

Stability of the Indirect Utility Process*

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Abstract. We investigate the dynamic stability of the indirect utility process associated with a (possibly sub-optimal) trading strategy under perturbations of the market. Establishing the reverse conjugacy characterizations first, we prove continuity and first-order convergence of the indirect utility process under simultaneous perturbations of the finite variation and martingale parts of the return of the risky asset.

Key words. stability, indirect utility, arbitrage of the first kind, no unbounded profit with bounded risk, local martingale deflator, duality theory, incomplete market

AMS subject classifications. 91G10, 93E20, C61, G11

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1. Introduction. Indirect utility appears in mathematical finance as one of the primary criteria of the quality of a portfolio. Therefore, the regularity of the solution to certain classical problems in mathematical finance is associated with the stability or continuity of the indirect utility under perturbations of the initial data. From the practical viewpoint, as every statistical procedure allows for an only approximate determination of the model parameters, implementation of algorithms of optimal investment and utility-based pricing and hedging hinges on continuity of the indirect utility under model perturbations. The results of this paper show that under reasonably natural assumptions the indirect utility is a stable criterion of quality of the portfolio. Further, this also holds in dynamic settings, where the dynamic characterizations are usually harder to establish, and even for suboptimal portfolios.

Mathematically, the results below hinge on the dual characterization of the indirect utility associated with a possibly suboptimal trading strategy. Both the dynamic formulation and such a suboptimality lead to multiple difficulties related to establishing convex-analytic results for functions whose codomain is a space of random variables \mathbb{L}^0 (and not \mathbb{R}), i.e., for random elements. It turns out that even fundamental theorems of convex analysis are harder to establish in such settings, and, for example, the classical Fenchel–Moreau (or biconjugation) theorem over a *dual pair of Banach spaces* has been proven only recently; see [DJK]. In the present settings, however, we need not work with a dual pair but rather with a pair of *polar sets* of stochastic processes, where polarity has to be understood appropriately. We identify precisely such polar sets and use the classical characterizations of the sets of wealth processes

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from [DS97] combined with changes of numéraire and results from [Mos15] to obtain the biconjugation result.

In the process of proving the biconjugacy, we establish a result which is closely related to the conditional minimax theorem. Another version of this theorem can be found in [BK10]. Note that in our formulation we also do not require compactness of either domain, but only *boundedness in probability* is needed. Such boundedness often appears in the mathematical finance literature in both primal and dual domains under natural no-arbitrage conditions, for example, in [KS99]. Note that minimax without compactness is a classical subject of analysis (see, e.g., [Fan79], [Ha81], [LQ91]); a version of the minimax theorem that is helpful in financial applications, in particular below, and that does not require compactness of either domain can be found in [BK17].

For the stability analysis, we introduce a parametrization of perturbations, which allows considering distortions of the drift or volatility of the risky asset together or separately. Then we identify certain primal and dual feasible elements under such perturbations and prove convergence of the indirect utility process (in the sense of Theorem 4.10 below) and complement this with finding its associated derivative with respect to a parameter (also in Theorem 4.10). Dual characterization is key here.

Our construction of the dual domain is consistent with the weak no-arbitrage condition, no unbounded profit with bounded risk (NUPBR) introduced in [KK07], that still allows for the meaningful structure of the underlying problem. As the formulation of the indirect utility in a dynamic formulation is closely related to forward performance processes (FPPs) of the form [ZŽ10, Definition 4.3], one of the contributions of the present paper is in showing that FPPs on a finite time horizon can be considered under NUPBR and possibly without a stronger no free lunch with vanishing risk (NFLVR) condition, which was predominantly used for the investigation of FPPs in the past. Note that FPPs were originally introduced in [MZ07] and [MZ08] to measure performances of portfolios in a way that allows for dynamic adaptation of the investor's preferences. Thus, the results of this paper provide an approach for the analysis of FPPs under market perturbations. They also imply the robustness of the indirect utility (and therefore the FPP in finite-horizon settings) as a dynamic criterion of the quality of a portfolio, and thus this paper complements the research of many authors, in particular, [SSZ16], in non-Markovian settings. However, the complete analysis of general FPPs goes beyond the scope of the current paper. Note that, in static settings, questions related to stability were investigated in [LŽ07], [AŽ10], and [KŽ11], among others.

The remainder of the paper is organized as follows: in section 2 we specify the model, section 3 contains the dual characterization, and in section 4 we show the stability of the indirect utility process.

2. Model. We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where the filtration satisfies the usual conditions, and \mathcal{F}_0 is trivial. There is a riskless asset, whose price equals to 1 at all times, and a risky one. The conditions on the risky asset will alter in different sections, thus in section 3, it can be considered to be a general multidimensional semimartingale, and for stability analysis in section 4 and below, we will work with a one-dimensional continuous process.

2.1. Utility field and maximization problem. Let us consider an Inada stochastic utility field $U: [0, \infty) \times \Omega \times [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$, i.e., a stochastic field, which satisfies the following assumption.

Assumption 2.1. For every $(t, \omega) \in [0, \infty) \times \Omega$, $U(t, \omega, \cdot)$ is an Inada utility function, that is, a strictly increasing, strictly concave, differentiable function, which satisfies the Inada conditions:

$$\lim_{x \searrow 0} U'(t, \omega, x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} U'(t, \omega, x) = 0,$$

where U' denotes the derivative with respect to the last argument. At $x = 0$, we suppose that $U(t, \omega, 0) = \lim_{x \searrow 0} U(t, \omega, x)$. This value may be $-\infty$. We also suppose that $U(\cdot, \cdot, x)$ is optional for every $x > 0$. As is common in the probability literature, the symbol ω will usually be omitted.

We refer to [Jar18, Chapter 9] for an overview of utility functions. In this and the following sections, we consider only one market, where there is a d -multidimensional risky asset with a return process R^0 and a riskless asset, whose price equals to 1 at all times. Following [KS99], we denote by $\mathcal{X}(x)$ the set of nonnegative wealth processes:

$$\mathcal{X}(x) = \{X \geq 0 : X = x + H \cdot R^0 \text{ for some } R^0\text{-integrable } H\}, \quad x \geq 0.$$

In this market, we fix an initial wealth $\bar{x} \geq 0$ and a predictable and R^0 -integrable process π (up to $t \in [0, T]$), which specifies the proportions of wealth invested in corresponding risky assets and such that $X^\pi = \bar{x}\mathcal{E}(\pi \cdot R^0) \geq 0$, where, here and below, $\mathcal{E}(\cdot)$ denotes the stochastic exponential. The set of wealth processes in $\mathcal{X}(\bar{x})$, which equal to X^π on $[0, t]$, is denoted by $\mathcal{A}(X_t^\pi, t)$, that is,

$$(1) \quad \mathcal{A}(X_t^\pi, t) := \left\{ \tilde{X} \in \mathcal{X}(\bar{x}) : \tilde{X}_s = X_s^\pi \text{ for } s \in [0, t], \mathbb{P}\text{-a.s.} \right\}.$$

In such settings, we define an indirect utility up to T of π as

$$(2) \quad u(X_t^\pi, t, T) := \operatorname{ess\,sup}_{\tilde{X} \in \mathcal{A}(X_t^\pi, t)} \mathbb{E} \left[U \left(T, \tilde{X}_T \right) \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

Note that this definition is closely related to the definition of FPPs; see, e.g., [BRT09], [NZ14], and [ASS18] on a finite time horizon. However, (2) does not require the existence of the optimizer (instead, it is proven below), and the supermartingale structure of $u(X_t^\pi, t, T)$, $t \in [0, T]$, i.e., $\mathbb{E} [u(X_{t_2}^\pi, t_2, T) | \mathcal{F}_{t_1}] \leq u(X_{t_1}^\pi, t_1, T)$ for $0 \leq t_1 \leq t_2 \leq T$, can also be shown. Further, (2) is also forward performances in the sense of [ZŽ10, Definition 4.3], where the difference is in the exact form of the domain for the optimization problem (2).

2.2. Reformulation of (2). For the analysis below, we need to extend the definition of (2) to the closure of the convex solid hull of $\{X_t : X \in \mathcal{X}(x)\}$. This is done in the following two-step procedure. First, we define

$$(3) \quad \mathcal{C}_t(x) \triangleq \{g \in m\mathcal{F}_t : g \leq X_t \text{ for some } X \in \mathcal{X}(x)\}, \quad x \geq 0,$$

where $m\mathcal{F}_t$ stands for \mathcal{F}_t measurability of g ; see, e.g., [Wil95, p. 29] for notation of this kind. Now, we set

$$(4) \quad \mathcal{C}_{tT}(x) := \left\{ g \in m\mathcal{F}_T : g \leq x + \int_{t+}^T H_u dR_u^0 \text{ for some } R^0\text{-integrable } H \right\}, \quad x \geq 0,$$

and we will denote $\mathcal{C}_t(1)$ by \mathcal{C}_t as well as $\mathcal{C}_{tT}(1)$ by \mathcal{C}_{tT} , respectively.

Remark 2.2. The idea behind such definitions is that every stochastic integral of the form $x + \int_0^T H dR^0$ can be represented as $(x + \int_0^t H_u dR_u^0)(1 + \int_{t+}^T \tilde{H}_u dR_u^0)$ for an appropriate \tilde{H} . Also, similarly to the argument in [KS99, section 4, p. 926] one can show that the sets \mathcal{C}_t and \mathcal{C}_{tT} are closed with respect to the topology of convergence in measure. This, in particular, allows for the following representation of \mathcal{C}_T :

$$\mathcal{C}_T = \mathcal{C}_t \mathcal{C}_{tT} = \{ \xi \rho : \xi \in \mathcal{C}_t, \rho \in \mathcal{C}_{tT} \}, \quad t \in [0, T].$$

Further, as U is increasing, an optimizer to (2) is a maximal element of \mathcal{C}_{tT} . Thus, by enlarging the domain of (2) as above, we do not lose the structure of the solution to (2). On the other hand, by passing from the set of wealth processes to the closure of the (convex) solid hull of such processes we gain the properties needed, in particular, for the conjugacy characterization below.

Thus, we extend the definition of u in (2) from X^π to \mathcal{C}_t as

$$(5) \quad u(\xi, t, T) := \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi \rho)], \quad \xi \in \bigcup_{x \geq 0} \mathcal{C}_t(x),$$

with the version of an effective domain of $u(\cdot, t, T)$ being the set

$$(6) \quad \mathcal{E}_t^u := \left\{ \xi \in \bigcup_{x \geq 0} \mathcal{C}_t(x) : \text{there exists } \rho \in \mathcal{C}_{tT} \text{ such that } \mathbb{E} [U^-(T, \xi \rho)] < \infty \right\}.$$

2.3. Technical assumptions. We will impose the following conditions.

Assumption 2.3 (NUBPR up to T). Let $T > 0$ be fixed. The set $\{X_T : X \in \mathcal{X}(1)\}$ is bounded in probability.

Assumption 2.4 (fin value at T). Let $T > 0$ be fixed. We suppose that

$$u(z, 0, T) > -\infty \quad \text{and} \quad \sup_{x > 0} (u(x, 0, T) - xz) < \infty, \quad z > 0.$$

These conditions are necessary for the model to be nondegenerate; see, e.g., the abstract theorems in [Mos15] and [CCFM17]. In the notation of section 4, these conditions will be imposed on the base or, equivalently, 0-model.

Remark 2.5. Assumption 2.4 will imply that the conditional expectations below are well-defined, where we adopt the definition [Shi84, Definition 1, p. 211], which does not require integrability.

3. Dual characterization. In this section, we will suppose that the prices process for the risky asset is a general multidimensional semimartingale, not necessarily continuous. The main contributions of this section are in finding the right structure of the dual problem that, in dynamic settings, allows for the existence and uniqueness results and a biconjugacy characterization of the indirect utility under no unbounded profit with bounded risk. Further, the results of this section provide a version of the Fenchel–Moreau theorem for random elements, i.e., functions whose codomains are a space of random variables. Note that this topic is not well-studied, and a version of a Fenchel–Moreau theorem over a pair of Banach spaces was only recently proven in [DJK]. Additionally, by using a change of numéraire approach below, a minimax type of result is established without the compactness of either domain. The polar structure of the primal and dual domains is key here, though.

3.1. Existence and uniqueness of a solution to (2). Let us denote

$$(7) \quad \mathcal{Z}_t := \left\{ (Z_s)_{s \in [t, T]} \geq 0 : \right. \\ \left. Z_t \leq 1 \text{ and } (Z_s X_s)_{s \in [t, T]} \text{ is a supermartingale for every } X \in \mathcal{X}(1) \right\}$$

and recall that the notion of Fatou-convergence of stochastic processes is introduced in [FK97, Definition 5.2]. The following lemma shows existence of an optimizer to (5).

Lemma 3.1. *Let $T > 0$ be fixed and let us suppose that Assumptions 2.1, 2.3, and 2.4 hold. Then for every $\xi \in \mathcal{E}_t^u$, where \mathcal{E}_t^u is defined in (6), there exists $\rho \in \mathcal{C}_{tT}$ such that*

$$u(\xi, t, T) = \mathbb{E}_t [U(T, \xi\rho)].$$

Moreover, if $\xi > 0$, \mathbb{P} -a.s., then such a ρ is unique.

