AN APPROACH TO THE GREEKS FOR INDIFFERENCE PRICING

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Abstract. We consider the problem of sensitivity of indifference pricing to the dynamics of the underlying assets. In the context of arbitrage-free pricing (AFP), such sensitivities are known as the Greeks. Here, in multidimensional semimartingale settings of incomplete models, we obtain the Greeks and corrections to the associated trading strategies for indifference pricing in the sense of [Dav97] and [KK21]. Unlike the traditional AFP, e.g., in the Black-Scholes model, where the Greeks represent the sensitivity of a *linear* pricing problem to perturbations of the stock price dynamics, as indifference prices are given via solutions to non-linear stochastic control problems, their sensitivities to perturbations of model parameters, that is the Greeks, are also represented by value functions of auxiliary quadratic stochastic control problems, which we introduce too. The proposed approach also allows for the hedging of *nonreplicable* contingent claims. This contrasts with the hedging based on the Greeks for AFPs in incomplete markets, where the AFPs for nonreplicable claims form intervals, and their derivatives are not defined in the usual sense. The proposed framework allows us to consider the sensitivity to the perturbations of the jump part of the stock price process - these are the settings where the AFPs are usually intervals. In turn, multidimensional settings are needed, in particular, to characterize the indifference ρ, the sensitivity to perturbations of the interest rate.

Contents

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1. Introduction

In the questions of pricing and hedging of contingent claims, the Greeks play an important role as they allow quantifying how one should adjust the portfolio under small perturbations of the model parameters. In the context of arbitrage-free pricing, this methodology has been known for decades and is used by both theoretical researchers and practitioners. Even the efficient computation of the Greeks, relying on Malliavin calculus, has been developed; see [FLL$^+$99] and [FLLL01].

One of the main limitations of the Greeks for the arbitrage-free pricing approach is the replicability assumption of a contingent claim (or even the completeness of the market model). As many of the stock price models, such as stochastic volatility models and models with jumps, exhibit incompleteness, most of the contingent claims become non-replicable, and therefore, their arbitrage-free prices are intervals. The derivatives of such prices with respect to model parameters, therefore, are not defined in the usual sense, and so the methodology of using the Greeks for hedging becomes inapplicable.

In order to remedy the restrictive replicability assumption, this paper proposes a methodology of computing the Greeks for indifference prices. As the analysis below shows, such indifference prices are not only differentiable in the usual sense (in fairly general settings below), making such Greeks well-defined, but also allow for the specification of corrections to hedging strategies. Moreover, for the replicable contingent claims, the proposed methodology of computation of the Greeks agrees with the one for arbitrage-free prices, as both pricing methodologies agree, see, e.g., [MS24a, Section 6]. The corrections to the hedging strategy are obtained below without Malliavin calculus, which requires stringent model assumptions.

The proposed parametrization of perturbations permits us to consider the distortions of the finite variation part, continuous martingale part, and the purely discontinuous martingale parts of the stock price return. If the dynamics of the risky assets allow for jumps of random size, typically, the Greeks for arbitrage-free pricing are not applicable in view of the non-replicability of the contingent claim in most of the jump models, see [Shr04, Chapter 11]. The class of nonreplicable claims, in such settings, typically includes even the most vanilla instruments, such as European put options. With the proposed approach below, it is possible to construct the indifference-price-based hedging strategies associated with small perturbations of the parameter governing the jumps.

In the process, we also establish results of independent interest on the stability and asymptotic analysis of optimal investment without random endowment with respect to model perturbations. Here, we consider the framework of [Sch01] for the base or unperturbed model, which ensures that the dual optimizer for the base model is a true martingale under the physical measure. The martingale property of the dual minimizer allows for the uniqueness and representation formula of the indifference prices in terms of the expectation of the discounted payoff under the dual-optimal measure, provided that the discount factor is deterministic.

Mathematically, the proofs are based on the simultaneous primal-dual expansions of the value functions. One of the main technical difficulties stemmed from the fact that, under the perturbations below, the primal and dual value functions are neither convex nor concave in the perturbation parameter ε. This, in particular, complicates the proof of Lemma 5.5, which is central in the analysis, and its proof relies on a number of preceding characterizations to bypass the lack of the joint concavity of u in x and ε.

In order to handle the jumps of the risky asset, we need to invoke the elements of the change of numéraire calculus and the implicit differentiation formulas related to the ones in [MS24b]. However, in contrast to [MS24b], the settings below allow the relaxation of the quasi-left continuity of the driving martingale M assumed in [MS24b]. This is done via identifying the appropriate *natural processes* as in [DM82, Section VI.61] at the core of the proofs and using their properties to complete the analysis. To the best of our knowledge, *natural processes have not been used even in the context of asymptotic analysis of optimal investment, let alone the stability and asymptotic analysis of the indifference pricing.*

The remainder of this paper is organized as follows. In Section 2, we introduce the model, and in Section 3, we present the asymptotic expansion of indifference prices under small perturbations of the dynamics of driving risky assets; Section 4 contains the stability and sensitivity of the optimal investment to perturbations of the dynamics of the risky asset. Section 5 contains the proofs of the results of Section 3. In Section 6, characterizations of the optimizers to auxiliary minimization problems as projections are given when the risk tolerance wealth process exists. Section 7 contains the explicit representations of the particular Greeks, such as Vega, Rho, and Delta.

We conclude this section by commenting on the notations used below. For an $\mathbb{R}^d$-valued semimartingale $X = (X^i)_{i=1,\dots,d}$ and predictable d-by-d matrix-valued process with uniformly bounded in t and ω components $\phi^{i,j}$, $i = 1,\dots,d, j = 1,\dots,d$, we use the row-by-column rule

$$(\phi \cdot X)^i = \sum_{j=1}^{d} \phi^{i,j} \cdot X^j, \quad i = 1,\dots,d.$$

For the stochastic integration of a predictable $\mathbb{R}^d$-valued and componentwise bounded process H with respect to X, we follow [JS03]. As a consequence, and in a consistent way, we introduce the following notation

$$H \cdot \phi \cdot X := (\phi^\top H) \cdot X = H \cdot (\phi \cdot X).$$

2. Model

Let us consider a complete stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, where $T \in (0,T)$ is the time horizon, the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions, and $\mathcal{F}_0$ is trivia. We suppose that there are $(d+1)$ traded assets, one riskless whose price process equals 1 at all times, and d risky securities. To characterize the sensitivity of the indifference pricing to small perturbations of the dynamics of the risky assets, we need to consider the base and perturbed models. For the base model, we assume the following dynamics of the return process

$$R = M + \int_0^\cdot d\langle M\rangle_s \lambda_s, \quad R_0 = 0,$$

where M is an $\mathbb{R}^d$-valued locally $\mathbb{P}$-square-integrable martingale and λ is an $\mathbb{R}^d$-valued predictable process, such that

$$\int_0^T \lambda_s^\top d\langle M\rangle_s \lambda_s < \infty, \quad \mathbb{P}\text{–a.s.}. \tag{1}$$

2.1. **Parametrization of perturbations.** In order to incorporate models with jumps, such as the ones in [Mer76] and [Kou02], in our analysis, following [JS03], let us further consider a decomposition of the martingale part M into the continuous part and the purely discontinuous part, that is, we write

$$M = M^c + M^d,$$

and we suppose that the process driving perturbations is

$$\tilde{R} = \phi \cdot M^c + \psi \cdot M^d + \int_0^\cdot d\langle M\rangle_s \zeta_s, \quad \tilde{R}_0 = 0, \tag{2}$$

for componentwise bounded predictable $\mathbb{R}^{d\times d}$-valued processes ϕ and ψ, and an $\mathbb{R}^d$-valued predictable process ζ, such that

$$|\zeta_t| \le C'|\lambda_t|, \quad t \in [0,T], \quad \mathbb{P}\text{–a.s.}. \tag{3}$$

We parametrize perturbations by ε, and suppose that

$$R^\varepsilon = R + \varepsilon \tilde{R}, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

for some constant $\varepsilon_0 > 0$. With such a parametrization, the dynamics of the return of the stock price dynamics

$$R^\varepsilon = (I + \varepsilon\phi) \cdot M^c + (I + \varepsilon\psi) \cdot M^d + \int_0^\cdot d\langle M\rangle \left(\lambda + \varepsilon\zeta\right), \quad R_0^\varepsilon = 0, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0), \tag{4}$$

for some $\varepsilon_0 > 0$, where I is the d-by-d identity matrix, ϕ and ψ are predictable $\mathbb{R}^{d\times d}$-(matrix)-valued process, whose components are uniformly bounded, M is a d-dimensional locally $\mathbb{P}$-square-intergrable martingale, λ and ζ are predictable d-dimensional processes, such that

$$\int_0^T \lambda_s^\top d\langle M\rangle_s \lambda_s + \int_0^T \zeta_s^\top d\langle M\rangle_s \zeta_s < \infty, \quad \mathbb{P}\text{–a.s.}. \tag{5}$$

We note that condition (5) ensures that, for every ε sufficiently close to 0, R^ε in (4) satisfies the structure condition from [FS10].

Let us fix a utility function $U\colon \mathbb{R}\to\mathbb{R}$ satisfying the following assumption.

Assumption 2.1. $U\colon \mathbb{R}\to\mathbb{R}$, is a strictly increasing, strictly concave, two times continuously differentiable on $\mathbb{R}$, and its absolute risk aversion

$$A(x) := -\frac{U''(x)}{U'(x)}, \quad x\in\mathbb{R},$$

is bounded away from 0 and ∞, that is, there exist constants $c_1>0$ and $c_2<\infty$, such that

$$c_1 \le A(x) \le c_2, \quad x\in\mathbb{R}.$$

Following [MS24b], let us set $\kappa := \sum\limits_{i=1}^{d}\langle M^i\rangle$, we have that $\langle M\rangle = \tilde{A}\cdot\kappa$ for some process $\tilde{A}$.

Assumption 2.2. We suppose that $\tilde{A}_t$ is invertible for every $t\in[0,T]$, $\mathbb{P}$–a.s..

Next, we can look for $X^{\Delta x,\varepsilon}$ in the form

$$X^{\Delta x,\varepsilon} = (x+\Delta x) + \left(\hat{H} + \varepsilon H^\varepsilon + \Delta x H^{\Delta x}\right)\cdot R^\varepsilon.$$

Commonly in the literature (see, e.g., [DS06]), wealth processes that are bounded from below by a constant are called admissible, and, for every $(x,\varepsilon)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)$, we set

$$\begin{aligned}\mathcal{X}(x,\varepsilon) := \{X = x + H\cdot R^\varepsilon : &\textit{ for some } R^\varepsilon\textit{-integrable } H,\\ &\textit{and such that } X \textit{ is bounded from below by a constant}\}.\end{aligned} \tag{6}$$

2.2. **Primal problem.**

$$u(x,\varepsilon) := \sup_{X\in\mathcal{X}(x,\varepsilon)} \mathbb{E}\left[U(X_T)\right], \quad (x,\varepsilon)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0), \tag{P}$$

where we use the convention

$$\mathbb{E}\left[U(X_T)\right] := -\infty, \quad \textit{if} \quad \mathbb{E}\left[U^-(X_T)\right] = \infty. \tag{7}$$

By a contingent claim we mean any bounded random variable. As its payoff might depend on the dynamics of the traded assets, as in European put options, for example, thus the payoff depend on ε, we will consider a family of contingent claims f^ε, $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$. Following [KK21, Section 3.4, page 157], we adapt the definition of indifference prices as follows.

Definition 2.3. A number p is call an indifference price for f^ε corresponding to the initial wealth $x\in\mathbb{R}$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, if we have

$$\mathbb{E}[U(X_T + qf^\varepsilon)] \le u(x,\varepsilon), \quad q\in\mathbb{R} \quad \textit{and} \quad X\in\mathcal{X}(x-qp,\varepsilon).$$

Let us define

$$V(y) := \sup_{x\in\mathbb{R}}\left(U(x) - xy\right), \quad y>0.$$

Then V is two times continuously differentiable function, therefore

$$B(y) := V''(y)y, \quad y>0, \tag{8}$$

is well-defined. Next, we observe that, as $V''(U'(x)) = -\frac{1}{U''(x)}$, $x \in \mathbb{R}$, we have

$$B(U'(x)) = \frac{1}{A(x)}, \quad x \in \mathbb{R}.$$

Therefore, Assumption 2.1 implies that

$$\frac{1}{c_2} \leq B(y) = V''(y)y \leq \frac{1}{c_1}, \quad y > 0. \tag{9}$$

Absence of arbitrage. Following [Sch01], we suppose that

$$\mathcal{Q}^e(0) \neq \emptyset, \tag{10}$$

where $\mathcal{Q}^e(\varepsilon)$ $(\mathcal{Q}^a(\varepsilon))$ is a set of equivalent (absolutely continuous) local martingale measures for R^ε, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

2.3. **Dual problem.**

$$v(y,\varepsilon) = \inf_{\mathbb{Q}\in\mathcal{Q}^e(\varepsilon)} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \quad (y,\varepsilon) \in (0,\infty)\times(-\varepsilon_0,\varepsilon_0), \tag{D}$$

where we use the convention

$$\mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] := \infty, \quad if \quad \mathbb{E}\left[V^+\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] = \infty. \tag{11}$$

As usual, we denote

$$\begin{aligned}\mathcal{Y}(y,\varepsilon) := \{\ (Y_t)_{0\leq t\leq T} \geq 0 : Y_0 = y \ and \ (X_tY_t)_{0\leq t\leq T} \ is\ a\ \mathbb{P}\text{-}supermartingale \\ for\ every\ X \in \mathcal{X}(1,\varepsilon)\}\,, \quad (y,\varepsilon) \in (0,\infty)\times(-\varepsilon_0,\varepsilon_0),\end{aligned} \tag{12}$$

so that every element of $\mathcal{Q}^e(\varepsilon)$ can be represented as a terminal value of an element of $\mathcal{Y}(1,\varepsilon)$.

2.4. **Existence and uniqueness results for** (P) **and** (D)**.** As demonstrated in [Sch01], the optimizers (associated with different $x \in \mathbb{R}$) to (P) are not necessarily bounded from below, and thus, they are not the elements of $\mathcal{X}(x,\varepsilon)$'s, in general. These sets have to be enlarged properly.

Thus, following [Sch01], we define the following sets.

Definition 2.4. For $(x,\varepsilon) \in \mathbb{R}\times(-\varepsilon_0,\varepsilon_0)$, define the set $\mathcal{C}^b(x,\varepsilon)$ by

$$C^b_U(x,\varepsilon) = \left\{\Gamma_T \in \mathbb{L}^0 :\ \Gamma_T \leq X_T \ for\ some\ X \in \mathcal{X}(x,\varepsilon)\ and\ \mathbb{E}\left[|U(\Gamma_T)|\right] < \infty\right\},$$

and let $\mathcal{C}_U(x,\varepsilon)$ denote the set

$$C_U(x,\varepsilon) = \left\{\Phi_T \in \mathbb{L}^0 :\ U(\Phi_T)\ is\ in\ the\ \mathbb{L}^1(\mathbb{P})\text{-}closure\ of\ \{U(\Gamma_T) : \Gamma_T \in \mathcal{C}^b_U(x,\varepsilon)\}\right\}.$$

For the existence and uniqueness results for the base model, we suppose that the assumptions of [Sch01, Theorem 2.2] hold. The following theorem is proven in [Sch01, Theorem 2.2], namely,

- locally bounded d-dimensional semimartingale S,
- the Inada conditions, they follow from Assumption 2.1,
- reasonable asymptotic elasticity, also seem to follow from Assumption 2.1,
- NFLVR for the base model - (10).

Assumption 2.5. There exists $x \in \mathbb{R}$, such that

$$u(x, 0) < \infty.$$

We note that Assumption 2.5 implies that $u(x, 0) < \infty$ for every $x \in \mathbb{R}$. By [Sch01], finiteness of v on $(0, \infty)$ will follow.

First, we observe that under $u(x, 0) < \infty$, for some $x \in \mathbb{R}$, we have

$$u(x,0) = \sup_{X \in \mathcal{X}(x,0)} \mathbb{E}\left[U(X_T)\right] = \sup_{\Gamma_T \in \mathcal{C}_U^b(x,0)} \mathbb{E}\left[U(\Gamma_T)\right] = \sup_{\Phi_T \in \mathcal{C}_U(x,\varepsilon)} \mathbb{E}\left[U(\Phi_T)\right], \quad x \in \mathbb{R}. \tag{13}$$

We remark that, under these conditions, [Sch01, Theorem 2.2] implies that

- $u(\cdot, 0)$ and $v(\cdot, 0)$ are finite-valued, strictly concave (resp. convex), differentiable functions on $\mathbb{R}$ (resp. $\mathbb{R}_+$), they are conjugate and satisfy the Inada conditions.
- For $y > 0$, the optimal solution $\hat{\mathbb{Q}}(y, 0) \in \mathcal{Q}^a(0)$ to the dual problem (D) exists, is unique and the map $y \to \hat{\mathbb{Q}}(y, 0)$ is continuous in the variation norm.
- For $x \in \mathbb{R}$, the optimal solution $\hat{\Phi}_T(x, 0) \in \mathcal{C}_U(x, 0)$ to the primal problem (P) exists, is unique and is given by
$$\hat{\Phi}_T(x,0) = -V'\left(y\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\right),$$
where $y = u_x(x, 0)$.
- If $\hat{\mathbb{Q}}(u_x(x, 0), 0) \in \mathcal{Q}^e(0)$, then $\hat{\Phi}_T(x, 0)$ equals the terminal value $\hat{X}_T(x, 0)$ for a process of the form $\hat{X}(x, 0) = x + H \cdot R$, where H is predictable and R-integrable, such that $\hat{X}(x, 0)$ is a uniformly integrable martingale under $\hat{\mathbb{Q}}(u_x(x, 0), 0)$.

Lemma 2.6. *Let $x \in \mathbb{R}$ be fixed, $M \in \mathcal{H}^2_{loc}(\mathbb{P})$, and suppose that* (1), (3), (10), *and Assumption 3.1 hold true. Then, NFLVR holds for every $\varepsilon \in (-\tilde{\varepsilon}_0', \tilde{\varepsilon}_0')$ for some $\tilde{\varepsilon}_0' > 0$.*

Without loss of generality, below we suppose that $\tilde{\varepsilon}_0' = \varepsilon_0$.

Corollary 2.7. *As a corollary to Lemma 2.6, we obtain that the assertions of* [Sch01, Theorem 2.2] *hold for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.*

2.5. **Key representation formula.**

Lemma 2.8. *Let us fix $x \in \mathbb{R}$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then, under the conditions of* [Sch01, Theorem 2.2], *with $y = u_x(x, \varepsilon)$, where y is well-defined by* [Sch01, Theorem 2.2], *for a bounded f^ε, we have*

$$p(x,\varepsilon) = \mathbb{E}_{\hat{\mathbb{Q}}(u_x(x,\varepsilon),\varepsilon)}\left[f^\varepsilon\right], \tag{14}$$

where $p(x, \varepsilon)$ is specified in Definition 2.3.

Proof. The proof follows from the adaptation of the argument in [MS24a, proof of Theorem 4.2] to the present settings. $\square$

Assumption 2.9. The processes R and $\tilde{R}$ are locally bounded.

Assumption 2.10. There exists a predictable symmetric positive semidefinite matrix-valued processes γ^0 and ν^0 with bounded components, exactly one of which is the identity matrix[1] $(d\kappa\times\mathbb{P})$-a.e., such that

$$\nu^0\cdot\langle M^d\rangle = \gamma^0\cdot\langle M^c\rangle.$$

Assumption 2.11. Let us suppose that $\hat{\mathbb{Q}}(u_x(x,0),0)\in\mathcal{Q}^e(0)$ and we have[2]

$$\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} = \frac{\hat{Y}_T(y,0)}{y} = \mathcal{E}\left(H\right).$$

where

$$H = -\lambda\cdot M + \beta\cdot(-\gamma^0\cdot M^c + \nu^0\cdot M^d) + L,$$

where $L\in\mathcal{H}^2_{loc}(\mathbb{P})$ is orthogonal to both M^c and M^d, β is $\left(-\gamma^0\cdot M^c+\nu^0\cdot M^d\right)$-integrable and $\beta\cdot\left(-\gamma^0\cdot M^c+\nu^0\cdot M^d\right)\in\mathcal{H}^2_{loc}(\mathbb{P})$.

3. Asymptotic expansions of the indifference prices

Let us denote by $\mathcal{H}^2_0(\mathbb{Q})$ the set of square-integrable martingales under the probability measure $\mathbb{Q}$ with the initial value 0. Next, we define

$$\mathcal{M}^2 := \{\tilde{M}\in\mathcal{H}^2_0(\mathbb{Q}):\ \tilde{M} = H\cdot R,\ for\ some\ R\text{-}integrable\ H\}. \tag{15}$$

The complement of $\mathcal{M}^2$ in $\mathcal{H}^2_0(\mathbb{Q})$ is denoted by $\mathcal{N}^2$, that is

$$\mathcal{N}^2 := \{\tilde{N}\in\mathcal{H}^2_0(\mathbb{Q}):\ \tilde{M}\tilde{N}\ is\ a\ \mathbb{Q}\ martingale\ for\ every\ \tilde{M}\in\mathcal{M}^2\}.$$

3.1. **Transformation $\cdot^H$.** Let H be the stochastic logarithm of $\frac{\hat{Y}}{y}$, that is $\hat{Y} = y\mathcal{E}\left(H\right)$. For a semimartingale K, we set

$$K^H := K - [K^c, H^c] - \sum_{s\leq\cdot}\Delta K_s\frac{\Delta H_s}{1+\Delta H_s}, \tag{16}$$

which is also a semimartingale. One can see that, if $H+K$ is non-vanishing, K^H satisfies

$$\mathcal{E}\left(K^H\right) = \frac{\mathcal{E}\left(K+H\right)}{\mathcal{E}\left(H\right)},$$

that is K^H is the excessive return of K under the dual numéraire $\mathcal{E}(H)$. We also observe that $\cdot H$ is linear in the sense that for semimartingales K^1 and K^2 and a constant $\tilde{c}$, we have

$$(K^1+\tilde{c}K^2)^H = K^{1,H}+\tilde{c}K^{2,H}.$$

With

$$\begin{aligned}\tilde{A}^c_t &:= {\nu^0}_t(I+{\nu^0}_t)^{-1}\tilde{A}_t 1_{\{{\gamma^0}_t\equiv I\}} + (I+{\gamma^0}_t)^{-1}\tilde{A}_t 1_{\{{\nu^0}_t\equiv I\}},\\ \tilde{A}^d_t &:= {\gamma^0}_t(I+{\gamma^0}_t)^{-1}\tilde{A}_t 1_{\{{\nu^0}_t\equiv I\}} + (I+{\nu^0}_t)^{-1}\tilde{A}_t 1_{\{{\gamma^0}_t\equiv I\}},\quad t\in[0,T],\end{aligned}$$

let us set

$$\begin{aligned}g^c_t &= \zeta - (\tilde{A}^c_t)^{-1}\phi_t\tilde{A}^c_t\lambda_t - (\tilde{A}^c_t)^{-1}\phi_t\tilde{A}^c_t{\gamma^0}_t\hat{\beta}_t,\\ g^d_t &= \zeta - (\tilde{A}^d_t)^{-1}\psi_t\tilde{A}^d_t\lambda_t + (\tilde{A}^d_t)^{-1}\psi_t\tilde{A}^d_t{\nu^0}_t\hat{\beta}_t,\quad t\in[0,T].\end{aligned} \tag{17}$$

[1]This allows to include both the situation with no jumps, in which case $\gamma^0\equiv 0$, and the discrete-time case, when $\nu^0\equiv 0$.

[2]We will denote below $\hat{\mathbb{Q}}(u_x(x,0),0)$ by $\mathbb{Q}$ and $\hat{Y}(u_x(x,0),0)$ by $\hat{Y}$.

and

$$F := -g^c \cdot M^{c,H} - g^d \cdot M^{d,H} \quad \textit{and} \quad G := \hat{H} \cdot \tilde{R}. \tag{18}$$

In the one-dimensional case, that is, if only one risky asset is available, we have

$$g^c = \zeta - \phi\lambda - \phi\gamma^0\beta \quad \textit{and} \quad g^d = \zeta - \psi\lambda + \psi\nu^0\beta.$$

Assumption 3.1. There exists a constant $c > 0$, such that

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(c|G_T|\right) + \exp\left(c\left(|F_T| + [F]_T\right)\right)\left(1 + \hat{X}_T^2\right) + \hat{Y}_T\right] < \infty,$$

and the jumps of F are bounded, where processes F and G are defined in (18).

3.2. **Quadratic minimization problems.** Let us consider auxiliary minimization problems

$$u_{xx} := -y \inf_{\tilde{M}\in\mathcal{M}^2} \mathbb{E}_{\mathbb{Q}}\left[A\left(\hat{X}_T\right)\left(1 + \tilde{M}_T\right)^2\right], \tag{19}$$

$$u_{\varepsilon\varepsilon} := -y \inf_{\tilde{M}\in\mathcal{M}^2} \mathbb{E}_{\mathbb{Q}}\left[A\left(\hat{X}_T\right)\left(\tilde{M}_T + G_T\right)^2 + 2\tilde{M}_T F_T\right], \tag{20}$$

$$v_{yy} := y \inf_{\tilde{N}\in\mathcal{N}^2} \mathbb{E}_{\mathbb{Q}}\left[B(\hat{Y}_T)\left(\frac{1}{y} + \tilde{N}_T\right)^2\right], \tag{21}$$

$$v_{\varepsilon\varepsilon} := y \inf_{\tilde{N}\in\mathcal{N}^2} \mathbb{E}_{\mathbb{Q}}\left[B(\hat{Y}_T)\left(F_T + \tilde{N}_T\right)^2 + 2\left(F_T + \tilde{N}_T\right)G_T\right]. \tag{22}$$

Under the conditions of Theorem 4.1, via the direct method in the calculus of variations, see [FL07], and Komlos' lemma, one can show that there exist unique minimizers to (19), (20), (21), and (22). Let us denote these minimizers by M^x, M^ε, N^y, and N^ε, respectively. Next, let us set

$$u_{x\varepsilon} := -y\mathbb{E}_{\mathbb{Q}}\left[A\left(\hat{X}_T\right)\left(1 + M_T^x\right)\left(M_T^\varepsilon + G_T\right) + M_T^x F_T\right], \tag{23}$$

$$v_{y\varepsilon} := y\mathbb{E}_{\mathbb{Q}}\left[B(\hat{Y}_T)\left(\frac{1}{y} + N_T^y\right)\left(F_T + N_T^\varepsilon\right) + G_T\left(\frac{1}{y} + N_T^y\right)\right]. \tag{24}$$

3.3. **Conditions on f^ε.** The following assumption imposes sufficient integrability for the proofs below to hold, and it allows to include the perturbations of the payoffs, e.g., of the put options under perturbations of the stock price dynamics, particularly in the Black-Scholes model.

Assumption 3.2. There exists a constant $C > 0$ such that

$$|f^\varepsilon| \le C \quad \mathbb{P}\text{–a.s.}, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

and such that

$$\mathbb{L}^1(\mathbb{Q})\text{-}\lim_{\varepsilon\to 0} \frac{f^\varepsilon - f^0}{\varepsilon} = f'.$$

We also denote f^0 by f for brevity.

Remark 3.3. If we consider the following perturbations of the dynamics of the volatility in the Black-Scholes model

$$R_t^\varepsilon = \mu t + (\sigma + \varepsilon)W_t, \quad t \in [0, T],$$

for the put option on $S^\varepsilon = s_0\mathcal{E}(R^\varepsilon)$, that if for $(K - S^\varepsilon)^+$, we have that $f^\varepsilon = (K - S^\varepsilon)^+$, and

$$\left.\frac{\partial f^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = Ce^{\sigma W^{\mathbb{Q}}_T} W^{\mathbb{Q}}_T + C,$$

where $W^{\mathbb{Q}}$ is a Brownian motion under the minimal martingale measure for R^0, $\mathbb{Q}$. so

$$\left|\frac{\partial f^\varepsilon}{\partial \varepsilon}\right|\Bigg|_{\varepsilon=0} \leq C + C' e^{2\sigma W^{\mathbb{Q}}_T} \in \mathbb{L}^1(\mathbb{Q}).$$

One can also see that Assumption 3.2 holds.

