SECOND-ORDER ENVELOPE THEOREMS IN PORTFOLIO OPTIMIZATION

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ABSTRACT. We develop a second-order sensitivity theory for stochastic control problems arising in portfolio optimization under perturbations of the objective function. Specifically, we consider utility maximization in incomplete semimartingale markets subject to smooth perturbations of investor preferences and establish second-order envelope theorems for the associated primal and dual value functions.

Our approach combines a perturbation theory for convex conjugates with the duality methods of stochastic control. We derive explicit first- and second-order asymptotic expansions for perturbed conjugate functions and use them to obtain second-order expansions of the value functions together with first-order expansions of the corresponding optimizers. The resulting second-order coefficients are characterized by quadratic minimization problems and admit alternative representations in terms of risk-tolerance wealth processes, when available, and Kunita–Watanabe decompositions. The perturbation theory for convex conjugates developed here may be of independent interest in convex and variational analysis.

1. INTRODUCTION

Envelope theorems are a cornerstone of optimization theory, providing sensitivity formulas for optimal values under perturbations of the objective function. While such results are classical in finite-dimensional settings, their extension to stochastic control problems is significantly more delicate. From a broader perspective, the problem can be viewed as a parametrized stochastic control problem, where perturbations act at the level of the objective functional rather than the system dynamics. In continuous-time portfolio optimization, even first-order envelope theorems require a careful interplay between duality methods and martingale techniques, as perturbations affect both the objective function and the optimizer. Obtaining second-order information is substantially more challenging and requires a refined combination of asymptotic analysis and structural properties of the problem.

This paper develops a second-order sensitivity theory for stochastic control problems arising in portfolio optimization in incomplete semimartingale markets under perturbations of investor preferences. More specifically, we consider smooth perturbations of the utility function and

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establish second-order envelope theorems for the associated primal and dual value functions, together with first-order expansions of the corresponding optimizers.

The motivation for such results is twofold. First, second-order sensitivity plays a central role in the asymptotic analysis and stability theory of utility maximization, complementing the well-developed first-order framework. Second, these expansions provide a natural mathematical framework for sensitivity of indifference prices and value functions with respect to preference parameters (often referred to as preference-based "Greeks"), where sensitivities arise alongside classical model perturbations. A growing body of work has studied sensitivity with respect to perturbations of market inputs, including changes in model coefficients and semimartingale characteristics (see, e.g., [MS19, MS24]). In contrast, perturbations of preferences act directly at the level of the objective function, leading to fundamentally different structures and requiring new analytical tools. The present paper develops a complementary perspective by focusing on preference perturbations and establishing a second-order sensitivity framework tailored to this setting.

A key feature of our approach is the interplay between two components. On the deterministic side, we develop a perturbation theory for convex conjugates, deriving explicit first- and second-order asymptotic expansions for perturbed conjugate functions. Classical stability properties of convex conjugation and infimal convolution under variational convergence are well understood; see, for example, [Mor70, Att84, RW98, Mor06], while perturbation analysis and envelope theorems have been extensively studied in optimization theory; see [BS00, MS02]. However, explicit second-order expansions for convex conjugates—equivalently, for infimal convolutions—appear to be largely unavailable in this generality. Related notions of second-order epi-differentiability and proto-differentiability have been extensively investigated in variational analysis; see, e.g., [Roc88, PR96, PRT00]. The results obtained here provide such expansions in a form suitable for stochastic control applications and yield a perturbation theory of independent interest in convex and variational analysis, with explicit first- and second-order expansions for the associated convex conjugates derived in Appendix A.

On the stochastic side, we combine these deterministic expansions with duality methods for portfolio optimization and the quadratic analysis of [KS06a]. This interaction allows us to lift second-order expansions to the level of stochastic control and to obtain tractable characterizations of the resulting sensitivity coefficients. The analysis reveals a Hilbert space structure underlying second-order sensitivity, governed by orthogonal decompositions of square-integrable martingale spaces under suitable endogenous probability measures and numéraires.

Our main results establish:

(1) A perturbation theory for convex conjugates, providing explicit first- and second-order expansions and derivative formulas for the associated conjugates;

(2) Second-order expansions for the primal and dual value functions with respect to both initial wealth and the preference perturbation parameter;
(3) Quadratic minimization representations for the second-order coefficients in terms of orthogonal projections in martingale spaces;
(4) First-order expansions of the optimal terminal wealth and state-price density, with derivatives characterized by the same quadratic minimization problems;
(5) Alternative representations of the sensitivity coefficients via risk-tolerance wealth processes and Kunita–Watanabe decompositions;

The perturbations are naturally encoded by the ratio of marginal utilities evaluated at the optimal terminal wealth, which serves as a fundamental perturbation variable. Remarkably, the second-order coefficients depend only on its orthogonal projections, revealing a unified Hilbert space structure underlying the sensitivity analysis. The results provide explicit second-order envelope theorems for stochastic control problems under preference perturbations in a general semimartingale setting, together with a systematic perturbation theory for convex conjugates adapted to this context.

The remainder of the paper is organized as follows. Section 2 contains the model formulation. Section 3 contains the main results, including second-order envelope theorems for the value functions and first-order expansions of the optimizers. Section 4 contains the proofs. Section 5 provides an alternative description of the second-order coefficients in terms of risk-tolerance wealth processes and Kunita–Watanabe decompositions. Appendix A contains first- and second-order expansions for perturbed convex conjugates.

2. Model formulation

We consider a financial market consisting of one riskless asset, whose discounted price is normalized to one, and (d) risky assets whose discounted price process $S = (S^1, \dots, S^d)$ is a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, where the filtration satisfies the usual conditions.

For $x > 0$, let

$$\mathcal{X}(x) := \{X \geq 0 : \ X = x + H \cdot S \text{ for some predictable } S\text{-integrable process } H\}$$

denote the family of nonnegative wealth processes with initial wealth x. For $y > 0$, let

$$\mathcal{Y}(y) := \{Y \geq 0 : \ Y_0 = y, \ XY \text{ is a supermartingale for every } X \in \mathcal{X}(1)\}$$

denote the family of supermartingale deflators. Throughout the paper we assume the absence of arbitrage in the sense of [KK07] that is equivalent to

$$\mathcal{Y}(1) \neq \emptyset. \tag{1}$$

Let $U : (0,\infty) \to \mathbb{R}$ be a utility function satisfying the Inada conditions

$$U'(0) = \infty, \qquad U'(\infty) = 0,$$

and let

$$V(y) := \sup_{x>0}\big(U(x) - xy\big), \qquad y > 0,$$

be its convex conjugate.

We consider perturbations of preferences of the form

$$U^\varepsilon := U + \varepsilon \widetilde{U}, \qquad \varepsilon \in \mathbb{R}, \tag{2}$$

where $\widetilde{U}$ is another utility function. The corresponding convex conjugate is given by

$$V^\varepsilon(y) := \sup_{x>0}\big(U^\varepsilon(x) - xy\big), \qquad y > 0.$$

For $(x,\varepsilon) \in (0,\infty) \times \mathbb{R}$, we define the primal value function

$$u(x,\varepsilon) := \sup_{X \in \mathcal{X}(x)} E[U^\varepsilon(X_T)], \tag{3}$$

and the dual value function

$$v(y,\varepsilon) := \inf_{Y \in \mathcal{Y}(y)} E[V^\varepsilon(Y_T)]. \tag{4}$$

For $\varepsilon = 0$, we write

$$u(x) := u(x,0), \qquad v(y) := v(y,0),$$

and assume that

$$v(y) = v(y,0) < \infty, \quad y > 0. \tag{5}$$

Assumption 2.1. There exists $\varepsilon_0 > 0$ such that, for every $|\varepsilon| < \varepsilon_0$, the utility function $U^\varepsilon := U + \varepsilon \widetilde{U}$ defined in (2) satisfies the Inada conditions and has relative risk aversion

$$A^\varepsilon(x) := -\frac{x(U^\varepsilon)''(x)}{(U^\varepsilon)'(x)}, \qquad x > 0,$$

uniformly bounded away from zero and infinity. That is, there exist constants $0 < a \le b < \infty$ such that

$$a \le A^\varepsilon(x) \le b, \qquad x > 0, \quad |\varepsilon| < \varepsilon_0.$$

For fixed $x > 0$ and $y = u_x(x,0)$, we denote by

$$\widehat{X} = \widehat{X}(x)$$

and

$$\widehat{Y} = \widehat{Y}(y)$$

the primal and dual optimizers corresponding to the unperturbed problem.

3. Second-order analysis

3.1. **Perturbation directions.** The key role in the second-order analysis is played by the sets $\mathcal{M}^2$ and $\mathcal{N}^2$ or complementary and orthogonal martingales under the probability measure defined via

$$\frac{d\mathbb{R}}{d\mathbb{P}} = \frac{\widehat{X}_T(x,0)\widehat{Y}_T(y,0)}{xy}.$$

$\mathcal{M}^2$ and $\mathcal{N}^2$ are introduced in [KS06a] in a two-step procedure. First, while the original assets are $\left(1, S^1, \dots, S^d\right)$, we change numéraire to

$$S^X := \left(\frac{x}{\widehat{X}}, \frac{xS^1}{\widehat{X}}, \dots, \frac{xS^d}{\widehat{X}}\right), \tag{6}$$

and second, we define

$$\mathcal{M}^2 := \left\{M \in \mathcal{H}_0^2(\mathbb{R}) : \ M = H \cdot S^X \ for\ some\ H\right\},$$

where $\mathcal{H}_0^2(\mathbb{R})$ is the set of square-integrable martingales under $\mathbb{R}$ with the initial value 0. The complement of $\mathcal{M}^2$ in $\mathcal{H}_0^2(\mathbb{R})$ is denoted by $\mathcal{N}^2$, that is

$$\mathcal{N}^2 := \{N \in \mathcal{H}_0^2(\mathbb{R}) : \ MN\ is\ a\ \mathbb{R}\ martingale\ for\ every\ M \in \mathcal{M}^2\}.$$

The second-order expansions naturally depend only on the induced perturbations under the numéraire $\widehat{X}$ and the measure $\mathbb{R}$. Therefore, it is convenient to formulate the expansions in terms of the Hilbert spaces $\mathcal{M}^2$ and $\mathcal{N}^2$. In order for $\mathcal{M}^2$ and $\mathcal{N}^2$ be complementary in $\mathcal{H}_0^2(\mathbb{R})$, we need the following assumption.