Proof. Let us fix $\xi \in \mathcal{E}_t^u$. First, we will show that the set

$$(8) \quad \mathcal{U}_t \triangleq \{ \mathbb{E}_t [U(T, \xi\rho)] : \rho \in \mathcal{C}_{tT} \}$$

is closed under pairwise maximization. Let ρ^1 and ρ^2 be some elements of \mathcal{C}_{tT} and let H^1 and H^2 be such that

$$\rho^i \leq 1 + \int_{t+}^T H_u^i dR_u^0, \quad i = 1, 2.$$

We define

$$A := \{ \mathbb{E}_t [U(T, \xi\rho^1)] > \mathbb{E}_t [U(T, \xi\rho^2)] \} \in \mathcal{F}_t$$

and set $H := H^1 1_A + H^2 1_{A^c}$. Then we obtain for $\rho := \rho^1 1_A + \rho^2 1_{A^c}$ that

$$(9) \quad \rho \leq 1 + \int_{t+}^T H_u dR_u^0$$

as

$$\rho = \begin{cases} \rho^1 \leq 1 + \int_{t+}^T H_u^1 dR_u^0 & \text{on } A, \\ \rho^2 \leq 1 + \int_{t+}^T H_u^2 dR_u^0 & \text{on } A^c. \end{cases}$$

Since $1 + \int_{t+}^T H_u dS_u \geq 0$, \mathbb{P} -a.s., we conclude from (9) that $\rho \in \mathcal{C}_{tT}$. Consequently,

$$\mathbb{E}_t [U(T, \xi\rho)] = \max \left(\mathbb{E}_t [U(T, \xi\rho^1)], \mathbb{E}_t [U(T, \xi\rho^2)] \right),$$

i.e., \mathcal{U}_t defined in (8) is closed under pairwise maximization. Applying [KS98, Theorem A.3, p. 324] (or rather an extension of this theorem to extended real-valued random variables; see e.g., [CP15, Proposition 2.6.1]), we deduce that there exists a sequence $(\rho^n)_{n \in \mathbb{N}} \subset \mathcal{C}_{tT}$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}_t [U(T, \xi\rho^n)] = \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\rho)].$$

By a Komlós-type lemma (see e.g., [DS94, Lemma A1.1]), we may find a sequence of convex combinations $\tilde{\rho}^n \in \operatorname{conv}(\rho^n, \rho^{n+1}, \dots)$, $n \in \mathbb{N}$, and a random variable \hat{g} , such that $(\tilde{\rho}_T^n)_{n \in \mathbb{N}}$ converges to \hat{g} , \mathbb{P} -a.s. By concavity of $U(T, \cdot)$, $(\tilde{\rho}_T^n)_{n \in \mathbb{N}}$ is also a maximizing sequence in the sense that

$$(10) \quad \lim_{n \rightarrow \infty} \mathbb{E}_t [U(T, \xi\tilde{\rho}^n)] = \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\rho)].$$

Similarly to the argument in [KS99, Proof of Proposition 3.1], one can show that \mathcal{C}_{tT} is closed in probability. Therefore, $\hat{g} \in \mathcal{C}_{tT}$. Via [Mos15, Lemma 3.5]¹ and the symmetry between primal and dual problems in [Mos15], one can show that $U^+(T, \xi\tilde{\rho}^n)$, $n \in \mathbb{N}$, is a uniformly integrable sequence, as is $\mathbb{E}_t [U^+(T, \xi\tilde{\rho}^n)]$, $n \in \mathbb{N}$, and therefore we have

$$\operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\rho)] = \lim_{n \rightarrow \infty} \mathbb{E}_t [U(T, \xi\tilde{\rho}^n)] \leq \mathbb{E}_t [U(T, \xi\hat{g})], \quad \mathbb{P}\text{-a.s.},$$

and thus \hat{g} is the maximizer to (5). Further, if $\xi > 0$, \mathbb{P} -a.s., the uniqueness of the maximizer follows from the strict concavity of $U(T, \cdot)$. \blacksquare

Remark 3.2. If the risky assets have continuous paths, after (10), one can apply an argument based on a version of the optional decomposition theorem for arbitrary filtrations; see [KK15, Optional Decomposition Theorem 1.4]. Suppose that \tilde{X}^n , $n \geq 1$, are nonnegative processes of the form $1 + \int_{t+}^s H_u^n dR_u^0$, $s \in (t, T]$, and $\tilde{X}_t^n = 1$, for some S -integrable H^n 's, such that

$$\tilde{X}_T^n \geq \tilde{\rho}^n, \quad \mathbb{P}\text{-a.s.}, \quad n \in \mathbb{N}.$$

We pick a strictly positive $Y \in \mathcal{Z}_t$ (whose existence follows from Assumption 2.3) and consider $\tilde{X}^n Y$, $n \in \mathbb{N}$. Let

$$\mathcal{T} := (\mathbb{Q} \cap (t, T)) \cup \{t\} \cup \{T\},$$

where \mathbb{Q} is the set of rational numbers.

Then, passing to convex combinations, we may find a subsequence of convex combinations $\tilde{\tilde{X}}^n$, $n \in \mathbb{N}$, such that $\tilde{\tilde{X}}^n Y$, $n \in \mathbb{N}$, is Fatou convergent to a supermartingale VY on \mathcal{T} and such that

$$(11) \quad V_t \leq \liminf_{s \searrow t, s \in \mathcal{T}} \liminf_{n \rightarrow \infty} \mathbb{E} \left[\tilde{\tilde{X}}_s^n \frac{Y_s}{Y_t} \middle| \mathcal{F}_t \right] \leq 1 \quad \text{and} \quad V_T \geq \hat{g}.$$

¹Section 3.5 contains a brief summary of the results from [Mos15] used in this paper.

One can show that $(V_s Z_s)_{s \in [t, T]}$ is a supermartingale for every $Z \in \mathcal{Z}_t$. We stress that the supermartingale property is only required on $[t, T]$. One can see that² for every supermartingale deflator Z , the process \bar{Z} of the form

$$\bar{Z}_s = \frac{Z_{t+s}}{Z_t}, \quad s \in [0, T - t],$$

is an element of \mathcal{Z}_t . We also set

$$\mathcal{G}_s := \mathcal{F}_{t+s}, \quad s \in [0, T - t].$$

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{G}_s)_{s \in [0, T-t]}$, $(V_s \bar{Z}_s)_{s \in [0, T-t]}$ is a supermartingale for every $\bar{Z} \in \mathcal{Z}_t$. Therefore V satisfies the conditions of (item 1 of) [KK15, Optional Decomposition Theorem 1.4]. As a result, there exists a decomposition of V of the form

$$(12) \quad V_s = V_0 + \int_{0+}^s \bar{H}_u dR_u^0 - A_s, \quad s \in (0, T - t],$$

where V_0 is $\mathcal{G}_0 = \mathcal{F}_t$ -measurable random variable, \bar{H} is predictable and S -integrable, and A is a nondecreasing, right-continuous, adapted process, such that $A_0 = 0$.

We denote $\hat{H}_{t+s} := \bar{H}_s$, $s \in [0, T - t]$. Therefore, using (11) and (12), we deduce that

$$\hat{X}_T = 1 + \int_{t+}^T \hat{H}_u dR_u^0 \geq \hat{g},$$

that is, $\hat{X}_T \in \mathcal{C}_{tT}$, by the definition of \mathcal{C}_{tT} in (4). It follows from Assumption 2.4 and Lemma 3.21 that $(U^+(T, X_T))_{X \in \mathcal{X}(x)}$ is uniformly integrable, therefore, so is $(\mathbb{E}_t [U^+(T, X_T)])_{X \in \mathcal{X}(x)}$, and using the monotonicity of $U(T, \cdot)$, we get

$$\operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi \rho)] = \lim_{n \rightarrow \infty} \mathbb{E}_t [U(T, \xi \tilde{\rho}^n)] \leq \mathbb{E}_t [U(T, \xi \hat{g})] \leq \mathbb{E}_t [U(T, \xi \hat{X}_T)].$$

We deduce that \hat{X}_T is the maximizer to (5). If $\xi > 0$, \mathbb{P} -a.s., the uniqueness of the maximizer follows from the strict concavity of U .

3.2. Structure of the dual process. First, we set

$$V(t, \omega, y) := \sup_{x > 0} (U(t, \omega, x) - xy), \quad (t, \omega, y) \in [0, T] \times \Omega \times [0, \infty).$$

For $t \in [0, T]$, we set

$$(13) \quad \mathcal{D}_t(y) := \{\eta \in m\mathcal{F}_t : \eta \leq yz_t \text{ for some } z \in \mathcal{Z}_0\}, \quad y \geq 0,$$

i.e., $\mathcal{D}_t(y)$ is a subset of the closure of the convex solid hull of the elements of $y\mathcal{Z}_0$ sampled at time t . We define

$$(14) \quad \mathcal{N}_t := \bigcup_{y \geq 0} \mathcal{D}_t(y).$$

²We use the convention $\frac{0}{0} = 0$.

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For $t \in [0, T]$, let \mathcal{Z}_t be given by (7) and we set

$$(15) \quad v(\eta, t, T) := \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V(T, \eta z_T)], \quad \eta \in \mathcal{N}_t.$$

Lemma 3.3. *Under the conditions of Lemma 3.1, for every $\eta \in \mathcal{N}_t$, there exists $\widehat{z} \in \mathcal{Z}_t$ such that*

$$(16) \quad v(\eta, t, T) = \mathbb{E} [V(T, \eta \widehat{z}_T) | \mathcal{F}_t].$$

Proof. Let us consider \bar{z}^1 and \bar{z}^2 in \mathcal{Z}_t , and let

$$A := \{\omega : \mathbb{E} [V(T, \eta \bar{z}_T^1) | \mathcal{F}_t](\omega) < \mathbb{E} [V(T, \eta \bar{z}_T^2) | \mathcal{F}_t](\omega)\} \in \mathcal{F}_t,$$

and

$$\bar{z}_{t'} := \bar{z}_{t'}^1 1_A + \bar{z}_{t'}^2 1_{A^c}, \quad t' \in [t, T].$$

Then on $[t, T]$, $\bar{z}X$ is a supermartingale deflator for every $X \in \mathcal{X}(1)$. Therefore, $\bar{z} \in \mathcal{Z}_t$. By direct computations, we have

$$\mathbb{E} [V(T, \eta \bar{z}_T) | \mathcal{F}_t] = \min(\mathbb{E} [V(T, \eta \bar{z}_T^1) | \mathcal{F}_t], \mathbb{E} [V(T, \eta \bar{z}_T^2) | \mathcal{F}_t]).$$

Therefore, from [CP15, Proposition 2.6.1], we deduce that there exists a sequence $(z^n)_{n \in \mathbb{N}}$ such that

$$(17) \quad \lim_{n \rightarrow \infty} \mathbb{E} [V(T, \eta z_T^n) | \mathcal{F}_t] = v(\eta, t, T), \quad \mathbb{P}\text{-a.s.}$$

By passing to convex combinations, we obtain a subsequence, which we do not relabel, such that $\lim_{n \rightarrow \infty} z^n = \widehat{z}$, where the limit is considered in the Fatou sense (in the terminology of [FK97, Definition 5.2]) on the set of rational numbers on (t, T) augmented with t and T . Note that by convexity of $V(T, \cdot)$, such a subsequence will also satisfy (17). It follows from the definition of \mathcal{N}_t and Lemma 3.21 that $(V^-(T, \eta z_T^n))_{n \in \mathbb{N}}$ is uniformly integrable. Therefore, we get

$$(18) \quad \lim_{n \rightarrow \infty} \mathbb{E} [V(T, \eta z_T^n) | \mathcal{F}_t] \geq \mathbb{E} [V(T, \eta \widehat{z}_T) | \mathcal{F}_t].$$

By direct computations, we deduce that for every $X \in \mathcal{X}(1)$, $(\widehat{z}_{t'} X_{t'})_{t' \in [t, T]}$ is a nonnegative supermartingale on $[t, T]$ such that $\widehat{z}_t \leq 1$ by properties of Fatou-convergence. Therefore $\widehat{z} \in \mathcal{Z}_t$. Via (17) and (18), we conclude that (16) holds. \blacksquare

Let

$$(19) \quad \mathcal{B}_t^+ := \{X \in m\mathcal{F}_t : X \in [0, 1], \mathbb{P}\text{-a.s.}\}$$

and

$$(20) \quad \mathcal{E}_t^v := \{\eta \in \mathcal{N}_t : \text{there exists } z \in \mathcal{Z}_t \text{ such that } \mathbb{E} [V^+(T, \eta z_T)] < \infty\},$$

which corresponds to the effective domain of $v(\cdot, t, T)$.