Let us set

$$p_x := u_{xx}\mathbb{E}_{\mathbb{Q}}\left[N^y_T f\right], \quad p_\varepsilon := \mathbb{E}_{\mathbb{Q}}\left[\{(u_{x\varepsilon}N^y_T + N^\varepsilon_T + F_T\}f + f'\right]. \tag{25}$$

Remark 3.4. In view of Theorem 4.2 below, p_ε in (25) can be represented as

$$p_\varepsilon = v_{y\varepsilon}p_x + \mathbb{E}_{\mathbb{Q}}\left[\left(N^\varepsilon_T + F_T\right)f + f'\right].$$

Theorem 3.5. *Let $x \in \mathbb{R}$ be fixed, $M \in \mathcal{H}^2_{loc}(\mathbb{P})$, and suppose that* (10), (1), (3) *and Assumptions 2.1, 2.2, 2.9, 2.10, 2.11, 3.1, and 3.2 hold, and denote $y = u_x(x, 0)$, which is well-defined by* [Sch01, Theorem 2.2]. *Then, we have*

$$\lim_{|\Delta x|+|\varepsilon|\to 0} \frac{|p(\varepsilon, \Delta x) - p(0,0) - \varepsilon p_\varepsilon - \Delta x p_x|}{|\Delta x| + |\varepsilon|} = 0, \tag{26}$$

where p_x and p_ε are given in (25).

The corrections of trading strategies are given in Theorem 4.7 below.

4. Asymptotic analysis of (P) and (D)

4.1. **First-order analysis.** We start from the first-order expansion theorem.

Theorem 4.1. *Let $x \in \mathbb{R}$ be fixed, $M \in \mathcal{H}^2_{loc}(\mathbb{P})$, and suppose that* (10), (1), (3) *and Assumptions 2.1, 2.2, 2.9, 2.10, 2.11, 3.1 hold, and denote $y = u_x(x, 0)$. Then, there exists $\tilde\varepsilon_0 > 0$, such that for every $\varepsilon \in (-\tilde\varepsilon_0, \tilde\varepsilon_0)$, we have*

$$u(x, \varepsilon) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad \textit{and} \quad v(y, \varepsilon) \in \mathbb{R}, \quad y > 0.$$

Further, u and v are jointly differentiable (thus, continuous) at $(x, 0)$ and $(y, 0)$, respectively, and we have

$$\nabla u(x,0) = \begin{pmatrix} y \\ u_\varepsilon(x,0)\end{pmatrix} \quad \textit{and} \quad \nabla v(y,0) = \begin{pmatrix} -x \\ v_\varepsilon(y,0)\end{pmatrix}, \tag{27}$$

where

$$u_\varepsilon(x, 0) = y\mathbb{E}_{\mathbb{Q}}\left[G_T\right] = v_\varepsilon(y, 0) = -y\mathbb{E}_{\mathbb{Q}}\left[\hat X_T F_T\right], \tag{28}$$

where G and F are defined in (18).

4.2. **Second-order analysis.** Here we establish second-order expansions of the value functions appearing (P) and (D), as well as the first-order expansions of the optimizers for these problems.

Theorem 4.2. *Let $x \in \mathbb{R}$ be fixed, suppose that the assumptions of Theorem 4.1 hold, and denote $y = u_x(x,0)$. Then, we have*

$$\begin{pmatrix} u_{xx} & 0 \\ u_{x\varepsilon} & 1 \end{pmatrix} \begin{pmatrix} v_{yy} & 0 \\ v_{y\varepsilon} & -1 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{29}$$

and

$$u_{\varepsilon\varepsilon} - v_{\varepsilon\varepsilon} = u_{x\varepsilon} v_{y\varepsilon}. \tag{30}$$

Furthermore, with M^x, M^ε, N^y, and N^ε denoting the optimizers to (19)*,* (20)*,* (21)*, and* (22)*, respectively, we have*

$$A(\hat{X}_T) \begin{pmatrix} 1 + M^x_T \\ G_T + M^\varepsilon_T \end{pmatrix} = - \begin{pmatrix} u_{xx} & 0 \\ u_{x\varepsilon} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{y} + N^y_T \\ F_T + N^\varepsilon_T \end{pmatrix}; \tag{31}$$

equivalently,

$$B(\hat{Y}_T) \begin{pmatrix} \frac{1}{y} + N^y_T \\ F_T + N^\varepsilon_T \end{pmatrix} = \begin{pmatrix} v_{yy} & 0 \\ v_{y\varepsilon} & -1 \end{pmatrix} \begin{pmatrix} 1 + M^x_T \\ G_T + M^\varepsilon_T \end{pmatrix}. \tag{32}$$

Theorem 4.3. *Let $x \in \mathbb{R}$ be fixed, suppose that the assumptions of Theorem 4.1 hold, and denote $y = u_x(x,0)$. With*

$$H_u := \begin{pmatrix} u_{xx} & u_{x\varepsilon} \\ u_{x\varepsilon} & u_{\varepsilon\varepsilon} \end{pmatrix}, \tag{33}$$

we have

$$u(x + \Delta x, \varepsilon) = u(x,0) + (\Delta x \ \varepsilon) \nabla u(x,0) + \frac{1}{2} (\Delta x \ \varepsilon) H_u \begin{pmatrix} \Delta x \\ \varepsilon \end{pmatrix} + o(\Delta x^2 + \varepsilon^2),$$

where ∇u is given by (27)*. Similarly, with*

$$H_v := \begin{pmatrix} v_{yy} & v_{y\varepsilon} \\ v_{y\varepsilon} & v_{\varepsilon\varepsilon} \end{pmatrix}, \tag{34}$$

we have

$$v(y + \Delta y, \varepsilon) = v(y,0) + (\Delta y \ \varepsilon) \nabla v(y,0) + \frac{1}{2} (\Delta y \ \varepsilon) H_v \begin{pmatrix} \Delta y \\ \varepsilon \end{pmatrix} + o(\Delta y^2 + \varepsilon^2),$$

where $\nabla v(y,0)$ is given by (27)*.*

Remark 4.4. In view of the concavity of u in the x-variable, Theorem 4.3 implies that u_{xx} given by (19) is the second-order partial derivative of u with respect to x. Moreover, (19) implies that

$$c_1 \le - \frac{u_{xx}(x,0)}{u_x(x,0)} \le c_2,$$

that is, the absolute risk aversion of the indirect utility u at $(x,0)$ is bounded by the same constants as in Assumption 2.1.

Moreover, considering only perturbations of the initial wealth, under the assumptions of Theorem 4.1, similarly to the proof below, one can show that the quadratic expansion of u in x holds at every $(x,\varepsilon)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)$, that is, similarly to (19), we have

(35)
$$-u_{xx}(x+\Delta x,\varepsilon)=u_x(x+\Delta x,\varepsilon)\inf_{\tilde M\in\mathcal{M}^2(x+\Delta x,\varepsilon)}\mathbb{E}_{\hat{\mathbb{Q}}(u_x(x+\Delta x,\varepsilon),\varepsilon)}\left[A\left(\hat X_T(x+\Delta x,\varepsilon)\right)\left(1+\tilde M_T\right)^2\right],$$

where $\hat{\mathbb{Q}}(u_x(x+\Delta x,\varepsilon),\varepsilon)$ is the dual optimal measure at $(x+\Delta x,\varepsilon)$, and $\mathcal{M}^2(x+\Delta x,\varepsilon)$ is the space of square-integrable martingales starting from 0 under $\widehat{\mathbb{Q}}(u_x(x+\Delta x,\varepsilon),\varepsilon)$ that are stochastic integrals with respect to R^ε, that is $\mathcal{M}^2(x+\Delta x,\varepsilon)$ is defined entirely similarly to (15), but at $(x+\Delta x,\varepsilon)$. By the concavity of u in x, (35) implies the two times differentiability of u in x at $(x+\Delta x,\varepsilon)$. Finally, from (35), we can obtain the following bounds for the absolute risk aversion of u

$$c_1\le-\frac{u_{xx}(x,\varepsilon)}{u_x(x,\varepsilon)}\le c_2,\quad (x,\varepsilon)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0),$$

where c_1 and c_2 are given by Assumption 2.1. These bounds will be used in the proof of Lemma 5.4 below.

Theorem 4.5. *Let $x\in\mathbb{R}$ be fixed, suppose that the assumptions of Theorem 4.1 hold, and denote $y=u_x(x,0)$. With M^x_T and M^ε denoting the optimizers to* (19) *and* (20)*, respectively, we have*

$$\lim_{|\Delta x|+|\varepsilon|\to0}\frac{1}{|\Delta x|+|\varepsilon|}\left|\hat X_T(x+\Delta x,\varepsilon)-\left\{\hat X_T(x,0)+\Delta x(1+M^x_T)+\varepsilon\left(G_T+M^\varepsilon_T\right)\right\}\right|=0,\tag{36}$$

where the convergence takes place in $\mathbb{P}$-probability. Likewise, with N^y_T and N^ε denoting the optimizers to (21) *and* (22)*, respectively, we have*

$$\mathbb{L}^1(\mathbb{P})\text{-}\lim_{|\Delta y|+|\varepsilon|\to0}\frac{\left|\hat Y_T(y+\Delta y,\varepsilon)-\hat Y_T(y,0)\left(1+\Delta y\left(\frac1y+N^y_T\right)+\varepsilon\left(F_T+N^\varepsilon_T\right)\right)\right|}{|\Delta y|+|\varepsilon|}=0.\tag{37}$$

Corollary 4.6. *Let $x\in\mathbb{R}$ be fixed, suppose that the assumptions of Theorem 4.1 hold, and denote $y=u_x(x,0)$. With N^y_T and N^ε_T denoting the optimizers to* (21) *and* (22)*, respectively, we have*

$$\mathbb{L}^1(\mathbb{P})\text{-}\lim\frac{1}{|\Delta y|+|\varepsilon|}\left|\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}-\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\left\{1+\Delta yN^y_T+\varepsilon(F_T+N^\varepsilon_T)\right\}\right|=0.\tag{38}$$

4.3. **Corrections to optimal strategies.** With M^x and M^ε be the optimizers to (19) and (20) that are the elements of $\mathcal{M}^2$, respectively, let us approximate them by bounded $\mathbb{Q}$-martingales $\overline{M}^{x,n}$ and $\overline{M}^{\varepsilon,n}$, such that

$$\lim_{n\to\infty}\overline{M}^{x,n}_T=M^x_T\quad and\quad\lim_{n\to\infty}\overline{M}^{\varepsilon,n}_T=M^\varepsilon_T,\quad\mathbb{P}-a.s.$$

By the local boundedness of R (which implies the σ-boundedness of R), we can further approximate $\overline{M}^{x,n}$'s and $\overline{M}^{\varepsilon,n}$'s, so that that there exist sequences of bounded predictable processes $H^{x,n}$ and $H^{\varepsilon,n}$, $n\in\mathbb{N}$, such that $H^{x,n}\cdot R$, $H^{x,n}\cdot\tilde R$, $H^{\varepsilon,n}\cdot R$, $H^{\varepsilon,n}\cdot\tilde R$ are bounded, and we have

$$\lim_{n\to\infty}H^{\Delta x,n}\cdot R_T=M^x_T\quad and\quad\lim_{n\to\infty}H^{\varepsilon,n}\cdot R_T=M^\varepsilon_T,\quad\mathbb{P}-a.s.\tag{39}$$

Let us consider

$$\tilde X^{\Delta x,\varepsilon,n}=(x+\Delta x)+(H+\Delta xH^{\Delta x,n}+\varepsilon H^{\varepsilon,n})\cdot R^\varepsilon,\quad(\Delta x,\varepsilon,n)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)\times\mathbb{N}.\tag{40}$$

We note that $\tilde X\in\mathcal{X}(x+\Delta x,\varepsilon)$, for every $(x+\Delta x,\varepsilon,n)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)\times\mathbb{N}$.

Theorem 4.7. *Let $x \in \mathbb{R}$ be fixed, and suppose that the assumptions of Theorem 4.1 hold. Then, there exists a function $n = n(\Delta x, \varepsilon): \ \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{N}$, such that*

$$\mathbb{E}\left[U\left(\tilde{X}_T^{\Delta x, \varepsilon, n}\right)\right] = u(x + \Delta x, \varepsilon) - o(\Delta x^2 + \varepsilon^2).$$

The process $\tilde{X}^{\Delta x, \varepsilon, n}$ has the following positions in the risky assets

$$H + \Delta x H^{\Delta x, n} + \varepsilon H^{\varepsilon, n}, \tag{41}$$

and $H^{\Delta x, n}$'s and $H^{\varepsilon, n}$'s satisfy (39).

4.4. **Technical estimates.**

Lemma 4.8. *Let us suppose that Assumption 2.1 holds. Then, we have*

$$U'(x)e^{-c_1 z} \leq U'(x+z) \leq U'(x)e^{-c_2 z}, \quad \text{for every} \quad (x, z) \in \mathbb{R} \times (-\infty, 0). \tag{42}$$

and

$$U'(x+z) \leq U'(x)\left(1 + e^{-c_2 z}\right), \quad \text{for every} \quad (x, z) \in \mathbb{R}^2. \tag{43}$$

Proof. Let $h(\tilde{x}) := \log(U'(\tilde{x}))$, $\tilde{x} \in \mathbb{R}$. Then, we have

$$h'(\tilde{x}) = \frac{U''(\tilde{x})}{U'(\tilde{x})}, \quad \tilde{x} \in \mathbb{R}.$$

Therefore, for every $x \in \mathbb{R}$ and $z < 0$, using Assumption 2.1, we obtain

$$c_1(-z) \leq \int_{z+x}^{x} (-h'(t))dt = h(x+z) - h(x) = \int_{z+x}^{x} (-h'(t))dt \leq c_2(-z).$$

Next, from the definition of h, we get

$$\log(U'(x)) + c_1(-z) \leq \log(U'(z+x)) \leq \log(U'(x)) + c_2(-z).$$

Exponentiating both sides, we obtain (42). In turn, as U' is decreasing, from (42), we deduce (43). $\square$

Lemma 4.9. *Let $U\colon \mathbb{R} \to \mathbb{R}$ satisfies Assumption 2.1. Then, we have*

$$V'(y) + \frac{1}{c_1}\log z \geq V'(zy) \geq V'(y) + \frac{1}{c_2}\log z, \quad \text{for every} \quad y > 0 \quad \text{and} \quad z > 0. \tag{44}$$

Proof. Let us fix $y > 0$ and $z > 0$. Then, with for $x = -V'(y)$, we recall that $U''(x) = -\frac{1}{V''(y)}$, and (9) gives

$$\frac{1}{c_2} \leq \ B(y) = V''(y)y \leq \frac{1}{c_1}, \quad y > 0.$$

As a consequence, by direct computations, we get

$$V'(zy) - V'(y) = \int_y^{zy} V''(t)dt = \int_y^{zy} V''(t)t\frac{dt}{t} \geq \frac{1}{c_2}\log z. \tag{45}$$

Similarly, we can show that

$$V'(zy) - V'(y) \leq \frac{1}{c_1}\log z. \tag{46}$$

As (45) and (46) hold for every $y > 0$ and $z > 0$, (44) follows. $\square$

Proof of Lemma 2.6. As in the proof of Lemma 4.20, one can show that

$$\frac{\hat{Y}}{y}\mathcal{E}\left(J^{0,\varepsilon,H}\right) \in \mathcal{Y}(1,\varepsilon),$$

where $J^{0,\varepsilon}$'s are defined in (77). As $\frac{\hat{Y}}{y}\mathcal{E}\left(J^{0,\varepsilon,H}\right)$ is a positive local martingale, thus supermartingale under $\mathbb{P}$, it is enough to show that, for every ε sufficiently close to 0, we have

$$\mathbb{E}\left[\frac{\hat{Y}_T}{y}\mathcal{E}\left(J^{0,\varepsilon,H}\right)_T\right] = 1. \tag{47}$$

Let us consider the left-hand side in (47) and rewrite it as

$$\mathbb{E}\left[\frac{\hat{Y}_T}{y}\mathcal{E}\left(J^{0,\varepsilon,H}\right)_T\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{E}\left(J^{0,\varepsilon,H}\right)_T\right]. \tag{48}$$

Now, $\mathcal{E}\left(J^{0,\varepsilon,H}\right)$ is a nonnegative local martingale under $\mathbb{Q}$, thus supermartingale. It follows from Assumption 3.1 that there exists $\tilde{\varepsilon}_0' > 0$ such that

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\mathcal{E}\left(J^{0,\varepsilon,H}\right)_T\right)^2\right] < \infty, \quad \varepsilon \in (-\tilde{\varepsilon}_0', \tilde{\varepsilon}_0').$$

Therefore, $\varepsilon \in (-\tilde{\varepsilon}_0', \tilde{\varepsilon}_0')$, $\mathcal{E}\left(J^{0,\varepsilon,H}\right)$ is a square-integrable martingale under $\mathbb{Q}$, and so, using (48), we get

$$1 = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{E}\left(J^{0,\varepsilon,H}\right)_T\right] = \mathbb{E}\left[\frac{\hat{Y}_T}{y}\mathcal{E}\left(J^{0,\varepsilon,H}\right)_T\right],$$

which implies (47), thus $\frac{Y}{y}\mathcal{E}\left(J^{0,\varepsilon,H}\right)$ is a true $\mathbb{P}$ martingale, and so

$$\mathcal{Q}^e(\varepsilon) \neq \emptyset, \quad \varepsilon \in (-\tilde{\varepsilon}_0', \tilde{\varepsilon}_0').$$

□

4.5. **Results needed for reformulations of the auxiliary minimization problems.**

Lemma 4.10. *Let $x \in \mathbb{R}$ be fixed and assumptions of Theorem 4.1 hold. Then, with*

$$\lambda^c := \lambda + (\gamma^0)^\top\beta \quad \textit{and} \quad \lambda^d := \lambda - (\nu^0)^\top\beta, \tag{49}$$

and

$$R^c := M^c + \int_0^\cdot d\langle M^c\rangle_s\lambda_s^c \quad \textit{and} \quad R^d := M^d + \int_0^\cdot d\langle M^d\rangle_s\lambda_s^d \tag{50}$$

the processes R^c and R^d satisfy

$$R = R^c + R^d, \tag{51}$$

and form a decomposition of a $\mathbb{Q}$-local martingale R into a continuous and purely discontinuous parts.

Proof. First, using Assumption 2.10, we obtain (51). Next, let us consider a bounded predictable d-dimensional process a. Then, from [Pro04, Theorem II.38], we have

$$\mathcal{E}\left(a\cdot R^c\right)\mathcal{E}\left(H\right) = \mathcal{E}\left(a\cdot R^c + H + [a\cdot R^c, H]\right),$$

where H is given by Assumption 2.11 is a $\mathbb{P}$-local martingale. By [JS03, Proposition I.4.49(c)], we have that $\left[\int_0^\cdot a_s^\top d\langle M^c\rangle_s\lambda_s^c, H\right]$ is a local martingale. The local martingale property of $(a\cdot R^c + H + [a\cdot R^c, H])$ follows, which implies, via [Pro04, Theorem III.29], that $\mathcal{E}\left(a\cdot R^c\right)\mathcal{E}\left(H\right)$ is a local martingale under

$\mathbb{P}$. Therefore, by [JS03, Proposition III.3.8], we deduce that $\mathcal{E}\left(a\cdot R^c\right)$ is a continuous local martingale under $\mathbb{Q}$. Consequently, $a\cdot R^c$ is a local martingale under $\mathbb{Q}$, as a stochastic logarithm of a continuous local martingale. Since a is an arbitrary bounded and predictable process, we deduce that R^c is a local martingale under $\mathbb{Q}$.

Now, having the local martingale property of R^c under $\mathbb{Q}$, using the local boundedness of R, similarly to the argument above, by taking a bounded predictable a, such that the jumps of $a\cdot R^d$ are strictly greater than -1, we can show that local martingale property of R^d. To show that R^d is a purely discontinuous locally bounded martingale under $\mathbb{Q}$, let us consider an arbitrary continuous local martingale K under $\mathbb{Q}$. Furthermore, in

$$K(a\cdot R^d) = (a\cdot R^d)_-\cdot K + (K_-a)\cdot R^d + \left[K, a\cdot R^d\right], \tag{52}$$

by [Pro04, Theorem III.29], $(a\cdot R^d)_-\cdot K$ is a local martingale under $\mathbb{Q}$ and, by [Pro04, Theorem IV.29], $(K_-a)\cdot R^d$ is a local martingale under $\mathbb{Q}$. Therefore, in (52), to show that $L(a\cdot R^d)$ is a local martingale under $\mathbb{Q}$, it is enough to show that $\left[K, a\cdot R^d\right]$ is a local martingale under $\mathbb{Q}$. We have

$$\left[K, a\cdot R^d\right] = \left[K, a\cdot M^d\right] + \left[K, \int_0^\cdot a_s^\top d\langle M^d\rangle_s\lambda^d\right],$$

where, by [JS03, Proposition I.4.49(c)], $\left[K, \int_0^\cdot a_s^\top d\langle M^d\rangle_s\lambda^d\right]$ is a local martingale under $\mathbb{Q}$. So, we are left to show that $\left[K, a\cdot M^d\right]$ is a local martingale under $\mathbb{Q}$, which via [JS03, Proposition III.3.8], holds if

$$\mathcal{E}\left(\left[K, a\cdot M^d\right]\right)\mathcal{E}\left(H\right) = \mathcal{E}\left(\left[K, a\cdot M^d\right] + H + \left[\left[K, a\cdot M^d\right], H\right]\right)$$

is a $\mathbb{P}$-local martingale. Since K is continuous, so is $\left[K, a\cdot M^d\right]$, therefore, by and [JS03, Proposition I.4.49(c)], $\left[\left[K, a\cdot M^d\right], H\right]$ is a $\mathbb{P}$-local martingale. Decomposing K into a continuous local martingale under $\mathbb{P}$ and a predictable finite variation part, using [JS03, Proposition I.4.49(c)] and the purely discontinuous martingale property of M^d under $\mathbb{P}$, we can show that $\left[K, a\cdot M^d\right]$ is a local martingale. $\square$

Lemma 4.11. *Let $x\in\mathbb{R}$ be fixed, and the assumptions of Theorem 4.1 hold. Then, there exists $\tilde\varepsilon > 0$*[3]*, such that for every $\varepsilon\in(-\tilde\varepsilon,\tilde\varepsilon)$, the families of vector-valued processes $\lambda^{\varepsilon,c}$ and $\lambda^{\varepsilon,d}$ given implicitly via*

$$\begin{aligned}\int_0^\cdot d\langle M^c\rangle_s\left(\lambda_s+\varepsilon\zeta_s\right) &= \int_0^\cdot\left(I+\varepsilon\phi_s\right)d\langle M^c\rangle_s\lambda_s^{\varepsilon,c},\\ \int_0^\cdot d\langle M^d\rangle_s\left(\lambda_s+\varepsilon\zeta_s\right) &= \int_0^\cdot\left(I+\varepsilon\psi_s\right)d\langle M^d\rangle_s\lambda_s^{\varepsilon,d},\end{aligned} \tag{53}$$

are well-defined. Furthermore, the matrix-valued processes γ^ε and ν^ε given via

$$\begin{aligned}\gamma^\varepsilon &:= \left(I+\varepsilon\psi\right)^{-1}\gamma^0\left(I+\varepsilon\phi\right), \quad \nu^\varepsilon = I, \quad if\ \nu^0\equiv I,\\ \nu^\varepsilon &:= \left(I+\varepsilon\phi\right)^{-1}\nu^0\left(I+\varepsilon\psi\right), \quad \gamma^\varepsilon = I, \quad if\ \gamma^0\equiv I,\end{aligned} \tag{54}$$

satisfy

$$\begin{aligned}\int_0^\cdot \gamma^\varepsilon{}_s d\langle M^c\rangle_s\lambda_s^{\varepsilon,c} &= \int_0^\cdot d\langle M^d\rangle_s\lambda_s^{\varepsilon,d}, \quad if \quad \nu^0\equiv I,\\ \int_0^\cdot d\langle M^c\rangle_s\lambda_s^{\varepsilon,c} &= \int_0^\cdot \nu^\varepsilon{}_s d\langle M^d\rangle_s\lambda_s^{\varepsilon,d}, \quad if \quad \gamma^0\equiv I.\end{aligned} \tag{55}$$

[3]Below, we suppose that $\tilde\varepsilon = \varepsilon_0$, without loss of generality.

Proof. By the uniform boundedness of the components of ϕ and ψ, we deduce that there exists $\tilde{\varepsilon} > 0$, such that for every $\varepsilon \in (-\tilde{\varepsilon}, \tilde{\varepsilon})$, $(I+\varepsilon\phi)^{-1}$ and $(I+\varepsilon\psi)^{-1}$ exist and are bounded. Using the matrix-valued version of the Radon-Nikodym Theorem (see, e.g., [RR68]), we deduce that for every $\varepsilon \in (-\tilde{\varepsilon}, \tilde{\varepsilon})$, the vector-valued processes $\lambda^{\varepsilon,c}$ and $\lambda^{\varepsilon,d}$ given as solutions to (53) are well-defined. Now, (55) follows from (53) and (54). □

4.6. **Implicit differentiation.**

Lemma 4.12 (First-order implicit differentiation)**.** *Let $x \in \mathbb{R}$ be fixed, and the assumptions of Theorem 4.1 hold. Then, for every predictable process $\tilde{H}$ and $\tilde{\beta}$, such that $\tilde{H}$ is R integrable and $\tilde{\beta}$ is M-integrable and the integrals in* (56) *are well-defined and finite-valued $\mathbb{P}$–a.s., we have*

$$\begin{aligned}&\int_0^T (\tilde{H}_s)^\top d\langle M\rangle_s \zeta_s - \int_0^T (\tilde{H}_s)^\top \phi_s d\langle M^c\rangle_s \left(\lambda_s + (\gamma^0_s)^\top \tilde{\beta}\right) - \int_0^T (\tilde{H}_s)^\top \psi_s d\langle M^d\rangle_s \left(\lambda_s - (\nu^0_s)^\top \tilde{\beta}_s\right)\\ &= \int_0^T (\tilde{H}_s)^\top d\langle M^c\rangle_s \left((\lambda^{0,c}_s)' + \left((\gamma^0_s)^\top\right)' \tilde{\beta}_s\right) + \int_0^T (\tilde{H}_s)^\top d\langle M^d\rangle_s \left(\left(\lambda^{0,d}_s\right)' - \left((\nu^0_s)^\top\right)' \tilde{\beta}_s\right),\end{aligned} \tag{56}$$

where the derivatives above are given by

$$\begin{aligned}&(\lambda^{0,c}_t)' = \zeta - (\tilde{A}^c_t)^{-1}\phi_t \tilde{A}^c_t \lambda_t, \quad (\gamma^0_t)' = -\left((\tilde{A}^c_t)^{-1}\phi_t \tilde{A}^c_t {\gamma^0}_t\right)^\top,\\ &(\lambda^{0,d}_t)' = \zeta - (\tilde{A}^d_t)^{-1}\psi_t \tilde{A}^d_t \lambda_t, \quad (\nu^0_t)' = -\left((\tilde{A}^d_t)^{-1}\psi_t \tilde{A}^d_t {\nu^0}_t\right)^\top, \quad t \in [0,T].\end{aligned} \tag{57}$$

Moreover, for a predictable and R-integrable $\overline{H}$, such that the integrals in (58) *are well-defined and finite-valued $\mathbb{P}$–a.s., we have*

$$\begin{aligned}&\int_0^T (\overline{H}_s)^\top d\langle M\rangle_s \zeta_s - \int_0^T (\overline{H}_s)^\top \phi_s d\langle M^c\rangle_s \lambda^c_s - \int_0^T (\overline{H}_s)^\top \psi_s d\langle M^d\rangle_s \lambda^d_s\\ &\qquad = \int_0^T (\overline{H}_s)^\top d\langle M^c\rangle_s g^c_s + \int_0^T (\overline{H}_s)^\top d\langle M^d\rangle_s g^d_s.\end{aligned} \tag{58}$$

Remark 4.13. Lemma 4.12 allows to characterize g^c and g^d defined in (17) as

$$g^c_t = (\lambda^{0,c}_t)' + \left((\gamma^0_t)'\right)^\top \beta_t \quad \textit{and} \quad g^d = (\lambda^{0,d}_t)' - \left((\nu^0_t)'\right)^\top \beta_t, \quad t \in [0,T],$$

where $(\lambda^{0,c}_t)'$, $(\lambda^{0,d}_t)'$, $\gamma^0_t)'$, and $(\nu^0_t)'$ are given by (57) in Lemma 4.12.