Assumption 3.1. Let us suppose that S^X is sigma-bounded in the sense of [KS06a, Definition 1].

The primal perturbations correspond to martingales in $\mathcal{M}^2$, whereas the dual perturbations correspond to orthogonal martingales in $\mathcal{N}^2$. The orthogonality representation

$$\mathcal{H}_0^2(\mathbb{R}) = \mathcal{M}^2 \oplus \mathcal{N}^2$$

plays a central role in the duality relations below.

Let us denote by $\mathcal{M}^\infty$ the family of uniformly bounded wealth processes under the numéraire $\widehat{X}$ with initial value 0, that is the family of semimartingales M such that for some $\delta = \delta(M) > 0$, we have

$$X(1+\delta M) \in \mathcal{X}(x) \quad and \quad X(1-\delta M) \in \mathcal{X}(x).$$

By $\mathcal{N}^\infty$ we denote the family of semimartingales N such that for some $\delta = \delta(N) > 0$, we have

$$Y(1+\delta N) \in \mathcal{Y}(y) \quad and \quad Y(1-\delta N) \in \mathcal{Y}(y).$$

Let

$$f_T := \frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)}. \tag{7}$$

We assume throughout that

$$f_T \in L^2(\mathbb{R}). \tag{8}$$

This is natural, since the quadratic minimization problems below (10) and (14) defining $u_{\varepsilon\varepsilon}$ and $v_{\varepsilon\varepsilon}$ contain f_T as the forcing term.

3.2. **Second-order derivatives and quadratic expansions.** Let us set

$$u_{xx}(x,0) = -\inf_{M\in\mathcal{M}^2} \frac{y}{x}\mathbb{E}^{\mathbb{R}}\left[A(X_T(x,0))\left(1+M_T\right)^2\right]. \tag{9}$$

$$u_{\varepsilon\varepsilon}(x,0) = -\inf_{M\in\mathcal{M}^2} \frac{y}{x}\mathbb{E}^{\mathbb{R}}\left[A(X_T(x,0))M_T^2 - 2x\frac{\widetilde{U}'(X_T(x,0))}{U'(X_T(x,0))}M_T\right]. \tag{10}$$

With $\widehat{M}^x$ and $\widehat{M}^\varepsilon$ denoting the optimizers to (9) and (10), respectively, we get

$$u_{x\varepsilon}(x,0) = -\frac{y}{x}\mathbb{E}^{\mathbb{R}}\left[A(X_T(x,0))(1+\widehat{M}^x_T)\widehat{M}^\varepsilon_T - x\frac{\widetilde{U}'(X_T(x,0))}{U'(X_T(x,0))}(1+\widehat{M}^x_T)\right]. \tag{11}$$

Remark 3.2. (Consistency check) In the particular case $\widetilde{U} = U$, as $\frac{\widetilde{U}'(X_T(x,0))}{U'(X_T(x,0))} \equiv 1$, for every $M \in \mathcal{M}^2$, the second term in (10) becomes

$$\frac{y}{x}\mathbb{E}^{\mathbb{R}}\left[2x\frac{\widetilde{U}'(X_T(x,0))}{U'(X_T(x,0))}M_T\right] = 0,$$

as M is a square-integrable martingale under $\mathbb{R}$, (10), in this case, becomes

$$u_{\varepsilon\varepsilon}(x,0) = -\inf_{M\in\mathcal{M}^2} \frac{y}{x}\mathbb{E}^{\mathbb{R}}\left[A(X_T(x,0))M_T^2\right] = 0,$$

where the infimum is reached at $\widehat{M}^\varepsilon \equiv 0$. Thus, the quadratic coefficient $u_{\varepsilon\varepsilon}(x,0)$ vanishes, as expected from the identity

$$U^\varepsilon = (1+\varepsilon)U.$$

This provides a simple consistency check for the representation (10).

Now, let us specify the quadratic expansion terms for the dual value function. Let us first introduce

$$B(y) := -\frac{(V^0)''(y)y}{(V^0)'(y)}, \quad y > 0,$$

and observe that if $y = U'(x)$, then

$$A(x)B(y) = 1. \tag{12}$$

Observe that by the envelope theorem type computations, we get

$$V^\varepsilon(y) = V^0(y) + \varepsilon\widetilde{U}(I(y)) - \frac{\varepsilon^2}{2}\frac{\left(\widetilde{U}'(I(y))\right)^2}{U''(I(y))} + o(\varepsilon^2), \quad y > 0.$$

$$v_{yy}(y,0) = \inf_{N\in\mathcal{N}^2} \frac{x}{y}\mathbb{E}^{\mathbb{R}}\left[B\left(\widehat{Y}_T(y,0)\right)(1+N_T)^2\right]. \tag{13}$$

$$v_{\varepsilon\varepsilon}(y,0) = \frac{x}{y}\inf_{N\in\mathcal{N}^2} \mathbb{E}^{\mathbb{R}}\left[B(\widehat{Y}_T)\left(N_T - y\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)}\right)^2\right]. \tag{14}$$

With $\widehat{N}^y$ and $\widehat{N}^\varepsilon$ denoting the optimizers to (13) and (14), respectively, let us set

$$v_{y\varepsilon}(y,0) = \frac{x}{y}\mathbb{E}^{\mathbb{R}}\left[B(\widehat{Y}_T)\left(1+\widehat{N}^y_T\right)\left(\widehat{N}^\varepsilon_T - y\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)}\right)\right]. \tag{15}$$

Remark 3.3. (Consistency check) In particular, if $\widetilde{U} = U$, then

$$U^\varepsilon = (1+\varepsilon)U,$$

and therefore

$$V^\varepsilon(y) = (1+\varepsilon)V\left(\frac{y}{1+\varepsilon}\right). \tag{16}$$

Consequently,

$$v(y,\varepsilon) = (1+\varepsilon)v\left(\frac{y}{1+\varepsilon},0\right).$$

To show the heuristic consistency, assuming enough differentiability of v and differentiating twice with respect to ε at $\varepsilon = 0$, yields

$$v_{\varepsilon\varepsilon}(y,0) = y^2 v_{yy}(y,0). \tag{17}$$

On the other hand, if $\widetilde{U} = U$, then

$$\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)} \equiv 1,$$

and (14) becomes

$$v_{\varepsilon\varepsilon}(y,0) = \frac{x}{y}\inf_{N\in\mathcal{N}^2} \mathbb{E}^R\left[B(\widehat{Y}_T)(N_T - y)^2\right]. \tag{18}$$

Factoring out y^2, using the fact that $N \to -\frac{N}{y}$ is a bijection of $\mathcal{N}^2$ onto itself, and by comparing to (13), we can rewrite the latter equality as

$$v_{\varepsilon\varepsilon}(y,0) = y^2\left(\frac{x}{y}\inf_{N\in\mathcal{N}^2} \mathbb{E}^R\left[B(\widehat{Y}_T)(N_T+1)^2\right]\right) = y^2 v_{yy}(y,0),$$

in agreement with (17). In particular, comparing (13) and (18), we conclude that the minimum in (18) is attained for

$$\widehat{N}^\varepsilon_T = -y\widehat{N}^y_T, \tag{19}$$

where $\widehat{N}^y$ is the minimizer to (13). Thus, if $\widetilde{U} = U$, (14) becomes analogous to (13), as perturbations in ε for the primal objective, translate into the perspective mapping of the dual objective as in (16).

We summarize that, if $\widetilde{U} = U$, similarly to (16), (14) reproduces the identity

$$v(y,\varepsilon) = (1+\varepsilon)v\left(\frac{y}{1+\varepsilon}, 0\right), \tag{20}$$

and, additionally, (13) and (14) satisfy

$$v_{\varepsilon\varepsilon}(y,0) = y^2\left(\frac{x}{y}\inf_{N\in\mathcal{N}^2}\mathbb{E}^R\left[B(\widehat{Y}_T)\left(N_T+1\right)^2\right]\right) = y^2 v_{yy}(y,0),$$

which can also be heuristically obtained from differentiation in (20). This provides a consistency check for the dual representation.

Theorem 3.4. *Let $x>0$ be fixed. Suppose that* (1), (5), (8), *and Assumptions 2.1 and 3.1 hold. Assume, in addition, that*

$$\widetilde{U}(\widehat{X}_T) \in L^1(P), \qquad \widetilde{U}'(\widehat{X}_T)\widehat{X}_T \in L^1(P). \tag{21}$$

Then, we have

$$\begin{aligned} u(x+\Delta x,\varepsilon) =& u(x,0) + \begin{pmatrix} y \\ \mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right]\end{pmatrix}^\top \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix} \\ &+\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^\top \begin{pmatrix} u_{xx}(x,0) & u_{x\varepsilon}(x,0)\\ u_{x\varepsilon}(x,0) & u_{\varepsilon\varepsilon}(x,0)\end{pmatrix}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix} + o(\Delta x^2+\varepsilon^2).\end{aligned} \tag{22}$$

$$\begin{aligned} v(y+\Delta y,\varepsilon) =& v(y,0) + \begin{pmatrix} -x \\ \mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right]\end{pmatrix}^\top \begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix} \\ &+\frac{1}{2}\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}^\top \begin{pmatrix} v_{yy}(y,0) & v_{y\varepsilon}(y,0)\\ v_{y\varepsilon}(y,0) & v_{\varepsilon\varepsilon}(y,0)\end{pmatrix}\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix} + o(\Delta y^2+\varepsilon^2).\end{aligned} \tag{23}$$

The first-order sensitivity of the value functions with respect to perturbations of preferences admits a particularly simple form. The following envelope-type theorem is the corollary of Theorem 3.4, and follows immediately from the second-order expansions established in Theorem 3.4.