Lemma 3.4. *Let the condition of Lemma 3.1 hold, $\lambda \in \mathcal{B}_t^+$, and η^1 and η^2 are some elements of \mathcal{E}_t^v ; then $\eta := \lambda\eta^1 + (1 - \lambda)\eta^2 \in \mathcal{E}_t^v$ and we have*

$$(21) \quad v(\eta, t, T) \leq \lambda v(\eta^1, t, T) + (1 - \lambda)v(\eta^2, t, T).$$

Proof. As $\eta^i \in \mathcal{D}_t(y^i)$, for some $y^i > 0$, $i = 1, 2$, one can see that $\eta \leq Y_t$ for some $Y \in (y^1 + y^2)\mathcal{Z}_0$. Next we will show (21). By Lemma 3.3, we deduce the existence of \hat{z}^1 and \hat{z}^2 , the optimizers to (15) corresponding to η^1 and η^2 , respectively. With

$$z_s := \frac{\lambda\eta^1\hat{z}_s^1 + (1 - \lambda)\eta^2\hat{z}_s^2}{\eta} 1_{\{\eta \neq 0\}} + (\lambda\hat{z}_s^1 + (1 - \lambda)\hat{z}_s^2) 1_{\{\eta=0\}}, \quad s \in [t, T],$$

by direct computations, one can see that $(z_{t'}X_{t'})_{t' \in [t, T]}$ is a supermartingale for every $X \in \mathcal{X}(1)$. By construction, $z_t \leq 1$. We conclude that $z \in \mathcal{Z}_t$. To show that $\eta \in \mathcal{E}_t^v$, first we observe that on $\{\eta = 0\}$, we have $\lambda\hat{\eta}^1 = (1 - \lambda)\eta^2 = 0$. Therefore, we obtain

$$\begin{aligned} \eta z_T &= \eta z_T 1_{\{\eta > 0\}} + 0 \cdot z_T 1_{\{\eta=0\}} \\ &= (\lambda\eta^1\hat{z}_T^1 + (1 - \lambda)\eta^2\hat{z}_T^2) 1_{\{\eta > 0\}} + 0 \cdot (\lambda\hat{z}_T^1 + (1 - \lambda)\hat{z}_T^2) 1_{\{\eta=0\}} \\ &= (\lambda\eta^1\hat{z}_T^1 + (1 - \lambda)\eta^2\hat{z}_T^2) 1_{\{\eta > 0\}} + (\lambda\eta^1\hat{z}_T^1 + (1 - \lambda)\eta^2\hat{z}_T^2) 1_{\{\eta=0\}} \\ &= \lambda\eta^1\hat{z}_T^1 + (1 - \lambda)\eta^2\hat{z}_T^2. \end{aligned}$$

Therefore, using the convexity of $V^+(T, \cdot)$, we get

$$\begin{aligned} \mathbb{E}_t [V^+(T, \eta z_T)] &= \mathbb{E}_t [V^+(T, \lambda\eta^1\hat{z}_T^1 + (1 - \lambda)\eta^2\hat{z}_T^2)] \\ &\leq \lambda\mathbb{E}_t [V^+(T, \eta^1\hat{z}_T^1)] + (1 - \lambda)\mathbb{E}_t [V^+(T, \eta^2\hat{z}_T^2)]. \end{aligned}$$

Therefore, as η^1 and η^2 are in \mathcal{E}_t^v , we deduce that so is η . Likewise, using the convexity of V , we get

$$(22) \quad \begin{aligned} v(\eta, t, T) &\leq \mathbb{E}_t [V(T, \eta z_T)] = \mathbb{E}_t [V(T, \lambda\eta^1\hat{z}_T^1 + (1 - \lambda)\eta^2\hat{z}_T^2)] \\ &\leq \lambda\mathbb{E}_t [V(T, \eta^1\hat{z}_T^1)] + (1 - \lambda)\mathbb{E}_t [V(T, \eta^2\hat{z}_T^2)] = \lambda v(\eta^1, t, T) + (1 - \lambda)v(\eta^2, t, T). \end{aligned}$$

Thus, (21) holds. ■

An important role in the proofs below will be played by the set

$$(23) \quad \mathcal{G}_t := \{\eta z_T : \eta \in \mathcal{N}_t, z \in \mathcal{Z}_t\},$$

which is characterized in Lemma 3.5 below.

Lemma 3.5. *Let the condition of Lemma 3.1 hold. Then the set \mathcal{G}_t is closed under convex concatenations in the following sense: for a weight $\lambda \in \mathcal{B}_t^+$, and $\eta^i z^i \in \mathcal{G}_t$, $i = 1, 2$, we set*

$$(24) \quad \eta := \lambda\eta^1 + (1 - \lambda)\eta^2.$$

Then $\eta \in \mathcal{N}_t$, and we have

$$(25) \quad z_s := \frac{\lambda \eta^1 z_s^1 + (1 - \lambda) \eta^2 z_s^2}{\eta} 1_{\{\eta \neq 0\}} + (\lambda z_s^1 + (1 - \lambda) z_s^2) 1_{\{\eta = 0\}}, \quad s \in [t, T],$$

is an element of \mathcal{Z}_t ,

$$(26) \quad \lambda \eta^1 z_T^1 + (1 - \lambda) \eta^2 z_T^2 = \eta z_T \in \mathcal{G}_t.$$

In particular, (24) holds if λ is a constant taking values in $[0, 1]$, i.e., \mathcal{G}_t is a convex set and if η^i 's are in \mathcal{E}_t^v , then $\eta \in \mathcal{E}_t^v$.

Proof. Let us consider η and z are defined in (24) and (25), respectively. As $\eta \leq \eta^1 + \eta^2$, we deduce (trivially) that $\eta \in \mathcal{N}_t$. It follows from Lemma 3.4 that if, additionally, η^1 and η^2 are some elements of \mathcal{E}_t^v , then $\eta \in \mathcal{E}_t^v$. Following the proof of the same lemma, $z \in \mathcal{Z}_t$. The validity of

$$\lambda \eta^1 z_T^1 + (1 - \lambda) \eta^2 z_T^2 = \eta z_T$$

is a consequence of the definitions of η and z . As, by the argument above, $\eta \in \mathcal{N}_t$ and $z \in \mathcal{Z}_t$, we deduce that (26) holds. Thus, in particular, \mathcal{G}_t is convex. ■

3.3. Conjugacy of u and v . We recall that $\mathcal{C}(x)$, $x > 0$, are defined in (3). The goal is to show that

$$u(\xi, t, T) = \operatorname{ess\,inf}_{\eta \in \mathcal{N}_t} (v(\eta, t, T) + \xi \eta), \quad \xi \in \bigcup_{x>0} \mathcal{C}(x),$$

and

$$v(\eta, t, T) = \operatorname{ess\,sup}_{\xi \in \bigcup_{x>0} \mathcal{C}_t(x)} (u(\xi, t, T) - \xi \eta), \quad \eta \in \mathcal{N}_t.$$

Lemma 3.6. *Let the condition of Lemma 3.1 hold. Then for every $\eta \in \mathcal{N}_t$ and $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$, we have*

$$u(\xi, t, T) \leq v(\eta, t, T) + \xi \eta.$$

Proof. Let $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$ and $\eta \in \mathcal{N}_t$ be fixed. We need to show that

$$(27) \quad \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}[U(T, \xi \rho) | \mathcal{F}_t] - \xi \eta \leq \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t].$$

From the definition of the conjugate function, for every $\rho \in \mathcal{C}_{tT}$ and every $z \in \mathcal{Z}_t$, we have

$$(28) \quad \mathbb{E}_t[U(T, \xi \rho)] \leq \mathbb{E}_t[V(T, \eta z_T) + \xi \rho \eta z_T].$$

As $z \in \mathcal{Z}_t$, $\mathbb{E}_t[\rho z_T] \leq 1$ and we have

$$(29) \quad \mathbb{E}_t[\xi \rho \eta z_T] = \xi \eta \mathbb{E}_t[\rho z_T] \leq \xi \eta.$$

Combining (28) and (29), we get

$$\mathbb{E}_t[U(T, \xi \rho)] \leq \mathbb{E}_t[V(T, \eta z_T)] + \xi \eta,$$

which implies (27). This completes the proof of the lemma. ■

Lemma 3.7. *Let the condition of Lemma 3.1 hold and $\eta \in \mathcal{N}_t$. Then we have*

$$(30) \quad \lim_{x \rightarrow \infty} \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \operatorname{ess\,sup}_{\xi \in \mathcal{C}_T(x)} \mathbb{E}_t [(U(T, \xi) - \xi \eta z_T)] \geq \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V(T, \eta z_T)] = v(\eta, t, T).$$

Further, for every $A \in \mathcal{F}_t$, we have

$$(31) \quad \lim_{x \rightarrow \infty} \operatorname{inf}_{z \in \mathcal{Z}_t} \operatorname{sup}_{\xi \in \mathcal{C}_T(x)} \mathbb{E} [(U(T, \xi) - \xi \eta z_T) 1_A] \geq \operatorname{inf}_{z \in \mathcal{Z}_t} \mathbb{E} [V(T, \eta z_T) 1_A].$$

Proof. *Step 1.* First, we suppose that

$$(32) \quad \mathbb{E}[U(T, 1)] > -\infty.$$

Let us set

$$V^n(T, y) := \sup_{x \in (0, n]} (U(T, x) - xy), \quad y \geq 0, \quad n \in \mathbb{N}.$$

Next, we fix $z \in \mathcal{Z}_t$. Then, for every $n \in \mathbb{N}$, as $n \in \mathcal{C}_T(n)$, we get

$$(33) \quad \operatorname{ess\,sup}_{\xi \in \mathcal{C}_T(n)} \mathbb{E}_t [(U(T, \xi) - \xi \eta z_T)] \geq \mathbb{E}_t [V^n(T, \eta z_T)].$$

Next we set

$$v^n(\eta, t, T) := \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V^n(T, \eta z_T)], \quad \eta \in \mathcal{N}_t,$$

and observe that $v^n(\eta, t, T)$, $n \in \mathbb{N}$, in an increasing sequence. From (33), we obtain

$$(34) \quad \lim_{n \rightarrow \infty} \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \operatorname{ess\,sup}_{\xi \in \mathcal{C}_T(n)} \mathbb{E}_t [(U(T, \xi) - \xi \eta z_T)] \geq \lim_{n \rightarrow \infty} v^n(\eta, t, T).$$

As $V^n(T, y) \leq V(T, y)$, $y \geq 0$, similarly to Lemma 3.3, we deduce that there exists $\hat{z}^n \in \mathcal{Z}_t$ such that

$$v^n(\eta, t, T) = \mathbb{E}_t [V^n(T, \eta \hat{z}_T^n)], \quad n \in \mathbb{N}.$$

One can pass to convex combinations, which we denote \tilde{z}^n , $n \in \mathbb{N}$, to obtain a Fatou-limit of \tilde{z}^n 's, which we denote $\hat{z} \in \mathcal{Z}_t$. As

$$V^n(T, y) \geq V^2(T, y) \geq V(T, y) 1_{\{y \geq U'(T, 2)\}} + (U(T, 2) - 2U'(T, 2)) 1_{\{y < U'(T, 2)\}}, \quad y \geq 0, n \geq 2,$$

using convexity of $U(T, \cdot)$, we get $U'(T, 2) \leq U(T, 2) - U(T, 1)$, and thus

$$V^n(T, y) \geq \min(V(T, y), 2U(T, 1) - U(T, 2)), \quad y \geq 0, n \geq 2,$$

and uniform integrability of $(V^n)^-(T, \tilde{z}_T^n \eta)$, $n \geq 2$, follows from Lemma 3.21 and (32). As a consequence, using convexity of $V^n(T, \cdot)$'s and Fatou's lemma, we get

$$(35) \quad \lim_{n \rightarrow \infty} v^n(\eta, t, T) = \lim_{n \rightarrow \infty} \mathbb{E}_t [V^n(T, \hat{z}_T^n \eta)] \geq \lim_{n \rightarrow \infty} \mathbb{E}_t [V^n(T, \tilde{z}_T^n \eta)] \geq \mathbb{E}_t [V(T, \hat{z}_T \eta)] \geq v(\eta, t, T).$$

Combining (34) and (35), we obtain (30). In turn, (31) can be proven similarly.

Step 2. Here we do not suppose that (32) holds. This case can be reduced to the one above by taking $\hat{\rho} = \arg \max_{\xi \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \frac{1}{2}\xi\rho)]$ and by setting $\rho = \max(\hat{\rho}, \frac{1}{2})$. Then $0 < \rho \in \mathcal{C}_{tT}$ and for

$$\begin{aligned} \tilde{U}(T, x) &:= U(T, \rho x), & \tilde{V}(T, y) &:= V\left(T, \frac{y}{\rho}\right), \\ \tilde{\mathcal{C}}_{tT} &:= \{\tilde{\rho} : \tilde{\rho}\rho \in \mathcal{C}_{tT}\}, & \tilde{\mathcal{Z}}_t &:= \left\{\tilde{z} : \frac{z}{\rho} \in \mathcal{Z}_t\right\}. \end{aligned}$$

Then we can represent u and v as

$$u(\xi, t, T) = \operatorname{ess\,sup}_{\tilde{\rho} \in \tilde{\mathcal{C}}_{tT}} \mathbb{E}_t \left[\tilde{U}(T, \tilde{\rho}) \right], \quad v(\eta, t, T) = \operatorname{ess\,inf}_{\tilde{z} \in \tilde{\mathcal{Z}}_t} \mathbb{E}_t \left[\tilde{V}(T, \eta\tilde{z}) \right]$$

and \tilde{U} satisfies (32). Then, (30) and (31) follow from Step 1. ■

Let us define

$$(36) \quad \mathcal{C}'_{tT} := \left\{ \rho \in \mathcal{C}_{tT} : \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t[\rho z_T] = 1 \right\}$$

and adapt to our settings the notation from [DS97],

$$(37) \quad \mathcal{K}_{tT}^{\max} := \{ \text{maximal elements of } \mathcal{X}_{tT}(1) \}.$$

The following lemma extends some characterization of maximal 1-admissible contingent claims from [DS97] to the present settings, mainly to Assumption 2.3.