Proof of Lemma 4.12. With $\lambda^{\varepsilon,c}$ and $\lambda^{\varepsilon,d}$ being given by (53) and γ^ε and ν^ε given by (54), using [Pro04, Theorem II.38], one can see that for every predictable process $\tilde{H}$ and $\tilde{\beta}$, such that the integrals in (56) are well-defined and finite-valued $\mathbb{P}$–a.s., the process

$$\mathcal{E}\left(\tilde{H}\cdot R^\varepsilon\right)\mathcal{E}\left(-\lambda^{\varepsilon,c}\cdot M^c - \lambda^{\varepsilon,d}\cdot M^d + \tilde{\beta}\cdot\left(-\gamma^\varepsilon\cdot M^c + \nu^\varepsilon \cdot M^d\right)\right)$$

is a local martingale. The local martingale property of this process implies that

$$\begin{aligned}\int_0^\cdot \tilde{H}_s^\top d\langle M\rangle_s(\lambda_s + \varepsilon\zeta_s) = &\int_0^\cdot \tilde{H}_s^\top (I+\varepsilon\phi)\, d\langle M^c\rangle_s \left(\lambda^{\varepsilon,c}_s + (\gamma^\varepsilon_s)^\top \tilde{\beta}_s\right)\\ &+ \int_0^\cdot \tilde{H}_s^\top (I+\varepsilon\psi)\, d\langle M^d\rangle_s \left(\lambda^{\varepsilon,d}_s - (\nu^\varepsilon_s)^\top \tilde{\beta}_s\right).\end{aligned} \tag{59}$$

Let us consider a sequence ε_n, $n \in \mathbb{N}$, converging to 0. From (59), we have

$$\begin{aligned}\int_0^\cdot \tilde{H}_s^\top d\langle M\rangle_s \zeta_s =& \frac{1}{\varepsilon_n}\left(\int_0^\cdot \tilde{H}_s^\top (I+\varepsilon\phi)\, d\langle M^c\rangle_s \left(\lambda_s^{\varepsilon,c} + (\gamma_s^\varepsilon)^\top \tilde{\beta}_s\right)\right. \\ &\qquad \left. - \int_0^\cdot \tilde{H}_s^\top d\langle M^c\rangle_s \left(\lambda_s^c + (\gamma_s^0)^\top \tilde{\beta}_s\right)\right) \\ &+ \frac{1}{\varepsilon_n}\left(\int_0^\cdot \tilde{H}_s^\top (I+\varepsilon\psi)\, d\langle M^d\rangle_s \left(\lambda_s^{\varepsilon,d} - (\nu_s^\varepsilon)^\top \tilde{\beta}_s\right)\right. \\ &\qquad \left. - \int_0^\cdot \tilde{H}_s^\top d\langle M^d\rangle_s \left(\lambda_s^d - (\nu_s^0)^\top \tilde{\beta}_s\right)\right).\end{aligned}$$

Taking the limit as $n \to 0$, we obtain (56). (58) can be proven similarly. □

Similarly to the proof of the previous lemma, we can establish the second-order implicit differentiation formulas.

Lemma 4.14 (Second-order implicit differentiation)**.** *Let $x \in \mathbb{R}$ be fixed, and the assumptions of Theorem 4.1 hold. Then, for every predictable process H^ε and $\tilde{\beta}$, such that the integral below are well-defined and finite-valued $\mathbb{P}$–a.s., we have*

(60)
$$\begin{aligned}&2\int_0^T (H_s^\varepsilon)^\top \phi_s d\langle M^c\rangle_s \left((\lambda_s^{0,c})' + \left((\gamma^0{}_s)^\top\right)' \tilde{\beta}_s\right) + 2\int_0^T (H_s^\varepsilon)^\top \psi_s d\langle M^d\rangle_s \left(\left(\lambda_s^{0,d}\right)' - \left((\nu_s^0)^\top\right)' \tilde{\beta}_s\right) \\ &= -\int_0^T (H_s^\varepsilon)^\top d\langle M^c\rangle_s \left((\lambda_s^{0,c})'' + \left((\gamma^0{}_s)^\top\right)'' \tilde{\beta}_s\right) - \int_0^T (H_s^\varepsilon)^\top d\langle M^d\rangle_s \left(\left(\lambda_s^{0,d}\right)'' - \left((\nu_s^0)^\top\right)'' \tilde{\beta}_s\right).\end{aligned}$$

4.7. **Characterization of key $\mathbb{Q}$-martingales.**

Lemma 4.15. *Let $x \in \mathbb{R}$ be fixed, and the assumptions of Theorem 4.1 hold. Let H^ε be a predictable process, such that*

$$H^\varepsilon \cdot R \in \mathcal{H}^2_{loc}(\mathbb{Q}), \tag{61}$$

and $g^{\varepsilon,c}$ and $g^{\varepsilon,d}$ be predictable processes, such that $\frac{g^{\varepsilon,c}}{g^c}1_{\{g^c\neq 0\}}$ and $\frac{g^{\varepsilon,d}}{g^d}1_{\{g^d\neq 0\}}$ are locally bounded and $g^{\varepsilon,c}=0$ on $\{g^c=0\}$ as well as $g^{\varepsilon,d}=0$ on $\{g^d=0\}$. Then, the following processes are local martingales under $\mathbb{Q}$:

$$(H^\varepsilon \cdot R)\left(g^{\varepsilon,c}\cdot M^{c,H} + g^{\varepsilon,d}\cdot M^{d,H}\right) - \int_0^\cdot H_s^\varepsilon \left(d\langle M^c\rangle_s g_s^{\varepsilon,c} + d\langle M^d\rangle_s g_s^{\varepsilon,d}\right); \tag{62}$$

$$(H^\varepsilon \cdot R^c)\left(g^\varepsilon \cdot M^{c,H}\right) - \int_0^\cdot H_s^\varepsilon d\langle M^c\rangle_s g_s^\varepsilon; \tag{63}$$

$$\left(H^\varepsilon \cdot R^d\right)\left(g^\varepsilon \cdot M^{d,H}\right) - \int_0^\cdot H_s^\varepsilon d\langle M^d\rangle_s g_s^\varepsilon. \tag{64}$$

If additionally

$$H^\varepsilon \cdot R \quad \textit{and} \quad g^{\varepsilon,c}\cdot M^{c,H} + g^{\varepsilon,d}\cdot M^{d,H} \quad \textit{are in } \mathcal{H}^2(\mathbb{Q}),$$

then the processes in (62), (63), *and* (64) *are true martingales under $\mathbb{Q}$.*

Proof. First, in view of Assumption 3.1, we have

$$K := g^{\varepsilon,c}\cdot M^{c,H} + g^{\varepsilon,d}\cdot M^{d,H} \quad \textit{is locally bounded.}$$

As $\frac{\hat{Y}_-}{y}$ is left-continuous, by [Pro04, Theorem III.29], we have that

$$\frac{\hat{Y}_-}{y}\cdot\left(\int_0^\cdot H^{\varepsilon\top}_s d\left[M^d\right]_s g^{\varepsilon,d}_s - \int_0^\cdot H^{\varepsilon\top}_s d\langle M^d\rangle_s g^{\varepsilon,d}_s\right)\ \textit{is a } \mathbb{P}-\textit{local martingale}. \tag{65}$$

Let τ'_n, $n\in\mathbb{N}$, be the localizing sequence of this local martingale, $\tilde\tau_n$, $n\in\mathbb{N}$, be a sequence of stopping times such that $\int_0^{t\wedge\tilde\tau_n} H^{\varepsilon\top}_s d\langle M^d\rangle_s g^{\varepsilon,d}_s\in\mathbb{L}^1(\mathbb{Q})$ $t\in[0,T]$, $n\in\mathbb{N}$. Note that such as sequence $\tilde\tau_n$, $n\in\mathbb{N}$, exists as every component of $M\in\mathcal{H}^2_{loc}(\mathbb{P})$, and every component of R is locally bounded, so every component of $\int_0^\cdot\langle M\rangle_s\lambda_s$ is locally square-integrable under $\mathbb{P}$; therefore, by Assumption 3.1 and the Cauchy-Schwartz inequality, we have

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tilde\tau_n} H^\varepsilon_s d\langle M\rangle_s\lambda_s\right]\leq\frac{1}{\sqrt{y}}\left(\mathbb{E}\left[\int_0^{\tilde\tau_n} H^\varepsilon_s d\langle M\rangle_s\lambda_s\right]\right)^{\frac12}\mathbb{E}_{\mathbb{Q}}\left[\hat{Y}_T\right]^{\frac12}<\infty.$$

Also, let $\overline\tau_n$, $n\in\mathbb{N}$, be a sequence of stopping times, such that $g^{\varepsilon,c}\cdot M^{c,H}$, $g^{\varepsilon,d}\cdot M^{d,H}$, $[g^{\varepsilon,c}\cdot M^{c,H}]$, and $[g^{\varepsilon,d}\cdot M^{d,H}]$ are bounded on $[0,\overline\tau_n]$, $n\in\mathbb{N}$. The existence of such a sequence $\overline\tau_n$, $n\in\mathbb{N}$ follows from Assumption 3.1. It follows from [Pro04, Theorem 3.11] that $\int_0^{\cdot\wedge\overline\tau_n\wedge\tilde\tau_n} H^{\varepsilon\top}_s d\langle M\rangle_s\lambda_s$ is natural. Therefore, in view of the definition of the natural processes as in [Pro04, page 111], we have

$$\mathbb{E}\left[\left[\int_0^\cdot H^{\varepsilon\top}_s d\langle M\rangle_s\lambda_s, K\right]_{\overline\tau_n\wedge\tilde\tau_n}\right]=0,\quad n\in\mathbb{N}.$$

Let τ''_n, $n\in\mathbb{N}$, be a localizing sequence for $H^\varepsilon\cdot R$. Let us fix $n\in\mathbb{N}$, consider an arbitrary stopping time σ and set $\tau:=\sigma\wedge\tau'_n\wedge\tilde\tau_n\wedge\overline\tau_n\wedge\tau''_n$. Then we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(H^\varepsilon\cdot R_\tau)(K_\tau)\right]\\
&=\mathbb{E}_{\mathbb{Q}}\left[\left[H^\varepsilon\cdot\left(M+\int_0^\cdot d\langle M\rangle_s\lambda_s\right), K-[g^{\varepsilon,c}\cdot M^c, H^c]-\sum_{s\leq\cdot}g^{\varepsilon,d}_s\Delta M_s\frac{\Delta H_s}{1+\Delta H_s}\right]_\tau\right]\\
&=\mathbb{E}_{\mathbb{Q}}\left[\left[H^\varepsilon\cdot M, K-[g^{\varepsilon,c}\cdot M^c, H^c]-\sum_{s\leq\cdot}g^{\varepsilon,d}_s\Delta M_s\frac{\Delta H_s}{1+\Delta H_s}\right]_\tau\right],
\end{aligned}\tag{66}$$

where, in the second equality, we used the (true) martingale property of $\left[\int_0^\cdot H^{\varepsilon\top}_s d\langle M\rangle_s\lambda_s, K\right]$ on $[0,\tau]$. Further, by direct computations, we can rewrite the last expression in (66) as

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^\tau H^{\varepsilon\top}_s d\langle M^c\rangle_s g^{\varepsilon,c}_s+\sum_{s\leq\tau}H^\varepsilon_s\Delta M_s\frac{g^{\varepsilon,d}_s\Delta M_s}{1+\Delta H_s}\right].$$

Let us denote

$$\tilde T_1:=\mathbb{E}_{\mathbb{Q}}\left[\int_0^\tau H^{\varepsilon\top}_s d\langle M^c\rangle_s g^{\varepsilon,c}_s\right]\quad \textit{and}\quad \tilde T_2:=\mathbb{E}_{\mathbb{Q}}\left[\sum_{s\leq\tau}H^\varepsilon_s\Delta M_s\frac{g^{\varepsilon,d}_s\Delta M_s}{1+\Delta H_s}\right].$$

It follows from the assumption of the lemma that both $\tilde T_1$ and $\tilde T_2$ are well-defined and finite-valued, and using integration by parts, one can rewrite $\tilde T_2$ as

$$\tilde T_2=\mathbb{E}\left[\sum_{s\leq\tau}\frac{\hat{Y}_{s-}}{y}\left(H^\varepsilon_s\Delta M_s\right)\left(g^{\varepsilon,d}_s\Delta M_s\right)\right].\tag{67}$$

By [Pro04, Theorem II.28, page 75], we have

$$\int_0^{\cdot} H_s^{\varepsilon\top} d\left[M^d\right]_s g_s^{\varepsilon,d} = \left[H^\varepsilon \cdot M^d, g^{\varepsilon,d}\cdot M^d\right] = \sum_{s\leq\cdot} \left(H_s^\varepsilon \Delta M_s\right)\left(g_s^{\varepsilon,d}\Delta M_s\right). \tag{68}$$

In view of (65), (68) allows to further rewrite $\tilde{T}_s$ in (67) as

$$\tilde{T}_2 = \mathbb{E}\left[\int_0^\tau \frac{\hat{Y}_{s-}}{y} H_s^{\varepsilon\top} d\langle M^d\rangle_s g_s^{\varepsilon,d}\right]. \tag{69}$$

Further, using localization, [JS03, Theorem I.4.49], we can rewrite $\tilde{T}_2$ as

$$\tilde{T}_2 = \mathbb{E}_{\mathbb{Q}}\left[\int_0^\tau H_s^{\varepsilon\top} d\langle M^d\rangle_s g_s^{\varepsilon,d}\right].$$

We recapitulate that (66) can be rewritten as

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[(H^\varepsilon\cdot R_\tau)\left(g^{\varepsilon,c}\cdot M_\tau^{c,H} + g^{\varepsilon,d}\cdot M_\tau^{d,H}\right)\right] = \tilde{T}_1 + \tilde{T}_2\\ =&\mathbb{E}_{\mathbb{Q}}\left[\int_0^\tau H_s^{\varepsilon\top} d\langle M^c\rangle_s g_s^{\varepsilon,d}\right] + \mathbb{E}_{\mathbb{Q}}\left[\int_0^\tau H_s^{\varepsilon\top} d\langle M^d\rangle_s g_s^{\varepsilon,d}\right].\end{aligned} \tag{70}$$

As τ is an arbitrary stopping time on $[0,\tau_n]$ and τ_n, $n\in\mathbb{N}$, is a localizing sequence, we conclude that $\int_0^\tau H_s^{\varepsilon\top}\left(d\langle M^c\rangle_s g_s^{\varepsilon,c} + d\langle M^d\rangle_s g_s^{\varepsilon,d}\right)$ is the predictable quadratic covariation under $\mathbb{Q}$ of the pair $(H^\varepsilon\cdot R, g^{\varepsilon,c}\cdot M^{c,H} + g^{\varepsilon,d}\cdot M^{d,H})$.

If additionally both $H^\varepsilon\cdot R$ and $g^{\varepsilon,c}\cdot M^{c,H} + g^{\varepsilon,d}\cdot M^{d,H}$ are in $\mathcal{H}^2(\mathbb{Q})$, we deduce from [JS03, Theorem I.4.2, p. 38] that

$$(H^\varepsilon\cdot R)(g^{\varepsilon,c}\cdot M^{c,H} + g^{\varepsilon,d}\cdot M^{d,H}) - \int_0^{\cdot} H_s^{\varepsilon\cdot}\left(d\langle M^c\rangle_s g_s^{\varepsilon,c} + d\langle M^d\rangle_s g_s^{\varepsilon,d}\right)$$

is a true martingale under $\mathbb{Q}$, and so (70) implies (62). (63) and (64) can be proven similarly. $\square$

4.8. Bound for u.

Lemma 4.16. *Let $x\in\mathbb{R}$ be fixed. Under the conditions of Theorem 4.1 with $y = u_x(x,0)$, let $H^{\Delta x}$ and H^ε be bounded predictable processes such that*

$$H^{\Delta x}\cdot\left(\phi\cdot\left(M^c + \int_0^{\cdot}\langle M^c\rangle_s\lambda_s^c\right)\right),\quad H^{\Delta x}\cdot\left(\psi\cdot\left(M^d + \int_0^{\cdot}\langle M^d\rangle_s\lambda_s^d\right)\right),$$
$$H^{\varepsilon}\cdot\left(\phi\cdot\left(M^c + \int_0^{\cdot}\langle M^c\rangle_s\lambda_s^c\right)\right),\quad H^{\varepsilon}\cdot\left(\psi\cdot\left(M^d + \int_0^{\cdot}\langle M^d\rangle_s\lambda_s^d\right)\right),$$
$$H^{\Delta x}\cdot R,\quad H^\varepsilon\cdot R,\quad H^{\Delta x}\cdot\tilde{R},\quad \textit{and}\quad H^\varepsilon\cdot\tilde{R}\quad \textit{are bounded.}$$

Let $X^{\Delta x,\varepsilon}$ be given by (40) for $(x,\varepsilon)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)$ and let us define

$$w(\Delta x,\varepsilon) := \mathbb{E}\left[U\left(X_T^{\Delta x,\varepsilon}\right)\right],\quad (x,\varepsilon)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0).$$

Then, w admits the following expansion

$$w(\Delta x,\varepsilon) = w(0,0) + (\Delta x\quad \varepsilon)\nabla w(x,0) + \frac{1}{2}(\Delta x\quad\varepsilon)H_w\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix} + o(\Delta x^2+\varepsilon^2),$$

where

$$w_{\Delta x}(0,0) = u_x(x,0),\quad w_\varepsilon(x,0) = \mathbb{E}_{\mathbb{Q}}\left[G_T\right],$$

and

$$H_w := \begin{pmatrix} w_{\Delta x\Delta x} & w_{\Delta x\varepsilon} \\ w_{\Delta x\varepsilon} & w_{\varepsilon\varepsilon} \end{pmatrix},$$

where, with processes F and G defined in (18)*, we have*

$$\begin{aligned} w_{\Delta x\Delta x} &= -y\mathbb{E}_{\mathbb{Q}}\left[A\left(\hat{X}_T\right)\left(1 + H^{\Delta x}\cdot R_T\right)^2\right], \\ w_{\Delta x\varepsilon} &= -y\mathbb{E}_{\mathbb{Q}}\left[A\left(\hat{X}_T\right)\left(1 + H^{\Delta x}\cdot R_T\right)\left(H^{\varepsilon}\cdot R_T + G_T\right) + (H^{\Delta x}\cdot R_T)F_T\right], \\ w_{\varepsilon\varepsilon} &= -y\mathbb{E}_{\mathbb{Q}}\left[A\left(\hat{X}_T\right)\left(H^{\varepsilon}\cdot R_T + G_T\right)^2 + 2(H^{\varepsilon}\cdot R_T)F_T\right]. \end{aligned}$$

Lemma 4.17. *Let $x \in \mathbb{R}$ be fixed. Under the conditions of Theorem 4.1 with $y = u_x(x, 0)$, let and H^ε be bounded predictable processes such that satisfying the assumptions of Lemma 4.16. Then, we have*

$$\mathbb{E}_{\mathbb{Q}}\left[H^{\varepsilon}\cdot\tilde{R}_T\right] = \mathbb{E}_{\mathbb{Q}}\left[\left(H^{\varepsilon}\cdot R_T\right)\left(g^c\cdot M^{c,H}_T + g^d\cdot M^{d,H}_T\right)\right] = -\mathbb{E}_{\mathbb{Q}}\left[\left(H^{\varepsilon}\cdot R_T\right)F_T\right],$$

where g^c and g^d are defined in (17) *and F is defined in* (18).

Proof. Let us recall that, in (49), $\lambda^c = \lambda + (\gamma^0)^\top\beta$, $\lambda^d = \lambda - (\nu^0)^\top\beta$. Next, one can see that

$$H^{\varepsilon}\cdot\left(\phi\cdot\left(M^c + \int_0^{\cdot}\langle M^c\rangle_s\lambda^c_s\right)\right) \quad \textit{and} \quad H^{\varepsilon}\cdot\left(\psi\cdot\left(M^d + \int_0^{\cdot}\langle M^d\rangle_s\lambda^d_s\right)\right) \quad \textit{are} \quad \mathbb{Q}\textit{-martingales.}$$

Completing $\tilde{R}$ to a local martingale under $\mathbb{Q}$, we get

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[H^{\varepsilon}\cdot\tilde{R}_T\right] &= \mathbb{E}_{\mathbb{Q}}\left[H^{\varepsilon}\cdot\left(\phi\cdot M^c + \psi\cdot M^d + \int_0^{\cdot} d\langle M\rangle_s\zeta_s\right)_T\right] \\ &= \mathbb{E}_{\mathbb{Q}}\Bigg[H^{\varepsilon}\cdot\Bigg(\phi\cdot\left(M^c + \int_0^{\cdot}\langle M^c\rangle_s\lambda^c_s\right) + \psi\cdot\left(M^d + \int_0^{\cdot}\langle M^d\rangle_s\lambda^d_s\right) \\ &\qquad - \int_0^{\cdot}\phi_s d\langle M^c\rangle_s\lambda^c_s - \int_0^{\cdot}\psi_s d\langle M^d\rangle_s\lambda^d_s + \int_0^{\cdot} d\langle M\rangle_s\zeta_s\Bigg)_T\Bigg]. \end{aligned}$$

Now, using the martingale property of $H^{\varepsilon}\cdot\left(\phi\cdot\left(M^c + \int_0^{\cdot}\langle M^c\rangle_s\lambda^c_s\right)\right)$ and $H^{\varepsilon}\cdot\left(\psi\cdot\left(M^d + \int_0^{\cdot}\langle M^d\rangle_s\lambda^d_s\right)\right)$ under $\mathbb{Q}$, we can rewrite the latter expression as

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[H^{\varepsilon}\cdot\left(-\int_0^{\cdot}\phi_s d\langle M^c\rangle_s\lambda^c_s - \int_0^{\cdot}\psi_s d\langle M^d\rangle_s\lambda^d_s + \int_0^{\cdot} d\langle M\rangle_s\zeta_s\right)_T\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^T (H^{\varepsilon})^\top\left(d\langle M^c\rangle_s g^c_s + d\langle M^d\rangle_s g^d_s\right)\right], \end{aligned}$$

where, in the last equality, we have used Lemma 4.12 and the definition of g^c and g^d in (17). Now, the assertion of the lemma follows from Lemma 4.15 (particularly from (63) and (64)) and the definition of process F in (18). □

Proof of Lemma 4.16. Let us first fix $\delta > 0$, then fix $(\Delta x, \varepsilon) \in B_\delta(0, 0)$, where $B_\delta(0, 0)$ is a ball of radius δ in $\mathbb{R}^2$ centered at $(0, 0)$. Let us consider

$$X^{z\Delta x, z\varepsilon} = (x + z\Delta x) + (\hat{H} + z\Delta x H^{\Delta x} + z\varepsilon H^{\varepsilon})\cdot R^{z\varepsilon}, \quad z \in (-1, 1).$$

By direct computations, we get

$$
\begin{aligned}
\frac{\partial X^{z\Delta x, z\varepsilon}}{\partial z} &= \Delta x + \left(\Delta x H^{\Delta x} + \varepsilon H^{\varepsilon}\right) \cdot R^{\varepsilon} + (\hat{H} + z\Delta x H^{\Delta x} + z\varepsilon H^{\varepsilon}) \cdot \tilde{R}, \\
\frac{\partial^2 X^{z\Delta x, z\varepsilon}}{\partial z^2} &= 2\varepsilon \left(\Delta x H^{\Delta x} + \varepsilon H^{\varepsilon}\right) \cdot \tilde{R}.
\end{aligned} \tag{71}
$$

Next, we set

$$
W(z) := U\left(X_T^{z\Delta x, z\varepsilon}\right), \quad z \in (-1, 1).
$$

By direct computations, we get

$$
\begin{aligned}
W'(z) &= U'\left(X_T^{z\Delta x, z\varepsilon}\right) \frac{\partial X_T^{z\Delta x, z\varepsilon}}{\partial z}, \\
W''(z) &= U''\left(X_T^{z\Delta x, z\varepsilon}\right) \left(\frac{\partial X_T^{z\Delta x, z\varepsilon}}{\partial z}\right)^2 + U'\left(X_T^{z\Delta x, z\varepsilon}\right) \frac{\partial^2 X_T^{z\Delta x, z\varepsilon}}{\partial z^2},
\end{aligned} \tag{72}
$$

With process G being defined in (18), let us introduce

$$
J := 1 + |G_T|.
$$

From (72), via (71), Assumption 2.1, and Lemma 4.8, we deduce that there exists a constant $b > 0$, which does not depend on δ, such that

$$
\sup_{z \in (-1,1)} \left|W'(z)\right| + \sup_{z \in (-1,1)} \left|W'(z)\right| \leq bU'(\hat{X}_T) \exp(b\delta J) \left(J + J^2\right). \tag{73}
$$

As $1 \leq J \leq J^2$, we deduce from (73) that, for every z_1 and z_2 in $(-1, 1)$, we have

$$
\left|\frac{W(z_1) - W(z_2)}{z_1 - z_2}\right| + \left|\frac{W'(z_1) - W'(z_2)}{z_1 - z_2}\right| \leq 2bU'(\hat{X}_T) \exp(b\delta J) J^2. \tag{74}
$$

Now, by choosing a sufficiently small δ, we obtain from Assumption 3.1 via Holder's inequality that the right-hand side of (74) is integrable. Now, the assertion of the lemma follows from the dominated convergence theorem and Lemma 4.17. $\square$

Lemma 4.18. *Let $x \in \mathbb{R}$ be fixed, suppose that the assumptions of Theorem 4.1 hold, and denote $y = u_x(x, 0)$. Then, we have*

$$
u(x + \Delta x, \varepsilon) \geq u(x, 0) + (\Delta x\ \varepsilon)\nabla u(x, 0) + \frac{1}{2}(\Delta x\ \varepsilon) H_u \begin{pmatrix} \Delta x \\ \varepsilon \end{pmatrix} + o(\Delta x^2 + \varepsilon^2),
$$

Proof. Assumption 2.9 and [KS06a, Lemma 7 and Lemma 8] implies, in the terminology of [KS06a], the sigma-boundedness of R. Therefore, [KS06a, Lemma 6] asserts that we can approximate elements of $\mathcal{M}^2$ by *bounded* martingales in $\mathcal{M}^2$. Now, the assertion of the lemma follows from Lemma 4.16. $\square$

4.9. Preliminary results for the bound for v.

Lemma 4.19. *Let $x \in \mathbb{R}$ be fixed and Assumptions of Theorem 4.1 hold. Then, we have*

$$
-\mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T F_T\right] = \mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T\left(g^c \cdot M_T^{c,H} + g^d \cdot M_T^{d,H}\right)\right] = \mathbb{E}_{\mathbb{Q}}\left[\hat{H} \cdot \tilde{R}_T\right] = \mathbb{E}_{\mathbb{Q}}\left[G_T\right].
$$