Corollary 3.5. *Let $x>0$ be fixed. Suppose that the assumptions of Theorem 3.4 hold. Then, the primal value function satisfies*

$$u(x,\varepsilon) = u(x,0) + \varepsilon E[\widetilde{U}(\widehat{X}_T(x))] + o(\varepsilon), \qquad \varepsilon\to 0,$$

where $\widehat{X}(x)$ is the optimizer corresponding to the utility function U. In particular,

$$u_\varepsilon(x,0) = E[\widetilde{U}(\widehat{X}_T(x))].$$

Similarly, if $y = u_x(x,0)$*, then*

$$v(y,\varepsilon) = v(y,0) + \varepsilon E[\widetilde{U}(\widehat{X}_T(x))] + o(\varepsilon), \qquad \varepsilon \to 0,$$

and

$$v_\varepsilon(y,0) = E[\widetilde{U}(\widehat{X}_T(x))].$$

Remark 3.6. The identities

$$u_\varepsilon(x,0) = v_\varepsilon(y,0) = E[\widetilde{U}(\widehat{X}_T)]$$

may be viewed as envelope theorems for the primal and dual optimization problems. In particular, the derivative of the value function with respect to the preference parameter is obtained by differentiating the objective function while keeping the optimizer fixed. A related envelope theorem for convex conjugates is presented in Appendix A.

Theorem 3.7. *Let* $x > 0$ *be fixed and suppose that assumptions of Theorem 3.4 hold. Then, for* $y = u_x(x,0)$*, we have*

$$\begin{aligned} u_{xx}(x,0) &= -\frac{1}{v_{yy}(y,0)}, \\ u_{x\varepsilon}(x,0) &= -\frac{v_{y\varepsilon}(y,0)}{v_{yy}(y,0)} = u_{xx}(x,0)v_{y\varepsilon}(y,0), \\ u_{\varepsilon\varepsilon}(x,0) &= v_{\varepsilon\varepsilon}(y,0) + v_{y\varepsilon}(y,0)u_{x\varepsilon}(x,0), \end{aligned} \tag{24}$$

and

$$1 + \widehat{N}^y_T = -A(\widehat{X}_T)(1 + \widehat{M}^x_T), \tag{25}$$

and

$$\widehat{N}^\varepsilon_T - y\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)} = -A(\widehat{X}_T)\widehat{M}^\varepsilon_T + x\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)}. \tag{26}$$

Theorem 3.8 (First-order derivatives of the optimizers)**.** *Let* $x > 0$ *be fixed and suppose that assumptions of Theorem 3.4 hold. With* $\widehat{X}_T(x,\varepsilon)$ *and* $\widehat{Y}_T(y,\varepsilon)$ *denoting the primal and dual optimizers, as* $(\Delta x, \varepsilon) \to (0,0)$*, we have*

$$\frac{1}{|\Delta x| + |\varepsilon|}\left|\widehat{X}_T(x+\Delta x,\varepsilon) - \widehat{X}_T\left(1 + \frac{\Delta x}{x}(1+\widehat{M}^x_T) + \frac{\varepsilon}{x}\widehat{M}^\varepsilon_T\right)\right| \to 0$$

in $\mathbb{P}$*-probability. Similarly, as* $(\Delta y, \varepsilon) \to (0,0)$*,*

$$\frac{1}{|\Delta y| + |\varepsilon|}\left|\widehat{Y}_T(y+\Delta y,\varepsilon) - \widehat{Y}_T\left(1 + \frac{\Delta y}{y}(1+\widehat{N}^y_T) + \frac{\varepsilon}{y}\widehat{N}^\varepsilon_T\right)\right| \to 0$$

in $\mathbb{P}$*-probability.*

4. Proofs of the main results

We will need the following lemma, which is a consequences of [MS19, Lemma 4.12].

Lemma 4.1. *Suppose U^θ has relative risk aversion uniformly bounded away from zero and infinity. Then for every compact interval $K \subset (0,\infty)$, there exists $C_K > 0$ such that*

$$I^\theta(\lambda y) \le C_K I^\theta(y), \qquad \lambda \in K.$$

Lemma 4.2. *Let $x > 0$ be fixed and suppose that assumptions of Theorem 3.4 hold. Then, we have*

$$\begin{aligned} u(x+\Delta x, \varepsilon) \ge & u(x,0) + \begin{pmatrix} y \\ \mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right]\end{pmatrix}^\top \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix} \\ & + \frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^\top \begin{pmatrix} u_{xx}(x,0) & u_{x\varepsilon}(x,0) \\ u_{x\varepsilon}(x,0) & u_{\varepsilon\varepsilon}(x,0)\end{pmatrix}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix} + o(\Delta x^2 + \varepsilon^2).\end{aligned}$$

Proof. Let us fix predictable S-integrable H^x and H^ε such that $\left(\frac{x(1+H^x\cdot S)}{\widehat{X}}\right)$ and $\frac{xH^\varepsilon \cdot S}{\widehat{X}}$ are bounded.

Let

$$X_T(\Delta x, \varepsilon) := \widehat{X}_T + \Delta x(1 + H^x\cdot S_T) + \varepsilon(H^\varepsilon \cdot S_T).$$

Equivalently, writing

$$P^x := \frac{x(1+H^x\cdot S_T)}{\widehat{X}_T}, \qquad P^\varepsilon := \frac{x(H^\varepsilon\cdot S_T)}{\widehat{X}_T},$$

we have

$$X_T(\Delta x, \varepsilon) = \widehat{X}_T\left(1 + \frac{\Delta x}{x}P^x + \frac{\varepsilon}{x}P^\varepsilon\right).$$

For fixed Δx and ε, define

$$\eta := \frac{\Delta x}{x}P^x + \frac{\varepsilon}{x}P^\varepsilon$$

and consider

$$g(z) := U^{z\varepsilon}\big(\widehat{X}_T(1+z\eta)\big), \qquad z\in[0,1].$$

Since

$$U^{z\varepsilon} = U + z\varepsilon\widetilde{U},$$

we have

$$g(z) = U\big(\widehat{X}_T(1+z\eta)\big) + z\varepsilon\widetilde{U}\big(\widehat{X}_T(1+z\eta)\big).$$

Then

$$g(0) = U(\widehat{X}_T),$$

and

$$g(1) = U^\varepsilon(X_T^{\Delta x,\varepsilon}).$$

A direct differentiation gives

$$g'(z) = U'\big(\widehat{X}_T(1+z\eta)\big)\widehat{X}_T\eta + \varepsilon\widetilde{U}\big(\widehat{X}_T(1+z\eta)\big) + z\varepsilon\widetilde{U}'\big(\widehat{X}_T(1+z\eta)\big)\widehat{X}_T\eta.$$

In particular,

$$g'(0) = U'(\widehat{X}_T)\widehat{X}_T\eta + \varepsilon\widetilde{U}(\widehat{X}_T).$$

Since $U'(\widehat{X}_T) = \widehat{Y}_T$, this yields

$$\mathbb{E}[g'(0)] = y\Delta x + \varepsilon\mathbb{E}[\widetilde{U}(\widehat{X}_T)].$$

Next,

$$\begin{aligned} g''(z) = {} & U''\big(\widehat{X}_T(1+z\eta)\big)\widehat{X}_T^2\eta^2 \\ & + 2\varepsilon\widetilde{U}'\big(\widehat{X}_T(1+z\eta)\big)\widehat{X}_T\eta \\ & + z\varepsilon\widetilde{U}''\big(\widehat{X}_T(1+z\eta)\big)\widehat{X}_T^2\eta^2. \end{aligned}$$

Therefore,

$$g''(0) = U''(\widehat{X}_T)\widehat{X}_T^2\eta^2 + 2\varepsilon\widetilde{U}'(\widehat{X}_T)\widehat{X}_T\eta.$$

By Taylor's formula with integral remainder,

$$g(1) = g(0) + g'(0) + \int_0^1 (1-z)g''(z)\,dz.$$

Equivalently,

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(0) + r(\Delta x, \varepsilon),$$

where

$$r(\Delta x, \varepsilon) := \int_0^1 (1-z)\big(g''(z) - g''(0)\big)\,dz.$$

Thus,

$$\begin{aligned} U^\varepsilon(X_T^{\Delta x,\varepsilon}) = {} & U(\widehat{X}_T) + U'(\widehat{X}_T)\widehat{X}_T\eta + \varepsilon\widetilde{U}(\widehat{X}_T) \\ & + \frac{1}{2}U''(\widehat{X}_T)\widehat{X}_T^2\eta^2 + \varepsilon\widetilde{U}'(\widehat{X}_T)\widehat{X}_T\eta + r(\Delta x, \varepsilon). \end{aligned}$$

Since

$$\eta = \frac{\Delta x}{x}P^x + \frac{\varepsilon}{x}P^\varepsilon,$$

the second-order part expands as

$$\begin{aligned} \frac{1}{2}U''(\widehat{X}_T)\widehat{X}_T^2\eta^2 &= \frac{1}{2}U''(\widehat{X}_T)\big(\Delta x(1 + H^x\cdot S_T) + \varepsilon(H^\varepsilon\cdot S_T)\big)^2 \\ &= \frac{1}{2}U''(\widehat{X}_T)(1 + H^x\cdot S_T)^2\Delta x^2 \\ &\quad + U''(\widehat{X}_T)(1 + H^x\cdot S_T)(H^\varepsilon\cdot S_T)\Delta x\,\varepsilon \\ &\quad + \frac{1}{2}U''(\widehat{X}_T)(H^\varepsilon\cdot S_T)^2\varepsilon^2. \end{aligned}$$

Also,

$$\varepsilon\widetilde{U}'(\widehat{X}_T)\widehat{X}_T\eta = \widetilde{U}'(\widehat{X}_T)(1 + H^x\cdot S_T)\Delta x\,\varepsilon + \widetilde{U}'(\widehat{X}_T)(H^\varepsilon\cdot S_T)\varepsilon^2.$$