Lemma 3.8. *Let Assumption 2.3 hold. Then we have*

$$(38) \quad \mathcal{K}_{tT}^{\max} \subseteq \mathcal{C}'_{tT} \subset \mathcal{C}_{tT}.$$

As a consequence, for every $x > 0$ and $\xi \in \mathcal{C}_t(x)$, we have

$$(39) \quad u(\xi, t, T) = \operatorname{ess\,sup}_{\phi \in \mathcal{C}'_{tT}} \mathbb{E}_t [U(T, \xi\phi)] = \operatorname{ess\,sup}_{\phi \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\phi)],$$

and for every $A \in \mathcal{F}_t$, we have

$$(40) \quad \sup_{\phi \in \mathcal{C}'_{tT}} \mathbb{E} [U(T, \xi\phi)1_A] = \sup_{\phi \in \mathcal{C}_{tT}} \mathbb{E} [U(T, \xi\phi)1_A].$$

Proof. The proof is based on a change of numéraire idea. By the results of [KK07], Assumption 2.3 implies the existence of the numéraire portfolio N . Let us assume that for some trading strategy G , $X^{tT} := 1 + \int_{t+}^T G_s dR_s^0$ is a (maximal) element of \mathcal{K}_{tT}^{\max} ; then, as we can extend G by 0 on $[0, t]$ to obtain an element of \mathcal{K}^{\max} , one can see that $\frac{X^{tT}}{N}$ is a maximal element under the numéraire N , and NFLVR holds for $(\frac{1}{N}, \frac{R^0}{N})$. As densities of locally equivalent martingale measures under the new numéraire can be represented as $z'zN$, where z' is a supermartingale deflator for S on $[0, t]$ and z is an element of \mathcal{Z}_t , we deduce from [DS97, Theorem 2.5] the existence of $z'z$ such that

$$1 = \mathbb{E} \left[z'_t z_T N_T \frac{X^{tT}}{N_T} \right] = \mathbb{E} \left[z'_t z_T X^{tT} \right] = \mathbb{E} \left[z'_t \mathbb{E}_t \left[z_T X^{tT} \right] \right].$$

As $\mathbb{E}_t[z_T X_T^{tT}] \leq 1$, by construction, it follows that $\mathbb{E}_t[z_T X_T^{tT}] = 1$, \mathbb{P} -a.s., and therefore we have

$$(41) \quad K_{tT}^{\max} \subseteq \mathcal{C}'_{tT}.$$

Similarly to [KS99, Proposition 3.1], we deduce that

$$\mathcal{C}_{tT} = \{c \in m\mathcal{F}_T : c \leq h, \text{ for some } h \in \mathcal{K}_{tT}^{\max}\} = \left\{c \in m\mathcal{F}_T : \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t[cz_T] \leq 1\right\},$$

and thus (38) holds. Moreover, as $U(T, \cdot)$ is nondecreasing, we get

$$\operatorname{ess\,sup}_{\phi \in \mathcal{K}_{tT}^{\max}} \mathbb{E}_t[U(T, \xi\phi)] = \operatorname{ess\,sup}_{\phi \in \mathcal{C}_{tT}} \mathbb{E}_t[U(T, \xi\phi)].$$

Combining the latter equality with (38), we obtain (39). Finally, (40) can be obtained similarly to (39). ■

Lemma 3.9. *Let the condition of Lemma 3.1 hold and $\eta \in \mathcal{N}_t$ be fixed. Then for every $A \in \mathcal{F}_t$, we have*

$$\begin{aligned} \inf_{z \in \mathcal{Z}_t} \mathbb{E}[V(T, \eta z_T) 1_A] &= \lim_{x \rightarrow \infty} \inf_{z \in \mathcal{Z}_t} \sup_{\xi \rho \in \mathcal{C}_T(x)} \mathbb{E}[(U(T, \xi\rho) - \eta\xi\rho z_T) 1_A] \\ &= \sup_{\substack{\xi \rho \in \bigcup_{x>0} \mathcal{C}_T(x) \\ x>0}} \inf_{z \in \mathcal{Z}_t} \mathbb{E}[(U(T, \xi\rho) - \eta\xi\rho z_T) 1_A]. \end{aligned}$$

Proof. The first equality follows from Lemma 3.7 (see (31)) and the definition of V , whereas the second one is a consequence of the minimax theorem; see [BK17, Theorem B.3]. ■

Remark 3.10. The proof of Lemma 3.9 follows the structure of the proof of Lemma 3.22 given in [Mos15, Lemma 3.9]. However, in view of [BK17, Theorem B.3], one does not need to truncate the domain of u and to invoke the Banach–Alaoglu theorem.

Remark 3.11 (on the multiplicative decomposition of \mathcal{C}_{tT}). Let us consider

$$\alpha(\rho) := \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t[\rho z_T], \quad \rho \in \mathcal{C}_{tT}.$$

Then, for every $\rho \in \mathcal{C}_{tT}$, $\alpha(\rho)$ takes values in $[0, 1]$, and it follows from Assumption 2.3 that

$$(42) \quad \mathbb{P}[\{\rho > 0\} \cap \{\alpha(\rho) = 0\}] = 0,$$

and we recall that \mathcal{C}'_{tT} and \mathcal{B}_t^+ are defined in (36) and (19), respectively. Then using (42), we get

$$\rho = 1_{\{\alpha(\rho) > 0\}} \alpha(\rho) \frac{\rho}{\alpha(\rho)} + 1_{\{\alpha(\rho) = 0\}} \rho = 1_{\{\alpha(\rho) > 0\}} \alpha(\rho) \frac{\rho}{\alpha(\rho)},$$

i.e., a multiplicative decomposition of ρ into an element of \mathcal{B}_t^+ and an element of \mathcal{C}'_{tT} , which holds on $\{\alpha(\rho) > 0\}$, and which we can extend to $\{\alpha(\rho) = 0\}$ by $\alpha(\rho)$ multiplied by any element of \mathcal{C}'_{tT} (restricted to $\{\alpha(\rho) = 0\}$).

Lemma 3.12. *Let the condition of Lemma 3.1 hold. Then we have*

$$(43) \quad \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E}[(U(T, \xi\rho) - \xi\eta) 1_A] = \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \inf_{z \in \mathcal{Z}_t} \mathbb{E}[(U(T, \xi\rho) - \xi\eta\rho z_T) 1_A].$$

Proof. With \mathcal{B}_t^+ and \mathcal{C}'_{tT} defined in (36) and (19), respectively, and following the argument of Remark 3.11, we can rewrite the right-hand side of (43) as

$$(44) \quad \begin{aligned} & \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \inf_{z \in \mathcal{Z}_t} \mathbb{E}[(U(T, \xi\rho) - \xi\eta\rho z_T) 1_A] \\ &= \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \left(\mathbb{E}[(U(T, \xi\rho) 1_A] - \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\rho z_T 1_A] \right) \\ &= \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\alpha \in \mathcal{B}_t^+} \sup_{\phi \in \mathcal{C}'_{tT}} \left(\mathbb{E}[(U(T, \xi\alpha\phi) 1_A] - \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha\phi z_T 1_A] \right). \end{aligned}$$

Let us consider the latter term, $\sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha\phi z_T 1_A]$, which we can rewrite as

$$(45) \quad \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha\phi z_T 1_A] = \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha 1_A \mathbb{E}_t[\phi z_T]],$$

where by the respective definitions of $\mathcal{C}_t(x)$, \mathcal{N}_t , and \mathcal{B}_t^+ , we deduce that

$$(46) \quad 0 \leq \mathbb{E}[\xi\eta\alpha 1_A] < \infty,$$

and from the definition of \mathcal{C}'_{tT} , for every $\phi \in \mathcal{C}'_{tT}$, we have

$$\operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t[\phi z_T] = 1, \quad \mathbb{P}\text{-a.s.}$$

Therefore, for every $z \in \mathcal{Z}_t$, we obtain

$$\mathbb{E}[\xi\eta\alpha 1_A \mathbb{E}_t[\phi z_T]] \leq \mathbb{E} \left[\xi\eta\alpha 1_A \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t[\phi z_T] \right] = \mathbb{E}[\xi\eta\alpha 1_A].$$

Consequently, for every $\xi \in \mathcal{C}_t(x)$, $\eta \in \mathcal{N}_t$, $\alpha \in \mathcal{B}_t^+$, $\phi \in \mathcal{C}'_{tT}$, we get

$$(47) \quad \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha 1_A \mathbb{E}_t[\phi z_T]] \leq \mathbb{E}[\xi\eta\alpha 1_A].$$

On the other hand, let us fix $\phi \in \mathcal{C}'_{tT}$ and two arbitrary elements of \mathcal{Z}_t , \bar{z}^1 , and \bar{z}^2 . With

$$B := \{ \mathbb{E}_t[\phi \bar{z}_T^1] > \mathbb{E}_t[\phi \bar{z}_T^2] \},$$

one can see that

$$\bar{z} := 1_B \bar{z}^1 + 1_{B^c} \bar{z}^2 \in \mathcal{Z}_t$$

is such that

$$\mathbb{E}_t[\phi \bar{z}_T] = \max(\mathbb{E}_t[\phi \bar{z}_T^1], \mathbb{E}_t[\phi \bar{z}_T^2]),$$

and therefore by [Pha09, Theorem A.2.3, p. 215], we deduce that there exists a sequence $(z^n)_{n \in \mathbb{N}} \subset \mathcal{Z}_t$ such that

$$(48) \quad \lim_{n \rightarrow \infty} \mathbb{E}_t [\phi z_T^n] = \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t [\phi z_T] = 1, \quad \mathbb{P}\text{-a.s.},$$

where the last equality follows from the definition of \mathcal{C}'_{tT} . Therefore, the left-hand side in (45) can be bounded from below as follows:

$$(49) \quad \sup_{z \in \mathcal{Z}_t} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T]] \geq \lim_{n \rightarrow \infty} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T^n]].$$

As $\mathbb{E}_t [\phi z_T^n] \leq 1$, $n \in \mathbb{N}$, \mathbb{P} -a.s., and in view of (46), an application of the dominated convergence theorem in the right-hand side of (49) gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T^n]] &= \mathbb{E} \left[\xi \eta \alpha 1_A \lim_{n \rightarrow \infty} \mathbb{E}_t [\phi z_T^n] \right] \\ &= \mathbb{E} \left[\xi \eta \alpha 1_A \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t [\phi z_T^n] \right] = \mathbb{E} [\xi \eta \alpha 1_A], \end{aligned}$$

where we used (48) in the last equality. Combining these equalities with (49), we get

$$\sup_{z \in \mathcal{Z}_t} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T]] \geq \mathbb{E} [\xi \eta \alpha 1_A],$$

which together with (47) imply that

$$\sup_{z \in \mathcal{Z}_t} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T]] = \mathbb{E} [\xi \eta \alpha 1_A].$$

Plugging this equality into (44), we obtain

$$(50) \quad \begin{aligned} &\sup_{\xi \in \mathcal{C}_t(x)} \sup_{\alpha \in \mathcal{B}_t^+} \sup_{\phi \in \mathcal{C}'_{tT}} \left(\mathbb{E} [(U(T, \xi \alpha \phi) 1_A) - \sup_{z \in \mathcal{Z}_t} \mathbb{E} [\xi \eta \alpha \phi z_T 1_A]] \right) \\ &= \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\alpha \in \mathcal{B}_t^+} \sup_{\phi \in \mathcal{C}'_{tT}} (\mathbb{E} [(U(T, \xi \alpha \phi) 1_A) - \mathbb{E} [\xi \eta \alpha 1_A]]). \end{aligned}$$

Note that $\mathcal{C}_t(x) = \mathcal{B}_t^+ \mathcal{C}_t(x)$, that is, for every $\alpha \in \mathcal{B}_t^+$ and $\xi \in \mathcal{C}_t(x)$, we have $\alpha \xi \in \mathcal{C}_t(x)$. It follows from Lemma 3.8 (see (40)), that in (50) the latter equality can be rewritten as

$$\begin{aligned} &\sup_{\xi \in \mathcal{C}_t(x)} \sup_{\alpha \in \mathcal{B}_t^+} \left(\sup_{\phi \in \mathcal{C}'_{tT}} \mathbb{E} [(U(T, \xi \alpha \phi) 1_A) - \mathbb{E} [\xi \eta \alpha 1_A]] \right) \\ &= \sup_{\xi \in \mathcal{C}_t(x)} \left(\sup_{\phi \in \mathcal{C}'_{tT}} \mathbb{E} [(U(T, \xi \phi) 1_A) - \mathbb{E} [\xi \eta 1_A]] \right). \end{aligned}$$

Finally, combining the latter equality with (chains of equalities) (44) and (50), we conclude that

$$\sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \inf_{z \in \mathcal{Z}_t} \mathbb{E} [(U(T, \xi \rho) - \xi \eta \rho z_T) 1_A] = \sup_{\xi \in \mathcal{C}_t(x)} \left(\sup_{\phi \in \mathcal{C}_{tT}} \mathbb{E} [(U(T, \xi \phi) 1_A) - \mathbb{E} [\xi \eta 1_A]] \right),$$

i.e., (43) holds. This completes the proof of the lemma. ■

Corollary 3.13. *Let the condition of Lemma 3.1 hold and $\eta \in \mathcal{N}_t$ be fixed. Then for every $A \in \mathcal{F}_t$, we have*

$$\inf_{z \in \mathcal{Z}_t} \mathbb{E} [V(T, \eta z_T) 1_A] = \sup_{\xi \in \bigcup_{x>0} \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E} [(U(T, \xi \rho) - \xi \eta) 1_A].$$