Proof. Let us recall R^c and R^d defined in (50) and consider

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\hat{H}\cdot\tilde{R}_T\right] &= \mathbb{E}_{\mathbb{Q}}\left[\hat{H}\cdot\left(\phi\cdot M^c+\psi\cdot M^d+\int_0^{\cdot} d\langle M\rangle_s\zeta_s\right)_T\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\hat{H}\cdot\left(\phi\cdot R^c-\int_0^{\cdot}\phi_s d\langle M^c\rangle_s\lambda^c_s\right.\right.\\
&\qquad\left.\left.+\psi\cdot R^d-\int_0^{\cdot}\psi_s d\langle M^d\rangle_s\lambda^d_s+\int_0^{\cdot} d\langle M\rangle_s\zeta_s\right)_T\right].
\end{aligned}\tag{75}
$$

We observe that, by Assumption 3.1, we have the square-integrability of both $\hat{H}\cdot R^c_T$ and $\hat{H}\cdot R^d_T$. Therefore, in view of the uniform componentwise boundedness of ϕ and ψ, $\left(\phi^\top\hat{H}\right)\cdot(R^c)$ and $\left(\psi^\top\hat{H}\right)\cdot R^d$ are in $\mathcal{H}^2(\mathbb{Q})$. Therefore, we can rewrite the latter expression in (75) as

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\hat{H}\cdot\left(-\int_0^{\cdot}\phi_s d\langle M^c\rangle_s\lambda^c_s-\int_0^{\cdot}\psi_s d\langle M^d\rangle_s\lambda^d_s+\int_0^{\cdot} d\langle M\rangle_s\zeta_s\right)_T\right]\\
&=\mathbb{E}_{\mathbb{Q}}\left[\int_0^T\hat{H}^\top_s\left(d\langle M^c\rangle_s g^c_s+d\langle M^d\rangle_s g^d_s\right)\right],
\end{aligned}
$$

where, in the last equality, we have used the first-order implicit differentiation Lemma 4.12. Next, from Lemma 4.15, we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\int_0^T\hat{H}^\top_s\left(d\langle M^c\rangle_s g^c_s+d\langle M^d\rangle_s g^d_s\right)\right] &= \mathbb{E}_{\mathbb{Q}}\left[\left(\hat{H}\cdot R_T\right)\left(g^c\cdot M^{c,H}_T+g^d\cdot M^{d,H}_T\right)\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\left(x+\hat{H}\cdot R_T\right)\left(g^c\cdot M^{c,H}_T+g^d\cdot M^{d,H}_T\right)\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T\left(g^c\cdot M^{c,H}_T+g^d\cdot M^{d,H}_T\right)\right],
\end{aligned}
$$

which, in view of definitions of processes F and G in (18), completes the proof of this lemma. □

Lemma 4.20. *Let $x\in\mathbb{R}$ be fixed and conditions of Theorem 4.1 hold, $y=u_x(x,0)$. Let $\overline{L}$ and $\tilde{L}$ be locally square-integrable $\mathbb{P}$-martingales that are orthogonal to both M^c and M^d. Let $\beta^{\Delta y}$ and β^{ε} be bounded predictable processes. Let us denote*

$$
\tilde{N}^{\Delta y}:=\beta^{\Delta y}\cdot(-\gamma^0\cdot M^{c,H}+\nu^0\cdot M^{d,H})+\overline{L}^H\quad\text{and}\quad\tilde{N}^{\varepsilon}:=\beta^{\varepsilon}\cdot(-\gamma^0\cdot M^{c,H}+\nu^0\cdot M^{d,H})+\tilde{L}^H,\tag{76}
$$

and suppose that the following processes are bounded:

(1) $\overline{L}^H$, $\tilde{L}^H$, $[\overline{L}^H]$, and $[\tilde{L}^H]$;
(2) $\beta^{\Delta y}\cdot M^{c,H}$ and $\beta^{\varepsilon}\cdot M^{c,H}$;
(3) $\beta^{\Delta y}\cdot M^{d,H}$ and $\beta^{\varepsilon}\cdot M^{d,H}$;
(4) $[\beta^{\Delta y}\cdot M^H]$ and $[\beta^{\varepsilon}\cdot M^H]$;
(5) $[F,\tilde{N}^{\Delta y}]$ and $[F,\tilde{N}^{\varepsilon}]$;
(6) $(g^d\cdot M^{d,H})_-\cdot\tilde{N}^{\Delta y}_T$ and $(g^c\cdot M^{c,H})_-\cdot\tilde{N}^{\Delta y}_T$;
(7) $\left(\int_0^{\cdot}\hat{H}^\top_s d\langle M^d\rangle_s g^d_s\right)_-\cdot\tilde{N}^{\Delta y}_T$ and $\left(\int_0^{\cdot}\hat{H}^\top_s d\langle M^c\rangle_s g^c_s\right)_-\cdot\tilde{N}^{\Delta y}_T$;
(8) $\left(F+\tilde{N}^{\varepsilon}\right)_-\cdot\tilde{N}^{\varepsilon}$.

Then, with

$$
\begin{aligned}
J^{\Delta y,\varepsilon} :=& -\left(\lambda^{\varepsilon,c}-\lambda\right)\cdot M^c - \left(\lambda^{\varepsilon,d}-\lambda\right)\cdot M^d \\
&+\beta\cdot\left(\left(-\gamma^{\varepsilon}+\gamma^0\right)\cdot M^c + \left(\nu^{\varepsilon}-\nu^0\right)\cdot M^d\right) \\
&+\left(\Delta y\beta^{\Delta y}+\varepsilon\beta^{\varepsilon}\right)\cdot\left(-\gamma^{\varepsilon}\cdot M^c+\nu^{\varepsilon}\cdot M^d\right)+\Delta y\overline{L}+\varepsilon\tilde{L},
\end{aligned}
\tag{77}
$$

there exists $\delta > 0$, such that for every $(\Delta y, \varepsilon) \in B_\delta(0,0)$, we have

$$\frac{\hat{Y}}{y}\mathcal{E}\left(J^{\Delta y,\varepsilon,H}\right) \in \mathcal{Y}(1,\varepsilon).$$

Remark 4.21. For $\beta^{\Delta y}$ and β^{ε} as in Lemma 4.20, with $\lambda^{\varepsilon,c}$ and $\lambda^{\varepsilon,d}$ being given by (53) and γ^{ε} and ν^{ε} by (54), $(\Delta y\beta^{\Delta y}+\varepsilon\beta^{\varepsilon})\cdot\left(-\gamma^{\varepsilon}\cdot M^c+\nu^{\varepsilon}\cdot M^d\right)$ is orthogonal to $\lambda^{\varepsilon,c}\cdot M^c+\lambda^{\varepsilon,d}\cdot M^d$.

Lemma 4.22. *Let $x \in \mathbb{R}$ be fixed, assumptions of Theorem 4.1 hold, and let us denote $y = u_x(x,0)$. With $\overline{L}$ and $\beta^{\Delta y}$ satisfying the assumptions of Lemma 4.20, and $K := \beta^{\Delta y}\cdot(-\gamma^0\cdot M^c+\nu^0\cdot M^d)+\overline{L}$, we have*

$$
\begin{aligned}
&y\mathbb{E}_{\mathbb{Q}}\Bigg[B(\hat{Y}_T)\left(\frac{1}{y}+K_T-\left[\hat{H},K\right]^c_T-\sum_{s\le T}\frac{\Delta\hat{H}_s\Delta K_s}{1+\Delta\hat{H}_s}\right)^2 \\
&\qquad -\hat{X}_T\left(\frac{1}{y}+K_T-\left[\hat{H},K\right]^c_T-\sum_{s\le T}\frac{\Delta\hat{H}_s\Delta K_s}{1+\Delta\hat{H}_s}\right)^2+\hat{X}_T\left(\frac{1}{y^2}+[K]^c+\sum_{s\le T}\frac{(\Delta K_s)^2}{\left(1+\Delta\hat{H}_s\right)^2}\right)\Bigg] \\
&= y\mathbb{E}_{\mathbb{Q}}\left[B(\hat{Y}_T)\left(\frac{1}{y}+K_T-\left[\hat{H},K\right]^c_T-\sum_{s\le T}\frac{\Delta\hat{H}_s\Delta K_s}{1+\Delta\hat{H}_s}\right)^2\right].
\end{aligned}
$$

Proof. Let us recall the transformation $\cdot^H$ in (16). As K^H is a bounded martingale under $\mathbb{Q}$ that is orthogonal to $\hat{X}$, we deduce that

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[-\hat{X}_T\left(\frac{1}{y}+K_T-\left[\hat{H},K\right]^c_T-\sum_{s\le T}\frac{\Delta\hat{H}_s\Delta K_s}{1+\Delta\hat{H}_s}\right)^2+\hat{X}_T\left(\frac{1}{y^2}+[K]^c+\sum_{s\le T}\frac{(\Delta K_s)^2}{\left(1+\Delta\hat{H}_s\right)^2}\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[-\hat{X}_T\left(\frac{1}{y}+K^H_T\right)^2+\hat{X}_T\left(\frac{1}{y^2}+\left[K^H\right]_T\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[-\frac{2}{y}\hat{X}_TK^H_T+\hat{X}_T\left(\left[K^H\right]_T-\left(K^H_T\right)^2\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T\left(\left[K^H\right]_T-\left(K^H_T\right)^2\right)\right].
\end{aligned}
$$

Now, by the boundedness of K^H and $[K^H]$ and Assumption 3.1 and the orthogonality of K^H to $\hat{X}$ under $\mathbb{Q}$ we have that $\hat{X}\left(\left[K^H\right]-\left(K^H\right)^2\right)$ is a uniformly integrable $\mathbb{Q}$-martingale, we conclude that

$$\mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T\left(\left[K^H\right]_T-\left(K^H_T\right)^2\right)\right]=0,$$

and the assertion of the lemma follows. □

Lemma 4.23. *Let $x \in \mathbb{R}$ be fixed, assumptions of Theorem 4.1 hold, and let us denote $y = u_x(x,0)$. Let $\beta^{\Delta y}$, β^{ε}, $\overline{L}$, $\tilde{L}$, $\tilde{N}^{\Delta y}$ and $\tilde{N}^{\varepsilon}$ be as in Lemma 4.20. Then, we have*

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\Bigg[-\hat{X}_T\Bigg\{\left(\frac{1}{y}+\tilde{N}^{\Delta y}_T\right)\left(F_T+\tilde{N}^{\varepsilon}_T\right)-\left[\frac{1}{y}+\tilde{N}^{\Delta y}, F+\tilde{N}^{\varepsilon}\right]_T \\ +\beta^{\Delta y}\cdot\left((-\gamma^0)'\cdot M^{c,H}+(\nu^0)'\cdot M^{d,H}\right)_T\Bigg\}\Bigg] = \mathbb{E}_{\mathbb{Q}}\left[G_T\left(\frac{1}{y}+\tilde{N}^{\Delta y}_T\right)\right],\end{aligned} \tag{78}$$

where processes F and G are defined in (18).

Proof. First, to match the terms containing $\frac{1}{y}$ in (78), we need to check that

$$\mathbb{E}_{\mathbb{Q}}\left[\hat{H}\cdot\tilde{R}_T\right] = \mathbb{E}_{\mathbb{Q}}\left[-\hat{X}_T\left(-g^c\cdot M^{c,H}_T - g^d\cdot M^{d,H}_T+\tilde{N}^{\varepsilon}_T\right)\right], \tag{79}$$

which follows from Lemma 4.19 and the orthogonality of $\hat{X}$ and $\tilde{N}^{\varepsilon}$ and which implies that

$$\mathbb{E}_{\mathbb{Q}}\left[-\hat{X}_T\left(-g^c\cdot M^{c,H}_T - g^d\cdot M^{d,H}_T+\tilde{N}^{\varepsilon}_T\right)\right] = \mathbb{E}_{\mathbb{Q}}\left[-\hat{X}_T\left(-g^c\cdot M^{c,H}_T - g^d\cdot M^{d,H}_T\right)\right].$$

With λ^c and λ^d defined in (49), R^c and R^d are defined in (50), as $H\cdot R\in\mathcal{H}^2(\mathbb{Q})$ by Assumption 3.1, from Lemma 4.10, we deduce that both $\hat{H}\cdot R^c$ and $\hat{H}\cdot R^d$ are in $\mathcal{H}^2_0(\mathbb{Q})$ and are orthogonal. Using Lemma 4.12, we can rewrite the right-hand side of (78) as

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[\hat{H}\cdot\left(\phi\cdot M^c+\psi\cdot M^d+\int_0^{\cdot} d\langle M\rangle_s\zeta_s\right)_T\tilde{N}^{\Delta y}_T\right]\\ =&\mathbb{E}_{\mathbb{Q}}\left[\hat{H}\cdot\left(\phi\cdot R^c+\psi\cdot R^d+\int_0^{\cdot} d\langle M^c\rangle_s g^c_s+\int_0^{\cdot} d\langle M^d\rangle_s g^d_s\right)_T\tilde{N}^{\Delta y}_T\right].\end{aligned} \tag{80}$$

Now, using Lemma 4.15, we can rewrite the right-hand side of (80) as

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[\hat{H}\cdot\left(\phi\cdot R^c+\psi\cdot R^d+\int_0^{\cdot} d\langle M^c\rangle_s g^c_s+\int_0^{\cdot} d\langle M^d\rangle_s g^d_s\right)_T\tilde{N}^{\Delta y}_T\right]\\ =&\mathbb{E}_{\mathbb{Q}}\Bigg[\int_0^T\hat{H}^\top_s\phi_s d\langle M^c\rangle_s\left(-\gamma^0{}_s\right)^\top\beta^{\Delta y}_s+\int_0^T\hat{H}^\top_s\psi_s d\langle M^c\rangle_s\left(\nu^0_s\right)^\top\beta^{\Delta y}_s\\ &+\left(\int_0^T\hat{H}^\top_s d\langle M^c\rangle_s g^c_s+\int_0^T\hat{H}^\top_s d\langle M^d\rangle_s g^d_s\right)\tilde{N}^{\Delta y}_T\Bigg].\end{aligned} \tag{81}$$

Now, let us consider the term $\mathbb{E}_{\mathbb{Q}}\left[-\hat{X}_T\left(\beta^{\Delta y}\cdot\left((-\gamma^0)'\cdot M^{c,H}+(\nu^0)'\cdot M^{d,H}\right)_T\right)\right]$ in the left-hand side of (78). Using the martingale property of $\beta^{\Delta y}\cdot\left((-\gamma^0)'\cdot M^{c,H}+(\nu^0)'\cdot M^{d,H}\right)$ under $\mathbb{Q}$ and Lemma 4.15, we can rewrite it as follows.

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[-\hat{X}_T\left(\beta^{\Delta y}\cdot\left((-\gamma^0)'\cdot M^{c,H}+(\nu^0)'\cdot M^{d,H}\right)_T\right)\right]\\ =&\mathbb{E}_{\mathbb{Q}}\left[-\hat{H}\cdot R_T\left(\beta^{\Delta y}\cdot\left((-\gamma^0)'\cdot M^{c,H}+(\nu^0)'\cdot M^{d,H}\right)_T\right)\right]\\ =&\mathbb{E}_{\mathbb{Q}}\left[\int_0^T\hat{H}^\top_s d\langle M^c\rangle_s\left((\gamma^0_s)'\right)^\top\beta^{\Delta y}_s+\int_0^T\hat{H}^\top_s d\langle M^d\rangle_s\left((-\nu^0_s)'\right)^\top\beta^{\Delta y}_s\right].\end{aligned} \tag{82}$$

Now, applying Lemma 4.12, we can rewrite the latter term in (82) as

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[\int_0^T\hat{H}^\top_s d\langle M^c\rangle_s\left((\gamma^0_s)'\right)^\top\beta^{\Delta y}_s+\int_0^T\hat{H}^\top_s d\langle M^d\rangle_s\left((-\nu^0_s)'\right)^\top\beta^{\Delta y}_s\right]\\ =&\mathbb{E}_{\mathbb{Q}}\left[\int_0^T\hat{H}^\top_s\phi_s d\langle M^c\rangle_s\left(-\gamma^0_s\right)^\top\beta^{\Delta y}_s+\int_0^T\hat{H}^\top_s\psi_s d\langle M^d\rangle_s\left(\nu^0_s\right)^\top\beta^{\Delta y}_s\right],\end{aligned} \tag{83}$$

which up to

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T \hat{H}_s^\top d\langle M^c\rangle_s g_s^c + \int_0^T \hat{H}_s^\top d\langle M^d\rangle_s g_s^d\right)\tilde{N}_T^{\Delta y}\right] \tag{84}$$

term coincides with the right-hand side of (81).

Now, in the left-hand side of (78), let us consider the term

$$\mathbb{E}_{\mathbb{Q}}\left[-\hat{X}_T\left\{\left(\tilde{N}_T^{\Delta y}\right)\left(-g^c\cdot M_T^{c,H} - g^d\cdot M_T^{d,H} + \tilde{N}_T^{\varepsilon}\right) - \left[N^{\Delta y}, -g^c\cdot M^{c,H} - g^d\cdot M^{d,H} + \tilde{N}^{\varepsilon}\right]_T\right\}\right]. \tag{85}$$

As $\left(\tilde{N}^{\Delta y}\right)\left(-g^c\cdot M^{c,H} - g^d\cdot M^{d,H} + \tilde{N}^{\varepsilon}\right) - \left[\tilde{N}^{\Delta y}, -g^c\cdot M^{c,H} - g^d\cdot M^{d,H} + \tilde{N}^{\varepsilon}\right]$ is a $\mathbb{Q}$-martingale and since both $N^{\Delta y}$ and N^{ε} are orthogonal to $\hat{H}\cdot R$, we deduce that (85) can be rewritten as

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[-\hat{H}\cdot R_T\left\{\left(\tilde{N}_T^{\Delta y}\right)\left(-g^c\cdot M_T^{c,H} - g^d\cdot M_T^{d,H} + \tilde{N}_T^{\varepsilon}\right) - \left[\tilde{N}^{\Delta y}, -g^c\cdot M_T^{c,H} - g^d\cdot M_T^{d,H} + \tilde{N}^{\varepsilon}\right]_T\right\}\right]\\ &= \mathbb{E}_{\mathbb{Q}}\left[-\hat{H}\cdot R_T\left\{\left(\tilde{N}_T^{\Delta y}\right)\left(-g^c\cdot M_T^{c,H} - g^d\cdot M_T^{d,H}\right) - \left[\tilde{N}^{\Delta y}, -g^c\cdot M^{c,H} - g^d\cdot M^{d,H}\right]_T\right\}\right].\end{aligned} \tag{86}$$

Let us observe that

$$\begin{aligned}&\left(\tilde{N}_T^{\Delta y}\right)\left(-g^c\cdot M_T^{c,H} - g^d\cdot M_T^{d,H}\right) - \left[\tilde{N}^{\Delta y}, -g^c\cdot M^{c,H} - g^d\cdot M^{d,H}\right]_T\\ &= -(\tilde{N}_-^{\Delta y}g^c)\cdot M_T^{c,H} - (g^c\cdot M^{c,H})_-\cdot\tilde{N}_T^{\Delta y} - (\tilde{N}_-^{\Delta y}g^d)\cdot M_T^{d,H} - (g^d\cdot M^{d,H})_-\cdot\tilde{N}_T^{\Delta y}.\end{aligned}$$

Consequently, we can rewrite the right-hand side of (86) as

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\Big[-\hat{H}\cdot R_T\Big\{&-(\tilde{N}_-^{\Delta y}g^c)\cdot M_T^{c,H} - (g^c\cdot M^{c,H})_-\cdot\tilde{N}_T^{\Delta y}\\ &-(\tilde{N}_-^{\Delta y}g^d)\cdot M_T^{d,H} - (g^d\cdot M^{d,H})_-\cdot\tilde{N}_T^{\Delta y}\Big\}\Big].\end{aligned} \tag{87}$$

By the assumption of the lemma $(g^d\cdot M^{d,H})_-\cdot\tilde{N}_T^{\Delta y}$ and $(g^c\cdot M^{c,H})_-\cdot\tilde{N}_T^{\Delta y}$ are bounded. Therefore, taking into account the orthogonality of $\tilde{N}^{\Delta y}$ and $\hat{H}\cdot R$ under $\mathbb{Q}$, and lemma 4.15, we can rewrite (87) as

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \hat{H}_s^\top d\langle M^c\rangle_s(\tilde{N}_{s-}^{\Delta y}g_s^c) + \int_0^T \hat{H}_s^\top d\langle M^d\rangle_s(\tilde{N}_{s-}^{\Delta y}g_s^d)\right]. \tag{88}$$

Now, as $\int_0^T \hat{H}_s^\top d\langle M^d\rangle_s g_s^d$ is integrable (in view of Lemma 4.15 and Assumption 3.1) and $\int_0^\cdot \hat{H}_s^\top d\langle M^d\rangle_s g_s^d$ is predictable, by [Pro04, Theorem III.11], $\int_0^\cdot \hat{H}_s^\top d\langle M^d\rangle_s g_s^d$ is natural. Therefore, by the definition of natural processes as in [Pro04, page 111], we have

$$\mathbb{E}\left[\left[\int_0^\cdot \hat{H}_s^\top d\langle M^d\rangle_s g_s^d, \tilde{N}^{\Delta y}\right]_T\right] = 0. \tag{89}$$

Next, using [JS03, Theorem I.4.49], we deduce that

$$\begin{aligned}\int_0^T \hat{H}_s^\top d\langle M^d\rangle_s(\tilde{N}_{s-}^{\Delta y}g_s^d) &= \left(\int_0^T \hat{H}_s^\top d\langle M^d\rangle_s g_s^d\right)\tilde{N}_T^{\Delta y}\\ &\quad+\left(\int_0^\cdot \hat{H}_s^\top d\langle M^d\rangle_s g_s^d\right)_-\cdot\tilde{N}_T^{\Delta y} + \left[\int_0^\cdot \hat{H}_s^\top d\langle M^d\rangle_s g_s^d, \tilde{N}^{\Delta y}\right]_T\end{aligned}$$

As a result, from (89) and the assumptions of the lemma, which imply that

$$\mathbb{E}\left[\left(\int_0^\cdot \hat{H}_s^\top d\langle M^d\rangle_s g_s^d\right)_-\cdot\tilde{N}_T^{\Delta y}\right] = 0,$$

we obtain

$$\mathbb{E}\left[\int_0^T \hat{H}_s^\top d\langle M^d\rangle_s (\tilde{N}^{\Delta y}_{s-} g^d_s)\right] = \mathbb{E}\left[\left(\int_0^T \hat{H}_s^\top d\langle M^d\rangle_s g^d_s\right) \tilde{N}^{\Delta y}_T\right].$$

Similarly, we get

$$\mathbb{E}\left[\int_0^T \hat{H}_s^\top d\langle M^c\rangle_s (\tilde{N}^{\Delta y}_{s-} g^c_s)\right] = \mathbb{E}\left[\left(\int_0^T \hat{H}_s^\top d\langle M^c\rangle_s g^c_s\right) \tilde{N}^{\Delta y}_T\right].$$

We conclude that, we can rewrite (88) as

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T \hat{H}_s^\top d\langle M^c\rangle_s g^c_s + \int_0^T \hat{H}_s^\top d\langle M^d\rangle_s g^d_s\right) \tilde{N}^{\Delta y}_T\right], \tag{90}$$

which is precisely the (84) term. The assertion of the lemma follows from combining the estimates above.

□

Lemma 4.24. *Let $x \in \mathbb{R}$ be fixed, assumptions of Theorem 4.1 hold. Let β^ε, $\tilde{L}$, and $\tilde{N}^\varepsilon$ be as in Lemma 4.20. Then, we have*

$$\begin{aligned} &-\mathbb{E}_{\mathbb{Q}}\Big[\hat{X}_T\Big\{\left(F_T + \tilde{N}^\varepsilon_T\right)^2 - \left[F + \tilde{N}^\varepsilon\right]_T + 2\beta^\varepsilon \cdot \left((-\gamma^0)' \cdot M^{c,H} + (\nu^0)' \cdot M^{d,H}\right)_T \\ &\qquad -(\lambda^{0,c})'' \cdot M^{c,H}_T - (\lambda^{0,d})'' \cdot M^{d,H}_T + \beta \cdot \left((-\gamma^0)'' \cdot M^{c,H} + (\nu^0)'' \cdot M^{d,H}\right)_T\Big\}\Big] \\ &= 2\mathbb{E}_{\mathbb{Q}}\left[\left(F_T + \tilde{N}^\varepsilon_T\right) G_T\right], \end{aligned} \tag{91}$$

where processes F and G are defined in (18).