Combining these terms gives

$$\begin{aligned}
U^\varepsilon(X_T(\Delta x,\varepsilon)) &= U(\widehat{X}_T) + U'(\widehat{X}_T)(1+H^x\cdot S_T)\Delta x \\
&+ \Big(U'(\widehat{X}_T)(H^\varepsilon\cdot S_T) + \widetilde{U}(\widehat{X}_T)\Big)\varepsilon \\
&+ \frac{1}{2}U''(\widehat{X}_T)(1+H^x\cdot S_T)^2\Delta x^2 \\
&+ \Big[U''(\widehat{X}_T)(1+H^x\cdot S_T)(H^\varepsilon\cdot S_T) + \widetilde{U}'(\widehat{X}_T)(1+H^x\cdot S_T)\Big]\Delta x\,\varepsilon \\
&+ \frac{1}{2}\Big[U''(\widehat{X}_T)(H^\varepsilon\cdot S_T)^2 + 2\widetilde{U}'(\widehat{X}_T)(H^\varepsilon\cdot S_T)\Big]\varepsilon^2 \\
&+ r(\Delta x,\varepsilon).
\end{aligned}$$

Here, let us observe that with

$$X^z_T = \widehat{X}_T\left(1+\frac{z}{x}(\Delta x P^x + \varepsilon P^\varepsilon)\right),$$

we have

$$\begin{aligned}
r(\Delta x,\varepsilon) &= \frac{1}{x^2}\int_0^1 (1-z)\Big[U''(X^z_T) - U''(\widehat{X}_T)\Big]\widehat{X}^2_T\big(\Delta x P^x + \varepsilon P^\varepsilon\big)^2\,dz \\
&+ \frac{2\varepsilon}{x}\int_0^1 (1-z)\Big[\widetilde{U}'(X^z_T) - \widetilde{U}'(\widehat{X}_T)\Big]\widehat{X}_T\big(\Delta x P^x + \varepsilon P^\varepsilon\big)\,dz \\
&+ \frac{\varepsilon}{x^2}\int_0^1 (1-z)z\widetilde{U}''(X^z_T)\widehat{X}^2_T\big(\Delta x P^x + \varepsilon P^\varepsilon\big)^2\,dz.
\end{aligned}$$

Let

$$\rho := \sqrt{\Delta x^2 + \varepsilon^2}.$$

Since P^x and P^ε are bounded, there is a constant $C>0$ such that

$$|\eta| = \left|\frac{\Delta x}{x}P^x + \frac{\varepsilon}{x}P^\varepsilon\right| \le C\rho.$$

For $(\Delta x,\varepsilon)$ sufficiently small, we also have

$$1+z\eta \in [1-\delta, 1+\delta], \qquad z\in[0,1].$$

Recall that

$$X^z_T = \widehat{X}_T(1+z\eta).$$

We write

$$r(\Delta x,\varepsilon) = r_1(\Delta x,\varepsilon) + r_2(\Delta x,\varepsilon) + r_3(\Delta x,\varepsilon),$$

where

$$\begin{aligned}
r_1(\Delta x,\varepsilon) &:= \int_0^1 (1-z)\left[U''(X^z_T) - U''(\widehat{X}_T)\right]\widehat{X}^2_T\eta^2\,dz, \\
r_2(\Delta x,\varepsilon) &:= 2\varepsilon\int_0^1 (1-z)\left[\widetilde{U}'(X^z_T) - \widetilde{U}'(\widehat{X}_T)\right]\widehat{X}_T\eta\,dz, \\
r_3(\Delta x,\varepsilon) &:= \varepsilon\int_0^1 (1-z)z\widetilde{U}''(X^z_T)\widehat{X}^2_T\eta^2\,dz.
\end{aligned}$$

We first consider r_1. Since $X_T^z/\widehat{X}_T \in [1-\delta, 1+\delta]$, the boundedness of the relative risk aversion of U implies

$$|U''(X_T^z)|(X_T^z)^2 \le CU'(\widehat{X}_T)\widehat{X}_T.$$

Equivalently,

$$|U''(X_T^z)|\widehat{X}_T^2 \le CU'(\widehat{X}_T)\widehat{X}_T = C\widehat{Y}_T\widehat{X}_T.$$

The same bound holds with X_T^z replaced by $\widehat{X}_T$. Hence

$$\left|U''(X_T^z) - U''(\widehat{X}_T)\right| \widehat{X}_T^2 \le C\widehat{X}_T\widehat{Y}_T.$$

Since

$$E[\widehat{X}_T\widehat{Y}_T] = xy < \infty,$$

and Since

$$X_T^z \to \widehat{X}_T \quad \text{a.s.},$$

and U'' is continuous,

$$U''(X_T^z) \to U''(\widehat{X}_T) \quad \text{a.s.}$$

Vitali's theorem gives

$$E\left[\left|U''(X_T^z) - U''(\widehat{X}_T)\right| \widehat{X}_T^2\right] \to 0.$$

Since $\eta^2 \le C\rho^2$, it follows that

$$E[|r_1(\Delta x, \varepsilon)|] = o(\rho^2).$$

For the second term, since $X_T^z/\widehat{X}_T \in [1-\delta, 1+\delta]$, the power estimates implied by bounded relative risk aversion of $\widetilde{U}$ give

$$\widetilde{U}'(X_T^z)X_T^z \le C\widetilde{U}'(\widehat{X}_T)\widehat{X}_T.$$

Thus,

$$\left|\widetilde{U}'(X_T^z) - \widetilde{U}'(\widehat{X}_T)\right| \widehat{X}_T \le C\widetilde{U}'(\widehat{X}_T)\widehat{X}_T.$$

By assumption,

$$\widetilde{U}'(\widehat{X}_T)\widehat{X}_T \in L^1(\mathbb{P}).$$

Since

$$X_T^z \to \widehat{X}_T \quad \text{a.s.},$$

and U' is continuous,

$$\widetilde{U}'(X_T^z) \to \widetilde{U}'(\widehat{X}_T) \quad \text{a.s.},$$

Vitali's theorem yields

$$E\left[\left|\widetilde{U}'(X_T^z) - \widetilde{U}'(\widehat{X}_T)\right| \widehat{X}_T\right] \to 0.$$

Moreover,

$$|\varepsilon|\,|\eta| \le C\rho^2.$$

Therefore,

$$E[|r_2(\Delta x, \varepsilon)|] = o(\rho^2).$$

Finally, for r_3, using bounded relative risk aversion of $\widetilde{U}$, we have

$$|\widetilde{U}''(X^z_T)|(X^z_T)^2 \leq C\widetilde{U}'(X^z_T)X^z_T.$$

Since $X^z_T/\widehat{X}_T \in [1-\delta, 1+\delta]$, it follows that

$$|\widetilde{U}''(X^z_T)|\widehat{X}^2_T \leq C\widetilde{U}'(\widehat{X}_T)\widehat{X}_T.$$

Hence

$$|r_3(\Delta x, \varepsilon)| \leq C|\varepsilon|\rho^2\widetilde{U}'(\widehat{X}_T)\widehat{X}_T.$$

Therefore,

$$E[|r_3(\Delta x, \varepsilon)|] = O(|\varepsilon|\rho^2) = o(\rho^2).$$

Combining the estimates for r_1, r_2, r_3, we obtain

$$E[|r(\Delta x, \varepsilon)|] = o(\Delta x^2 + \varepsilon^2), \qquad (\Delta x, \varepsilon) \to (0,0).$$

Consequently,

$$\begin{aligned}
E\left[U^\varepsilon(X_T(\Delta x,\varepsilon))\right] = {} & u(x,0) + y\Delta x + \varepsilon E[\widetilde{U}(\widehat{X}_T)] \\
& + \frac{1}{2}E\left[U''(\widehat{X}_T)(1+H^x\cdot S_T)^2\right]\Delta x^2 \\
& + \left(E\left[U''(\widehat{X}_T)(1+H^x\cdot S_T)(H^\varepsilon\cdot S_T)\right] + E\left[\widetilde{U}'(\widehat{X}_T)(1+H^x\cdot S_T)\right]\right)\Delta x\,\varepsilon \\
& + \frac{1}{2}\left(E\left[U''(\widehat{X}_T)(H^\varepsilon\cdot S_T)^2\right] + 2E\left[\widetilde{U}'(\widehat{X}_T)(H^\varepsilon\cdot S_T)\right]\right)\varepsilon^2 \\
& + o(\Delta x^2+\varepsilon^2).
\end{aligned}$$

By the sigma-boundedness assumption, bounded wealth ratios are dense in the corresponding weighted $L^2(\mathbb{R})$ spaces. Therefore, applying the preceding expansion to bounded approximating sequences for the optimizers $\widehat{H}^x$ and $\widehat{H}^\varepsilon$, and passing to the limit in the quadratic forms, yields the asserted lower bound. □

Lemma 4.3. *Under the conditions of Lemma 4.2, we have*

$$\begin{aligned}
v(y+\Delta y,\varepsilon) \leq {} & v(y,0) + \begin{pmatrix} -x \\ \mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right]\end{pmatrix}^\top \begin{pmatrix}\Delta y \\ \varepsilon\end{pmatrix} \\
& + \frac{1}{2}\begin{pmatrix}\Delta y \\ \varepsilon\end{pmatrix}^\top \begin{pmatrix} v_{yy}(y,0) & v_{y\varepsilon}(y,0) \\ v_{y\varepsilon}(y,0) & v_{\varepsilon\varepsilon}(y,0)\end{pmatrix}\begin{pmatrix}\Delta y \\ \varepsilon\end{pmatrix} + o(\Delta y^2+\varepsilon^2).
\end{aligned}$$

Proof of Lemma 4.3. Let us fix (bounded) $N^y \in \mathcal{N}^\infty$ and $N^\varepsilon \in \mathcal{N}^\infty$, and consider

$$Y(y+\Delta y,\varepsilon) = \widehat{Y}\left(1 + \frac{\Delta y}{y}(1+N^y) + \frac{\varepsilon}{y}N^\varepsilon\right). \tag{27}$$