Proof. The assertion of the corollary follows from Lemmas 3.9 and 3.12. ■

Lemma 3.14. *Let the condition of Lemma 3.1 hold and $\eta \in \mathcal{N}_t$ be fixed. Then, we have*

$$(51) \quad \operatorname{ess\,sup}_{\xi \in \bigcup_{x>0} \mathcal{C}_t(x)} (u(\xi, t, T) - \xi \eta) = v(\eta, t, T).$$

Proof. First, it follows from Lemma 3.6 that for every $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$, $\rho \in \mathcal{C}_{tT}$, and $z \in \mathcal{Z}_t$, we have

$$(52) \quad \mathbb{E}_t [U(T, \xi \rho)] - \xi \eta \leq \mathbb{E}_t [V(T, \eta z_T)].$$

Let us fix $m \in \mathbb{N}$ and set

$$A_m := \left\{ \operatorname{ess\,sup}_{\xi \in \bigcup_{x>0} \mathcal{C}_t(x)} \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} (\mathbb{E}_t [U(T, \xi \rho)] - \xi \eta) \leq \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V(T, \eta z_T)] - \frac{1}{m} \right\} \in \mathcal{F}_t.$$

Then, in view of (52), for every $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$, $\rho \in \mathcal{C}_{tT}$, and $z \in \mathcal{Z}_t$, we get

$$\mathbb{E}_t [U(T, \xi \rho)] - \xi \eta \leq \mathbb{E}_t [V(T, \eta z_T)] - \frac{1}{m} 1_{A_m}.$$

Multiplying both sides by 1_{A_m} , we obtain

$$\mathbb{E}_t [U(T, \xi \rho) 1_{A_m}] - \xi \eta 1_{A_m} \leq \mathbb{E}_t [V(T, \eta z_T) 1_{A_m}] - \frac{1}{m} 1_{A_m}.$$

Taking the expectation, we deduce that

$$\mathbb{E} [(U(T, \xi \rho) - \xi \eta) 1_{A_m}] \leq \mathbb{E} [V(T, \eta z_T) 1_{A_m}] - \frac{1}{m} \mathbb{P}[A_m].$$

As the above inequality holds for every $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$, $\rho \in \mathcal{C}_{tT}$, and $z \in \mathcal{Z}_t$, we get

$$\sup_{\xi \in \bigcup_{x>0} \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E} [(U(T, \xi \rho) - \xi \eta) 1_A] \leq \inf_{z \in \mathcal{Z}_t} \mathbb{E} [V(T, \eta z_T) 1_A] - \frac{1}{m} \mathbb{P}[A_m].$$

Combining the latter inequality with the assertion of Corollary 3.13, we obtain that $\mathbb{P}[A_m] = 0$. As $m \in \mathbb{N}$ is arbitrary, we conclude that

$$\left\{ \operatorname{ess\,sup}_{\xi \in \bigcup_{x>0} \mathcal{C}_t(x)} \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} (\mathbb{E}_t [U(T, \xi \rho)] - \xi \eta) < \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V(T, \eta z_T)] \right\} = \bigcup_{m \in \mathbb{N}} A_m$$

has measure 0. Equivalently, we have

$$\operatorname{ess\,sup}_{\substack{\xi \in \bigcup_{x>0} \mathcal{C}_t(x) \\ x>0}} \left(\operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\rho)] - \xi\eta \right) = \operatorname{ess\,inf}_{z \in \mathcal{Z}_T} \mathbb{E}_t [V(T, \eta z_T)], \quad \mathbb{P}\text{-a.s.},$$

and thus (51) holds. This completes the proof of the lemma. ■

3.4. The reverse conjugacy. The reverse conjugacy, or biconjugacy, between u and v is a subject closely related to the Fenchel–Moreau theorem. In the present context, this is a delicate topic, as u and $-v$ are defined as essential suprema, and thus they take values in space of \mathcal{F}_t -measurable extended real-valued functions. Therefore, we cannot apply the standard biconjugacy results from convex analysis, e.g., of Rockafellar [Roc70], directly. The topic of the Fenchel–Moreau theorem for $\bar{\mathbf{L}}^0$ -valued functions has been studied recently; see [DJK]. However, the domains of u and v do not form a dual pair of Banach spaces, and thus these domains do not satisfy the assumptions of [DJK]. Therefore, we have to prove biconjugacy by hand.

Lemma 3.15. *Under the conditions of Lemma 3.14, for every $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$, we have*

$$u(\xi, t, T) = \operatorname{ess\,inf}_{\eta \in \mathcal{N}_t} (v(\eta, t, T) + \xi\eta).$$

Proof. The proof follows the proof of Lemma 3.14 above with some minor modifications. Therefore, we do not present the complete proof and only highlight the differences. First, we need to pass from Z_t to the closure of the convex and solid hull of $\{z_T : z \in \mathcal{Z}_t\}$, which by the Fatou-convergence-type argument above, similarly to the proof of [KS99, Proposition 3.1], can be constructed as

$$(53) \quad \mathcal{D}_{tT} := \{h \in m\mathcal{F}_T : h \leq z_T \text{ for some } z \in \mathcal{Z}_t\}.$$

Then (and this is the main step) we need to show that for a given $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$ and $A \in \mathcal{F}_t$, we have

$$\limsup_{y \rightarrow \infty} \sup_{\rho \in \mathcal{C}_{tT}} \inf_{\eta z \in \mathcal{D}_T(y)} \mathbb{E} [(V(T, \eta z) + \eta z \xi \rho) 1_A] \leq \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E} [U(T, \xi \rho) 1_A].$$

The latter can be obtained as follows. Let us consider strictly positive elements $\eta \in \mathcal{D}_t$ and $z \in \mathcal{D}_{tT}$ (where \mathcal{D}_t and \mathcal{D}_{tT} are defined in (13) and (53), respectively) such that

$$(54) \quad \mathbb{E} [V(T, \eta z)] < \infty.$$

The existence of such elements follows from Assumption 2.4 (combined with the argument in [KS03, Proposition 1], where it can be proven that the infimum can be reached over the densities of the equivalent martingale measures under NFLVR, this argument combined with passing to the numéraire portfolio as a numéraire and stochastic utility, or equivalently, by treating the dual problem as in the proof of Theorem 3.20 contained in [Mos15, Theorem 3.3]).

Further, by Assumption 2.4, there exist $\xi \in \mathcal{C}_t$ and $\rho \in \mathcal{C}_{tT}$ such that $\mathbb{E}[U(T, \xi\rho)] > -\infty$. Then for such ξ, ρ, η , and z , from the definition of V and since $0 \leq \mathbb{E}[\xi\rho\eta z] < \infty$ (which is a consequence of the respective definitions of \mathcal{D}_t and \mathcal{D}_{tT}), we get

$$(55) \quad -\infty < \mathbb{E}[U(T, \xi\rho)] - y\mathbb{E}[\xi\rho\eta z] \leq \mathbb{E}[V(T, y\eta z)], \quad y > 0,$$

whereas from the monotonicity of $V(T, \cdot)$ and (54), we get

$$(56) \quad \mathbb{E}[V(T, y\eta z)] \leq \mathbb{E}[V(T, \eta z)] < \infty, \quad y \geq 1.$$

Combining (55) and (56), we deduce that (strictly positive elements) $\eta \in \mathcal{N}_t$ and $z \in \mathcal{D}_{tT}$ satisfy

$$-\infty < \mathbb{E}[V(T, y\eta z)] \leq \mathbb{E}[V(T, \eta z)] < \infty, \quad y \geq 1.$$

Next, along the lines of the proof of Lemma 3.7, we define

$$U^n(T, x) := \inf_{0 < y \leq n\eta z} (V(T, y) + xy), \quad (x, \omega) \in (0, \infty) \times \Omega,$$

i.e., from U , we pass to a sequence of truncated stochastic fields (similar to the ones in the proof of [Mos15, Lemma 3.9]). With such U^n 's, as in the proof of Lemma 3.7, we can show that

$$\lim_{n \rightarrow \infty} \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E}[U^n(T, \xi\rho)1_A] = \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E}[U(T, \xi\rho)1_A].$$

Here the only challenge is to establish the uniform integrability of $(U^n)^+(T, \xi\rho)$, $\rho \in \mathcal{C}_{tT}$. This, however, follows from the following estimates: for every $n \geq 2$, one can see that

$$(57) \quad \begin{aligned} U^n(T, x) &\leq U(T, x)1_{\{x > -V'(T, 2\eta z)\}} \\ &\quad + 2(V(T, \eta z) - V(T, 2\eta z))1_{\{x \leq -V'(T, 2\eta z)\}}, \quad (x, \omega) \in (0, \infty) \times \Omega, \end{aligned}$$

(57) implies that for every $n \geq 2$, we have

$$U^n(T, x) \leq \max(U(T, x), 2V(T, \eta z) - V(T, 2\eta z)), \quad (x, \omega) \in (0, \infty) \times \Omega,$$

and the uniform integrability of $(U^n)^+(T, \xi\rho)$, $\rho \in \mathcal{C}_{tT}$, follows from Lemma 3.21. The remaining parts of the proof of this lemma are very similar to the proof of Lemma 3.14 and, therefore, they are skipped. ■

Lemma 3.16. *Let the condition of Lemma 3.1 hold, and $\xi \in \mathcal{E}_t^u$. Then there exist $\hat{\eta}$ in is the closure of \mathcal{N}_t in \mathbb{L}^0 and $\hat{z} \in \mathcal{Z}_t$ such that*

$$(58) \quad \begin{aligned} u(\xi, t, T) &= \operatorname{ess\,inf}_{\eta \in \mathcal{N}_t} (v(\eta, t, T) + \xi\eta) \\ &= \mathbb{E}[V(T, \hat{\eta}\hat{z}_T) | \mathcal{F}_t] + \hat{\eta}\xi. \end{aligned}$$

Further, let $\hat{\rho}$ be the optimizer to (5) corresponding to ξ . Then we have

$$(59) \quad \mathbb{E}_t[\hat{\rho}\hat{z}_T] = 1, \quad \text{on } \{\xi > 0\}, \quad \mathbb{P}\text{-a.s.},$$

and

$$(60) \quad U'(T, \xi\hat{\rho}) = \hat{\eta}\hat{z}_T \quad \text{and} \quad \xi\hat{\rho} = -V'(T, \hat{\eta}\hat{z}_T) \quad \text{on } \{\xi > 0\}, \quad \mathbb{P}\text{-a.s.}$$

Proof. Let η^1, η^2 in \mathcal{N}_t and z^1 and z^2 be the corresponding minimizers to (15); then with

$$(61) \quad A := \{\mathbb{E}[V(T, \eta^1 z_T^1) | \mathcal{F}_t] + \eta^1 \xi < \mathbb{E}[V(T, \eta^2 z_T^2) | \mathcal{F}_t] + \eta^2 \xi\} \in \mathcal{F}_t,$$

and via Lemmas 3.4 and 3.5, we get

$$\begin{aligned} \eta &:= \eta^1 1_A + \eta^2 1_{A^c} \in \mathcal{N}_t, \quad \text{as well as} \\ z &:= \frac{1_A \eta^1 z^1 + 1_{A^c} \eta^2 z^2}{\eta} 1_{\{\eta \neq 0\}} + (1_A z^1 + 1_{A^c} z^2) 1_{\{\eta=0\}} \in \mathcal{Z}_t. \end{aligned}$$

In turn, by concavity of V , and using Lemma 3.5, we obtain

$$\begin{aligned} \mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t] + \eta \xi &= (\mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t] + \eta \xi) 1_A + (\mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t] + \eta \xi) 1_{A^c} \\ &= (\mathbb{E}[V(T, \eta z_T 1_A) | \mathcal{F}_t] + \eta^1 \xi) 1_A + (\mathbb{E}[V(T, \eta z_T 1_{A^c}) | \mathcal{F}_t] + \eta^2 \xi) 1_{A^c} \\ &= (\mathbb{E}[V(T, \eta^1 z_T^1) | \mathcal{F}_t] + \eta^1 \xi) 1_A + (\mathbb{E}[V(T, \eta^2 z_T^2) | \mathcal{F}_t] + \eta^2 \xi) 1_{A^c} \\ &= \min(\mathbb{E}[V(T, \eta^1 z_T^1) | \mathcal{F}_t] + \eta^1 \xi, \mathbb{E}[V(T, \eta^2 z_T^2) | \mathcal{F}_t] + \eta^2 \xi), \end{aligned}$$

where in the last equality, we have used (61). Therefore, using [Pha09, Theorem A.2.3, p. 215], we deduce the existence of a sequence $\eta^n z^n$, $n \in \mathbb{N}$, such that

$$(62) \quad \lim_{n \rightarrow \infty} (\mathbb{E}[V(T, \eta^n z_T^n) | \mathcal{F}_t] + \eta^n \xi) = \text{ess inf}_{\eta z \in \mathcal{G}_t} (\mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t] + \eta \xi), \quad \mathbb{P}\text{-a.s.},$$

where we recall that \mathcal{G}_t is defined in (23). By passing to convex combinations, and by applying Lemma 3.5, which asserts that a convex combination of elements of \mathcal{G}_t is an element of \mathcal{G}_t , we may obtain a subsequence of elements of \mathcal{G}_t , which we still denote $\eta^n z^n$, $n \in \mathbb{N}$, and which converges a.s. to a limit, which we denote by ψ .