Proof. Using Lemma 4.12, one can show that, on the right-hand side of (91), we have

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[(-g^c \cdot M^{c,H}_T - g^d \cdot M^{d,H}_T)(\hat{H} \cdot \tilde{R}_T)\right] \\ =&\mathbb{E}_{\mathbb{Q}}\left[(-g^c \cdot M^{c,H}_T - g^d \cdot M^{d,H}_T)(\hat{H}_s^\top d\langle M^c\rangle_s g^c_s + \hat{H}_s^\top d\langle M^d\rangle_s g^d_s + \hat{H} \cdot \phi \cdot R^c_T + \hat{H} \cdot \psi \cdot R^d_T)\right] \\ =&\mathbb{E}_{\mathbb{Q}}\Big[(-g^c \cdot M^{c,H}_T - g^d \cdot M^{d,H}_T)(\hat{H}_s^\top d\langle M^c\rangle_s g^c_s + \hat{H}_s^\top d\langle M^d\rangle_s g^d_s) \\ &\qquad -\hat{H}_s^\top \phi_s d\langle M^c\rangle_s g^c_s - \hat{H}_s^\top \psi_s d\langle M^d\rangle_s g^d_s\Big], \end{aligned} \tag{92}$$

where, in the last equality, we used Lemma 4.15. Next, using the martingale property of $\Big(-g^c \cdot M^{c,H} - g^d \cdot M^{d,H} +$ $\Big[-g^c \cdot M^{c,H} - g^d \cdot M^{d,H} + \tilde{N}^\varepsilon\Big]$ under $\mathbb{Q}$ and the integration by parts formula, we can rewrite the

$$-\mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T\left\{\left(-g^c \cdot M^{c,H}_T - g^d \cdot M^{d,H}_T + \tilde{N}^\varepsilon_T\right)^2 - \left[-g^c \cdot M^{c,H} - g^d \cdot M^{d,H} + \tilde{N}^\varepsilon\right]_T\right\}\right]$$

term in the left-hand side of (91) as

$$\begin{aligned} &-\mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T\left\{\left(-g^c \cdot M^{c,H}_T - g^d \cdot M^{d,H}_T + \tilde{N}^\varepsilon_T\right)^2 - \left[-g^c \cdot M^{c,H} - g^d \cdot M^{d,H} + \tilde{N}^\varepsilon\right]_T\right\}\right] \\ =&2\mathbb{E}_{\mathbb{Q}}\left[-\hat{H} \cdot R_T\left\{\left(-g^c \cdot M^{c,H} - g^d \cdot M^{d,H} + \tilde{N}^\varepsilon\right)_- \cdot \left(-g^c \cdot M^{c,H} - g^d \cdot M^{d,H} + \tilde{N}^\varepsilon\right)_T\right\}\right]. \end{aligned} \tag{93}$$

In turn, using the orthogonality of $\tilde{N}^\varepsilon$ and $\hat{H} \cdot R$ under $\mathbb{Q}$, we can rewrite the right-hand side of (95) as

$$2\mathbb{E}_{\mathbb{Q}}\left[-\hat{H} \cdot R_T\left\{\left(-g^c \cdot M^{c,H} - g^d \cdot M^{d,H} + \tilde{N}^\varepsilon\right)_- \cdot \left(-g^c \cdot M^{c,H} - g^d \cdot M^{d,H}\right)_T\right\}\right],$$

which, in turn, via Lemma 4.15 can be restated as

$$2\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \hat{H}_s^\top \left(d\langle M^c\rangle_s g_s^c + d\langle M^d\rangle_s g_s^d\right)\left(-g^c\cdot M^{c,H} - g^d\cdot M^{d,H} + \tilde{N}^\varepsilon\right)_{s-}\right]. \tag{94}$$

Using Lemma 4.15 and Assumption 3.1, we deduce that $\int_0^\cdot \hat{H}_s^\top \langle M^c\rangle_s g_s^c + \int_0^\cdot \hat{H}_s^\top \langle M^d\rangle_s g_s^d$ is the predictable quadratic covariation of the pair $(\hat{H}\cdot R, g^c\cdot M^{c,H} + g^d\cdot M^{d,H})$. Via Assumption 3.1, Doob's maximal inequality (see [KS98, Theorem I.3.8(iv)]) and the Burkholder-Davis-Gundy inequality (see [CE15, Theorem 11.5.5 and Remark 11.5.8]), we deduce that there exists $\delta > 0$, such that $\left|\int_0^\cdot \hat{H}_s^\top \langle M^c\rangle_s g_s^c + \int_0^\cdot \hat{H}_s^\top \langle M^d\rangle_s g_s^d\right|^{1+\delta}$ is of class D in the sense of [KS98, Definition I.4.8]. Therefore, in view of Assumption 3.1 and Hölder's inequality, we have

$$\left(-g^c\cdot M^{c,H} - g^d\cdot M^{d,H} + \tilde{N}^\varepsilon\right)\left|\int_0^\cdot \hat{H}_s^\top \langle M^c\rangle_s g_s^c + \int_0^\cdot \hat{H}_s^\top \langle M^d\rangle_s g_s^d\right| \quad \text{is of class } D.$$

Now, via localization, integration by parts and [JS03, Proposition I.4.49(c)], we can rewrite (94) as

$$2\mathbb{E}_{\mathbb{Q}}\left[\left(-g^c\cdot M_T^{c,H} - g^d\cdot M_T^{d,H} + \tilde{N}_T^\varepsilon\right)\int_0^T \hat{H}_s^\top \left(d\langle M^c\rangle_s g_s^c + d\langle M^d\rangle_s g_s^d\right)\right].$$

We recapitulate that

$$\begin{aligned} &-\mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T\left\{\left(-g^c\cdot M_T^{c,H} - g^d\cdot M_T^{d,H} + \tilde{N}_T^\varepsilon\right)^2 - \left[-g^c\cdot M^{c,H} - g^d\cdot M^{d,H} + \tilde{N}^\varepsilon\right]_T\right\}\right] \\ &= 2\mathbb{E}_{\mathbb{Q}}\left[\left(-g^c\cdot M_T^{c,H} - g^d\cdot M_T^{d,H} + \tilde{N}_T^\varepsilon\right)\int_0^T \hat{H}_s^\top \left(d\langle M^c\rangle_s g_s^c + d\langle M^d\rangle_s g_s^d\right)\right]. \end{aligned} \tag{95}$$

Next, similarly to the proof of Lemma 4.23 (and to the proof of [MS24b, Lemma 5.19]), we can show that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[-\hat{X}_T\beta^\varepsilon\cdot\left((-\gamma^0)'\cdot M^{c,H} + (\nu^0)'\cdot M^{d,H}\right)_T\right] \\ =&\mathbb{E}_{\mathbb{Q}}\left[(\tilde{N}_T^\varepsilon)\left(\hat{H}\cdot\tilde{R}_T\right)\right] - \mathbb{E}_{\mathbb{Q}}\left[(\tilde{N}_T^\varepsilon)\left(\int_0^T \hat{H}_s^\top\left(d\langle M^c\rangle_s g_s^c + d\langle M^d\rangle_s g_s^d\right)\right)\right]. \end{aligned} \tag{96}$$

Now, in the left-hand side of (91), let us consider

$$\begin{aligned} -\mathbb{E}_{\mathbb{Q}}\Big[\hat{X}_T\Big\{&-(\lambda^{0,c})''\cdot M_T^{c,H} - (\lambda^{0,d})''\cdot M_T^{d,H} \\ &+\beta\cdot\left((-\gamma^0)''\cdot M^{c,H} + (\nu^0)''\cdot M^{d,H}\right)_T\Big\}\Big]. \end{aligned} \tag{97}$$

Using Assumption 3.1 and [Pro04, Corollary II.3], we can show that

$$-(\lambda^{0,c})''\cdot M^{c,H} - (\lambda^{0,d})''\cdot M^{d,H} + \beta\cdot\left((-\gamma^0)''\cdot M^{c,H} + (\nu^0)''\cdot M^{d,H}\right) \in \mathcal{H}_0^2(\mathbb{Q}).$$

Therefore, using Lemma 4.15, we can rewrite (97) as

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[\left(-\hat{H}\cdot R_T\right)\left\{-(\lambda^{0,c})''\cdot M_T^{c,H} - (\lambda^{0,d})''\cdot M_T^{d,H} + \beta\cdot\left((-\gamma^0)''\cdot M^{c,H} + (\nu^0)''\cdot M^{d,H}\right)_T\right\}\right] \\ =&\mathbb{E}_{\mathbb{Q}}\left[-\int_0^T \hat{H}_s^\top d\langle M^c\rangle_s\left((-\lambda_s^{0,c})'' + \left((-\gamma^0{}_s)''\right)^\top\beta_s\right) - \int_0^T \hat{H}_s^\top d\langle M^d\rangle_s\left((-\lambda_s^{0,d})'' + \left((\nu_s^0)''\right)^\top\beta_s\right)\right]. \end{aligned} \tag{98}$$

Next, using Lemma 4.14, we can rewrite the right-hand side of (98) as

$$
\begin{aligned}
&2\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \hat{H}_s^\top \phi_s d\langle M^c\rangle_s \left((-\lambda_s^{0,c})' + \left((-\gamma^0{}_s)'\right)^\top \beta_s\right) + \int_0^T \hat{H}_s^\top \psi_s d\langle M^d\rangle_s \left((-\lambda_s^{0,d})' + \left((\nu_s^0)'\right)^\top \beta_s\right)\right]\\
=&-2\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \hat{H}_s^\top \phi_s d\langle M^c\rangle_s g_s^c + \int_0^T \hat{H}_s^\top \psi_s d\langle M^d\rangle_s g_s^d\right].
\end{aligned}
$$

We recapitulate that

$$
\begin{aligned}
&-\mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T\left\{-(\lambda^{0,c})''\cdot M^{c,H} - (\lambda^{0,d})''\cdot M^{d,H} + \beta\cdot\left((-\gamma^0)''\cdot M^{c,H} + (\nu^0)''\cdot M^{d,H}\right)\right\}\right]\\
&= -2\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \hat{H}_s^\top \phi_s d\langle M^c\rangle_s g_s^c + \int_0^T \hat{H}_s^\top \psi_s d\langle M^d\rangle_s g_s^d\right].
\end{aligned}
\tag{99}
$$

Now, from (96), (92), (95) and (99), we conclude that (91) holds.

□

4.10. Bound for v.

Lemma 4.25. *Let $x \in \mathbb{R}$ be fixed and conditions of Theorem 4.1 hold, $y = u_x(x,0)$. Let $\beta^{\Delta y}$, β^ε, $\overline{L}$, and $\tilde{L}$ as in Lemma 4.20 and consider*

$$\Psi(\Delta y,\varepsilon) := \left(1+\frac{\Delta y}{y}\right)\mathcal{E}\left(J^{\Delta y,\varepsilon,H}\right)_T, \quad (\Delta y,\varepsilon)\in B_\delta(0,0),$$

where $J^{\Delta y,\varepsilon}$ is given in (77), $\delta\in(0,y)$ is sufficiently small, so that the jumps of $J^{\Delta y,\varepsilon,H}$ take values in $[-1/2,1/2]$, for every $(\Delta y,\varepsilon)\in B_\delta(0,0)$. Let us define

$$\tilde{w}(\Delta y,\varepsilon) := \mathbb{E}\left[V\left(Y_T\Psi(\Delta y,\varepsilon)\right)\right], \quad (\Delta y,\varepsilon)\in B_\delta(0,0).$$

Then $\tilde{w}$ admits the following expansion at $(0,0)$:

$$\tilde{w}(\Delta y,\varepsilon) = \tilde{w}(0,0) + (\Delta y\ \varepsilon)\nabla\tilde{w}(y,0) + \frac{1}{2}(\Delta y\ \varepsilon)H_{\tilde{w}}\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix} + o(\Delta y^2+\varepsilon^2),$$

where

$$\tilde{w}_{\Delta y}(0,0) = -v_y(y,0), \quad \tilde{w}_\varepsilon(y,0) = -y\mathbb{E}_{\mathbb{Q}}\left[\hat{X}_T F_T\right],$$

and

$$H_w := \begin{pmatrix}\tilde{w}_{\Delta y\Delta y} & \tilde{w}_{\Delta y\varepsilon}\\ \tilde{w}_{\Delta y\varepsilon} & \tilde{w}_{\varepsilon\varepsilon}\end{pmatrix}, \tag{100}$$

where, with processes F and G being given by (18) and $\tilde{N}^{\Delta y}$ and $\tilde{N}^\varepsilon$ by (76), respectively, the components of H_w are given by

$$
\begin{aligned}
\tilde{w}_{\Delta y\Delta y} &= y\mathbb{E}_{\mathbb{Q}}\left[B(\hat{Y}_T)\left(\frac{1}{y}+\tilde{N}_T^{\Delta y}\right)^2\right],\\
\tilde{w}_{\Delta y\varepsilon} &= y\mathbb{E}_{\mathbb{Q}}\left[B(\hat{Y}_T)\left(\frac{1}{y}+\tilde{N}_T^{\Delta y}\right)\left(F_T+\tilde{N}_T^\varepsilon\right) + \left(\frac{1}{y}+\tilde{N}_T^{\Delta y}\right)G_T\right],\\
\tilde{w}_{\varepsilon\varepsilon} &= y\mathbb{E}_{\mathbb{Q}}\left[B(\hat{Y}_T)\left(F_T+\tilde{N}_T^\varepsilon\right)^2 + 2(F_T+\tilde{N}_T^\varepsilon)G_T\right].
\end{aligned}
$$

Proof. The proof follows [MS24b, proof of Lemma 5.16]. Let us consider

$$\mathcal{E}\left(J^{\Delta y,\varepsilon,H}\right)_T = \exp\left(J_T^{\Delta y,\varepsilon,H} - \tfrac{1}{2}\left[J^{\Delta y,\varepsilon,H}\right]^c_T + \sum_{s\le T}\left(\log(1+\Delta J_s^{\Delta y,\varepsilon,H}) - \Delta J_s^{\Delta y,\varepsilon,H}\right)\right). \tag{101}$$

As

$$|\log(1+x)-x| \le x^2, \quad \mathit{for\ every} \quad x\in\left[-\tfrac{1}{2},\tfrac{1}{2}\right],$$

we obtain that, in (101), the series $\sum\limits_{s\le T}\left(\log(1+\Delta J_s^{\Delta y,\varepsilon,H}) - \Delta J_s^{\Delta y,\varepsilon,H}\right)$ converges absolutely for every $(\Delta y,\varepsilon)\in B_\delta(0,0)$, $\mathbb{P}$–a.s., and we have

$$\sum_{s\le T}\left|\log(1+\Delta J_s^{\Delta y,\varepsilon,H}) - \Delta J_s^{\Delta y,\varepsilon,H}\right| \le \left[J^{\Delta y,\varepsilon,H}\right]_T.$$

With g^c and g^d being given by (17), next, we observe that for every $(\Delta y,\varepsilon)\in B_\delta(0,0)$, there exists $C>0$, which does not depend on $(\Delta y,\varepsilon)$, such that

$$\mathcal{E}\left(J^{\Delta y,\varepsilon,H}\right)_T \le \frac{y}{y-\delta}\Psi(\Delta y,\varepsilon) \le C\exp\left(C|\varepsilon|\left(|F_T|+[F]_T\right)\right). \tag{102}$$

Furthermore, the series of term-by-term partial derivatives of

$$\sum_{s\le T}\left|\log(1+\Delta J_s^{\Delta y,\varepsilon,H}) - \Delta J_s^{\Delta y,\varepsilon,H}\right| \le \left[J^{\Delta y,\varepsilon,H}\right]_T$$

converges uniformly in $(\Delta y,\varepsilon)\in B_\delta(0,0)$, $\mathbb{P}$–a.s., where additionally the term-by-term derivatives of $\sum\limits_{s\le T}\left(\log(1+\Delta J_s^{\Delta y,\varepsilon,H}) - \Delta J_s^{\Delta y,\varepsilon,H}\right)$ are continuous in $(\Delta y,\varepsilon)\in B_\delta(0,0)$, $\mathbb{P}$–a.s.. Therefore, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\Delta y}\sum_{s\le T}\left(\log(1+\Delta J_s^{\Delta y,\varepsilon,H}) - \Delta J_s^{\Delta y,\varepsilon,H}\right) = -\sum_{s\le T}\frac{\Delta J_s^{\Delta y,\varepsilon,H}}{1+\Delta J_s^{\Delta y,\varepsilon,H}}\left(\left((\nu^\varepsilon{}_s)^\top\beta_s^{\Delta y}\right)\Delta M_s^{d,H} + \Delta\overline{L}_s^H\right),\\
&\frac{\partial}{\partial\varepsilon}\sum_{s\le T}\left(\log(1+\Delta J_s^{\Delta y,\varepsilon,H}) - \Delta J_s^{\Delta y,\varepsilon,H}\right) = -\sum_{s\le T}\frac{\Delta J_s^{\Delta y,\varepsilon,H}}{1+\Delta J_s^{\Delta y,\varepsilon,H}}\\
&\times\left(-(\lambda_s^{\varepsilon,d})'\Delta M_s^{d,H} + \left(((\nu^\varepsilon{}_s)')^\top\left(\beta_s+\Delta y\beta_s^{\Delta y}+\varepsilon\beta_s^\varepsilon\right)\right)\Delta M_s^{d,H} + \left((\nu^\varepsilon{}_s)^\top\beta_s^\varepsilon\right)\Delta M_s^{d,H} + \Delta\tilde{L}_s^H\right).
\end{aligned}$$

For a fixed Δy and ε, let us denote $I := J^{\Delta y,\varepsilon,H}$. By direct computations, we get

$$\begin{aligned}
\frac{\Psi_{\Delta y}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)} &= \left(\frac{1}{y+\Delta y} + \frac{\partial I}{\partial\Delta y} - \left[I^c,\frac{\partial I^c}{\partial\Delta y}\right] - \sum_{s\le\cdot}\frac{\frac{\partial\Delta I_s}{\partial\Delta y}\Delta I_s}{1+\Delta I_s}\right)_T,\\
\frac{\Psi_{\varepsilon}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)} &= \left(\frac{\partial I}{\partial\varepsilon} - \left[I^c,\frac{\partial I^c}{\partial\varepsilon}\right] - \sum_{s\le\cdot}\frac{\frac{\partial\Delta I_s}{\partial\varepsilon}\Delta I_s}{1+\Delta I_s}\right)_T.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\left.\frac{\Psi_{\Delta y}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)}\right|_{(\Delta y,\varepsilon)=(0,0)} &= \frac{1}{y} + \tilde{N}_T^{\Delta y},\\
\left.\frac{\Psi_{\varepsilon}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)}\right|_{(\Delta y,\varepsilon)=(0,0)} &= F_T + \tilde{N}_T^{\varepsilon}.
\end{aligned}$$

Similarly, we can show that the series of term-by-term second-order partial derivatives of

$$\sum_{s\leq T}\left(\log(1+\Delta J^{\Delta y,\varepsilon,H}_s)-\Delta J^{\Delta y,\varepsilon,H}_s\right)$$

converges uniformly in $(\Delta y,\varepsilon)\in B_\delta(0,0)$, where moreover the term-by-term second-order partial drivatives of $\sum\limits_{s\leq T}\left(\log(1+\Delta J^{\Delta y,\varepsilon,H}_s)-\Delta J^{\Delta y,\varepsilon,H}_s\right)$ are continuous in $(\Delta y,\varepsilon)\in B_\delta(0,0)$. Therefore, we get

$$\begin{aligned}\frac{\Psi_{\Delta y\Delta y}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)} &= \left(\left(\frac{1}{y+\Delta y}+\frac{\partial I}{\partial \Delta y}-\left[I^c,\frac{\partial I^c}{\partial\Delta y}\right]-\sum_{s\leq\cdot}\frac{\frac{\partial\Delta I_s}{\partial\Delta y}\Delta I_s}{1+\Delta I_s}\right)_T\right)^2\\ &\quad-\frac{1}{(y+\Delta y)^2}+\frac{\partial^2 I_T}{\partial\Delta y^2}-\left[I^c,\frac{\partial^2 I^c}{\partial\Delta y^2}\right]_T-\left[\frac{\partial I^c}{\partial\Delta y}\right]_T\\ &\quad-\sum_{s\leq T}\frac{\frac{\partial^2\Delta I_s}{\partial\Delta y^2}\Delta I_s+\left(\frac{\partial\Delta I_s}{\partial\Delta y}\right)^2}{1+\Delta I_s}+\sum_{s\leq T}\left(\frac{\frac{\partial\Delta I_s}{\partial\Delta y}}{1+\Delta I_s}\right)^2\Delta I_s.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\left.\frac{\Psi_{\Delta y\Delta y}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)}\right|_{(\Delta y,\varepsilon)=(0,0)} &= \left(\frac{1}{y}+\tilde N^{\Delta y}_T\right)^2-\frac{1}{y^2}-\left[\frac{\partial I^c}{\partial\Delta y}\right]_T-\sum_{s\leq T}\left(\frac{\partial\Delta I_s}{\partial\Delta y}\right)^2\\ &= \left(\frac{1}{y}+\tilde N^{\Delta y}_T\right)^2-\frac{1}{y^2}-\left[\tilde N^{\Delta y}\right]_T.\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}\frac{\Psi_{\Delta y\varepsilon}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)} &= \left(\frac{1}{y+\Delta y}+\frac{\partial I}{\partial\Delta y}-\left[I^c,\frac{\partial I^c}{\partial\Delta y}\right]-\sum_{s\leq\cdot}\frac{\frac{\partial\Delta I_s}{\partial\Delta y}\Delta I_s}{1+\Delta I_s}\right)_T\\ &\quad\times\left(\frac{\partial I}{\partial\varepsilon}-\left[I^c,\frac{\partial I^c}{\partial\varepsilon}\right]-\sum_{s\leq\cdot}\frac{\frac{\partial\Delta I_s}{\partial\varepsilon}\Delta I_s}{1+\Delta I_s}\right)_T+\frac{\partial^2 I_T}{\partial\Delta y\partial\varepsilon}-\left[I^c,\frac{\partial^2 I^c}{\partial\Delta y\partial\varepsilon}\right]_T-\left[\frac{\partial I^c}{\partial\varepsilon},\frac{\partial I^c}{\partial\Delta y}\right]_T\\ &\quad-\sum_{s\leq T}\frac{\frac{\partial^2\Delta I_s}{\partial\Delta y\partial\varepsilon}\Delta I_s+\frac{\partial\Delta I_s}{\partial\Delta y}\frac{\partial\Delta I_s}{\partial\varepsilon}}{1+\Delta I_s}+\sum_{s\leq T}\frac{\frac{\partial\Delta I_s}{\partial\Delta y}\frac{\partial\Delta I_s}{\partial\varepsilon}\Delta I_s}{(1+\Delta I_s)^2}.\end{aligned}$$

As a result, we get

$$\begin{aligned}\left.\frac{\Psi_{\Delta y\varepsilon}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)}\right|_{(\Delta y,\varepsilon)=(0,0)} &= \left(\frac{1}{y}+\tilde N^{\Delta y}_T\right)\left(F_T+\tilde N^{\varepsilon}_T\right)-\left[\frac{1}{y}+\tilde N^{\Delta y},F+\tilde N^{\varepsilon}\right]_T\\ &\quad+\beta^{\Delta y}\cdot\left((-\gamma^0)'\cdot M^{c,H}+(\nu^0)'\cdot M^{d,H}\right)_T\end{aligned}$$

Likewise, we obtain

$$\begin{aligned}\frac{\Psi_{\varepsilon\varepsilon}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)} &= \left(\left(\frac{\partial I}{\partial\varepsilon}-\left[I^c,\frac{\partial I^c}{\partial\varepsilon}\right]-\sum_{s\leq\cdot}\frac{\frac{\partial\Delta I_s}{\partial\varepsilon}\Delta I_s}{1+\Delta I_s}\right)_T\right)^2+\frac{\partial^2 I_T}{\partial\varepsilon^2}-\left[I^c,\frac{\partial^2 I^c}{\partial\varepsilon^2}\right]_T-\left[\frac{\partial I^c}{\partial\varepsilon}\right]_T\\ &\quad-\sum_{s\leq T}\frac{\frac{\partial^2\Delta I_s}{\partial\varepsilon^2}\Delta I_s+\left(\frac{\partial\Delta I_s}{\partial\varepsilon}\right)^2}{1+\Delta I_s}+\sum_{s\leq T}\left(\frac{\frac{\partial\Delta I_s}{\partial\varepsilon}}{1+\Delta I_s}\right)^2\Delta I_s.\end{aligned}$$

and thus

$$\left.\frac{\Psi_{\varepsilon\varepsilon}(\Delta y,\varepsilon)}{\Psi(\Delta y,\varepsilon)}\right|_{(\Delta y,\varepsilon)=(0,0)} = \left(F_T+\tilde N^\varepsilon_T\right)^2 - \left[F+\tilde N^\varepsilon\right]_T + 2\beta^\varepsilon\cdot\left((-\gamma^0)'\cdot M^{c,H}+(\nu^0)'\cdot M^{d,H}\right)_T$$
$$-(\lambda^{0,c})''\cdot M^{c,H}_T-(\lambda^{0,d})''\cdot M^{d,H}_T+\beta\cdot\left((-\gamma^0)''\cdot M^{c,H}+(\nu^0)''\cdot M^{d,H}\right)_T.$$

Next, let us fix $(\Delta y,\varepsilon)\in B_\delta(0,0)$, and consider

$$W(z):=V\left(\hat Y_T\Psi(z\Delta y,z\varepsilon)\right),\quad z\in(-1,1).$$

Then, we have

$$\begin{aligned}W'(z)&=V'\left(\hat Y_T\Psi(z\Delta y,z\varepsilon)\right)\hat Y_T\frac{\partial}{\partial z}\Psi(z\Delta y,z\varepsilon)\\&=V'\left(\hat Y_T\Psi(z\Delta y,z\varepsilon)\right)\hat Y_T\left(\Psi_{\Delta y}(z\Delta y,z\varepsilon)\Delta y+\Psi_\varepsilon(z\Delta y,z\varepsilon)\varepsilon\right).\end{aligned}$$

Using Lemma 4.9, we get

$$|W'(z)|\le\left(|V'(\hat Y_T)|+\frac{1}{c_1}\left|\log\left(\Psi(\Delta y,\varepsilon)\right)\right|\right)\hat Y_T\left|\Psi_{\Delta y}(z\Delta y,z\varepsilon)\Delta y+\Psi_\varepsilon(z\Delta y,z\varepsilon)\varepsilon\right|.$$

With

$$J:=1+|F|+[F]_T,$$

from (102), can see that there exists a constant $b_1>0$, such that, for every $z\in(-1,1)$, we have

(103)
$$\Psi(z\Delta y,z\varepsilon)\le b_1\exp\left(b_1\delta J\right),\quad \frac{1}{c_1}\log\left(\Psi(z\Delta y,z\varepsilon)\right)\le b_1\delta J,\quad \textit{and}\qquad |\Psi'(z\Delta y,z\varepsilon)|\le b_1J\exp\left(b_1\delta J\right),$$

so that

$$\sup_{z\in(-1,1)}|W'(z)|\le\hat Y_T\left(|\hat X_T|+b_1\delta J\right)\exp\left(b_1\delta J\right)b_1J. \tag{104}$$

Similarly, we get

$$W''(z)=V''\left(\hat Y_T\Psi(z\Delta y,z\varepsilon)\right)\hat Y_T^2\left(\frac{\partial}{\partial z}\Psi(z\Delta y,z\varepsilon)\right)^2+V'\left(\hat Y_T\Psi(z\Delta y,z\varepsilon)\right)\hat Y_T\frac{\partial^2}{\partial z^2}\Psi(z\Delta y,z\varepsilon).$$

Using Assumption 2.1 and Lemma 4.9, we deduce that

$$|W''(z)|\le\frac{\hat Y_T}{c_1}\frac{\left(\frac{\partial}{\partial z}\Psi(z\Delta y,z\varepsilon)\right)^2}{\Psi(z\Delta y,z\varepsilon)}+\hat Y_T\left(|V'(\hat Y_T)|+\frac{1}{c_1}|\log\left(\Psi(z\Delta y,z\varepsilon)\right)|\right)\left|\frac{\partial^2}{\partial z^2}\Psi(z\Delta y,z\varepsilon)\right|. \tag{105}$$

As

$$\frac{\partial^2}{\partial z^2}\Psi(z\Delta y,z\varepsilon)=\Psi_{\Delta y\Delta y}(z\Delta y,z\varepsilon)\Delta y^2+2\Psi_{\Delta y\varepsilon}(z\Delta y,z\varepsilon)\Delta y\varepsilon+\Psi_{\varepsilon\varepsilon}(z\Delta y,z\varepsilon)\varepsilon^2,$$

using (102), we deduce that there exists a constant $\tilde b>0$, such that, for every $z\in(-1,1)$, we have

$$\left|\frac{\partial^2}{\partial z^2}\Psi(z\Delta y,z\varepsilon)\right|\le\tilde b\exp\left(\tilde b\delta J\right)J^2. \tag{106}$$

From (105), using (103) and (106), we deduce that there exists a constant $b_2>0$, such that

$$\sup_{z\in(-1,1)}|W''(z)|\le b_2\hat Y_T\exp\left(b_2\delta J\right)J^2+\hat Y_T\left(|\hat X_T|+b_1\delta J\right)b_2\exp\left(b_2\delta J\right)J^2. \tag{107}$$

Now, from (104) and (107), we obtain that there exists a constant $b > 0$, such that, for every z_1 and z_2 in $(-1, 1)$, we have

$$\left|\frac{W(z_1) - W(z_2)}{z_1 - z_2}\right| + \left|\frac{W'(z_1) - W'(z_2)}{z_1 - z_2}\right| \le \hat{Y}_T \left(|\hat{X}_T| + b\delta J\right) \exp\left(b\delta J\right) bJ. \tag{108}$$

By passing to a smaller δ, if necessary, and by using Höllder's inequality, we deduce from Assumption 3.1 that the right-hand side of (108) is integrable. Furthermore, as the bound given by the right-hand side in (108) is uniform in $(\Delta y, \varepsilon) \in B_\delta(0, 0)$, we deduce the assertion of the lemma from the dominated convergence theorem and Lemmas 4.22, 4.23, and 4.24. □

Similarly to Lemma 4.18, as a consequence of Lemma 4.25 and [KS06a, Lemma 6], we can establish the following bound for the dual value function.

Lemma 4.26. *Let $x \in \mathbb{R}$ be fixed, suppose that the assumptions of Theorem 4.1 hold, and denote $y = u_x(x, 0)$. Then, we have*

$$v(y + \Delta y, \varepsilon) \le v(y, 0) + (\Delta y\ \varepsilon)\nabla v(y, 0) + \frac{1}{2}(\Delta y\ \varepsilon)H_v \begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix} + o(\Delta y^2 + \varepsilon^2),$$

where $\nabla v(y, 0)$ is given by (27).