Since $N^y, N^\varepsilon \in \mathcal{N}^\infty$, there exists $\delta > 0$ such that for all sufficiently small $(\Delta y, \varepsilon)$, the process

$$1 + \frac{\Delta y}{y}(1+N^y) + \frac{\varepsilon}{y}N^\varepsilon$$

remains strictly positive and preserves the supermartingale property of XY for every $X \in \mathcal{X}(1)$, hence $Y(y+\Delta y, \varepsilon) \in \mathcal{Y}(y+\Delta y)$. That is, there is a ball B_δ in $\mathbb{R}^2$ of a positive radius δ such that for every $(\Delta y, \varepsilon) \in B_\delta$, we have that

$$Y(y+\Delta y, \varepsilon) \in \mathcal{Y}(y+\Delta y).$$

For a fixed $(\Delta y, \varepsilon) \in B_\delta$, let us define

$$h(z) := V^{z\varepsilon}\left(Y_T(y+z\Delta y, z\varepsilon)\right), \quad z \in [0,1].$$

Let us write

$$D^y := 1 + N^y, \qquad D^\varepsilon := N^\varepsilon,$$

and define

$$\eta := \frac{\Delta y}{y} D^y + \frac{\varepsilon}{y} D^\varepsilon. \tag{28}$$

Then

$$Y_T(y+z\Delta y, z\varepsilon) = \widehat{Y}_T(1+z\eta).$$

Therefore,

$$h(z) = V^{z\varepsilon}\Big(\widehat{Y}_T(1+z\eta)\Big).$$

Taylor's formula gives

$$h(1) = h(0) + h'(0) + \frac{1}{2}h''(0) + r(\Delta y, \varepsilon),$$

where

$$r(\Delta y, \varepsilon) = \int_0^1 (1-z)\big(h''(z) - h''(0)\big)\, dz.$$

Since

$$h''(z) = V^{z\varepsilon}_{yy}(Y^z_T)\widehat{Y}^2_T\eta^2 + 2\varepsilon V^{z\varepsilon}_{y\varepsilon}(Y^z_T)\widehat{Y}_T\eta + \varepsilon^2 V^{z\varepsilon}_{\varepsilon\varepsilon}(Y^z_T),$$

we write

$$r(\Delta y, \varepsilon) = r_1(\Delta y, \varepsilon) + r_2(\Delta y, \varepsilon) + r_3(\Delta y, \varepsilon), \tag{29}$$

where

$$r_1(\Delta y, \varepsilon) := \int_0^1 (1-z)\Big[V^{z\varepsilon}_{yy}(Y^z_T) - V^0_{yy}(\widehat{Y}_T)\Big]\widehat{Y}^2_T\eta^2\, dz, \tag{30}$$

$$r_2(\Delta y, \varepsilon) := 2\varepsilon\int_0^1 (1-z)\Big[V^{z\varepsilon}_{y\varepsilon}(Y^z_T) - V^0_{y\varepsilon}(\widehat{Y}_T)\Big]\widehat{Y}_T\eta\, dz, \tag{31}$$

and

$$r_3(\Delta y, \varepsilon) := \varepsilon^2 \int_0^1 (1-z)\left[V^{z\varepsilon}_{\varepsilon\varepsilon}(Y^z_T) - V^0_{\varepsilon\varepsilon}(\widehat{Y}_T)\right] dz. \tag{32}$$

Let

$$\rho := \sqrt{\Delta y^2 + \varepsilon^2}.$$

and recall that η is given in (28). Since D^y and D^ε are bounded, there is a constant $C > 0$ such that

$$|\eta| \le C\rho.$$

For $(\Delta y, \varepsilon)$ sufficiently small, we also have

$$1 + z\eta \in [1-\delta, 1+\delta], \qquad z \in [0,1].$$

Set

$$Y^z_T := \widehat{Y}_T(1+z\eta), \qquad \theta := z\varepsilon, \qquad X^z_T := I^\theta(Y^z_T).$$

By Assumption 2.1, there exists $\varepsilon_0 > 0$ such that, for every $|\theta| < \varepsilon_0$, U^θ is an Inada utility and

$$0 < a \le A^\theta(x) \le b < \infty, \qquad x > 0.$$

Then the standard power estimates for utilities with bounded relative risk aversion, e.g., as in Lemma 4.1, imply that, locally uniformly in

$$|\theta| < \varepsilon_0, \qquad \xi \in [1-\delta, 1+\delta],$$

we have

$$I^\theta(\xi y) \le C I^\theta(y), \qquad I^\theta(y) \le C I^0(y).$$

Consequently,

$$X^z_T = I^\theta(\widehat{Y}_T(1+z\eta)) \le C\widehat{X}_T.$$

Let us consider (29). First, using

$$V^\theta_{yy}(y) = -\frac{1}{(U^\theta)''(I^\theta(y))}$$

and

$$A^\theta(x) = -\frac{x(U^\theta)''(x)}{(U^\theta)'(x)},$$

we obtain

$$V^\theta_{yy}(Y^z_T)\widehat{Y}^2_T = \frac{1}{1+z\eta}\frac{X^z_T\widehat{Y}_T}{A^\theta(X^z_T)}.$$

Hence

$$V^\theta_{yy}(Y^z_T)\widehat{Y}^2_T \le C\widehat{X}_T\widehat{Y}_T.$$

Since

$$E[\widehat{X}_T\widehat{Y}_T] = xy < \infty,$$

the family

$$\left\{V^{\theta}_{yy}(Y^z_T)\widehat{Y}^2_T : z \in [0,1],\ |\Delta y| + |\varepsilon| < \delta\right\}$$

is uniformly integrable. Also,

$$V^{\theta}_{yy}(Y^z_T) \to V^0_{yy}(\widehat{Y}_T) \quad \text{a.s.}$$

Therefore, by Vitali's theorem,

$$E[|r_1(\Delta y, \varepsilon)|] = o(\rho^2).$$

For the second term, note that

$$V^{\theta}_{y\varepsilon}(Y^z_T) = \frac{\widetilde{U}'(X^z_T)}{(U^{\theta})''(X^z_T)} = -\frac{\widetilde{U}'(X^z_T)X^z_T}{A^{\theta}(X^z_T)Y^z_T}.$$

Thus

$$\left|V^{\theta}_{y\varepsilon}(Y^z_T)\right| \widehat{Y}_T \le C\widetilde{U}'(X^z_T)X^z_T.$$

By the bounded relative risk aversion of $\widetilde{U}$ and the comparison $X^z_T \le C\widehat{X}_T$, we have

$$\widetilde{U}'(X^z_T)X^z_T \le C\widetilde{U}'(\widehat{X}_T)\widehat{X}_T.$$

As, by assumption of this lemma, we have

$$\widetilde{U}'(\widehat{X}_T)\widehat{X}_T \in L^1(\mathbb{P}),$$

we obtain uniform integrability of

$$\left\{\left|V^{\theta}_{y\varepsilon}(Y^z_T)\right| \widehat{Y}_T : z \in [0,1],\ |\Delta y| + |\varepsilon| < \delta\right\}.$$

Since

$$V^{\theta}_{y\varepsilon}(Y^z_T) \to V^0_{y\varepsilon}(\widehat{Y}_T) \quad \text{a.s.,}$$

Vitali's theorem yields

$$E[|r_2(\Delta y, \varepsilon)|] = o(\rho^2).$$

Finally,

$$V^{\theta}_{\varepsilon\varepsilon}(Y^z_T) = -\frac{\left(\widetilde{U}'(X^z_T)\right)^2}{(U^{\theta})''(X^z_T)} = \frac{\left(\widetilde{U}'(X^z_T)\right)^2 X^z_T}{A^{\theta}(X^z_T)Y^z_T}.$$

Hence

$$\left|V^{\theta}_{\varepsilon\varepsilon}(Y^z_T)\right| \le C\frac{\left(\widetilde{U}'(X^z_T)\right)^2 X^z_T}{Y^z_T}.$$

Using again the local comparison $X^z_T \le C\widehat{X}_T$, and by the existence of a deterministic constant $C > 1$, such that

$$\frac{1}{C}\widehat{Y}_T \le Y^z_T \le C\widehat{Y}_T, \qquad z \in [0,1],$$

for all sufficiently small $(\Delta y, \varepsilon)$, and bounded relative risk aversion of $\widetilde{U}$, we get

$$\left|V^{\theta}_{\varepsilon\varepsilon}(Y^z_T)\right| \le C\frac{\left(\widetilde{U}'(\widehat{X}_T)\right)^2 \widehat{X}_T}{\widehat{Y}_T}.$$

But

$$\frac{\left(\widetilde{U}'(\widehat{X}_T)\right)^2 \widehat{X}_T}{\widehat{Y}_T} = \widehat{X}_T \widehat{Y}_T \left(\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)}\right)^2 .$$

Therefore, if

$$f_T := \frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)} \in L^2(\mathbb{R}),$$

then

$$E\left[\frac{\left(\widetilde{U}'(\widehat{X}_T)\right)^2 \widehat{X}_T}{\widehat{Y}_T}\right] = xy\, E^{\mathbb{R}}[f_T^2] < \infty.$$

Thus the family

$$\left\{\left|V^{\theta}_{\varepsilon\varepsilon}(Y^z_T)\right| : z \in [0,1],\ |\Delta y| + |\varepsilon| < \delta\right\}$$

is uniformly integrable. Since

$$V^{\theta}_{\varepsilon\varepsilon}(Y^z_T) \to V^0_{\varepsilon\varepsilon}(\widehat{Y}_T) \quad \text{a.s.},$$

Vitali's theorem gives

$$E[|r_3(\Delta y, \varepsilon)|] = o(\rho^2).$$

Combining the three estimates,

$$E[|r(\Delta y, \varepsilon)|] = o(\Delta y^2 + \varepsilon^2), \qquad (\Delta y, \varepsilon) \to (0,0).$$

Therefore, for $Y(y + \Delta y, \varepsilon)$ defined in (27), we obtain

$$\begin{aligned} E[V^{\varepsilon}(Y_T(\Delta y, \varepsilon))] &= v(y) - x\Delta y + \varepsilon E[\widetilde{U}(\widehat{X}_T)] \\ &+ \frac{1}{2} E\left[V''(\widehat{Y}_T)(1 + N^y_T)^2\right] \Delta y^2 \\ &+ \left(E\left[V''(\widehat{Y}_T)(1 + N^y_T) N^{\varepsilon}_T\right] + E\left[\widetilde{U}'(\widehat{X}_T)\frac{\widehat{X}_T}{\widehat{Y}_T}(1 + N^y_T)\right]\right) \Delta y\, \varepsilon \\ &+ \frac{1}{2}\left(E\left[V''(\widehat{Y}_T)(N^{\varepsilon}_T)^2\right] + 2E\left[\widetilde{U}'(\widehat{X}_T)\frac{\widehat{X}_T}{\widehat{Y}_T} N^{\varepsilon}_T\right]\right) \varepsilon^2 \\ &+ o(\Delta y^2 + \varepsilon^2). \end{aligned} \tag{33}$$

Therefore, applying the preceding expansion to bounded approximating sequences for the optimizers $\widehat{N}^y$ and $\widehat{N}^{\varepsilon}$ to (13) and (14), and passing to the limit in (33), yields the asserted upper bound.