Let us consider η^n , $n \in \mathbb{N}$. By passing to convex combinations, we may obtain a family of convex weights, $\lambda_k^n \in [0, 1]$, $k \in \{0, \dots, M_n\}$, $n \in \mathbb{N}$, where $M_n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, $\sum_{k=0}^{M_n} \lambda_k^n = 1$, and $\tilde{\eta}^n \triangleq \sum_{k=0}^{M_n} \lambda_k^n \eta^{n+k}$, $n \in \mathbb{N}$, converges along a subsequence to a limit, $\hat{\eta}$, a.s. Applying the same convex weights to $\eta^n z^n$, and passing to the same subsequence,³ via Lemma 3.5, we get

$$\sum_{k=0}^{M_n} \lambda_k^n \eta^{n+k} z_T^{n+k} = \tilde{\eta}^n z_T^n, \quad n \in \mathbb{N},$$

for some $\tilde{\eta}^n \in \mathcal{N}_t$ and $\tilde{z}^n \in \mathcal{Z}_t$, $n \in \mathbb{N}$. As $\eta^n z_T^n$, $n \in \mathbb{N}$, converges a.s. to ψ , $\tilde{\eta}^n \tilde{z}_T^n$, $n \in \mathbb{N}$, also converges a.s. to the same limit ψ . As both $\tilde{\eta}^n$ and $\tilde{\eta}^n \tilde{z}_T^n$, $n \geq 1$, converge, we additionally obtain that \tilde{z}_T^n , $n \in \mathbb{N}$, converges to a limit, which we denote \hat{z}_T . Therefore, we have

$$(63) \quad \psi = \lim_{n \rightarrow \infty} \tilde{\eta}^n \tilde{z}_T^n = \hat{\eta} \hat{z}_T.$$

As in the proof of Lemma 3.3, one may show that \hat{z}_T is the terminal value of the element of \mathcal{Z}_t , and thus $\hat{z} \in \mathcal{Z}_t$.

Now, for an arbitrary $\tilde{A} \in \mathcal{F}_t$, convexity of V , Lemma 3.15 and (62) imply that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mathbb{E}[(V(T, \tilde{\eta}^n \tilde{z}^n) + \tilde{\eta}^n \tilde{z}^n \xi \hat{\rho} - U(T, \xi \hat{\rho})) 1_{\tilde{A}}] \\ &\leq \lim_{n \rightarrow \infty} \mathbb{E}[(V(T, \tilde{\eta}^n \tilde{z}^n) + \tilde{\eta}^n \xi - U(T, \xi \hat{\rho})) 1_{\tilde{A}}] = 0. \end{aligned}$$

³ λ in (24) and (26) (of Lemma 3.5) is the same.

From the definition of V , for every $n \geq 1$, we have

$$0 \leq V(T, \tilde{\eta}^n \tilde{z}_T^n) + \tilde{\eta}^n \tilde{z}_T^n \xi \hat{\rho} - U(T, \xi \hat{\rho}), \quad \mathbb{P}\text{-a.s.}$$

Therefore, Fatou's lemma and (63) give

$$(64) \quad \mathbb{E}[(V(T, \hat{\eta} \hat{z}_T) + \hat{\eta} \hat{z}_T \xi \hat{\rho} - U(T, \xi \hat{\rho})) 1_{\tilde{A}}] = 0, \quad \tilde{A} \in \mathcal{F}_t.$$

(58) follows. In turn, (64) together with

$$V(T, \hat{\eta} \hat{z}_T) + \hat{\eta} \hat{z}_T \xi \hat{\rho} - U(T, \xi \hat{\rho}) \geq 0, \quad \mathbb{P}\text{-a.s.},$$

implies that

$$V(T, \hat{\eta} \hat{z}_T) + \hat{\eta} \hat{z}_T \xi \hat{\rho} = U(T, \xi \hat{\rho}), \quad \mathbb{P}\text{-a.s.},$$

which, via the definition of V , implies (60). In turn, (59) follows from the polar structure of \mathcal{C}_{tT} and \mathcal{Z}_t . ■

Lemma 3.17. *Let the condition of Lemma 3.1 hold, $\xi \in \mathcal{E}_t^u$. Then, with $\hat{\rho}$, $\hat{\eta}$, and \hat{z}_T being as in Lemma 3.16, we have that both $U(T, \xi \hat{\rho})$ and $V^+(T, \hat{\eta} \hat{z}_T)$ belong to $\mathbb{L}^1(\mathbb{P})$. Further, there exists A_m , $m \in \mathbb{N}$, an \mathcal{F}_t -measurable partition of Ω , such that $\hat{\eta} 1_{A_m} \in \mathcal{N}_t$ and $V(T, \hat{\eta} \hat{z}_T) 1_{A_m} \in \mathbb{L}^1(\mathbb{P})$, $m \in \mathbb{N}$.*

Proof. First, it follows from Assumption 2.4 that $U^+(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P})$. By (6), there exists $\rho \in \mathcal{C}_{tT}$ such that $\mathbb{E}[U^-(T, \xi \rho)] < \infty$. From optimality of $\hat{\rho}$, we get

$$(65) \quad \mathbb{E}_t[U^-(T, \xi \hat{\rho})] \leq \mathbb{E}_t[U^+(T, \xi \hat{\rho})] - \mathbb{E}_t[U^+(T, \xi \rho)] + \mathbb{E}_t[U^-(T, \xi \rho)].$$

As $\mathbb{E}[U^+(T, \xi \rho)] < \infty$ and $\mathbb{E}[U^-(T, \xi \rho)] < \infty$ (by (6)), we deduce from (65) that $U^-(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P})$. Therefore, $U(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P})$.

To show that $V(T, \hat{\eta} \hat{z}_T) \in \mathbb{L}^1(\mathbb{P})$, we use the equality

$$(66) \quad V(T, \hat{\eta} \hat{z}_T) + \xi \hat{\rho} \hat{\eta} \hat{z}_T = U(T, \xi \hat{\rho}), \quad \mathbb{P}\text{-a.s.}$$

By taking positive part in (66), we obtain

$$V^+(T, \hat{\eta} \hat{z}_T) \leq (V(T, \hat{\eta} \hat{z}_T) + \xi \hat{\rho} \hat{\eta} \hat{z}_T)^+ = U^+(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P}).$$

Likewise

$$(V(T, \hat{\eta} \hat{z}_T) + \xi \hat{\rho} \hat{\eta} \hat{z}_T)^- = U^-(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P}).$$

To show the existence of an \mathcal{F}_t -measurable partition of Ω such that $\hat{\eta} 1_{A_m} \in \mathcal{N}_t$, $m \geq 1$, first, we observe that it follows from Assumption 2.3 there exists a strictly positive element $\eta_0 \in \mathcal{N}_t$. Let $(\eta_m)_{m \in \mathbb{N}} \subset \mathcal{N}_t$ be a sequence, which converges to $\hat{\eta}$, \mathbb{P} -a.s. Let

$$A_0 := \emptyset \quad \text{and} \quad A_m := \{\hat{\eta} \leq 2\eta_m + \eta_0\} \setminus \bigcup_{k=0}^{m-1} A_k, \quad m \geq 1.$$

Then, by construction on each A_m , $\hat{\eta} 1_{A_m} \in \mathcal{N}_t$, and A_m 's are disjoint subsets \mathcal{F}_t . Finally, from \mathbb{P} -a.s. convergence of η_m , $m \geq 1$, to $\hat{\eta}$, we deduce that $\mathbb{P}[(\bigcup_{m \geq 1} A_m)^c] = 0$. Now, $V(T, \hat{\eta} \hat{z}_T) 1_{A_m} \in \mathbb{L}^1(\mathbb{P})$ by the integrability of $U(\xi \hat{\rho})$ and (66). ■

3.5. Some results from [Mos15] used above. Some results from [Mos15] used above are given below in an adjusted form. Let \mathbb{P} be a probability measure on a measurable space (Ω, \mathcal{F}) . Denote by $\mathbf{L}^0 = \mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ the vector space of (equivalence classes of) real-valued measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the topology of convergence in measure \mathbb{P} . Let \mathbf{L}_+^0 denote its positive orthant, i.e., the set of nonnegative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let \mathcal{C}, \mathcal{D} be polar subsets of \mathbf{L}_+^0 , that is,

$$(67) \quad \begin{aligned} \xi \in \mathcal{C} & \quad \text{if and only if} \quad \mathbb{E}[\xi\eta] \leq 1 \quad \text{for every } \eta \in \mathcal{D}, \\ \eta \in \mathcal{D} & \quad \text{if and only if} \quad \mathbb{E}[\xi\eta] \leq 1 \quad \text{for every } \xi \in \mathcal{C}, \end{aligned}$$

and we additionally suppose that

$$(68) \quad \begin{aligned} \text{there exists } \xi \in \mathcal{C} & \quad \text{such that } \xi > 0, \\ \text{there exists } \eta \in \mathcal{D} & \quad \text{such that } \eta > 0. \end{aligned}$$

Let us notice a symmetry between the sets \mathcal{C} and \mathcal{D} as well as their convexity and boundedness in $\mathbf{L}^0(\mathbb{P})$. For $x > 0$ and $y > 0$ one can define the sets

$$(69) \quad \begin{aligned} \mathcal{C}(x) & := x\mathcal{C} := \{x\xi : \xi \in \mathcal{C}\}, \\ \mathcal{D}(y) & := y\mathcal{D} := \{y\eta : \eta \in \mathcal{D}\}. \end{aligned}$$

Consider a *stochastic utility function* $U: \Omega \times [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$, which satisfies the following conditions.

Assumption 3.18. For every $\omega \in \Omega$, the function $x \rightarrow U(\omega, x)$ is strictly concave, strictly increasing, and continuously differentiable on $(0, \infty)$ and satisfies the Inada conditions:

$$(70) \quad \lim_{x \downarrow 0} U'(\omega, x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} U'(\omega, x) = 0,$$

where $U'(\cdot, \cdot)$ denotes the partial derivative with respect to the second argument. At $x = 0$ we have, by continuity, $U(\omega, 0) = \lim_{x \downarrow 0} U(\omega, x)$; this value may be $-\infty$. For every $x \geq 0$, we suppose that $U(\cdot, x)$ is measurable.

Define the *conjugate function* V to U as

$$V(\omega, y) := \sup_{x > 0} (U(\omega, x) - xy), \quad (\omega, y) \in \Omega \times [0, \infty).$$

Observe that $-V$ satisfies Assumption 3.18.

Now we can state the optimization problems:

$$(71) \quad u(x) = \sup_{\xi \in \mathcal{C}(x)} \mathbb{E}[U(\xi)], \quad x > 0,$$

$$(72) \quad v(y) = \inf_{\eta \in \mathcal{D}(y)} \mathbb{E}[V(\eta)], \quad y > 0.$$

Theorem 3.19 ([Mos15, Theorem 3.2]). Assume that \mathcal{C} and \mathcal{D} satisfy conditions (67) and (68). Let Assumption 3.18 hold and suppose

$$(73) \quad v(y) < \infty \quad \text{for all } y > 0 \quad \text{and} \quad u(x) > -\infty \quad \text{for all } x > 0.$$

Then we have the following:

- (1) $u(x) < \infty$ for all $x > 0$, $v(y) > -\infty$ for all $y > 0$. The functions u and v satisfy the biconjugacy relations, i.e.,

$$(74) \quad \begin{aligned} v(y) &= \sup_{x>0} (u(x) - xy), \quad y > 0, \\ u(x) &= \inf_{y>0} (v(y) + xy), \quad x > 0. \end{aligned}$$

The functions u and $-v$ are continuously differentiable on $(0, \infty)$, strictly increasing, and strictly concave and satisfy the Inada conditions:

$$\begin{aligned} \lim_{x \downarrow 0} u'(x) &= \infty, & \lim_{y \downarrow 0} -v'(y) &= \infty, \\ \lim_{x \rightarrow \infty} u'(x) &= 0, & \lim_{y \rightarrow \infty} -v'(y) &= 0. \end{aligned}$$

- (2) For every $x > 0$ the optimal solution $\hat{\xi}(x)$ to (71) exists and is unique. For every $y > 0$ the optimal solution $\hat{\eta}(y)$ to (72) exists and is unique. If $y = u'(x)$, we have the dual relations

$$\hat{\eta}(y) = U' \left(\hat{\xi}(x) \right), \quad \mathbb{P}\text{-a.s.},$$

and

$$\mathbb{E} \left[\hat{\xi}(x) \hat{\eta}(y) \right] = xy.$$

Let $\tilde{\mathcal{D}}$ be a subset of \mathcal{D} such that

- (i) $\tilde{\mathcal{D}}$ is closed under the countable convex combinations,
- (ii) for every $\xi \in \mathcal{C}$ we have

$$\sup_{\eta \in \mathcal{D}} \mathbb{E} [\xi \eta] = \sup_{\eta \in \tilde{\mathcal{D}}} \mathbb{E} [\xi \eta].$$

The statement of [Mos15, Theorem 3.3] (the part of [Mos15, Theorem 3.3] that was used above) is given below.