4.11. **Proofs of Theorems 4.2, 4.3, 4.5, and 4.7.**

Proof of Theorem 4.2. First, using the optimality conditions for the optimizers to (19) and (21), we get

$$\begin{aligned} B(\hat{Y}_T)\left(\frac{1}{y} + N^y_T\right) &= v_{yy}\left(1 + M^x_T\right), \\ A(\hat{X}_T)\left(1 + M^x_T\right) &= -u_{xx}\left(\frac{1}{y} + N^y_T\right). \end{aligned} \tag{109}$$

Multiplying these two equations and taking the expectation under $\mathbb{Q}$, we deduce that

$$u_{xx}(x, 0)v_{yy}(y, 0) = -1. \tag{110}$$

Next, using the standard techniques of calculus of variations, for some $M \in \mathcal{M}^2$ and $N \in \mathcal{N}^2$, we get

$$\begin{aligned} B(\hat{Y}_T)\left(N^\varepsilon_T + F_T\right) + G_T &= d\left(1 + M_T\right), \\ A(\hat{X}_T)\left(M^\varepsilon_T + G_T\right) + F_T &= c\left(\frac{1}{y} + N_T\right). \end{aligned} \tag{111}$$

Multiplying the first equation by $\left(\frac{1}{y} + N^y_T\right)$ and the second by $(1 + M^x_T)$ and taking the expectation under $\mathbb{Q}$, we deduce that

$$d = v_{y\varepsilon} \quad \textit{and} \quad c = -u_{x\varepsilon}.$$

So, we can rewrite (111) as

$$\begin{aligned} B(\hat{Y}_T)\left(N^\varepsilon_T + F_T\right) &= v_{y\varepsilon}\left(1 + M_T\right) - G_T, \\ A(\hat{X}_T)\left(M^\varepsilon_T + G_T\right) &= -u_{x\varepsilon}\left(\frac{1}{y} + N_T\right) - F_T. \end{aligned} \tag{112}$$

$$
\begin{aligned}
B(\hat{Y}_T)\left(N_T^\varepsilon + F_T\right) &= v_{y\varepsilon}\left(1 + M_T^x + M_T - M_T^x\right) - G_T, \\
A(\hat{X}_T)\left(M_T^\varepsilon + G_T\right) &= -u_{x\varepsilon}\left(\frac{1}{y} + N_T^y + N_T - N_T^y\right) - F_T,
\end{aligned} \tag{113}
$$

which, using (109), we can rewrite as

$$
\begin{aligned}
B(\hat{Y}_T)\left(N_T^\varepsilon + F_T\right) &= \frac{v_{y\varepsilon}}{v_{yy}} B(\hat{Y}_T)\left(\frac{1}{y} + N_T^y\right) + v_{y\varepsilon}\left(M_T - M_T^x\right) - G_T, \\
A(\hat{X}_T)\left(M_T^\varepsilon + G_T\right) &= \frac{u_{x\varepsilon}}{u_{xx}} A(\hat{X}_T)\left(1 + M_T^x\right) - u_{x\varepsilon}\left(N_T - N_T^y\right) - F_T,
\end{aligned} \tag{114}
$$

Multiplying the first equation by $A(\hat{X}_T)$ and the second by $B(\hat{Y}_T)$, respectively, further deduce that

$$
\begin{aligned}
\left(N_T^\varepsilon + F_T\right) &= \frac{v_{y\varepsilon}}{v_{yy}}\left(\frac{1}{y} + N_T^y\right) + A(\hat{X}_T)\left(v_{y\varepsilon}\left(M_T - M_T^x\right) - G_T\right), \\
\left(M_T^\varepsilon + G_T\right) &= \frac{u_{x\varepsilon}}{u_{xx}}\left(1 + M_T^x\right) - B(\hat{Y}_T)\left(u_{x\varepsilon}\left(N_T - N_T^y\right) + F_T\right).
\end{aligned}
$$

Next, by rearranging terms, we obtain

$$
\begin{aligned}
A(\hat{X}_T)\left(v_{y\varepsilon}\left(M_T^x - M_T\right) + G_T\right) + F_T &= \frac{v_{y\varepsilon}}{v_{yy}}\left(\frac{1}{y} + N_T^y\right) - N_T^\varepsilon, \\
B(\hat{Y}_T)\left(u_{x\varepsilon}\left(N_T - N_T^y\right) + F_T\right) + G_T &= \frac{u_{x\varepsilon}}{u_{xx}}\left(1 + M_T^x\right) - M_T^\varepsilon.
\end{aligned} \tag{115}
$$

Using characterizations of the unique minimizers to (20) and (22), respectively, in (111), we get

$$
\begin{aligned}
M^\varepsilon &= v_{y\varepsilon}\left(M^x - M\right), \quad -u_{x\varepsilon} = \frac{v_{y\varepsilon}}{v_{yy}}, \\
N^\varepsilon &= u_{x\varepsilon}\left(N_T - N_T^y\right), \quad v_{y\varepsilon} = \frac{u_{x\varepsilon}}{u_{xx}}.
\end{aligned} \tag{116}
$$

Now, (110) and (116) imply (29). Next, plugging characterizations in (116) formulas into (115), we get

$$
\begin{aligned}
A(\hat{X}_T)\left(M_T^\varepsilon + G_T\right) &= -u_{x\varepsilon}\left(\frac{1}{y} + N_T^y\right) - \left(N_T^\varepsilon + F_T\right), \\
B(\hat{Y}_T)\left(N_T^\varepsilon + F_T\right) &= v_{y\varepsilon}\left(1 + M_T^x\right) - \left(M_T^\varepsilon + G_T\right).
\end{aligned} \tag{117}
$$

Now, (109) and (117) imply (30) and (31). It remains to show (30). Plugging the representations from (117) into (20) and (22), respectively, we conclude that

$$
\begin{aligned}
\frac{v_{\varepsilon\varepsilon}}{y} &= \mathbb{E}_{\mathbb{Q}}\left[v_{y\varepsilon}\left(1 + M_T^x\right)\left(N_T^\varepsilon + F_T\right) - \left(M_T^\varepsilon + G_T\right)\left(N_T^\varepsilon + F_T\right) + 2G_T N_T^\varepsilon\right] + 2\mathbb{E}_{\mathbb{Q}}\left[F_T G_T\right], \\
-\frac{u_{\varepsilon\varepsilon}}{y} &= \mathbb{E}_{\mathbb{Q}}\left[-u_{x\varepsilon}\left(\frac{1}{y} + N_T^y\right)\left(M_T^\varepsilon + G_T\right) - \left(N_T^\varepsilon + F_T\right)\left(M_T^\varepsilon + G_T\right) + 2M_T^\varepsilon F_T\right].
\end{aligned} \tag{118}
$$

Adding these equations, we get

$$
\begin{aligned}
\frac{v_{\varepsilon\varepsilon} - u_{\varepsilon\varepsilon}}{y} =& \mathbb{E}_{\mathbb{Q}}\left[v_{y\varepsilon}\left(1 + M_T^x\right)\left(N_T^\varepsilon + F_T\right) - \left(M_T^\varepsilon + G_T\right)\left(N_T^\varepsilon + F_T\right) + 2G_T N_T^\varepsilon\right] + 2\mathbb{E}_{\mathbb{Q}}\left[F_T G_T\right] \\
&+ \mathbb{E}_{\mathbb{Q}}\left[-u_{x\varepsilon}\left(\frac{1}{y} + N_T^y\right)\left(M_T^\varepsilon + G_T\right) - \left(N_T^\varepsilon + F_T\right)\left(M_T^\varepsilon + G_T\right) + 2M_T^\varepsilon F_T\right].
\end{aligned} \tag{119}
$$

Canceling the $\mathbb{E}_{\mathbb{Q}}\left[2M_T^\varepsilon F_T\right]$ and $\mathbb{E}_{\mathbb{Q}}\left[2G_T N_T^\varepsilon\right]$ terms and using the orthogonality of M^ε and N^ε, we can rewrite the right-hand side of (119) as

$$
\mathbb{E}_{\mathbb{Q}}\left[v_{y\varepsilon}\left(1 + M_T^x\right)\left(N_T^\varepsilon + F_T\right) - 2G_T F_T\right] + 2\mathbb{E}_{\mathbb{Q}}\left[F_T G_T\right] + \mathbb{E}_{\mathbb{Q}}\left[-u_{x\varepsilon}\left(\frac{1}{y} + N_T^y\right)\left(M_T^\varepsilon + G_T\right)\right].
$$

Next, cancelling the $\mathbb{E}_{\mathbb{Q}}[2G_TF_T]$ terms and using the orthogonality of M^ε and N^ε again, we can rewrite the latter expression as

$$\mathbb{E}_{\mathbb{Q}}\left[v_{y\varepsilon}\left(1+M^x_T\right)F_T - u_{x\varepsilon}\left(\frac{1}{y}+N^y_T\right)G_T\right].$$

We recapitulate that

$$\frac{v_{\varepsilon\varepsilon}-u_{\varepsilon\varepsilon}}{y} = \mathbb{E}_{\mathbb{Q}}\left[v_{y\varepsilon}\left(1+M^x_T\right)F_T - u_{x\varepsilon}\left(\frac{1}{y}+N^y_T\right)G_T\right]. \tag{120}$$

On the other hand, by multiplying the equations in (117) and taking the expectation under $\mathbb{Q}$, we obtain

$$\frac{1}{y}u_{x\varepsilon}v_{y\varepsilon} = \mathbb{E}_{\mathbb{Q}}\left[u_{x\varepsilon}\left(\frac{1}{y}+N^y_T\right)G_T - v_{y\varepsilon}\left(1+M^x_T\right)F_T\right].$$

Comparing this to (120), we get

$$\frac{v_{\varepsilon\varepsilon}-u_{\varepsilon\varepsilon}}{y} = -\frac{1}{y}u_{x\varepsilon}v_{y\varepsilon},$$

which implies (30). □

Proof of Theorem 4.3. With Theorem 4.2 and Lemmas 4.18 and 4.26 proven, the remaining steps parallel [MS24b, proof of Theorem 4.15]. □

Proof of Theorem 4.5. Let $(\varepsilon^n, \Delta y^n)$, $n \in \mathbb{N}$, be a sequence convergent to $(0,0)$ and such that $|\varepsilon^n| < \varepsilon_0$, where ε_0 is given by Lemma 2.8. Let us set

$$\hat\eta^n := \hat Y_T(y+\Delta y^n, \varepsilon^n), \quad n \in \mathbb{N},$$

and observe that, by Lemma 2.8, $\hat Y(y+\Delta y^n,\varepsilon^n)$'s are nonnegative $\mathbb{P}$-martingales.

Next, let us fix bounded predictable processes $\beta^{\Delta y}$ and β^ε and locally square-integrable $\mathbb{P}$-martingales $\overline{L}$ and $\tilde L$ satisfying the assumptions of Lemma 4.20, and with $J^{\Delta y^n,\varepsilon^n}$ be given by (77), let us define

$$\eta^n := \frac{\hat Y_T}{y}\left(y+\Delta y^n\right)\mathcal{E}\left(J^{\Delta y^n,\varepsilon^n,H}\right)_T, \quad \theta^n := \mathcal{E}\left(J^{\Delta y^n,\varepsilon^n,H}\right)_T, \quad n\in\mathbb{N}, \tag{121}$$

Similarly to the proof of Lemma 2.6, one can show that there exists $n_0 \in \mathbb{N}$, such that

$$\eta^n \in \mathbb{L}^1(\mathbb{P}) \quad \textit{and} \quad \theta^n \in \mathbb{L}^1(\mathbb{Q}), \quad n \geq n_0. \tag{122}$$

Let us fix $n \geq n_0$ and consider a correspondence (in the sense of [AB06, Definition 17.1]) $\tilde\psi : \Omega \to \mathbb{R}$ defined as

$$\tilde\psi(\omega) := \begin{cases} [\eta^n(\omega), \hat\eta^n(\omega)], & if \quad \eta^n(\omega) < \hat\eta^n(\omega), \\ [\hat\eta^n(\omega), \eta^n(\omega)], & if \quad \hat\eta^n(\omega) < \eta^n(\omega), \\ \{0\}, & if \quad \hat\eta^n(\omega) = \eta^n(\omega). \end{cases}$$

Then, the distance function associated with $\tilde\psi$ is given by

$$\begin{aligned}\delta(\omega,y) = &\left((y-\hat\eta^n(\omega))^+ + (\eta^n(\omega)-y)^+\right)1_{\{\eta^n(\omega)<\hat\eta^n(\omega)\}} \\ &+ \left((y-\eta^n(\omega))^+ + (\hat\eta^n(\omega)-y)^+\right)1_{\{\hat\eta^n(\omega)<\eta^n(\omega)\}} \\ &+ |y|1_{\{\hat\eta^n(\omega)=\eta^n(\omega)\}}, \quad (\omega,y)\in\Omega\times\mathbb{R}.\end{aligned}$$

One can see that δ is continuous for every $\omega \in \Omega$ and is measurable for every $y \in \mathbb{R}$, that is, δ is a Caratheodory function in the sense of [AB06, Definition 4.50]. Therefore, by [AB06, Theorem 18.5], $\tilde\psi$

is weakly measurable in the sense of [AB06, Definition 18.1]. Additionally, $\tilde{\psi}$ has nonempty compact values by its construction and

$$g(\omega, y) := \frac{1}{2}\left(-V''(y)\right)(\hat{\eta}^n(\omega) - \eta^n(\omega))^2, \quad (\omega, y) \in \Omega \times (0, \infty),$$

is also a Caratheodory function by the continuity of V'', which, in turn, follows from Assumption 2.1. Consequently, we deduce from the Measurable Maximum Theorem, [AB06, Theorem 18.19], that there exists a random variable ξ^n taking values in $[\eta^n \wedge \hat{\eta}^n, \eta^n \vee \hat{\eta}^n]$, such that

$$\xi^n(\omega) \in \arg\max_{y \in \tilde{\psi}(\omega)} g(\omega, y), \quad \omega \in \Omega. \tag{123}$$

That is, ξ^n is a measurable selector of the arg max correspondence. Therefore, we have

$$V(\eta^n) - V(\hat{\eta}^n) \geq V'(\hat{\eta}^n)(\eta^n - \hat{\eta}^n) + \frac{1}{2}V''(\xi^n)\left(\eta^n - \hat{\eta}^n\right)^2, \quad n \geq n_0. \tag{124}$$

As $-V'(\hat{\eta}^n)$ are optimizers to the primal problem, by [Sch01, Theorem 2.2], we get

$$\mathbb{E}\left[V'(\hat{\eta}^n)\hat{\eta}^n\right] = (y + \Delta y^n)v_y(y + \Delta y^n, \varepsilon^n), \quad n \geq n_0. \tag{125}$$

For every $n \geq n_0$, as $\mathbb{E}\left[V'(\hat{\eta}^n)\eta^n\right] = -\mathbb{E}\left[\hat{X}^n_T\eta^n\right]$, following the definition of the admissible wealth processes in [Sch01, Theorem 2.2] and its consequence, (13), let us consider an approximating sequence of bounded from below wealth processes with the initial wealth at most $x + \Delta x^n$, and denote this sequence $\tilde{X}^{n,m}$, $m \in \mathbb{N}$, such that $U\left(\tilde{X}^{n,m}_T\right)$ converges to $U\left(\hat{X}^n_T\right)$ in $\mathbb{L}^1(\mathbb{P})$. By Fatou's lemma, we get

$$\mathbb{E}_{\mathbb{Q}}\left[\hat{X}^n_T 1_{\{\hat{X}^n_T \geq 0\}}\theta^n\right] \leq \liminf_{m\to\infty} \mathbb{E}_{\mathbb{Q}}\left[\tilde{X}^{n,m}_T 1_{\{\tilde{X}^{n,m}_T \geq 0\}}\theta^n\right], \quad n \geq n_0. \tag{126}$$

From Lemma 42, we deduce that there exist constants $\tilde{c}_0 > 1$, $\tilde{c}_1 > 0$, and $\tilde{c}_2 > 0$, such that

$$(-x)^{4\tilde{c}_0} \leq \tilde{c}_1(U^-(x)) + \tilde{c}_2, \quad x \leq 0. \tag{127}$$

Therefore, we obtain

$$2\mathbb{E}_{\mathbb{Q}}\left[\left(-\tilde{X}^{n,m}_T 1_{\{\tilde{X}^{n,m}_T < 0\}}\theta^n\right)^{\tilde{c}_0}\right] \leq \mathbb{E}_{\mathbb{Q}}\left[\left(-\tilde{X}^{n,m}_T 1_{\{\tilde{X}^{n,m}_T < 0\}}\right)^{2\tilde{c}_0}\right] + \mathbb{E}_{\mathbb{Q}}\left[(\theta^n)^{2\tilde{c}_0}\right], \tag{128}$$

Next, from Assumption 3.1, similarly to the proof of Lemma 2.6, one can show that there exists $\tilde{n}_0 \geq n_0$, such that, for every $n \geq \tilde{n}_0$, the term the $\mathbb{E}_{\mathbb{Q}}\left[(\theta^n)^{2\tilde{c}_0}\right] < \infty$. Let us bound $\mathbb{E}_{\mathbb{Q}}\left[\left(-\tilde{X}^{n,m}_T 1_{\{\tilde{X}^{n,m}_T < 0\}}\right)^{2\tilde{c}_0}\right]$ term in (128) as follows.

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\left(-\tilde{X}^{n,m}_T 1_{\{\tilde{X}^{n,m}_T < 0\}}\right)^{2\tilde{c}_0}\right] &= \mathbb{E}\left[\frac{\hat{Y}_T}{y}\left(-\tilde{X}^{n,m}_T 1_{\{\tilde{X}^{n,m}_T < 0\}}\right)^{2\tilde{c}_0}\right] \\ &\leq \frac{1}{y}\mathbb{E}_{\mathbb{Q}}\left[\hat{Y}_T\right] + \mathbb{E}\left[\left(-\tilde{X}^{n,m}_T 1_{\{\tilde{X}^{n,m}_T < 0\}}\right)^{4\tilde{c}_0}\right].\end{aligned} \tag{129}$$

Here $\mathbb{E}_{\mathbb{Q}}\left[\hat{Y}_T\right] < \infty$ by Assumption 3.1 and $\mathbb{E}\left[\left(-\tilde{X}^{n,m}_T 1_{\{\tilde{X}^{n,m}_T < 0\}}\right)^{4\tilde{c}_0}\right]$ is bounded uniformly in m by (127) and the boundedness in $\mathbb{L}^1(\mathbb{P})$ of $U\left(\tilde{X}^{n,m}_T\right)$, $m \in \mathbb{N}$. (128) and (129) imply the uniform integrability of $(-\tilde{X}^{n,m}_T 1_{\{\tilde{X}^{n,m}_T < 0\}})\theta^n$, $m \in \mathbb{N}$, under $\mathbb{Q}$, for every $n \geq \tilde{n}_0$.

Therefore, by the construction of η^n in (121), we deduce that, $\left(-\tilde X^{n,m}_T\right)1_{\{\tilde X^{n,m}_T<0\}}\eta^n$, $m\in\mathbb{N}$, is uniformly integrable under $\mathbb{P}$, for every $n\geq \tilde n_0$, which together with (126) imply that

$$\mathbb{E}_{\mathbb{Q}}\left[\hat X^n_T\theta^n\right]\leq \liminf_{k\to\infty}\mathbb{E}\left[\tilde X^{n,m_k}_T\eta^n\right]\leq -v_y(y+\Delta y^n,\varepsilon^n),\quad n\geq \tilde n_0.$$

and thus

$$-\mathbb{E}\left[V'(\hat\eta^n)\eta^n\right]\leq -(y+\Delta y^n)v_y(y+\Delta y^n,\varepsilon^n),\quad n\in\mathbb{N}. \tag{130}$$

Now, from (125) and (130), we conclude that

$$\mathbb{E}\left[V'(\hat\eta^n)\left(\eta^n-\hat\eta^n\right)\right]\geq 0,\quad n\geq\tilde n_0. \tag{131}$$

Therefore, from (124) and (131), we obtain

$$\mathbb{E}\left[\tfrac{1}{2}V''(\xi^n)\left(\eta^n-\hat\eta^n\right)^2\right]\leq \mathbb{E}\left[V(\eta^n)\right]-v(y+\Delta y^n,\varepsilon^n),\quad n\geq\tilde n_0. \tag{132}$$

With $H_v(y,0)$ given by (34) in Theorem 4.3 and H_w given by (100) in Lemma 4.25, from Theorem 4.3 and Lemma 4.25, we get

$$\limsup_{n\to\infty}\frac{\mathbb{E}[V(\eta^n)]-v(y+\Delta y^n,\varepsilon^n)}{(\Delta y^n)^2+(\varepsilon^n)^2}\leq \frac{1}{2}|H_w-H_v(y,0)|. \tag{133}$$

Combining (132) and (133), we deduce that

$$\limsup_{n\to\infty}\frac{\mathbb{E}\left[\frac{1}{2}V''(\xi^n)\left(\eta^n-\hat\eta^n\right)^2\right]}{(\Delta y^n)^2+(\varepsilon^n)^2}\leq\frac{1}{2}|H_w-H_v(y,0)|. \tag{134}$$

We remark that, by the choice of $\beta^{\Delta y}$, β^{ε}, $\overline{L}$, $\tilde L$, such that $\beta^{\Delta y}\cdot\left(-\gamma^0\cdot M^{c,H}+\nu^0\cdot M^{d,H}\right)+\overline{L}^H$ and $\beta^{\varepsilon}\cdot\left(-\gamma^0\cdot M^{c,H}+\nu^0\cdot M^{d,H}\right)+\tilde L^H$ are close in $\mathcal{H}^2(\mathbb{Q})$ to the optimizers to (21) and (22), N^y and N^ε, respectively, we can make the right-hand side of (133) arbitrarily small. Such an approximation of elements of $\mathcal{N}^2$ by the bounded ones is possible by [KS06a, Lemma 6].

Next, via Hölder's inequality, we deduce that

$$\frac{\mathbb{E}[|\eta^n-\hat\eta^n|]}{|\Delta y^n|+|\varepsilon^n|}\leq\frac{\left((\Delta y^n)^2+(\varepsilon^n)^2\right)^{\frac12}}{|\Delta y^n|+|\varepsilon^n|}\frac{\mathbb{E}\left[\frac{1}{2}V''(\xi^n)\left(\eta^n-\hat\eta^n\right)^2\right]^{\frac12}}{\left((\Delta y^n)^2+(\varepsilon^n)^2\right)^{\frac12}}\mathbb{E}\left[\frac{2}{V''(\xi^n)}\right]^{\frac12}. \tag{135}$$

As Assumption 2.1 implies (9), by taking into account that ξ^n is given by by (123) and takes values in $[\eta^n\wedge\hat\eta^n,\eta^n\vee\hat\eta^n]$, from (9), we obtain

$$\frac{2}{V''(\xi^n)}\leq 2c_2\xi^n\leq 2c_2\left(\eta^n+\hat\eta^n\right),\quad n\geq\tilde n_0.$$

Furthermore, since η^n and $\hat\eta^n$ are terminal values of $\mathbb{P}$-martingales, we get

$$\mathbb{E}\left[\frac{2}{V''(\xi^n)}\right]\leq\mathbb{E}\left[2c_2\left(\eta^n+\hat\eta^n\right)\right]=4c_2(y+|\varepsilon^n|),\quad n\geq\tilde n_0. \tag{136}$$

From (135) and (136), we deduce that, there exists a constant $\tilde c>0$, which does not depend on the choice of $\beta^{\Delta y}$, β^ε, $\overline{L}$, and $\tilde L$, such that

$$\limsup_{n\to\infty}\frac{\mathbb{E}[|\eta^n-\hat\eta^n|]}{|\Delta y^n|+|\varepsilon^n|}\leq\tilde c\limsup_{n\to\infty}\frac{\mathbb{E}\left[\frac{1}{2}V''(\xi^n)\left(\eta^n-\hat\eta^n\right)^2\right]^{\frac12}}{\left((\Delta y^n)^2+(\varepsilon^n)^2\right)^{\frac12}}. \tag{137}$$

Combining (134) and (138), we obtain that

$$\limsup_{n\to\infty} \frac{\mathbb{E}[|\eta^n - \hat{\eta}^n|]}{|\Delta y^n| + |\varepsilon^n|} \leq \frac{\tilde{c}}{2}|H_w - H_v(y,0)|. \tag{138}$$

Next, we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{\mathbb{E}\left[\left|\hat{\eta}^n - \frac{\hat{Y}_T}{y}\left(y + \Delta y^n(1 + yN_T^y) + \varepsilon^n\left(yN_T^\varepsilon + yF_T\right)\right)\right|\right]}{|\Delta y^n| + |\varepsilon^n|} \\ \leq &\limsup_{n\to\infty} \frac{\mathbb{E}[|\hat{\eta}^n - \eta^n|]}{|\Delta y^n| + |\varepsilon^n|} \\ &+ \limsup_{n\to\infty} \frac{\mathbb{E}\left[\left|\eta^n - \frac{\hat{Y}_T}{y}\left(y + \Delta y^n(1 + yN_T^y) + \varepsilon^n\left(yN_T^\varepsilon + yF_T\right)\right)\right|\right]}{|\Delta y^n| + |\varepsilon^n|}. \end{aligned} \tag{139}$$

If we choose $\beta^{\Delta y}$, β^ε, $\overline{L}$, and $\tilde{L}$, such that $\beta^{\Delta y}\cdot\left(-\gamma^0\cdot M^{c,H} + \nu^0\cdot M^{d,H}\right) + \overline{L}^H$ and $\beta^\varepsilon\cdot\left(-\gamma^0\cdot M^{c,H} + \nu^0\cdot M^{d,H}\right) + \tilde{L}^H$ are sufficiently close in $\mathcal{H}^2(\mathbb{Q})$ to N^y and N^ε, respectively, we can make the right-hand side of (139) arbitrarily small, as the first term in the right-hand side can be made arbitrarily small by (138) and the second by the construction of η^n's in (121), which ensures that

$$\limsup_{n\to\infty} \frac{1}{|\Delta y^n| + |\varepsilon^n|}\mathbb{E}\left[\left|\eta^n - \frac{\hat{Y}_T}{y}\left(y + \Delta y^n(1 + yN_T^y) + \varepsilon^n\left(yN_T^\varepsilon + yF_T\right)\right)\right|\right]$$

can be made arbitrarily small by the choice of $\beta^{\Delta y}$, β^ε, $\overline{L}$, and $\tilde{L}$. Thus, via (139), we deduce that

$$\mathbb{L}^1(\mathbb{P})\text{-}\lim_{n\to\infty} \frac{1}{|\Delta y^n| + |\varepsilon^n|}\left|\hat{\eta}^n - \frac{\hat{Y}_T}{y}\left(y + \Delta y^n(1 + yN_T^y) + \varepsilon^n\left(yN_T^\varepsilon + yF_T\right)\right)\right| = 0.$$

This shows (37). (36) can be proven similarly. □

Proof of Corollary 4.6. By direct computations, we have

$$\begin{aligned} &\frac{1}{|\Delta y| + |\varepsilon|}\left\{\frac{\hat{Y}_T(y+\Delta y,\varepsilon)}{y+\Delta y} - \frac{\hat{Y}_T(y,0)}{y}\left(1 + \Delta y N_T^y + \varepsilon(F_T + N_T^\varepsilon)\right)\right\} \\ =&\frac{1}{y}\frac{1}{|\Delta y| + |\varepsilon|}\left\{\hat{Y}_T(y+\Delta y,\varepsilon) - \hat{Y}_T(y,0)\left(1 + \Delta y\left(\frac{1}{y} + N_T^y\right) + \varepsilon(F_T + N_T^\varepsilon)\right)\right\} \\ &+ \frac{1}{|\Delta y| + |\varepsilon|}\left\{\hat{Y}(y+\Delta y,\varepsilon)\left(\frac{1}{y+\Delta y} - \frac{1}{y}\right) + \hat{Y}_T\frac{1}{y^2}\Delta y\right\}, \end{aligned} \tag{140}$$

where, in the right-hand side, the first term

$$\frac{1}{y}\frac{1}{|\Delta y| + |\varepsilon|}\left\{\hat{Y}_T(y+\Delta y,\varepsilon) - \hat{Y}_T(y,0)\left(1 + \Delta y\left(\frac{1}{y} + N_T^y\right) + \varepsilon(F_T + N_T^\varepsilon)\right)\right\} \to 0 \quad in\ \mathbb{L}^1(\mathbb{P}),$$

by Theorem 4.5, and the second term can be rewritten as

$$\begin{aligned} &\frac{1}{|\Delta y| + |\varepsilon|}\left\{\hat{Y}(y+\Delta y,\varepsilon)\left(\frac{1}{y+\Delta y} - \frac{1}{y}\right) + \hat{Y}_T(y,0)\frac{1}{y^2}\Delta y\right\} \\ =&\frac{\Delta y}{|\Delta y| + |\varepsilon|}\left\{\frac{1}{y(y+\Delta y)}\left(\hat{Y}_T(y,0) - \hat{Y}_T(y+\Delta y,\varepsilon)\right)\right\} + \frac{\Delta y^2}{|\Delta y| + |\varepsilon|}\frac{\hat{Y}_T(y,0)}{y^2(y+\Delta y)}. \end{aligned}$$

In the latter expression, the first term, $\frac{\Delta y}{|\Delta y|+|\varepsilon|}\left\{\frac{1}{y(y+\Delta y)}\left(\hat{Y}_T(y,0) - \hat{Y}_T(y+\Delta y,\varepsilon)\right)\right\}$, converges to 0 in $\mathbb{L}^1(\mathbb{P})$ as a consequence of Theorem 4.5 and the second term, $\frac{\Delta y^2}{|\Delta y|+|\varepsilon|}\frac{\hat{Y}_T(y,0)}{y^2(y+\Delta y)}$, also converges to 0 also in $\mathbb{L}^1(\mathbb{P})$, as $\hat{Y}_T(y,0)\in\mathbb{L}^1(\mathbb{P})$. Combining these estimates, we conclude that the left-hand side in (140) converges to 0 in $\mathbb{L}^1(\mathbb{P})$, and so, (38) holds.