□

Proof of Theorem 3.7. The first equality,

$$u_{xx}(x, 0) v_{yy}(y, 0) = -1,$$

follows from [KS06a, Lemma 1 and Theorem 1].To show the second, let us introduce the following notations. With

$$P^x := \frac{x(1+\widehat{H}^x \cdot S_T)}{\widehat{X}_T} = 1 + \widehat{M}^x_T, \quad P^\varepsilon := \frac{x\widehat{H}^\varepsilon \cdot S_T}{\widehat{X}_T} = \widehat{M}^\varepsilon_T,$$

and

$$D^y := 1 + \widehat{N}^y_T, \quad D^\varepsilon := \widehat{N}^\varepsilon_T - y\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)}.$$

Then, (9) and (11) can be rewritten as

$$u_{xx}(x,0) = -\frac{y}{x}\mathbb{E}^{\mathbb{R}}\left[A(\widehat{X}_T)(P^x)^2\right]. \tag{34}$$

$$u_{x\varepsilon}(x,0) = -\frac{y}{x}\mathbb{E}^{\mathbb{R}}\left[A(\widehat{X}_T)P^xP^\varepsilon - x\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)}P^x\right]. \tag{35}$$

and

$$v_{yy}(y,0) = \frac{x}{y}\mathbb{E}^{\mathbb{R}}\left[B(\widehat{Y}_T)(D^y)^2\right], \tag{36}$$

$$v_{y\varepsilon}(y,0) = \frac{x}{y}\mathbb{E}^{\mathbb{R}}\left[B(\widehat{Y}_T)D^yD^\varepsilon\right]. \tag{37}$$

Using

$$A(\widehat{X}_T)B(\widehat{Y}_T) = 1,$$

the first-order optimality conditions for the quadratic minimization problems imply the projection relations

$$D^y = -A(\widehat{X}_T)P^x,$$

and

$$D^\varepsilon = -A(\widehat{X}_T)P^\varepsilon + x\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)},$$

which give (25) and (26), respectively. We note that these identities correspond to the orthogonal projections associated with the decomposition

$$\mathcal{H}^2_0(\mathbb{R}) = \mathcal{M}^2 \oplus \mathcal{N}^2.$$

Substituting these into the formula for (37), we obtain

$$\begin{aligned} v_{y\varepsilon}(y,0) &= \frac{x}{y}\mathbb{E}^{\mathbb{R}}\left[B(\widehat{Y}_T)D^yD^\varepsilon\right] \\ &= \frac{x}{y}\mathbb{E}^{\mathbb{R}}\left[B(\widehat{Y}_T)\big(-A(\widehat{X}_T)P^x\big)\left(-A(\widehat{X}_T)P^\varepsilon + x\frac{\widetilde{U}'(\widehat{X}_T)}{U'(\widehat{X}_T)}\right)\right] \\ &= -\frac{v_{yy}(y,0)}{u_{x\varepsilon}(x,0)}, \end{aligned}$$

hence

$$u_{x\varepsilon}(x,0) = -\frac{v_{y\varepsilon}(y,0)}{v_{yy}(y,0)}.$$

Similarly one can show that

$$u_{\varepsilon\varepsilon}(x,0) = v_{\varepsilon\varepsilon}(y,0) - \frac{v_{y\varepsilon}(y,0)^2}{v_{yy}(y,0)},$$

which is equivalent to

$$u_{\varepsilon\varepsilon}(x,0) = v_{\varepsilon\varepsilon}(y,0) + v_{y\varepsilon}(y,0)\, u_{x\varepsilon}(x,0).$$

□

Proof of Theorem 3.4. First, from Lemma 4.2, we deduce that

$$\begin{aligned} u(x+\Delta x,\varepsilon) \geq & u(x,0) + \begin{pmatrix} y \\ \mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right]\end{pmatrix}^{\top} \begin{pmatrix}\Delta x \\ \varepsilon\end{pmatrix} \\ & + \frac{1}{2}\begin{pmatrix}\Delta x \\ \varepsilon\end{pmatrix}^{\top} \begin{pmatrix} u_{xx}(x,0) & u_{x\varepsilon}(x,0) \\ u_{x\varepsilon}(x,0) & u_{\varepsilon\varepsilon}(x,0)\end{pmatrix}\begin{pmatrix}\Delta x \\ \varepsilon\end{pmatrix} + o(\Delta x^2+\varepsilon^2). \end{aligned} \tag{38}$$

Likewise, from Lemma 4.3, we deduce that

$$\begin{aligned} v(y+\Delta y,\varepsilon) \leq & v(y,0) + \begin{pmatrix} -x \\ \mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right]\end{pmatrix}^{\top} \begin{pmatrix}\Delta y \\ \varepsilon\end{pmatrix} \\ & + \frac{1}{2}\begin{pmatrix}\Delta y \\ \varepsilon\end{pmatrix}^{\top} \begin{pmatrix} v_{yy}(y,0) & v_{y\varepsilon}(y,0) \\ v_{y\varepsilon}(y,0) & v_{\varepsilon\varepsilon}(y,0)\end{pmatrix}\begin{pmatrix}\Delta y \\ \varepsilon\end{pmatrix} + o(\Delta y^2+\varepsilon^2). \end{aligned} \tag{39}$$

Next, from conjugacy between u and v, we get

$$u(x+\Delta x,\varepsilon) \leq v(y+\Delta y,\varepsilon) + (x+\Delta x)(y+\Delta y). \tag{40}$$

Using (39), we can rewrite (40) as

$$\begin{aligned} u(x+\Delta x,\varepsilon) \leq & v(y,0) - x\Delta y + \varepsilon\mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right] \\ & + \frac{1}{2}v_{yy}(y,0)\Delta y^2 + v_{y\varepsilon}(y,0)\Delta y\varepsilon + \frac{1}{2}v_{\varepsilon\varepsilon}(y,0)\varepsilon^2 \\ & + xy + x\Delta y + y\Delta x + \Delta x\Delta y + o(\Delta y^2+\varepsilon^2). \end{aligned} \tag{41}$$

In turn, from (41) and the assertions of Theorem 3.7, we get

$$\begin{aligned} u(x+\Delta x,\varepsilon) \leq & u(x,0) + \varepsilon\mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right] + y\Delta x + \Delta x\Delta y \\ & - \frac{1}{2}\frac{\Delta y^2}{u_{xx}(x,0)} + \frac{u_{x\varepsilon}(x,0)}{u_{xx}(x,0)}\Delta y\varepsilon \\ & + \frac{1}{2}\left(u_{\varepsilon\varepsilon}(x,0) - v_{y\varepsilon}(y,0)u_{x\varepsilon}(x,0)\right)\varepsilon^2 + o(\Delta y^2+\varepsilon^2). \end{aligned} \tag{42}$$

Now, for

$$\Delta y = u_{xx}(x,0)\Delta x + u_{x\varepsilon}(x,0)\varepsilon,$$

(42) gives

$$
\begin{aligned}
u(x+\Delta x,\varepsilon) \le & u(x,0)+\varepsilon\mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right]+y\Delta x+\Delta x\left(u_{xx}(x,0)\Delta x+u_{x\varepsilon}(x,0)\varepsilon\right)\\
&-\frac{1}{2}\frac{(u_{xx}(x,0)\Delta x+u_{x\varepsilon}(x,0)\varepsilon)^2}{u_{xx}(x,0)}\\
&+\frac{u_{x\varepsilon}(x,0)}{u_{xx}(x,0)}(u_{xx}(x,0)\Delta x+u_{x\varepsilon}(x,0)\varepsilon)\varepsilon\\
&+\frac{1}{2}\left(u_{\varepsilon\varepsilon}(x,0)-v_{y\varepsilon}(y,0)u_{x\varepsilon}(x,0)\right)\varepsilon^2+o(\Delta x^2+\varepsilon^2).
\end{aligned}\tag{43}
$$

Consequently, from Theorem 3.7 (equation (24)), we can rewrite (43) as

$$
\begin{aligned}
u(x+\Delta x,\varepsilon) \le & u(x,0)+\varepsilon\mathbb{E}\left[\widetilde{U}(\widehat{X}_T(x,0))\right]+y\Delta x\\
&+\frac{1}{2}u_{xx}(x,0)\Delta x^2+u_{x\varepsilon}(x,0)\Delta x\varepsilon+\frac{1}{2}u_{\varepsilon\varepsilon}(x,0)\varepsilon^2+o(\Delta x^2+\varepsilon^2).
\end{aligned}\tag{44}
$$

Comparing (38) and (44), we deduce that these inequalities are in fact equalities, and therefore, (22) holds. (23) can be proven similarly. □

Proof of theorem 3.8. We prove the dual assertion; the primal assertion follows by the same argument, or equivalently by using the primal-dual relation

$$\widehat{X}_T(y,\varepsilon)=-V^{\varepsilon}_y(\widehat{Y}_T(y,\varepsilon)).$$

Let $(\Delta y_n,\varepsilon_n)\to(0,0)$, and set

$$\rho_n:=|\Delta y_n|+|\varepsilon_n|.$$

Let

$$\widehat{Y}^n_T:=\widehat{Y}_T(y+\Delta y_n,\varepsilon_n)$$

be the true dual optimizer, and define the first-order candidate

$$\widetilde{Y}^n:=\widehat{Y}\left(1+\frac{\Delta y_n}{y}(1+\widehat{N}^y)+\frac{\varepsilon_n}{y}\widehat{N}^{\varepsilon}\right).$$

By construction,

$$\widetilde{Y}^n\in\mathcal{Y}(y+\Delta y_n)$$

for all sufficiently large n.