Theorem 3.20 ([Mos15, Theorem 3.3]). *Under the conditions of Theorem 3.19, we have*

$$v(y) = \inf_{\eta \in \tilde{\mathcal{D}}} \mathbb{E} [V(y\eta)], \quad y > 0,$$

Lemma 3.21 ([Mos15, Lemma 3.5]). *Under the conditions of Theorem 3.19, for every $y > 0$ the family $(V^-(h))_{h \in \mathcal{D}(y)}$ is uniformly integrable.*

Lemma 3.22 ([Mos15, Lemma 3.9]). *Under the assumptions of Theorem 3.19, we have*

$$(75) \quad v(y) = \sup_{x>0} (u(x) - xy), \quad y > 0.$$

4. Stability analysis. Here we assume that the 0-market consists of one risky and one riskless asset (whose price still equals to 1 at all times). Let M be a one-dimensional continuous local martingale that drives the return process of the risky asset. Throughout this section, $T > 0$ is fixed. Then the dynamic of the risky asset is given by

$$(76) \quad R^0 = M + \lambda \cdot \langle M \rangle,$$

where λ is a predictable process such that

$$(77) \quad \lambda^2 \cdot \langle M \rangle_T < \infty, \quad \mathbb{P}\text{-a.s.}$$

Thus, the return of the risky asset (from section 2) has the form (76). For the absence of arbitrage in the sense of Assumption 2.3, the finite variation part of the return process has to be absolutely continuous with respect to the quadratic variation of its martingale part; see [HS10]. We suppose that the riskless asset stays unperturbed and consider a perturbed family of returns of risky assets of the form

$$(78) \quad R^\varepsilon = (1 + \varepsilon\psi) \cdot (M + \lambda(1 + \varepsilon\theta) \cdot \langle M \rangle), \quad \varepsilon \in \mathbb{R},$$

where ψ and θ are some predictable processes, such that

$$(79) \quad \theta^2 \cdot \langle M \rangle < \infty, \quad \mathbb{P}\text{-a.s.},$$

and $|\psi|$ is uniformly bounded. Perturbations of the input model parameters might appear due to errors in the estimation of the model parameters under a statistical procedure. In connection to many models of the stock price used in practice, ψ corresponds to perturbations of the volatility, and then, once ϕ is fixed, θ governs the distortions of the drift of the risky asset. We discuss a connection to a different parametrization of perturbations in the following remark. Mathematically, the closest papers where such perturbations occur are [Mos20] and [MS19] (the case of $\psi \equiv 0$).

Remark 4.1. Parametrization of perturbations in the form (78) is closely related to the ones that appear in the literature more often:

$$(80) \quad \begin{aligned} R^\varepsilon &= R^0 + (\varepsilon\psi) \cdot M + (\varepsilon\nu) \cdot \langle M \rangle \\ &= (1 + (\varepsilon\psi)) \cdot M + (\lambda + (\varepsilon\nu)) \cdot \langle M \rangle. \end{aligned}$$

Here $\varepsilon\psi$ amount to perturbations of the martingale part (of volatility in the simplest settings) and $\varepsilon\nu$ to perturbations of the finite variation part (or drift) of the return of the risky asset. From (80), one can arrive to (78) by assuming that ν (linearly) depends on ε and by making the following reparametrization:

$$(81) \quad \nu = \lambda\nu', \quad \nu' = \psi + \nu'', \quad \nu'' = \theta(1 + \varepsilon\psi).$$

The reason for imposing (78) instead of (80) is a simple structure of integrability, Assumption 4.4, and no issues related to differentiation with respect to a parameter under stochastic integration, as in [Mét82] and [HN84]. We give further details on how to get the stability results with (80) in Remarks 4.2 and 4.3 below.

Now we fix a proportion of the total wealth invested in the risky asset⁴ and investigate the dynamic behavior of the indirect utility under small perturbations of the drift and volatility

⁴Fixing the proportion of wealth invested in the risky asset leads to the admissibility of the corresponding wealth process for every $\varepsilon \neq 0$. If one fixes the number of shares of the risky asset in the portfolio, then for $\varepsilon \neq 0$, under both parametrizations (78) and (80), the associated wealth process can be negative with positive probability, in general.

of the underlying risky asset as in (78). To make this mathematically precise, we extend the definitions from section 2 in a natural way as follows:

$$\mathcal{X}^\varepsilon(x) \triangleq \{X \geq 0 : X = x + H \cdot R^\varepsilon \text{ for some } R^\varepsilon\text{-integrable } H\}, \quad x \geq 0, \varepsilon \in \mathbb{R}.$$

For every $\varepsilon \in \mathbb{R}$, the initial wealth $\bar{x} \geq 0$ and a predictable and locally bounded process π are the same, but the corresponding family of the wealth processes alters due to different integrators R^ε , i.e., we consider the family

$$X^{\pi, \varepsilon} = \bar{x} \mathcal{E}(\pi \cdot R^\varepsilon), \quad \varepsilon \in \mathbb{R}.$$

Likewise, for every $\varepsilon \in \mathbb{R}$, the set of wealth processes in $\mathcal{X}^\varepsilon(\bar{x})$, which equal to $X^{\pi, \varepsilon}$ on $[0, t]$, is denoted by $\mathcal{A}^\varepsilon(X_t^{\pi, \varepsilon}, t)$, that is,

$$(82) \quad \mathcal{A}^\varepsilon(X_t^{\pi, \varepsilon}, t) := \left\{ \tilde{X} \in \mathcal{X}^\varepsilon(\bar{x}) : \tilde{X}_s = X_s^{\pi, \varepsilon}, \text{ for } s \in [0, t], \mathbb{P}\text{-a.s.} \right\}, \quad \varepsilon \in \mathbb{R}.$$

Finally the family of dynamic indirect utilities associated with π up to T is defined as

$$u^\varepsilon(X_t^{\pi, \varepsilon}, t, T) := \operatorname{ess\,sup}_{\tilde{X} \in \mathcal{A}^\varepsilon(X_t^{\pi, \varepsilon}, t)} \mathbb{E} \left[U \left(T, \tilde{X}_T \right) \middle| \mathcal{F}_t \right], \quad t \in [0, T], \varepsilon \in \mathbb{R},$$

and for brevity, we denote

$$(83) \quad J_t^{\varepsilon, T} := u^\varepsilon(X_t^{\pi, \varepsilon}, t, T), \quad t \in [0, T], \varepsilon \in \mathbb{R}.$$

For every $\varepsilon \in \mathbb{R}$, let us set

$$(84) \quad \eta^\varepsilon = -\varepsilon \lambda \theta \quad \text{and} \quad L^\varepsilon := \mathcal{E}(\eta^\varepsilon \cdot R^0).$$

Note that $L^\varepsilon \in \mathcal{X}^0(1)$ for every ε . The processes L^ε drive the correction terms in Proposition 4.10.

Remark 4.2. If one chooses perturbations (80), the η^ε and L^ε should be defined as

$$\eta^\varepsilon := \left(\lambda - \frac{\lambda + \varepsilon \nu}{1 + \varepsilon \psi} \right) \quad \text{and} \quad L^\varepsilon := \mathcal{E}(\eta^\varepsilon \cdot R^0).$$

This leads to the same heuristic limiting formulas, but stronger integrability conditions (than the one in Assumption 4.4) are needed to complete proofs.

4.1. Heuristic derivative of L^ε .

We denote

$$(85) \quad \bar{R} = (\lambda \theta) \cdot R^0.$$

Then

$$L^\varepsilon = \mathcal{E}(\eta^\varepsilon \cdot R^0) = \mathcal{E}(-\varepsilon \cdot \bar{R}) = \exp\left(-\varepsilon \bar{R} - \frac{1}{2} \varepsilon^2 \langle \bar{R} \rangle\right).$$

We set $F := -\bar{R}_T$. Therefore, we get

$$\frac{\partial L_T^\varepsilon}{\partial \varepsilon} = L_T^\varepsilon (-\bar{R}_T - \varepsilon \langle \bar{R} \rangle_T), \quad \left. \frac{\partial L_T^\varepsilon}{\partial \varepsilon} \right|_{\varepsilon=0} = -\bar{R}_T = F.$$

Similarly, we obtain

$$\frac{\partial \frac{1}{L_T^\varepsilon}}{\partial \varepsilon} = \frac{1}{L_T^\varepsilon} (\bar{R}_T + \varepsilon \langle \bar{R} \rangle_T), \quad \left. \frac{\partial \frac{1}{L_T^\varepsilon}}{\partial \varepsilon} \right|_{\varepsilon=0} = \bar{R}_T = -F.$$

Remark 4.3. Here we show that under (80), we get the same heuristic formulas for the derivatives of L^ε . We recall that under (80), the corresponding η^ε and L^ε are given by

$$\eta^\varepsilon := \left(\lambda - \frac{\lambda + \varepsilon\nu}{1 + \varepsilon\psi} \right) \quad \text{and} \quad L^\varepsilon := \mathcal{E}(\eta^\varepsilon \cdot R^0).$$

Then, by direct computations and via the reparametrization formulas (81), we obtain

$$\frac{\partial L_T^\varepsilon}{\partial \varepsilon} \Big|_{\varepsilon=0} = F \quad \text{and} \quad \frac{\partial \frac{1}{L_T^\varepsilon}}{\partial \varepsilon} \Big|_{\varepsilon=0} = -F.$$

Thus, the asymptotic behavior of L^ε 's is similar under both parameterizations of perturbations.

4.2. Rigorous derivation. For $t \in [0, T]$, and $\hat{\rho}$ and \hat{z}_T being associated with $X_t^{\pi,0}$ (via Lemma 3.16), let us define the probability measure \mathbb{R}^t as

$$(86) \quad \frac{d\mathbb{R}^t}{d\mathbb{P}} \Big|_{\mathcal{F}_s} := 1_{\{s \in [0, t]\}} + \mathbb{E}_s[\hat{\rho}\hat{z}_T] 1_{\{s > t\}}, \quad s \in [0, T].$$

Note that Lemma 3.16 (via (59)) ensures that \mathbb{R}^t is a probability measure. Finally, we suppose that the perturbations are sufficiently bounded in the following sense.

Assumption 4.4. The process $|\psi|$ is uniformly bounded from above and there exists a constant $\bar{c} > 0$ such that

$$\exp(\bar{c}(|\bar{R}_T| + \langle \bar{R} \rangle_T)) \in \mathbb{L}^1(\mathbb{R}^t), \quad t \in [0, T],$$

where the probability measure \mathbb{R}^t and the process \bar{R} are defined in (86) and (85), respectively.

We also need to strengthen the assumptions on U .

Assumption 4.5. For every $\omega \in \Omega$, $U(T, \cdot)$ is strictly concave, strictly increasing, and continuously differentiable, and there exist positive constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that for every $x > 0$ and $z \in (0, 1]$, we have

$$(87) \quad U'(T, zx) \leq z^{-\gamma_1} U'(T, x) \quad \text{and} \quad -V'(T, zx) \leq -V'(T, x) z^{-\gamma_2}.$$

For every $x \geq 0$, $U(T, x)$ is measurable.

Remark 4.6. Assumption 4.5 holds if either relative risk aversion, $A(x) := -\frac{U''(T,x)x}{U'(T,x)}$, $x > 0$, or relative risk tolerance of $U(T, \cdot)$ at x , given by $-\frac{yV''(T,y)}{V'(T,y)}$ for $y = U'(T, x)$, $x > 0$, is bounded away from 0 and ∞ uniformly in $\omega \in \Omega$; see, e.g., [MS19, Lemma 5.12].

Remark 4.7. Condition (87) implies the Inada conditions. This can be shown as follows. Let us fix $\omega \in \Omega$. Applying $U'(T, \cdot)$ to both sides of the second inequality in (87), and since $U'(T, \cdot)$ is decreasing, we get

$$(88) \quad zx \geq U'(T, z^{-\gamma_2}(-V'(T, x))), \quad x > 0, z \in (0, 1].$$

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Now for $x = U'(T, 1)$, $-V'(T, x) = 1$, and, in (88), we have

$$zU'(T, 1) \geq U'(T, z^{-\gamma_2}), \quad z \in (0, 1].$$

Taking the limit as $z \searrow 0$, we deduce that

$$0 \geq \lim_{\tilde{x} \rightarrow \infty} U'(T, \tilde{x}).$$

Similarly, from the first inequality in (87), applying $-V'(T, \cdot)$ to both sides, and since $-V'(T, \cdot)$ is decreasing, we get

$$(89) \quad zx \geq -V'(T, z^{-\gamma_1} U'(T, x)), \quad x > 0, z \in (0, 1].$$

For $x = -V'(T, 1)$, we have $U'(T, x) = 1$, and therefore in (89), we obtain

$$-V'(T, 1)z \geq -V'(T, z^{-\gamma_1}), \quad z \in (0, 1].$$

Taking the limit as $z \searrow 0$, we deduce that

$$0 \geq - \lim_{\tilde{z} \rightarrow \infty} V'(T, \tilde{z}).$$

By conjugacy between $U(T, \cdot)$ and $V(T, \cdot)$, the latter inequality implies that $\lim_{\tilde{x} \rightarrow 0} U'(T, \tilde{x}) = \infty$.

Lemma 4.8. *Let $T > 0$ be fixed and consider a family of risky assets parametrized by $\varepsilon \in \mathbb{R}$, whose returns are given by (78). Let us suppose the validity of Assumptions 4.4 and 4.5, and*

$$(90) \quad u^0(z, 0, T) > -\infty \quad \text{and} \quad \sup_{x>0} (u^0(x, 0, T) - xz) < \infty, \quad z > 0.$$

Then, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the pair of traded assets, whose returns are given by 0 and R^ε , satisfy NUPBR, and

$$(91) \quad u^\varepsilon(z, 0, T) > -\infty \quad \text{and} \quad \sup_{x>0} (u^\varepsilon(x, 0, T) - xz) < \infty, \quad z > 0,$$

that is, both Assumptions 2.3 and 2.4 hold for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Remark 4.9. In particular, in view of Remark 4.7, under the conditions of Lemma 4.8, the results of section 3 apply to perturbed models for every ε in some neighborhood of the origin.