□

Proof of Theorem 4.7. The proof is very similar to the proof of [MS19, Theorem 3.1]; see also the proof of [Mos20, Theorem 4.2]. It is omitted for the brevity of the presentation. □

5. Proofs of the asymptotic expansion of the indifference prices

Lemma 5.1. *Let $x\in\mathbb{R}$ be fixed, suppose that the assumptions of Theorem 4.1 hold, and denote $y=u_x(x,0)$. Then, we have*

(141)
$$\mathbb{L}^1(\mathbb{P})\text{-}\lim \frac{1}{|\Delta y|+|\varepsilon|}\left|\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}f^\varepsilon-\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\left\{1+\Delta y N^y_T+\varepsilon(F_T+N^\varepsilon_T)\right\}f^0-\varepsilon\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}f'\right|=0.$$

Proof. Let us consider

$$\begin{aligned}&\frac{1}{|\Delta y|+|\varepsilon|}\left\{\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}f^\varepsilon-\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\left\{1+\Delta y N^y_T+\varepsilon(F_T+N^\varepsilon_T)\right\}f^0-\varepsilon\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}f'\right\}\\
=&\frac{1}{|\Delta y|+|\varepsilon|}\left\{\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}f^0-\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\left\{1+\Delta y N^y_T+\varepsilon(F_T+N^\varepsilon_T)\right\}f^0\right\}\\
&+\frac{1}{|\Delta y|+|\varepsilon|}\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}\left\{f^\varepsilon-f^0\right\}-\frac{\varepsilon}{|\Delta y|+|\varepsilon|}\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}f'.\end{aligned}\tag{142}$$

The first term in the right-hand side converges to 0 in $\mathbb{L}^1(\mathbb{P})$ by Corollary 4.6 and boundedness of f^0, which together imply that

$$\frac{1}{|\Delta y|+|\varepsilon|}\left\{\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}f^0-\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\left\{1+\Delta y N^y_T+\varepsilon(F_T+N^\varepsilon_T)\right\}f^0\right\},\quad (\Delta y,\varepsilon)\in B_\delta(0,0),$$

is uniformly integrable for some $\delta>0$. Therefore, as from Corollary 4.6, we also have

$$\mathbb{P}\text{-}\lim_{|\Delta y|+|\varepsilon|\to 0}\frac{1}{|\Delta y|+|\varepsilon|}\left\{\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}f^0-\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\left\{1+\Delta y N^y_T+\varepsilon(F_T+N^\varepsilon_T)\right\}f^0\right\}=0,$$

by the uniform integrability of this sequence, the convergence also holds in $\mathbb{L}^1(\mathbb{P})$.

Let us consider the other term on the right-hand side of (142). We can rewrite it as

$$\begin{aligned}&\frac{1}{|\Delta y|+|\varepsilon|}\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}\left\{f^\varepsilon-f^0\right\}-\frac{\varepsilon}{|\Delta y|+|\varepsilon|}\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}f'\\
=&\frac{1}{|\Delta y|+|\varepsilon|}\left\{\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}-\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\right\}\left\{f^\varepsilon-f^0\right\}+\frac{1}{|\Delta y|+|\varepsilon|}\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\left(f^\varepsilon-f^0-\varepsilon f'\right).\end{aligned}\tag{143}$$

By Corollary 4.6 and Assumption 3.2, which implies that $\left|f^\varepsilon-f^0\right|$ is bounded uniformly in $(\Delta y,\varepsilon)\in B_\delta(0,0)$, for some $\delta>0$, we deduce that, in the right-hand side of (143), we have

$$\mathbb{L}^1(\mathbb{P})\text{-}\lim_{|\Delta y|+|\varepsilon|\to 0}\frac{1}{|\Delta y|+|\varepsilon|}\left\{\frac{d\hat{\mathbb{Q}}(y+\Delta y,\varepsilon)}{d\mathbb{P}}-\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\right\}\left\{f^\varepsilon-f^0\right\}=0.$$

As for the remaining term on the right-hand side of (143), by Assumption 3.2, we have

$$\mathbb{L}^1(\mathbb{P})\text{-}\lim_{|\Delta y|+|\varepsilon|\to 0} \frac{1}{|\Delta y|+|\varepsilon|}\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\left(f^\varepsilon - f^0 - \varepsilon f'\right) = 0.$$

We conclude that every term in the right-hand side of (142) converges to 0 in $\mathbb{L}^1(\mathbb{P})$ as $(|\Delta y| + |\varepsilon|) \to 0$. ☐

Lemma 5.2. *Let $x \in \mathbb{R}$ be fixed and suppose that the assumptions of Theorem 4.1 hold. Then, we have*

$$\lim_{(\Delta x,\varepsilon)\to(0,0)} u_x(x+\Delta x,\varepsilon) = u_x(x,0). \tag{144}$$

Proof. By [Sch01, Theorem 2.2] and Lemma 2.6 we have

$$\hat{Y}_T(u_x(x+\Delta x,\varepsilon),\varepsilon) = U'(\hat{X}_T(x+\Delta x,\varepsilon)), \quad (\Delta y,\varepsilon)\in B_{\varepsilon_0}(0,0). \tag{145}$$

Since $\hat{X}_T(x+\Delta x, x) \to \hat{X}_T(x,0)$ in probability by Theorem 4.5, we deduce from (145) that

$$\lim_{(\Delta x,\varepsilon)\to(0,0)} \hat{Y}_T(u_x(x+\Delta x,\varepsilon),\varepsilon) = \hat{Y}_T(u_x(x,0),0), \quad \textit{in probability.} \tag{146}$$

As, by [Sch01, Theorem 2.2], we have $\hat{Y}_T(u_x(x+\Delta x,\varepsilon),\varepsilon) = u_x(x+\Delta x,\varepsilon)\frac{d\mathbb{Q}(u_x(x+\Delta x,\varepsilon),\varepsilon)}{d\mathbb{P}}$, that is, every dual minimizer in $B_{\varepsilon_0}(0,0)$ does not lose mass, we deduce from (146), the nonnegativity of the dual minimizers and Fatou's lemma that

$$\begin{aligned}\liminf_{(\Delta x,\varepsilon)\to(0,0)} u_x(x+\Delta x,\varepsilon) &= \liminf_{(\Delta x,\varepsilon)\to(0,0)} \mathbb{E}\left[\hat{Y}_T(u_x(x+\Delta x,\varepsilon),\varepsilon)\right] \\ &\geq \mathbb{E}\left[\hat{Y}_T(u_x(x,0),0)\right] = u_x(x,0).\end{aligned} \tag{147}$$

Next, let us observe that

$$\mathbb{E}\left[V\left(\hat{Y}_T(u_x(x+\Delta x,\varepsilon),\varepsilon)\right)\right] = v(u_x(x+\Delta x,\varepsilon),\varepsilon), \quad (\Delta x,\varepsilon)\in B_{\varepsilon_0}(0,0).$$

Therefore, from (146) and boundedness from below of V, using Fatou's lemma, we get

$$\begin{aligned}\limsup_{(\Delta x,\varepsilon)\to(0,0)} (-v(u_x(x+\Delta x,\varepsilon),\varepsilon)) &= -\liminf_{(\Delta x,\varepsilon)\to(0,0)} \mathbb{E}\left[V\left(\hat{Y}_T(u_x(x+\Delta x,\varepsilon),\varepsilon)\right)\right] \\ &\leq -\mathbb{E}\left[V\left(\hat{Y}_T(u_x(x,0),0)\right)\right] = -v(u_x(x,0),0),\end{aligned} \tag{148}$$

whereas, from Theorem 4.1, we obtain

$$\lim_{(\Delta x,\varepsilon)\to(0,0)} u(x+\Delta x,\varepsilon) = u(x,0). \tag{149}$$

Furthermore, from [Sch01, Theorem 2.2], we get

$$(x+\Delta x)u_x(x+\Delta x,\varepsilon) = u(x+\Delta x,\varepsilon) - v(u_x(x+\Delta x,\varepsilon),\varepsilon), \quad (\Delta x,\varepsilon)\in B_{\varepsilon_0}(0,0).$$

Therefore, using (148) and (149), we deduce that

$$\begin{aligned}\limsup_{(\Delta x,\varepsilon)\to(0,0)} (x+\Delta x)u_x(x+\Delta x,\varepsilon) &= \limsup_{(\Delta x,\varepsilon)\to(0,0)} \left(u(x+\Delta x,\varepsilon) - v(u_x(x+\Delta x,\varepsilon),\varepsilon)\right) \\ &= \lim_{(\Delta x,\varepsilon)\to(0,0)} u(x+\Delta x,\varepsilon) + \limsup_{(\Delta x,\varepsilon)\to(0,0)} (-v(u_x(x+\Delta x,\varepsilon),\varepsilon)) \\ &\leq u(x,0) - v(u_x(x,0),0) \\ &= xu_x(x,0),\end{aligned}$$

combining which with (147), we conclude that (144) holds.

□

Lemma 5.3. *Let $x \in \mathbb{R}$ be fixed, suppose that the assumptions of Theorem 4.1 hold, and $y = u_x(x, 0)$. Then, we have*

$$\liminf_{\varepsilon \to 0} (-v_y(y, \varepsilon)) \geq -v_y(y, 0) = x. \tag{150}$$

Proof. Let $\varepsilon_n \in (-\varepsilon_0, \varepsilon_0)$, $n \in \mathbb{N}$, be a sequence convergent to 0. For some bounded and predictable β^ε and bounded $\tilde{L}$ satisfying the assumptions of Lemma 4.20 and for J defined in (77), similarly to the proof of Theorem 4.5, let us consider

$$\hat{\eta}^n = \hat{Y}_T(y, \varepsilon_n) \quad \textit{and} \quad \eta^n := \frac{\hat{Y}_T}{y} \mathcal{E}\left(J^{0,\varepsilon^n,H}\right)_T, \quad n \in \mathbb{N}.$$

Then, we have

$$\hat{\eta}^n(-V'(\hat{\eta}^n)) = \hat{\eta}^n(-V'(\hat{\eta}^n)) - \eta^n(-V'(\eta^n)) + \eta^n(-V'(\eta^n)), \quad n \in \mathbb{N}. \tag{151}$$

Using Assumption 3.1, one can show that

$$\mathbb{L}^1(\mathbb{P})\text{-}\lim_{n \to \infty} \eta^n(-V'(\eta^n)) = \hat{Y}_T(-V'(\hat{Y}_T)) \tag{152}$$

and

$$\mathbb{L}^1(\mathbb{P})\text{-}\lim_{n \to \infty} \eta^n = \hat{Y}_T. \tag{153}$$

Therefore, using the mean value theorem for random variables, we get

$$\hat{\eta}^n(-V'(\hat{\eta}^n)) - \eta^n(-V'(\eta^n)) = -V''(\xi^n)\xi^n\left(\hat{\eta}^n - \eta^n\right) - V'(\xi^n)\left(\hat{\eta}^n - \eta^n\right), \quad n \in \mathbb{N}, \tag{154}$$

for some random variables ξ^n taking values between $\hat{\eta}^n$ and η^n. As by (9), we have

$$\frac{1}{c_2} \leq V''(\xi^n)\xi^n \leq \frac{1}{c_1},$$

we deduce from Theorem 4.5 and (153) that

$$\mathbb{L}^1(\mathbb{P})\text{-}\lim_{n \to \infty} \left|V''(\xi^n)\xi^n\left(\hat{\eta}^n - \eta^n\right)\right| \leq \mathbb{L}^1(\mathbb{P})\text{-}\lim_{n \to \infty} \frac{1}{c_1}\left|\hat{\eta}^n - \eta^n\right| = 0. \tag{155}$$

For the $V'(\xi^n)\left(\hat{\eta}^n - \eta^n\right)$ term in (154), by the monotonicity of V', we have

$$V'(\xi^n)\left(\hat{\eta}^n - \eta^n\right) \leq \left(V'(\hat{\eta}^n)\left(\hat{\eta}^n - \eta^n\right)\right) \vee \left(V'(\eta^n)\left(\hat{\eta}^n - \eta^n\right)\right). \tag{156}$$

Next, using the mean value theorem for random variables, we have

$$V'(\hat{\eta}^n)\left(\hat{\eta}^n - \eta^n\right) = V(\hat{\eta}^n) - V(\eta^n) + \frac{1}{2}V''(\tilde{\xi}^n)\left(\hat{\eta}^n - \eta^n\right)^2, \tag{157}$$

for some $\tilde{\xi}^n$ taking values between $\hat{\eta}^n$ and η^n.

Since $V(\hat{\eta}^n) \to V(\hat{Y}_T)$ and $V(\eta^n) \to V(\hat{Y}_T)$ in probability and

$$\mathbb{E}[V(\hat{\eta}^n)] \to \mathbb{E}[V(\hat{Y}_T)] \quad \textit{and} \quad \mathbb{E}[V(\eta^n)] \to \mathbb{E}[V(\hat{Y}_T)], \tag{158}$$

as V is bounded from below, by Schaffe's lemma, the convergence in probability and (158) imply that $V(\hat{\eta}^n) \to V(\hat{Y}_T)$ in $\mathbb{L}^1(\mathbb{P})$ and $V(\eta^n) \to V(\hat{Y}_T)$ in $\mathbb{L}^1(\mathbb{P})$. As for $\frac{1}{2}V''(\tilde{\xi}^n)\left(\hat{\eta}^n - \eta^n\right)^2$ term in the right-hand side of (157), it can also be proven to converge to 0 in $\mathbb{L}^1(\mathbb{P})$, similarly to the proof of Theorem 4.5. We deduce that, in the left-hand side of (157), we have

$$\lim_{n\to\infty} \mathbb{E}\left[V'(\hat{\eta}^n)\,(\hat{\eta}^n - \eta^n)\right] = 0.$$

Let us denote $A_n := \{V'(\eta^n)\,(\hat{\eta}^n - \eta^n) \geq 0\}$. Using the convexity of V, we have

$$(V(\hat{\eta}^n) - V(\eta^\varepsilon))\,1_{A_n} \geq V'(\eta^\varepsilon)\,(\hat{\eta}^n - \eta^\varepsilon)\,1_{A_n} \geq 0. \tag{159}$$

Therefore, as $V(\hat{\eta}^n) - V(\eta^n) \to 0$ in $\mathbb{L}^1(\mathbb{P})$, we obtain from (159) that $V'(\eta^n)\,(\hat{\eta}^n - \eta^n)\,1_{A_n} \to 0$ in $\mathbb{L}^1(\mathbb{P})$. As a result, by Fatou's lemma, we deduce that

$$\limsup_{n\to\infty} \mathbb{E}\left[V'(\eta^n)\,(\hat{\eta}^n - \eta^n)\right] \leq 0,$$

which via (156) implies that

$$\limsup_{n\to\infty} \mathbb{E}\left[V'(\xi^n)\,(\hat{\eta}^n - \eta^n)\right] \leq 0.$$

Combining this with (154) and (155), we get

$$\liminf_{n\to\infty} \mathbb{E}\left[\hat{\eta}^n(-V'(\hat{\eta}^n)) + \eta^n(-V'(\eta^n))\right] \geq 0. \tag{160}$$

As, from Theorem 4.5, we have that

$$\mathbb{P}\text{-}\lim_{n\to\infty} \hat{\eta}^n(-V'(\hat{\eta}^n)) = \hat{Y}_T(-V'(\hat{Y}_T)),$$

we conclude from (151), (152), (160), and Fatou's lemma that (150) holds. $\square$

Lemma 5.4. *Let $x \in \mathbb{R}$ be fixed and suppose that the assumptions of Theorem 4.1 hold. Then, we have*

$$\limsup_{|\Delta x|+|\varepsilon|\to 0} \frac{|u_x(x+\Delta x,\varepsilon) - u_x(x,0)|}{|\Delta x| + |\varepsilon|} < \infty. \tag{161}$$

Proof. First, following Remark 4.4, we observe that $u_{xx}(x+\Delta x,\varepsilon)$ is a second-order derivative of u with respect to x exists for every $(x,\varepsilon) \in \mathbb{R}\times(-\varepsilon_0,\varepsilon_0)$, and the absolute risk aversion of the indirect utility u satisfies the bounds of Assumption 3.1, that is

$$c_1 \leq -\frac{u_{xx}(x+\Delta x,\varepsilon)}{u_x(x+\Delta x,\varepsilon)} \leq c_2, \quad (\Delta x,\varepsilon) \in \mathbb{R}\times(-\varepsilon_0,\varepsilon_0).$$

Next, for every $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ let us set

$$\tilde{x}(\varepsilon) := -v_y(u_x(x,0),\varepsilon), \quad \varepsilon \in (-\varepsilon_0,\varepsilon_0),$$

and observe that by Lemma 5.3, we have

$$\liminf_{\varepsilon\to 0} \tilde{x}(\varepsilon) \geq -v_y(u_x(x,0),0) = x.$$

Therefore, by the mean-value theorem, there exists $\xi(\varepsilon) \in [(x+\Delta x)\wedge\tilde{x}(\varepsilon),(x+\Delta x)\vee\tilde{x}(\varepsilon)]$, such that

$$|u_x(x+\Delta x,\varepsilon) - u_x(\tilde{x},\varepsilon)| = -u_{xx}(\xi(\varepsilon),\varepsilon)|\Delta x| \leq c_2 u_x(\xi(\varepsilon),\varepsilon)|\Delta x| \leq c_2 u_x((x+\Delta x)\wedge\tilde{x}(\varepsilon),\varepsilon)|\Delta x|,$$

where, in the inequality, we have used (35). Therefore, we have

$$\begin{aligned}\frac{|u_x(x+\Delta x,\varepsilon)-u_x(x,0)|}{|\Delta x|+|\varepsilon|} &\leq \frac{|u_x(x+\Delta x,\varepsilon)-u_x(\tilde{x}(\varepsilon),\varepsilon)|+|u_x(\tilde{x}(\varepsilon),\varepsilon)-u_x(x,0)|}{|\Delta x|+|\varepsilon|}\\ &\leq \frac{c_2u_x((x+\Delta x)\wedge\tilde{x}(\varepsilon),\varepsilon)|\Delta x|+\left|\mathbb{E}\left[\hat{Y}_T(u_x(x,0),\varepsilon)-\hat{Y}_T(u_x(x,0),0)\right]\right|}{|\Delta x|+|\varepsilon|}.\end{aligned}$$

As a result, using Lemma 5.2, Lemma 5.3 and Theorem 4.5, we get

$$\begin{aligned}&\limsup_{|\Delta x|+|\varepsilon|\to 0}\frac{|u_x(x+\Delta x,\varepsilon)-u_x(x,0)|}{|\Delta x|+|\varepsilon|}\\ \leq&\limsup_{|\Delta x|+|\varepsilon|\to 0}\frac{c_2u_x((x+\Delta x)\wedge\tilde{x}(\varepsilon),\varepsilon)|\Delta x|+\left|\mathbb{E}\left[\hat{Y}_T(u_x(x,0),\varepsilon)-\hat{Y}_T(u_x(x,0),0)\right]\right|}{|\Delta x|+|\varepsilon|}\\ \leq&c_2u_x(x,0)+\mathbb{E}_{\mathbb{Q}}\left[|G+M^{\varepsilon}_T|\right],\end{aligned}$$

which implies (161). □

Lemma 5.5. *Let $x\in\mathbb{R}$ be fixed and suppose that the assumptions of Theorem 4.1 hold. Then, we have*

$$\lim_{|\Delta x|+|\varepsilon|\to 0}\frac{|u_x(x+\Delta x,\varepsilon)-u_x(x,0)-\Delta x u_{xx}(x,0)-\varepsilon u_{x\varepsilon}(x,0)|}{|\Delta x|+|\varepsilon|}=0. \tag{162}$$

Proof. Let ε_0 be as in Lemma 2.6 and let us denote

$$\begin{aligned}\Delta y&=\Delta y(\Delta x,\varepsilon):=u_{xx}\Delta x+u_{x\varepsilon}\varepsilon,\\ \overline{\Delta y}&=\overline{\Delta y}(\Delta x,\varepsilon):=u_x(x+\Delta x,\varepsilon)-u_x(x,0),\quad(\Delta x,\varepsilon)\in B_{\varepsilon_0}(0,0).\end{aligned} \tag{163}$$

and observe that, by Lemma 5.2, we have

$$\lim_{|\Delta x|+|\varepsilon|\to 0}\overline{\Delta y}(\Delta x,\varepsilon)=0,$$

and by Lemma 5.4, we have a stronger assertion

$$\limsup_{|\Delta x|+|\varepsilon|\to 0}\frac{|\overline{\Delta y}(\Delta x,\varepsilon)|}{|\Delta x|+|\varepsilon|}<\infty.$$

Next, by Lemma 2.6 and [Sch01, Theorem 2.2], we have the conjugacy relations between the value functions, that is

$$v(y+\Delta y(\Delta x,\varepsilon),\varepsilon)+(x+\Delta x)(y+\Delta y)=u(x+\Delta x,\varepsilon),\quad(\Delta x,\varepsilon)\in B_{\varepsilon_0}(0,0),$$

which implies that

$$v(y+\Delta y,\varepsilon)-v(y,0)+(x+\Delta x)\Delta y=u(x+\Delta x,\varepsilon)-u(x,0)-y\Delta x,\quad(\Delta x,\varepsilon)\in B_{\varepsilon_0}(0,0), \tag{164}$$

and, similarly, we get

$$v(y+\overline{\Delta y},\varepsilon)-v(y,0)+(x+\Delta x)\overline{\Delta y}=u(x+\Delta x,\varepsilon)-u(x,0)-y\Delta x,\quad(\Delta x,\varepsilon)\in B_{\varepsilon_0}(0,0). \tag{165}$$

From (164) and (165), we obtain

$$\begin{aligned}v(y+\overline{\Delta y},\varepsilon)-v(y,0)+(x+\Delta x)\overline{\Delta y}=v(y+\Delta y,\varepsilon)-v(y,0)+(x+\Delta x)\Delta y,\\ (\Delta x,\varepsilon)\in B_{\varepsilon_0}(0,0).\end{aligned} \tag{166}$$

Using Theorem 4.3 and Lemma 5.4, we deduce from (166) that

$$\begin{aligned}&(\overline{\Delta y}\quad \varepsilon)\nabla v(y,0)+\tfrac{1}{2}(\overline{\Delta y}\quad \varepsilon)H_v\begin{pmatrix}\overline{\Delta y}\\ \varepsilon\end{pmatrix}+(x+\Delta x)\overline{\Delta y}\\ =&(\Delta y\quad \varepsilon)\nabla v(y,0)+\tfrac{1}{2}(\Delta y\quad \varepsilon)H_v\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}+(x+\Delta x)\Delta y+o(\Delta x^2+\varepsilon^2),\end{aligned}$$

which simplifies to

$$\tfrac{1}{2}(\overline{\Delta y}\quad \varepsilon)H_v\begin{pmatrix}\overline{\Delta y}\\ \varepsilon\end{pmatrix}+\Delta x\overline{\Delta y}=\tfrac{1}{2}(\Delta y\quad \varepsilon)H_v\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}+\Delta x\Delta y+o(\Delta x^2+\varepsilon^2),$$

that is

$$v_{yy}((\overline{\Delta y}(\Delta x,\varepsilon))^2-\Delta y^2)+2v_{y\varepsilon}(\overline{\Delta y}(\Delta x,\varepsilon)-\Delta y)\varepsilon+2\Delta x(\overline{\Delta y}(\Delta x,\varepsilon)-\Delta y)=o(\Delta x^2+\varepsilon^2),$$

which implies that

$$\left(\overline{\Delta y}(\Delta x,\varepsilon)-\Delta y\right)\left(v_{yy}\left(\overline{\Delta y}(\Delta x,\varepsilon)+\Delta y\right)+2v_{y\varepsilon}\varepsilon+2\Delta x\right)=o(\Delta x^2+\varepsilon^2).$$

Using Theorem 4.2, we get

$$\left(\overline{\Delta y}(\Delta x,\varepsilon)-\Delta y\right)\left(\left(\overline{\Delta y}(\Delta x,\varepsilon)+\Delta y\right)-2\left(u_{x\varepsilon}\varepsilon+u_{xx}\Delta x\right)\right)=o(\Delta x^2+\varepsilon^2). \tag{167}$$

Since $u_{x\varepsilon}\varepsilon+u_{xx}\Delta x=\Delta y$, we can rewrite (167) as

$$\left(\overline{\Delta y}(\Delta x,\varepsilon)-\Delta y\right)^2=o(\Delta x^2+\varepsilon^2), \tag{168}$$

that is

$$|\overline{\Delta y}(\Delta x,\varepsilon)-\Delta y|=o(|\Delta x|+|\varepsilon|),$$

which, in view of (163), implies that (162) holds. ☐

Lemma 5.6. *Let $x\in\mathbb{R}$ be fixed, suppose that the assumptions of Theorem 4.1 hold and ε_0 is as in Lemma 2.6. Let us denote $y(x+\Delta x,\varepsilon)=u_x(x+\Delta x,\varepsilon)$, $(\Delta x,\varepsilon)\in B_{\varepsilon_0}(0,0)$. Then, we have*

$$\begin{aligned}\mathbb{L}^1(\mathbb{P})\text{-}\lim_{|\Delta x|+|\varepsilon|\to 0}\frac{1}{|\Delta x|+|\varepsilon|}&\left|\frac{d\hat{\mathbb{Q}}\left(y(x+\Delta x,\varepsilon),\varepsilon\right)}{d\mathbb{P}}\right.\\ &\left.-\frac{d\hat{\mathbb{Q}}\left(y(x,0),0\right)}{d\mathbb{P}}\left\{1+N^y_T\left(u_{xx}\Delta x+u_{x\varepsilon}\varepsilon\right)+(F_T+N^\varepsilon_T)\varepsilon\right\}\right|=0.\end{aligned} \tag{169}$$

Proof. For every $(\Delta x,\varepsilon)\in B_{\varepsilon_0}(0,0)$, let us denote

$$\Delta y=\Delta y(\Delta x,\varepsilon):=u_x(x+\Delta x,\varepsilon)-u_x(x,0). \tag{170}$$