First, by the stability of the dual optimizer, see, e.g., [MT24, Theorem 2.4], we have

$$\widehat{Y}^n_T\to\widehat{Y}_T\qquad\text{in probability.}$$

Next, for every $n\in\mathbb{N}$, by strict convexity of V^{ε_n}, define the random variable

$$\Theta_n:=\inf_{\lambda\in[0,1]}V^{\varepsilon_n}_{yy}\left(\lambda\widehat{Y}^n_T+(1-\lambda)\widetilde{Y}^n_T\right),$$

where the infimum is understood pointwise. Then

$$\Theta_n>0\qquad\mathbb{P}\text{-}a.s.,\qquad n\in\mathbb{N},$$

and Taylor's formula gives

$$V^{\varepsilon_n}(\widetilde{Y}^n_T) - V^{\varepsilon_n}(\widehat{Y}^n_T) \geq V^{\varepsilon_n}_y(\widehat{Y}^n_T)(\widetilde{Y}^n_T - \widehat{Y}^n_T) + \frac{1}{2}\Theta_n(\widetilde{Y}^n_T - \widehat{Y}^n_T)^2.$$

By the optimality of $\widehat{Y}^n$ and the bipolar relation,

$$E\left[V^{\varepsilon_n}_y(\widehat{Y}^n_T)(\widetilde{Y}^n_T - \widehat{Y}^n_T)\right] \geq 0.$$

Therefore,

$$E\left[\Theta_n(\widetilde{Y}^n_T - \widehat{Y}^n_T)^2\right] \leq 2\left(E[V^{\varepsilon_n}(\widetilde{Y}^n_T)] - v(y + \Delta y_n, \varepsilon_n)\right).$$

By Lemma 4.3 applied to the candidate direction $(\widehat{N}^y, \widehat{N}^\varepsilon)$, and by the expansion of v in Theorem 3.4, the right-hand side satisfies

$$E[V^{\varepsilon_n}(\widetilde{Y}^n_T)] - v(y + \Delta y_n, \varepsilon_n) = o(\rho_n^2).$$

Hence

$$E\left[\Theta_n\left(\frac{\widetilde{Y}^n_T - \widehat{Y}^n_T}{\rho_n}\right)^2\right] \to 0.$$

Set

$$Z_n := \frac{\widetilde{Y}^n_T - \widehat{Y}^n_T}{\rho_n \widehat{Y}_T}.$$

We claim that $Z_n \to 0$ in probability.

Indeed, since

$$\widehat{Y}^n_T \to \widehat{Y}_T, \qquad \widetilde{Y}^n_T \to \widehat{Y}_T$$

in probability, for every $\delta > 0$, the following events

$$A_n := \left\{\frac{1}{2}\widehat{Y}_T \leq \lambda\widehat{Y}^n_T + (1-\lambda)\widetilde{Y}^n_T \leq 2\widehat{Y}_T \text{ for all } \lambda \in [0,1]\right\}, \qquad n \in \mathbb{N},$$

satisfy

$$\mathbb{P}[A_n] \to 1.$$

On this event, using the uniform boundedness away from zero of the dual relative risk tolerance B^ε, we obtain

$$\Theta_n \geq c\frac{\widehat{X}_T}{\widehat{Y}_T}.$$

Therefore,

$$\Theta_n(\widetilde{Y}^n_T - \widehat{Y}^n_T)^2 \geq c\widehat{X}_T\widehat{Y}_T\left(\frac{\widetilde{Y}^n_T - \widehat{Y}^n_T}{\widehat{Y}_T}\right)^2 = c\rho_n^2\widehat{X}_T\widehat{Y}_T Z_n^2.$$

Since the previous estimate gives

$$E\left[\Theta_n(\widetilde{Y}^n_T - \widehat{Y}^n_T)^2\right] = o(\rho_n^2),$$

we get

$$E\left[\widehat{X}_T\widehat{Y}_T Z_n^2 \mathbf{1}_{A_n}\right] \to 0,$$

where A_n denotes the high-probability event above.

Because

$$\widehat{X}_T\widehat{Y}_T > 0 \quad \text{a.s.},$$

this implies

$$Z_n \to 0 \qquad \text{in probability.}$$

Hence

$$\frac{1}{\rho_n}\left(\frac{\widehat{Y}^n_T}{\widehat{Y}_T} - \frac{\widetilde{Y}^n_T}{\widehat{Y}_T}\right) \to 0 \qquad \text{in probability.}$$

This proves

$$\frac{1}{|\Delta y_n| + |\varepsilon_n|}\left|\widehat{Y}_T(y+\Delta y_n, \varepsilon_n) - \widehat{Y}_T\left(1 + \frac{\Delta y_n}{y}(1+\widehat{N}^y_T) + \frac{\varepsilon_n}{y}\widehat{N}^\varepsilon_T\right)\right| \to 0.$$

The primal proof is analogous. One defines

$$\widetilde{X}^n_T := \widehat{X}_T\left(1 + \frac{\Delta x_n}{x}(1+\widehat{M}^x_T) + \frac{\varepsilon_n}{x}\widehat{M}^\varepsilon_T\right),$$

uses strict concavity of U^{ε_n}, the optimality of $\widehat{X}_T(x+\Delta x_n, \varepsilon_n)$, and compares

$$u(x+\Delta x_n, \varepsilon_n) - E[U^{\varepsilon_n}(\widetilde{X}^n_T)].$$

The expansion in Lemma 4.4 and Theorem 4.2 imply that this difference is $o((|\Delta x_n| + |\varepsilon_n|)^2)$, and the local strict concavity estimate then yields the desired convergence in probability.

□

5. Risk tolerance

Following [KS06b], let us assume that there exists a strictly positive wealth process $\mathcal{R} = \mathcal{R}_0 + H\cdot S$, such that

$$\mathcal{R}_T = \frac{\widehat{X}_T}{A(\widehat{X}_T)} = -\frac{U'(\widehat{X}_T)}{U''(\widehat{X}_T)}. \tag{45}$$

Let us define the probability measure $\widetilde{\mathbb{R}}$ by

$$\frac{d\widetilde{\mathbb{R}}}{d\mathbb{P}} := \frac{\mathcal{R}_T\widehat{Y}_T}{\mathcal{R}_0 y}.$$

We also change numéraire to $\mathcal{R}$

$$S^{\mathcal{R}} = \left(\frac{\mathcal{R}_0}{\mathcal{R}}, \frac{\mathcal{R}_0 S^1}{\mathcal{R}}, \dots, \frac{\mathcal{R}_0 S^d}{\mathcal{R}}\right)$$

and introduce the closed subspace

$$\widetilde{\mathcal{M}}^2 := \left\{H\cdot S^{\mathcal{R}} \in \mathcal{H}^2_0(\widetilde{\mathbb{R}})\right\},$$

and denote by

$$\widetilde{\mathcal{N}}^2 := (\widetilde{\mathcal{M}}^2)^\perp$$

its orthogonal complement in $\mathcal{H}^2_0(\widetilde{\mathbb{R}})$. Then the quadratic minimization problems defining $u_{\varepsilon\varepsilon}(x,0)$ and $v_{\varepsilon\varepsilon}(y,0)$ reduce to orthogonal projection problems in $L^2_0(\widetilde{\mathbb{R}})$. Indeed, with f_T defined in (7) under (8)

$$f_T \in L^2(\mathbb{R}).$$

Therefore, we have

$$\begin{aligned}\infty > \mathbb{E}^{\mathbb{R}}\left[f_T^2\right] =& \mathbb{E}\left[\frac{\widehat{X}_T\widehat{Y}_T}{xy}f_T^2\right] = \mathbb{E}\left[\left(-\frac{U''(\widehat{X}_T)\widehat{X}_T}{U'(\widehat{X}_T)}\right)\left(-\frac{U'(\widehat{X}_T)}{U''(\widehat{X}_T)}\right)\frac{\widehat{Y}_T}{xy}f_T^2\right]\\ =&\frac{x}{\mathcal{R}_0}\mathbb{E}\left[A(\widehat{X}_T)\ \frac{\mathcal{R}_T\widehat{Y}_T}{R_0 y}f_T^2\right] = \frac{x}{\mathcal{R}_0}\mathbb{E}^{\widetilde{\mathbb{R}}}\left[A(\widehat{X}_T)f_T^2\right],\end{aligned}$$

and thus, in view of Assumption 2.1, there exists a constant $C>0$, such that

$$\mathbb{E}^{\widetilde{\mathbb{R}}}\left[f_T^2\right] < C\mathbb{E}^{\mathbb{R}}\left[f_T^2\right] < \infty.$$

Next, let us rewrite (10) as

$$\begin{aligned}u_{\varepsilon\varepsilon}(x,0) =& -\frac{y}{\mathcal{R}_0}\inf_{\widetilde{M}\in\widetilde{\mathcal{M}}^2} E^{\widetilde{\mathbb{R}}}\left[\left(\widetilde{M}_T\right)^2 - 2\mathcal{R}_0 f_T\widetilde{M}_T\right]\\ =& -\frac{y}{\mathcal{R}_0}\inf_{\widetilde{M}\in\widetilde{\mathcal{M}}^2} E^{\widetilde{\mathbb{R}}}\left[\left(\widetilde{M}_T - \mathcal{R}_0 f_T\right)^2\right] + y\mathcal{R}_0 E^{\widetilde{\mathbb{R}}}[f_T^2]\end{aligned} \tag{46}$$

and let us also restate (14) as

$$v_{\varepsilon\varepsilon}(y,0) = \frac{\mathcal{R}_0}{y}\inf_{\widetilde{N}\in\widetilde{\mathcal{N}}^2}\mathbb{E}^{\widetilde{\mathbb{R}}}\left[\left(\widetilde{N}_T - yf_T\right)^2\right]. \tag{47}$$

