays at the correct dates.** Let $v = x + d$, $\varepsilon = |d| + |\theta|$, and

$$s = \frac{x + d - |\theta| w_{\mathrm{res}}}{x}.$$

For small $\varepsilon$, $s \ge 1/2$. Scale the baseline plan from $x$ by $s$ and launch $|\theta|$ units of every dated superhedge. At $t_k$, liquidate the $k$th reserve, pay $\theta\mathcal{F}_k$, and retain the nonnegative surplus $|\theta|\mathcal{W}_{t_k}^k - \theta\mathcal{F}_k$ in cash until $T$. The resulting reserve balance is

$$\mathcal{R}_t^\theta = \sum_{t_k > t} |\theta|\mathcal{W}_t^k + \sum_{t_k \le t} (|\theta|\mathcal{W}_{t_k}^k - \theta\mathcal{F}_k) \ge 0.$$

The combined plan consumes $s\mathfrak{c}_k^*$ at $k < n$ and consumes

$$\mathfrak{c}_n^{d,\theta} = s\mathfrak{c}_n^* + \frac{|\theta|\mathcal{W}_T - \theta\sum_k \mathcal{F}_k}{a_n} \tag{68}$$

at $T$. Until final consumption, its financial wealth is the sum of the remaining scaled baseline wealth and the nonnegative reserve. Its initial cost is $sx + |\theta| w_{\mathrm{res}} = x + d$, and its terminal wealth is zero after settling all liabilities and consumption. To check trading feasibility, sum $s$ times the baseline stock holding with $|\theta|\sum_k \psi_t^k \mathbf{1}_{\{t \le t_k\}}$, the holdings of the still active superhedges. This is a predictable $\widetilde{S}$-integrable process. Dividing its risky dollar exposure by the positive combined pre-terminal wealth defines a predictable fraction; at zero wealth take zero exposure. Liquidation at a date produces no jump in the trading gain because $S$ has no fixed-time jump. Its surplus is credited in cash while the net liability and consumption are settled as in (37). This constructs an admissible plan for both signs of $\theta$, with every liability financed at its payment date.

**Joint expansion and uniqueness.** The uniqueness argument is the finite-date counterpart of the dual-martingale mechanism in Mostovyi and Siorpaes (2024, Theorem 4.2). Here the dual optimizer $yZ$ supplies the supporting price, while the datewise superhedges and their exact martingale budgets permit feasible perturbations of either sign and the first-order expansion below. We retain the direct argument to cover intermediate consumption and potentially unbounded dated cash flows under Assumption 5.1.

For $\Delta\mathfrak{c}_k = \mathfrak{c}_k^{d,\theta} - \mathfrak{c}_k^*$, the construction gives

$$\mathfrak{c}_k^{d,\theta} \ge \mathfrak{c}_k^*/2, \qquad \frac{|\Delta\mathfrak{c}_k|}{\varepsilon} \le C\mathfrak{c}_k^* + \frac{2\mathcal{W}_T}{a_n}\mathbf{1}_{\{k=n\}}.$$

Every point on the line segment between $\mathfrak{c}_k^*$ and $\mathfrak{c}_k^{d,\theta}$ is at least $\mathfrak{c}_k^*/2$. Write the Taylor remainder exactly as

$$\mathrm{Rem}_k = \Delta\mathfrak{c}_k \int_0^1 [U'(\mathfrak{c}_k^* + u\Delta\mathfrak{c}_k) - U'(\mathfrak{c}_k^*)]\,\mathrm{d}u.$$

The displayed bound on $|\Delta\mathfrak{c}_k|/\varepsilon$ and $U'(\mathfrak{c}) = \mathfrak{c}^{-p}$ show that $\sum_k a_k w_k |\,\mathrm{Rem}_k\,|/\varepsilon$ is bounded by a constant times

$$\sum_k a_k w_k (\mathfrak{c}_k^*)^{1-p} + w_n (\mathfrak{c}_n^*)^{-p} \mathcal{W}_T.$$

This is integrable: its first term has expectation $yx$ by (33), and the second has expectation $y\mathbb{E}[Z_T\mathcal{W}_T] = yw_{\mathrm{res}}$ by (65). For each sample point, the normalized remainder tends to zero along every sequence $(d,\theta) \to (0,0)$, since $\Delta\mathfrak{c}_k \to 0$ and $U'$ is continuous on $(0,\infty)$. The candidate utility is finite: the lower bound $\mathfrak{c}_k^{d,\theta} \geq \mathfrak{c}_k^*/2$ controls the negative part for $p > 1$, and the supporting-line inequality with the integrable marginal term controls its positive part for $p < 1$. Dominated convergence gives the joint expansion

$$\begin{aligned}
\mathbb{E}\sum_k a_k w_k U(\mathfrak{c}_k^{d,\theta}) &= u(x) + \mathbb{E}\sum_k a_k w_k (\mathfrak{c}_k^*)^{-p}\Delta\mathfrak{c}_k + o(\varepsilon) \\
&= u(x) + y\{(s-1)x + |\theta| w_{\mathrm{res}} - \theta\mathfrak{p}\} + o(\varepsilon) \\
&= u(x) + y(d - \theta\mathfrak{p}) + o(\varepsilon).
\end{aligned}$$

The admissible perturbation and the supporting upper bound (67) therefore give (44) for both signs of $\theta$.

This also establishes the local finiteness of the endowed value function. If $b \neq \mathfrak{p}$, that expansion implies

$$J(x + \theta b, \theta) - u(x) = y\theta(b - \mathfrak{p}) + o(|\theta|) > 0$$

for sufficiently small $\theta$ of the appropriate sign. Such a quote fails the no-trade criterion, proving uniqueness.

The proof uses only the nonnegativity and attainability of the dated reserves in Assumption 5.1 and their exact budgets (65); no cash-and-stock form is required.

## Acknowledgments

The research of Oleksii Mostovyi was partially supported by the National Science Foundation under grant DMS-1848339.

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