To show (169), let us first observe that

$$\begin{aligned}&\frac{\left|\frac{d\hat{\mathbb{Q}}(y(x+\Delta x,\varepsilon),\varepsilon)}{d\mathbb{P}}-\frac{d\hat{\mathbb{Q}}(y(x,0),0)}{d\mathbb{P}}\left\{1+N^y_T\left(u_{xx}\Delta x+u_{x\varepsilon}\varepsilon\right)+(F_T+N^\varepsilon_T)\varepsilon\right\}\right|}{|\Delta x|+|\varepsilon|}\\ \le&\frac{(|\Delta y|+|\varepsilon|)}{(|\Delta x|+|\varepsilon|)}\frac{\left|\frac{d\hat{\mathbb{Q}}(y(x+\Delta x,\varepsilon),\varepsilon)}{d\mathbb{P}}-\frac{d\hat{\mathbb{Q}}(y(x,0),0)}{d\mathbb{P}}\left\{1+N^y_T\Delta y+(F_T+N^\varepsilon_T)\varepsilon\right\}\right|}{(|\Delta y|+|\varepsilon|)}\\ &+\frac{d\hat{\mathbb{Q}}\left(y(x,0),0\right)}{d\mathbb{P}}|N^y_T|\frac{|\Delta y-\Delta x u_{xx}-\varepsilon u_{x\varepsilon}|}{|\Delta x|+|\varepsilon|}.\end{aligned} \tag{171}$$

As $\Delta x \to 0$, it follows from (170) and Lemma 5.2 that $\Delta y \to 0$, so the first term on the right-hand side of (171) converges to 0 in $\mathbb{L}^1(\mathbb{P})$ by Corollary 4.6 and since, by Lemma 5.5, we have

$$\limsup_{|\Delta x|+|\varepsilon|\to 0} \frac{|\Delta y| + |\varepsilon|}{|\Delta x| + |\varepsilon|} = \limsup_{|\Delta x|+|\varepsilon|\to 0} \frac{|u_x(x+\Delta x, \varepsilon) - u_x(x,0)| + |\varepsilon|}{|\Delta x| + |\varepsilon|} \leq -u_{xx} + |u_{x\varepsilon}| + 1 < \infty.$$

It remains to show that

$$\mathbb{L}^1(\mathbb{P})\text{-}\lim_{|\Delta x|+|\varepsilon|\to 0} \frac{d\hat{\mathbb{Q}}\left(y(x,0),0\right)}{d\mathbb{P}} |N^y_T| \frac{|\Delta y - \Delta x u_{xx} - \varepsilon u_{x\varepsilon}|}{|\Delta x| + |\varepsilon|} = 0,$$

which, in view of (170) and the square-integrability of N^y_T under $\mathbb{Q}$, follows from

$$\lim_{|\Delta x|+|\varepsilon|\to 0} \frac{|\Delta y - \Delta x u_{xx} - \varepsilon u_{x\varepsilon}|}{|\Delta x| + |\varepsilon|} = \lim_{|\Delta x|+|\varepsilon|\to 0} \frac{|u_x(x+\Delta x, \varepsilon) - u_x(x,0) - \Delta x u_{xx} - \varepsilon u_{x\varepsilon}|}{|\Delta x| + |\varepsilon|} = 0,$$

that is established in Lemma 5.5. $\square$

Proof of Theorem 3.5. Let ε_0 be as in Lemma 2.6 and let us set

$$y(x+\Delta x, \varepsilon) = u_x(x+\Delta x, \varepsilon), \quad \Delta y = u_x(x+\Delta x, \varepsilon) - u_x(x,0), \quad (\Delta x, \varepsilon) \in B_{\varepsilon_0}(0,0).$$

Next, let us consider

$$\begin{aligned} &\frac{1}{|\Delta x| + |\varepsilon|} \left(\frac{d\hat{\mathbb{Q}}(u_x(x+\Delta x, \varepsilon), \varepsilon)}{d\mathbb{P}} f^\varepsilon \right. \\ &\qquad \left. - \frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} \left\{ 1 + N^y_T(u_{xx}\Delta x + u_{x\varepsilon}\varepsilon) + \varepsilon(F_T + N^\varepsilon_T) \right\} f^0 - \varepsilon \frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} f' \right) \\ &= \frac{f^0}{|\Delta x| + |\varepsilon|} \left(\frac{d\hat{\mathbb{Q}}(u_x(x+\Delta x, \varepsilon), \varepsilon)}{d\mathbb{P}} - \frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} \left\{ 1 + N^y_T(u_{xx}\Delta x + u_{x\varepsilon}\varepsilon) + \varepsilon(F_T + N^\varepsilon_T) \right\} \right) \\ &\quad + \frac{\left(f^\varepsilon - f^0\right)}{|\Delta x| + |\varepsilon|} \frac{d\hat{\mathbb{Q}}(u_x(x+\Delta x, \varepsilon), \varepsilon)}{d\mathbb{P}} - \varepsilon \frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} f'. \end{aligned} \tag{172}$$

On the first term on the right-hand side of (172), from Lemma 5.6 and the boundedness of f^0 by Assumption 3.2, we have

$$\begin{aligned} &\mathbb{L}^1(\mathbb{P})\text{-}\lim_{|\Delta x|+|\varepsilon|\to 0} \frac{1}{|\Delta x| + |\varepsilon|} \left| \left(\frac{d\hat{\mathbb{Q}}(u_x(x+\Delta x, \varepsilon), \varepsilon)}{d\mathbb{P}} \right. \right. \\ &\qquad \left. \left. - \frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} \left\{ 1 + N^y_T(u_{xx}\Delta x + u_{x\varepsilon}\varepsilon) + \varepsilon(F_T + N^\varepsilon_T) \right\} \right) f^0 \right| = 0. \end{aligned} \tag{173}$$

For the remaining term on the right-hand side of (172), we can rewrite it as

$$\begin{aligned} &\frac{\left(f^\varepsilon - f^0\right)}{|\Delta x| + |\varepsilon|} \frac{d\hat{\mathbb{Q}}(u_x(x+\Delta x, \varepsilon), \varepsilon)}{d\mathbb{P}} - \varepsilon \frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} f' \\ &= \frac{1}{|\Delta x| + |\varepsilon|} \left(\frac{d\hat{\mathbb{Q}}(u_x(x+\Delta x, \varepsilon), \varepsilon)}{d\mathbb{P}} - \frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} \right) \left(f^\varepsilon - f^0\right) + \frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} \left(f^\varepsilon - f^0 - \varepsilon f'\right), \end{aligned} \tag{174}$$

where $\mathbb{L}^1(\mathbb{P})\text{-}\lim_{|\Delta x|+|\varepsilon|\to 0} \frac{1}{|\Delta x|+|\varepsilon|} \left(\frac{d\hat{\mathbb{Q}}(u_x(x+\Delta x,\varepsilon),\varepsilon)}{d\mathbb{P}} - \frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}} \right) \left(f^\varepsilon - f^0\right) = 0$ by Theorem 4.5 and the uniform in ε boundedness of f^ε that together imply, for some $\delta > 0$, the uniform integrability of the family

$$\left\{\frac{1}{|\Delta x|+|\varepsilon|}\left(\frac{d\hat{\mathbb{Q}}(u_x(x+\Delta x,\varepsilon),\varepsilon)}{d\mathbb{P}}-\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\right)\left(f^\varepsilon-f^0\right):\ (\Delta x,\varepsilon)\in B_\delta(0,0)\right\}.$$

As for the remaining term in the right-hand side of (174), $\frac{d\hat{\mathbb{Q}}(y,0)}{d\mathbb{P}}\left(f^\varepsilon-f^0-\varepsilon f'\right)$, it converges to 0 in $\mathbb{L}^1(\mathbb{P})$, by Assumption 3.2. Thus, the left-hand side of (174) converges to 0 as $|\Delta x|+|\varepsilon|\to 0$ in $\mathbb{L}^1(\mathbb{P})$, which together with (172) and (173) imply that the left-hand side of (172) converges to 0 as $|\Delta x|+|\varepsilon|\to 0$ in $\mathbb{L}^1(\mathbb{P})$, and so (26) follows, where p_x and p_ε are given by (25). □

6. Link to the risk tolerance wealth process

In this section, we will characterize the solutions to quadratic minimization problems (20) and (22) through a Kunita-Watanabe decomposition under a change of measure and numéraire, provided that there exists a maximal wealth process $\mathcal{R}$, such that

$$\mathcal{R}_T=-\frac{U'(\hat{X}_T)}{U''(\hat{X}_T)}.$$

In the context of the utility defined on the positive real line, this process was introduced in [KS06b]. If the risk-tolerance wealth process $\mathcal{R}$ exists, let us change the measure and numéraire to

$$\frac{d\tilde{\mathbb{R}}}{d\mathbb{P}}=\frac{\mathcal{R}_T}{\mathcal{R}_0}\frac{\hat{Y}_T}{y}\quad \textit{and}\quad S^{\mathcal{R}}:=\left(\frac{\mathcal{R}_0}{\mathcal{R}},\frac{\mathcal{R}_0\mathcal{E}\left(R^1\right)}{\mathcal{R}},\dots,\frac{\mathcal{R}_0\mathcal{E}\left(R^d\right)}{\mathcal{R}}\right).$$

This leads to the sets of orthogonal martingales under the measure $\tilde{\mathbb{R}}$ and numéraire $\mathcal{R}$

$$\begin{aligned}&\tilde{\mathcal{M}}^2:=\left\{M\in\mathcal{H}^2_0(\tilde{\mathbb{R}}):\ M=H\cdot S^{\mathcal{R}}\right\},\\ &\tilde{\mathcal{N}}^2 \textit{ is the orthogonal complement of } \tilde{\mathcal{M}}^2 \textit{ in } \mathcal{H}^2_0(\tilde{\mathbb{R}}).\end{aligned}$$

Similarly to [MS19, Lemma 9.1], we can establish the following characterization of $\tilde{\mathcal{M}}^2$ and $\tilde{\mathcal{N}}^2$.

Lemma 6.1. *Let us suppose R is locally bounded d-dimensional semimartingale satisfying* (10)*, Assumptions 2.1 and 2.5 hold*[4]*. Let us also fix an $x\in\mathbb{R}$ and assume that the risk tolerance wealth processes $\mathcal{R}$ exists (for the base model at x). Then we have*

$$\tilde{\mathcal{M}}^2=\frac{\mathcal{M}^2}{\mathcal{R}}:=\left\{\frac{M}{\mathcal{R}}:\ M\in\mathcal{M}^2\right\}\quad \textit{and}\quad \tilde{\mathcal{N}}^2=\mathcal{N}^2.$$

Similarly to [MS24b, Proposition 6.2], we can establish the following lemma.

Lemma 6.2. *Let us suppose that $x\in\mathbb{R}$ is fixed and assume that the conditions of Lemma 6.1 hold. Then, the minimizers to* (19) *an* (21) *are given by*

$$M^x=\frac{\mathcal{R}}{\mathcal{R}_0}-1\quad \textit{and}\quad N^y=0.$$

If additionally Assumption 3.1 hold. Then the following process[5]

$$P_t:=-\mathbb{E}_{\tilde{\mathbb{R}}}\left[\left(F_T+\frac{G_T}{\mathcal{R}_T}\right)|\mathcal{F}_t\right],\quad t\in[0,T],$$

[4] These conditions imply the assertions of [Sch01, Theorem 2.2].

[5] We recall that processes F and G are defined in (18).

is well-defined and in $\mathcal{H}^2(\tilde{\mathbb{R}})$*. The Kunita-Watanabe decomposition of* P *under* $\tilde{\mathbb{R}}$

$$P = P_0 + \tilde{M}^\varepsilon + \tilde{N}^\varepsilon, \quad \textit{where } \tilde{M}^\varepsilon \in \tilde{\mathcal{M}}^2 \quad \textit{and} \quad \tilde{N}^\varepsilon \in \tilde{\mathcal{N}}^2.$$

is related to the optimizers to (20) *and* (22) *via*

$$M^\varepsilon = \tilde{M}^\varepsilon \mathcal{R}, \quad \textit{and} \quad N^\varepsilon = \tilde{N}^\varepsilon.$$

7. Examples of the indifference Greeks

7.1. **Indifference Delta.** In arbitrage-free pricing, e.g., in the Black-Scholes settings, Delta denotes the sensitivity of the arbitrage-free price with respect to the perturbations of the stock price, which is a state variable, that is, this price can be represented as a function of the stock price (and other parameters). For indifference pricing, the role of Delta is played by the initial wealth, that is, the state variable in (14) is x. Therefore, we call indifference Delta the sensitivity of the indifference prices to small perturbations of the initial wealth. For its representation, from Theorem 3.5, we get

$$p_x = \lim_{\Delta x \to 0} \frac{p(x+\Delta x, 0) - p(x,0)}{\Delta x} = \mathbb{E}_{\hat{\mathbb{Q}}}\left[\hat{N}^0_T f\right].$$

For the hedging strategy, from Theorem 4.7, we get the following

$$\tilde{X}^{\Delta x, 0, n} = (x + \Delta x) + \left(\hat{H} + \Delta x H^{\Delta x, n}\right) \cdot R^0,$$

where

$$H^{\Delta x, n} \cdot R^0 = M^{\Delta x, n}, \quad n \in \mathbb{N},$$

where $M^{x,n}_T \to M^x_T$, $\mathbb{P}$–a.s.. $M^{\Delta x, n}$, $n \in \mathbb{N}$, is a sequence of *bounded* processes converging to M^x, the solution to (19), it is characterized in Theorem 4.7.

We conclude this part by noting that, for the exponential utility function, as the absolute risk aversion $A(x)$ does not depend on $x \in \mathbb{R}$, the minimizer to (19) is $M^x = 0$. Therefore, we have

$$H^{\Delta x, n} = 0 \quad \textit{and} \quad \tilde{X}^{\Delta x, 0, n} = (x + \Delta x) + \hat{H} \cdot R^0,$$

where the later is the optimizer to (P) at $x + \Delta x$.

7.2. **Indifference Vega.** The derivative of the option price with respect to volatility is called Vega. To compute indifference Vega, in 17, we have a possibility to only perturb the continuous martingale part of the stock price or the martingale part of the stock price. Let us consider the first scenario. In this case, in (2), we need to take

$$\phi \neq 0, \quad \zeta \equiv 0, \quad \textit{and} \quad \psi \equiv 0.$$

In the case of only one risky asset, in (17), we obtain

$$g^c = -\phi(\lambda + \gamma^0 \beta) \quad \textit{and} \quad g^d \equiv 0,$$

so (18) becomes

$$F = -g^c \cdot M^{c,H} \quad \textit{and} \quad G = \hat{H} \cdot \tilde{R} = H \cdot (\phi \cdot M^c).$$

With these of processes F and G, the optimizers to (21) and (22), N^y and N^ε govern the sensitivity of the indifference prices to small perturbations of the volatility, which is represented by p_ε in Theorem 3.5. It is given by

$$p_\varepsilon = \mathbb{E}_{\mathbb{Q}}\left[\{(u_{x\varepsilon}N^y_T + (N^\varepsilon_T + F_T)\}f + f'\right] = v_{y\varepsilon}p_x + \mathbb{E}_{\mathbb{Q}}\left[(N^\varepsilon_T + F_T)\, f + f'\right],$$

where $p_x = u_{xx}\mathbb{E}_{\mathbb{Q}}\left[N^y_T f\right]$, and, in turn, u_{xx}, $u_{x\varepsilon}$, and $v_{y\varepsilon}$ are given by (19), (23), and (24), respectively. The corrections to the optimal wealth process are given by $H^{\varepsilon,n}$'s in Theorem 4.7, see (41) and also (39).

7.3. **Sensitivity to perturbations of the jump part M^d.** First, to the best of the author's knowledge, there is no special Greek letter denoting such a sensitivity for the arbitrage-free prices, as when the dynamics of the stock price process includes a nontrivial pure jump martingale, the model is (typically) incomplete, and the class of replicable contingent claims is very narrow. Each of such nonreplicable contingent claims allows for an interval of arbitrage-free prices. Therefore, the differentiability in the usual sense of such prices is not possible, and so the Greeks for arbitrage-free prices are not defined (in the usual sense for such nonreplicable contingent claims). We refer to [Shr04, Chapter 11] and [CT04, Chapter 10] for more details.

Perturbations of the pure jump martingale M^d are exactly the settings where the approach of this paper works when the traditional Greeks for the arbitrage-free pricing is not applicable except for some very particular contingent claims that are replicable for every ε in some neighborhood of 0 and some very particular models of the jumps, where the jump sizes are allowed to take very particular values, as elaborated in [Shr04, Chapter 11]. In the literature, however, models admitting jumps are quite widespread, and they are typically parametrized by more than one constant, see, e.g., [Mer76], [Kou02], [CT04], and [CK11]. The setup of the present paper uses one process ψ in (2) to describe the perturbations of M^d, and one can also use perturbations of $\langle M\rangle$ to parametrize perturbations of the jumps. We leave the sensitivity of various parameters governing jumps considered in the literature for future research and illustrate the case for which the framework of this paper allows.

To compute the sensitivities of the indifference prices to small perturbations of the pure jump martingale part of M, in (2), we take

$$\phi \equiv 0, \quad \zeta \equiv 0, \quad \textit{and} \quad \psi \neq 0.$$

In the case of only one risky asset, in (17), we get

$$g^c \equiv 0 \quad \textit{and} \quad g^d = -\psi(\lambda - \nu^0\beta),$$

so that in (18), we have

$$F = -g^d \cdot M^{d,H} \quad \textit{and} \quad G = \hat{H}\cdot\tilde{R} = \hat{H}\cdot(\psi\cdot M^d).$$

With these specifications of processes G and F, the optimizers to (21) and (22), N^y and N^ε govern the sensitivity of the indifference prices to small perturbations of the pure discontinuous martingale part of the stock price in the sense that

$$p_\varepsilon = \mathbb{E}_{\mathbb{Q}}\left[\{(u_{x\varepsilon}N^y_T + (N^\varepsilon_T + F_T)\}f + f'\right] = v_{y\varepsilon}p_x + \mathbb{E}_{\mathbb{Q}}\left[(N^\varepsilon_T + F_T)\, f + f'\right],$$

where $p_x = u_{xx}\mathbb{E}_{\mathbb{Q}}\left[N^y_T f\right]$, and, in turn, u_{xx}, $u_{x\varepsilon}$, and $v_{y\varepsilon}$ are given by (19), (23), and (24), respectively, and as a consequence of Theorem 3.5. $H^{\varepsilon,n}$'s in Theorem 4.7, see (41) and also (39), give the corrections to the optimal wealth process under perturbations of the pure discontinuous martingale part of the risky asset.

7.4. **Sensitivity to small perturbations of the finite variation part of R.** These sensitivities correspond to the following choices in (2) of ζ, ϕ, and ψ

$$\zeta \neq 0, \quad and \quad \phi = 0 = \psi.$$

In the case of only one risky asset, in (17), we have

$$g^c = \zeta = g^d,$$

and (18) becomes

$$F = -\zeta \cdot M^H \quad and \quad G = \hat{H} \cdot \tilde{R} = \int_0^\cdot H_s^\top d\langle M\rangle_s \zeta_s.$$

Theorem 3.5, again, for a given contingent claim f satisfying Assumption 3.2, produces the following sensitivity

$$p_\varepsilon = \mathbb{E}_{\mathbb{Q}}\left[\{(u_{x\varepsilon}N^y_T + (N^\varepsilon_T + F_T)\}f + f'\right] = v_{y\varepsilon}p_x + \mathbb{E}_{\mathbb{Q}}\left[(N^\varepsilon_T + F_T)\, f + f'\right],$$

where $p_x = u_{xx}\mathbb{E}_{\mathbb{Q}}\left[N^y_T f\right]$, u_{xx}, $u_{x\varepsilon}$, and $v_{y\varepsilon}$ are given by (19), (23), and (24), respectively, and N^y and N^ε are the optimizers to (21) and (22). $H^{\varepsilon,n}$'s in Theorem 4.7, see (41) and also (39), give the corrections to optimal strategies.

7.5. **Indifference ρ.** The framework of this paper allows us to compute the indifference ρ, the sensitivity to small perturbations of the interest rate in the settings, where the latter is deterministic. For simplicity of the presentation, let us consider the following model of the evolution of undiscounted traded assets, where $\overline{R}^\varepsilon$ is the d-dimensional process representing the returns of the risky assets, and $\overline{R}^{0,\varepsilon}$ is the return of the riskless assets

$$\overline{R}^\varepsilon = \int_0^\cdot \mu_s ds + \Sigma \cdot W + M^d,$$
$$\overline{R}^{0,\varepsilon} = \int_0^\cdot r^\varepsilon_s ds,$$

where μ and Σ are predictable and sufficiently integrable processes, W is a d-dimensional Brownian motion, and M^d is a d-dimensional pure jump martingale, whose quadratic covariation is absolutely continuous with respect to time, that is of the form

$$\langle M^d\rangle = \int_0^\cdot m^d_s ds, \tag{175}$$

for some predictable process m^d, taking values in symmetric positive definite d-by-d matrices.

Remark 7.1. If we suppose that there is only one risky asset present on the market, that is, if $\overline{R}^\varepsilon$ is one-dimensional, by supposing that M^d is a compound Poisson process of the form $M^d_t = \sum\limits_{i=1}^{N_t} Y_i - ct$, $t \in [0,T]$, where N is a Poisson process with intensity $\tilde{\lambda} > 0$, and Y_i are IID random variables with particular properties, we can include the framework of [Mer76] by taking Y_i to be normal (and constant

c can be chosen appropriately to preserve the martingale structure), and the framework in [Kou02], if $Y_i \geq -1$ and $\log(Y_i + 1)$ has an asymmetric double exponential distribution, and the framework in [CK11] by taking Y_i to be mixed-exponential random variables. In each of these cases, in (175), we have $m^d = Var(Y_1)\tilde{\lambda}$.

For the r^ε, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we suppose that this is a family of *deterministic* nonnegative functions on $[0, T]$ of the form

$$r^\varepsilon_t = r^0_t + \varepsilon \tilde{r}_t, \quad t \in [0, T],$$

where both r^0 and $\tilde{r}$ are uniformly bounded on $[0, T]$. We denote

$$r' := \frac{\partial}{\partial \varepsilon}\left(\int_0^T r^\varepsilon_s ds\right)\bigg|_{\varepsilon=0} = \int_0^T \tilde{r}_s ds. \tag{176}$$

Remark 7.2. Extending the assertions of this example to stochastic interest rates would likely require extending the results in [Sch01, Theorem 2.2] to stochastic utilities.

Let $\overline{\mathcal{X}}(x, \varepsilon)$ be the set of self-financing wealth processes obtained by trading in R^ε and $R^{0,\varepsilon}$ starting from the initial wealth $x \in \mathbb{R}$ and that are bounded from below. us formulate the utility maximization problem in undiscounted terms as

$$\overline{u}(\overline{x}, \varepsilon) = \sup_{X \in \overline{\mathcal{X}}(\overline{x}, \varepsilon)} \mathbb{E}[U(X_T)], \quad (\overline{x}, \varepsilon) \in \mathbb{R} \times (-\varepsilon_0, \varepsilon_0). \tag{177}$$

We remark that this formulation is closely related to (P), yet (177) is an extension of (P) to undiscounted traded assets.

Definition 7.3. Let $\overline{x} \in \mathbb{R}$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ be fixed. For a bounded contingent claim f^ε, its indifference price $\overline{p} = \overline{p}(\overline{x}, \varepsilon)$ is defined as a constant $\overline{p} \in \mathbb{R}$ such that

$$\mathbb{E}\left[U(X_T + \overline{q} f^\varepsilon)\right] \leq \overline{u}(\overline{x}, \varepsilon), \quad for\ every \quad \overline{q} \in \mathbb{R} \quad and\ every \quad X \in \overline{\mathcal{X}}(\overline{x} - q\overline{p}, \varepsilon). \tag{178}$$

Let us us a change of numéraire and, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, let us use the riskless asset as a numéraire. Thus, the evolution of the discounted traded assets is

$$\begin{aligned} R^\varepsilon &= \int_0^\cdot (\mu_s - r^\varepsilon_s) ds + \Sigma \cdot W + M^d, \\ R^{0,\varepsilon} &= 0, \end{aligned} \tag{179}$$

Let us denote by $\mathcal{X}(x, \varepsilon)$ the set of bounded from below self-financing discounted wealth processes, that is, measured in the units of the riskless asset. The associate value function is

$$u(x, \varepsilon) = \sup_{X \in \mathcal{X}(x, \varepsilon)} \mathbb{E}[U(X_T)], \quad (x, \varepsilon) \in \mathbb{R} \times (-\varepsilon_0, \varepsilon_0), \tag{180}$$

which is the formulation in (P), for which the results of the previous sections apply.

Let us denote

$$B^\varepsilon = \exp\left(\int_0^\cdot r^\varepsilon_s ds\right), \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0). \tag{181}$$

A change of numéraire approach gives

$$\overline{X}(\overline{x}, \varepsilon) = \mathcal{X}(\overline{x}, \varepsilon) B^\varepsilon, \quad (\overline{x}, \varepsilon) \in \mathbb{R} \times (-\varepsilon_0, \varepsilon_0). \tag{182}$$

Since B^ε_T is deterministic, we can further obtain from (182) that

$$\left\{\overline{X}_T:\ \overline{X}\in\overline{X}(\overline{x},\varepsilon)\right\} = \left\{X_T:\ X\in\mathcal{X}(\overline{x}B^\varepsilon_T,\varepsilon)\right\},\quad (\overline{x},\varepsilon)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0), \tag{183}$$

which results in

$$\overline{u}(\overline{x},\varepsilon) = u(\overline{x}B^\varepsilon_T,\varepsilon),\quad (\overline{x},\varepsilon)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0), \tag{184}$$

where $\overline{u}$ and u are defined in (177) and (180), respectively. Further, (183) and (184) imply that (178) can be rewritten as

$$\mathbb{E}\left[U(X_T+qf^\varepsilon)\right]\leq u(\overline{x}B^\varepsilon_T,\varepsilon),\quad \textit{for every } q\in\mathbb{R} \textit{ and every } X\in\mathcal{X}(\overline{x}B^\varepsilon_T-q\overline{p},\varepsilon). \tag{185}$$

Comparing (185) to Definition 2.3, we conclude that $\overline{p}$ is an indifference price in the sense of Definition 7.3 at $(\overline{x},\varepsilon)$, if an only is $\overline{p}$ is an indifference price in the sense of Definition 2.3 at $(\overline{x}B^\varepsilon_T,\varepsilon)$. We deduce that

$$\overline{p}(\overline{x},\varepsilon) = p(\overline{x}B^\varepsilon_T,\varepsilon),\quad (\overline{x},\varepsilon)\in\mathbb{R}\times(-\varepsilon_0,\varepsilon_0), \tag{186}$$

where $\overline{p}$'s are given by Definition 7.3 and p's by Definition 2.3.

Next, let us fix $\overline{x}\in\mathbb{R}$ and set $x:=\overline{x}B^0_T$. Supposing that the conditions of Theorem 3.5 hold at x, we deduce from this theorem and (186) that

$$\overline{p}_\varepsilon := \lim_{\varepsilon\to 0}\frac{\overline{p}(\overline{x},\varepsilon)-\overline{p}(\overline{x},0)}{\varepsilon} = p_x x r' + p_\varepsilon, \tag{187}$$

where p_x and p_ε are given by (25) and r' by (176). The constant $\overline{p}_\varepsilon$ is the sensitivity of the indifference price to small perturbations of the interest rate, that is, $\overline{p}_\varepsilon$ is the indifference ρ. It corresponds to taking $\zeta_t = -(m^d_t+\Sigma_t\Sigma_t^\top)^{-1}\tilde{r}_t$, $t\in[0,T]$, in (2), where m^d is given by (175), and considering the joint perturbations of the finite-variation part of the return of the risky assets as described in (179) and (perturbations) of the initial wealth of a particular form $\Delta x=\Delta x(\varepsilon)=\overline{x}(B^\varepsilon_T-B^0_T)$, where B^ε_T's are given by (181). Finally, the corrections to the optimal strategies for ε sufficiently close to 0 are given by Theorem 4.7 with $\Delta x=\Delta x(\varepsilon)=\overline{x}(B^\varepsilon_T-B^0_T)$.

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