Let $f_t := \mathbb{E}^{\widetilde{\mathbb{R}}}\left[f_T|\mathcal{F}_t\right], t\in[0,T]$, and consider its Kunita–Watanabe decomposition

$$f = E^{\widetilde{\mathbb{R}}}[f_T] + \widetilde{M}^f + \widetilde{N}^f, \qquad \widetilde{M}^f \in \widetilde{\mathcal{M}}^2, \quad \widetilde{N}^f\in\widetilde{\mathcal{N}}^2,$$

be under $\widetilde{\mathbb{R}}$. Then the optimizers in the quadratic minimization problems satisfy

$$\widehat{M}^\varepsilon = \frac{x\mathcal{R}}{\widehat{X}}\widetilde{M}^f, \qquad \widehat{N}^\varepsilon = y\widetilde{N}^f, \tag{48}$$

Moreover, we have

$$||\widetilde{M}^f_T||^2_{\mathbb{L}^2(\widetilde{\mathbb{R}})} + ||\widetilde{N}^f_T||^2_{\mathbb{L}^2(\widetilde{\mathbb{R}})} = Var^{\widetilde{\mathbb{R}}}(f_T).$$

$$\begin{aligned}u_{\varepsilon\varepsilon}(x,0) =& -y\mathcal{R}_0 E^{\widetilde{\mathbb{R}}}\left[\left(\widetilde{M}^f_T - f_T\right)^2\right] + y\mathcal{R}_0 E^{\widetilde{\mathbb{R}}}[f_T^2]\\ =& y\mathcal{R}_0||\widetilde{M}^f_T||^2_{\mathbb{L}^2(\widetilde{\mathbb{R}})}\end{aligned} \tag{49}$$

$$v_{\varepsilon\varepsilon}(y,0) = y\mathcal{R}_0\mathbb{E}^{\widetilde{\mathbb{R}}}\left[\left(\widetilde{N}^f_T - f_T\right)^2\right] = y\mathcal{R}_0||\widetilde{M}^f_T||^2_{\mathbb{L}^2(\widetilde{\mathbb{R}})} + y\mathcal{R}_0\left(\mathbb{E}^{\widetilde{\mathbb{R}}}\left[f_T\right]\right)^2. \tag{50}$$

$$v_{\varepsilon\varepsilon}(y,0) - u_{\varepsilon\varepsilon}(x,0) = y\mathcal{R}_0\left(\mathbb{E}^{\widetilde{\mathbb{R}}}[f_T]\right)^2.$$

In particular, the primal and dual second-order coefficients coincide iff $\mathbb{E}^{\widetilde{\mathbb{R}}}[f_T] = 0$.

Appendix A. Perturbations of convex conjugates

This appendix develops a perturbation theory for convex conjugates underlying the second-order sensitivity analysis of the main text. While stability properties of convex conjugation and infimal convolution under variational convergence are classical (see, e.g., [Mor70, Att84, RW98, Mor06]), notions of second-order epi-differentiability and proto-differentiability have been extensively studied in variational analysis; see, e.g., [Roc88, PR96, PRT00]. Explicit second-order expansions for perturbed convex conjugates, however, appear to be less standard. The results below provide such expansions in a form suitable for applications in stochastic control.

Let

$$U^\varepsilon(x) := U(x) + \varepsilon \widetilde{U}(x), \qquad x > 0,$$

where U and $\widetilde{U}$ are C^2, strictly concave functions on $(0, \infty)$, and assume that U satisfies the Inada conditions.

For $y > 0$, define the convex conjugate

$$V^\varepsilon(y) := \sup_{x>0} \big(U^\varepsilon(x) - xy \big).$$

Denote by

$$x_0 := I(y) := (U')^{-1}(y)$$

the optimizer for the unperturbed problem, that is,

$$U'(x_0) = y.$$

Proposition A.1. *For every fixed $y > 0$,*

$$V^\varepsilon(y) = V(y) + \varepsilon \widetilde{U}(x_0) - \frac{\varepsilon^2}{2} \frac{(\widetilde{U}'(x_0))^2}{U''(x_0)} + o(\varepsilon^2),$$

where

$$V(y) = U(x_0) - x_0 y.$$

Equivalently,

$$V^\varepsilon(y) = V(y) + \varepsilon \widetilde{U}(I(y)) - \frac{\varepsilon^2}{2} \frac{(\widetilde{U}'(I(y)))^2}{U''(I(y))} + o(\varepsilon^2).$$

Proof. Let (x_ε) denote the optimizer in the definition of $(V^\varepsilon(y))$. Then

$$U'(x_\varepsilon) + \varepsilon \widetilde{U}'(x_\varepsilon) = y.$$

We seek a first-order expansion of the optimizer of the form

$$x_\varepsilon = x_0 + \varepsilon x_1 + o(\varepsilon).$$

Using Taylor expansion,

$$U'(x_\varepsilon) = U'(x_0) + \varepsilon U''(x_0)x_1 + o(\varepsilon),$$

and similarly,

$$\widetilde{U}'(x_\varepsilon) = \widetilde{U}'(x_0) + o(1).$$

Substituting into the first-order condition gives

$$U'(x_0) + \varepsilon U''(x_0)x_1 + \varepsilon\widetilde{U}'(x_0) + o(\varepsilon) = y.$$

Since $U'(x_0) = y$,

$$U''(x_0)x_1 + \widetilde{U}'(x_0) = 0,$$

hence

$$x_1 = -\frac{\widetilde{U}'(x_0)}{U''(x_0)}.$$

Therefore,

$$x_\varepsilon = x_0 - \varepsilon\frac{\widetilde{U}'(x_0)}{U''(x_0)} + o(\varepsilon).$$

Now,

$$V^\varepsilon(y) = U(x_\varepsilon) + \varepsilon\widetilde{U}(x_\varepsilon) - x_\varepsilon y.$$

Expanding each term around x_0,

$$U(x_\varepsilon) = U(x_0) + \varepsilon U'(x_0)x_1 + \frac{\varepsilon^2}{2}U''(x_0)x_1^2 + o(\varepsilon^2),$$
$$\widetilde{U}(x_\varepsilon) = \widetilde{U}(x_0) + \varepsilon\widetilde{U}'(x_0)x_1 + o(\varepsilon),$$

and

$$-x_\varepsilon y = -x_0 y - \varepsilon x_1 y.$$

Combining terms,

$$\begin{aligned} V^\varepsilon(y) = U(x_0) - x_0 y & \\ &+ \varepsilon\big(U'(x_0)x_1 + \widetilde{U}(x_0) - x_1 y\big) \\ &+ \varepsilon^2\left(\frac{1}{2}U''(x_0)x_1^2 + \widetilde{U}'(x_0)x_1\right) + o(\varepsilon^2). \end{aligned}$$

Using $U'(x_0) = y$, the linear terms involving x_1 cancel:

$$U'(x_0)x_1 - x_1 y = 0.$$

Thus,

$$V^\varepsilon(y) = V(y) + \varepsilon\widetilde{U}(x_0) + \varepsilon^2\left(\frac{1}{2}U''(x_0)x_1^2 + \widetilde{U}'(x_0)x_1\right) + o(\varepsilon^2).$$

Substituting

$$x_1 = -\frac{\widetilde{U}'(x_0)}{U''(x_0)},$$

we obtain

$$\frac{1}{2}U''(x_0)x_1^2 + \widetilde{U}'(x_0)x_1 = -\frac{1}{2}\frac{(\widetilde{U}'(x_0))^2}{U''(x_0)}.$$

Hence,

$$V^\varepsilon(y) = V(y) + \varepsilon\widetilde{U}(x_0) - \frac{\varepsilon^2}{2}\frac{(\widetilde{U}'(x_0))^2}{U''(x_0)} + o(\varepsilon^2).$$

□

We can now state the following corollary to Proposition A.1.

Corollary A.2. *For every $y > 0$ and every ε for which U^ε is an Inada utility, we have*

$$\frac{\partial}{\partial\varepsilon}V^\varepsilon(y) = \widetilde{U}(I^\varepsilon(y)),$$

where

$$I^\varepsilon(y) = ((U^\varepsilon)')^{-1}(y).$$

Furthermore,

$$\frac{\partial^2}{\partial\varepsilon^2}V^\varepsilon(y) = -\frac{\left(\widetilde{U}'(I^\varepsilon(y))\right)^2}{(U^\varepsilon)''(I^\varepsilon(y))}.$$

In particular, at $\varepsilon = 0$,

$$\frac{\partial}{\partial\varepsilon}V^\varepsilon(y)\Big|_{\varepsilon=0} = \widetilde{U}(I(y)),$$

and

$$\frac{\partial^2}{\partial\varepsilon^2}V^\varepsilon(y)\Big|_{\varepsilon=0} = -\frac{(\widetilde{U}'(I(y)))^2}{U''(I(y))}.$$

Proof. Differentiating the first-order condition

$$(U^\varepsilon)'(I^\varepsilon(y)) = y$$

with respect to ε yields

$$\frac{\partial I^\varepsilon(y)}{\partial\varepsilon} = -\frac{\widetilde{U}'(I^\varepsilon(y))}{(U^\varepsilon)''(I^\varepsilon(y))}.$$

Since

$$V^\varepsilon(y) = U^\varepsilon(I^\varepsilon(y)) - yI^\varepsilon(y),$$

the first identity follows from the envelope theorem:

$$\frac{\partial}{\partial\varepsilon}V^\varepsilon(y) = \widetilde{U}(I^\varepsilon(y)).$$

Differentiating once more gives the second identity. □

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