Proof of Lemma 4.8. Conditions (77) and (79) imply that no unbounded profit with bounded risk holds for both the unperturbed model (corresponding to $\varepsilon = 0$) and perturbed models ($\varepsilon \neq 0$), as $\mathcal{E}(-(\lambda(1 + \varepsilon\theta)) \cdot M)$ is a supermartingale deflator for $\mathcal{X}^\varepsilon(1)$, and [KK07, Theorem 4.12] applies.

To show (91), first let us fix $x > 0$ and consider $X^{\pi,0} \in \mathcal{X}^0(x)$ such that

$$\mathbb{E} \left[U(X_T^{\pi,0}) \right] = u^0(x, 0, T) \in \mathbb{R}.$$

The existence of such an $X^{\pi,0}$ follows from (90), no unbounded profit with bounded risk established above, and Theorem 3.19. An application of Ito's lemma shows that $\frac{X^{\pi,0}}{L^\varepsilon} = \left(\frac{X_t^{\pi,0}}{L_t^\varepsilon} \right)_{t \in [0, T]} \in \mathcal{X}^\varepsilon(x)$ for every ε in some neighborhood of 0. Using Assumption 4.5, we get

$$\begin{aligned} & \left| U' \left(T, \frac{X_T^{\pi,0}}{L_T^\varepsilon} \right) \frac{X_T^{\pi,0}}{L_T^\varepsilon} (\bar{R}_T + \varepsilon \langle \bar{R} \rangle_T) \right| \\ & \leq U' \left(T, X_T^{\pi,0} \right) X_T^{\pi,0} \max(1, (L_T^\varepsilon)^{-\gamma_1}) \frac{1}{L_T^\varepsilon} (|\bar{R}_T| + |\varepsilon| \langle \bar{R} \rangle_T). \end{aligned}$$

Therefore, Assumption 4.4 implies that for every ε in some neighborhood of 0, we have

$$(92) \quad \mathbb{E} \left[\int_0^\varepsilon \left| U' \left(T, \frac{X_T^{\pi,0}}{L_T^\varepsilon} \right) \frac{X_T^{\pi,0}}{L_T^\varepsilon} (\bar{R}_T + \tilde{\varepsilon} \langle \bar{R} \rangle_T) \right| d\tilde{\varepsilon} \right] < \infty.$$

Consequently, we obtain

$$\begin{aligned} (93) \quad u^\varepsilon(x, 0, T) & \geq \mathbb{E} \left[U \left(T, \frac{X_T^{\pi,0}}{L_T^\varepsilon} \right) \right] = u^0(x, 0, T) + \mathbb{E} \left[U \left(T, \frac{X_T^{\pi,0}}{L_T^\varepsilon} \right) - U \left(T, X_T^{\pi,0} \right) \right] \\ & = u^0(x, 0, T) + \mathbb{E} \left[\int_0^\varepsilon U' \left(T, \frac{X_T^{\pi,0}}{L_T^\varepsilon} \right) \frac{X_T^{\pi,0}}{L_T^\varepsilon} (\bar{R}_T + \tilde{\varepsilon} \langle \bar{R} \rangle_T) d\tilde{\varepsilon} \right] \\ & \geq u^0(x, 0, T) - \mathbb{E} \left[\int_0^\varepsilon \left| U' \left(T, \frac{X_T^{\pi,0}}{L_T^\varepsilon} \right) \frac{X_T^{\pi,0}}{L_T^\varepsilon} (\bar{R}_T + \tilde{\varepsilon} \langle \bar{R} \rangle_T) \right| d\tilde{\varepsilon} \right] > -\infty, \end{aligned}$$

where, in the second inequality, we have used (92).

Likewise, for a fixed $y > 0$ and $Z \in \mathcal{Z}_0$, such that $\mathbb{E}[V(yZ_T)] \in \mathbb{R}$, whose existence follows from (90), no unbounded profit with bounded risk, and Theorem 3.19. An application of Ito’s formula implies that for every ε in some neighborhood of 0 and $X^\varepsilon \in \mathcal{X}^\varepsilon(1)$, $X^\varepsilon Z L^\varepsilon = (X_t^\varepsilon Z_t L_t^\varepsilon)_{t \in [0, T]}$ is supermartingale, and thus $Z L^\varepsilon = (Z_t L_t^\varepsilon)_{t \in [0, T]}$ is a supermartingale deflator for the perturbed model. Therefore, similarly to (93), we can show that

$$\infty > \mathbb{E}[V(yZ_T L_T^\varepsilon)] \geq \sup_{x > 0} (u^\varepsilon(x, 0, T) - xz). \quad \blacksquare$$

Theorem 4.10. *Let $T > 0$ be fixed and suppose that Assumptions 2.3, 2.4, 4.4, and 4.5 hold as well as $\mathbb{E}[U^-(T, X_T^{\pi,0})] < \infty$. Then, we have*

$$(94) \quad \mathbb{P} - \lim_{\varepsilon \rightarrow 0} J_t^{\varepsilon, T} = J_t^{0, T}, \quad t \in [0, T].$$

Further, for each $t \in [0, T]$, with $\hat{\eta}_t$ being associated to $X_t^{\pi,0}$ via (60) and with

$$(95) \quad M^R := R^0 - \pi \cdot \langle R^0 \rangle,$$

we have

$$(96) \quad \mathbb{P} - \lim_{\varepsilon \rightarrow 0} \frac{J_t^{\varepsilon, T} - J_t^{0, T}}{\varepsilon} = X_t^{\pi,0} \hat{\eta}_t \left((\psi\pi - \lambda\theta) \cdot M_t^R + \mathbb{E}_t^{\mathbb{R}^t} [(\lambda\theta) \cdot R_T^0] \right).$$

Remark 4.11. Theorem 4.10 does not assert the stability nor provide the derivatives of the optimal trading strategies that are, in general, more difficult to obtain mathematically. However, Theorem 4.10 does show that under perturbations of the price process of the risky asset, the strategies that are optimal for the base model, which corresponds to $\varepsilon = 0$, drive the nearly optimal wealth processes for perturbed models.

Proof of Theorem 4.10. Let us fix $t \in [0, T]$. Via a direct application of Ito's formula, one can show that

$$(97) \quad \begin{aligned} X_t^{\pi, \varepsilon} &= x \mathcal{E}(\pi \cdot R^\varepsilon)_t = x \mathcal{E}(\pi \cdot R^0)_t \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t}{\mathcal{E}(-\varepsilon(\lambda\theta) \cdot R^0)_t} \\ &= X_t^{\pi, 0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_t^\varepsilon}. \end{aligned}$$

This implies that

$$u^\varepsilon(X_t^{\pi, \varepsilon}, t, T) \geq \mathbb{E}_t \left[U \left(T, X_t^{\pi, 0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{L_t^\varepsilon} \right) \right], \quad \mathbb{P}\text{-a.s.},$$

where $\hat{\rho}_t$ is the optimizer to (5) corresponding to $\xi = X_t^{\pi, 0}$ and $\varepsilon = 0$, that is, the base model for the risky asset. For $\varepsilon > 0$, let us consider

$$\begin{aligned} & \frac{1}{\varepsilon} \left(u^\varepsilon(X_t^{\pi, \varepsilon}, t, T) - u^0(X_t^{\pi, 0}, t, T) \right) \\ & \geq \frac{1}{\varepsilon} \left(\mathbb{E}_t \left[U \left(T, X_t^{\pi, 0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{L_t^\varepsilon} \right) \right] - \mathbb{E}_t \left[U \left(T, X_t^{\pi, 0} \hat{\rho}_t \right) \right] \right) \\ & = \frac{1}{\varepsilon} \left(\mathbb{E}_t \left[\int_0^\varepsilon U' \left(T, X_t^{\pi, 0} \hat{\rho}_t \frac{\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_T^\beta} \right) X_t^{\pi, 0} \hat{\rho}_t \frac{\partial}{\partial \beta} \left(\frac{\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_T^\beta} \right) d\beta \right] \right). \end{aligned}$$

From Assumption 4.5, we get

$$(98) \quad \begin{aligned} & U' \left(T, X_t^{\pi, 0} \hat{\rho}_t \frac{\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_T^\beta} \right) \\ & \leq U' \left(T, X_t^{\pi, 0} \hat{\rho}_t \right) \max \left(\left(\frac{\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_T^\beta} \right)^{-\gamma_1}, 1 \right). \end{aligned}$$

We recall that, in general (see, e.g., [Shi84, Definition 1, p. 211]), the definition of conditional expectation does not require integrability. This, in particular, allows us to circumvent any integrability conditions on $\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t$. Therefore, from (98), following [MS19, Lemma 5.14], and using Assumption 4.4, we obtain

$$(99) \quad \begin{aligned} & \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left(u^\varepsilon(X_t^{\pi, \varepsilon}, t, T) - u^0(X_t^{\pi, 0}, t, T) \right) \\ & \geq \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left(\mathbb{E}_t \left[U \left(T, X_t^{\pi, 0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{\mathcal{E}(-\varepsilon(\lambda\theta) \cdot R^0)_T} \right) \right] - \mathbb{E}_t \left[U \left(T, X_t^{\pi, 0} \hat{\rho}_t \right) \right] \right) \\ & = \mathbb{E}_t \left[U' \left(T, X_t^{\pi, 0} \hat{\rho}_t \right) X_t^{\pi, 0} \hat{\rho}_t \left((\psi\pi - \lambda\theta) \cdot M_t^R + (\lambda\theta) \cdot R_T^0 \right) \right] \\ & = \mathbb{E}_t \left[X_t^{\pi, 0} \hat{\rho}_t \hat{\eta}_t \hat{z}_{tT} \left((\psi\pi - \lambda\theta) \cdot M_t^R + (\lambda\theta) \cdot R_T^0 \right) \right] \\ & = X_t^{\pi, 0} \hat{\eta}_t \left((\psi\pi - \lambda\theta) \cdot M_t^R + \mathbb{E}_t^{\mathbb{R}^t} \left[(\lambda\theta) \cdot R_T^0 \right] \right), \end{aligned}$$

where $\hat{\eta}_t$ and \hat{z}_{tT} are given via Lemma 3.16.

To obtain the opposite inequality, from Lemma 3.16, we get

$$u^\varepsilon(X_t^{\pi,\varepsilon}, t, T) \leq \mathbb{E}_t [V(T, \hat{\eta}_t \hat{z}_{tT} L_T^\varepsilon)] + X_t^{\pi,\varepsilon} \hat{\eta}_t L_t^\varepsilon.$$

Therefore, for $\varepsilon > 0$, using Lemma 3.16 again, we deduce that

$$(100) \quad \begin{aligned} & \frac{1}{\varepsilon} \left(u^\varepsilon(X_t^{\pi,\varepsilon}, t, T) - u^0(X_t^{\pi,0}, t, T) \right) \\ & \leq \frac{1}{\varepsilon} \left(\mathbb{E}_t \left[V \left(T, \hat{\eta}_t L_t^\varepsilon \hat{z}_{tT} \frac{L_T^\varepsilon}{L_t^\varepsilon} \right) \right] - \mathbb{E}_t [V(T, \hat{\eta}_t \hat{z}_{tT})] \right) + \frac{1}{\varepsilon} \left(X_t^{\pi,\varepsilon} \hat{\eta}_t L_t^\varepsilon - X_t^{\pi,0} \hat{\eta}_t \right). \end{aligned}$$

Using Assumptions 4.4 and 4.5, and by passing into an \mathcal{F}_t -measurable partition of Ω as in Lemma 3.17, we get

$$(101) \quad \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbb{E}_t [V(T, \hat{\eta}_t \hat{z}_{tT} L_T^\varepsilon) - V(T, \hat{\eta}_t \hat{z}_{tT})] = X_t^{\pi,0} \hat{\eta}_t \mathbb{E}_t^{\mathbb{R}^t} [(\lambda\theta) \cdot R_T^0]$$

and

$$(102) \quad \begin{aligned} & \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left(X_t^{\pi,\varepsilon} \hat{\eta}_t L_t^\varepsilon - X_t^{\pi,0} \hat{\eta}_t \right) \\ & = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left(X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_t^\varepsilon} \hat{\eta}_t L_t^\varepsilon - X_t^{\pi,0} \hat{\eta}_t \right) \\ & = X_t^{\pi,0} \hat{\eta}_t \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left(\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t - 1 \right) \\ & = X_t^{\pi,0} \hat{\eta}_t (\psi\pi - \lambda\theta) \cdot M_t^R. \end{aligned}$$

From (99) and (100) using (101) and (102), we deduce that

$$\mathbb{P} - \lim_{\varepsilon \searrow 0} \frac{u^\varepsilon(X_t^{\pi,\varepsilon}, t, T) - u^0(X_t^{\pi,0}, t, T)}{\varepsilon} = X_t^{\pi,0} \hat{\eta}_t \left((\psi\pi - \lambda\theta) \cdot M_t^R + \mathbb{E}_t^{\mathbb{R}^t} [(\lambda\theta) \cdot R_T^0] \right).$$

Similarly, we can show that

$$\mathbb{P} - \lim_{\varepsilon \nearrow 0} \frac{u^\varepsilon(X_t^{\pi,\varepsilon}, t, T) - u^0(X_t^{\pi,0}, t, T)}{\varepsilon} = X_t^{\pi,0} \hat{\eta}_t \left((\psi\pi - \lambda\theta) \cdot M_t^R + \mathbb{E}_t^{\mathbb{R}^t} [(\lambda\theta) \cdot R_T^0] \right).$$

(94) and (96) follow. ■

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