

STABILITY OF THE INDIRECT UTILITY PROCESS

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ABSTRACT. We investigate the dynamic stability of the indirect utility process associated with a (possibly suboptimal) trading strategy under perturbations of the market. Establishing biconjugacy characterizations first, we prove continuity and first-order convergence of the indirect-utility process under simultaneous perturbations of the finite variation and martingale parts of the stock price return. We discuss the relation to forward performance process (FPP), and therefore the results of this paper provide an approach for the stability analysis of the FPPs under perturbations of the market.

1. INTRODUCTION

Indirect utility appears in mathematical finance as one of the primary criteria of the quality of a portfolio. Therefore, the regularity of the solution to certain classical problems in mathematical finance is associated with the stability or continuity of the indirect utility under perturbations of the initial data. From the practical viewpoint, as every statistical procedure allows for an only approximate determination of the model parameters, implementation of algorithms of optimal investment and utility-based pricing and hedging hinges on continuity of the indirect utility under model perturbations. The results of this paper show that under reasonably natural assumptions the indirect utility is a stable criterion of quality of the portfolio. Further, this also holds in dynamic settings (where the dynamic characterizations are usually much harder to establish, see, e.g., [MS05]) and even for suboptimal portfolios.

One of the primary motivations for studying stability for suboptimal strategies in dynamic settings is a connection with forward performance processes (FPPs), which were originally introduced in [MZ07] and [MZ08] to measure performances of portfolios in a way that allows for dynamic adaptation of the investor's preferences. Thus, the results of this paper provide an approach for the analysis of FPPs under market perturbations. They also imply the robustness of the indirect utility (and therefore the FPP in finite-horizon settings) as a dynamic criterion of the quality of a portfolio, and thus this paper complements the research of many authors, in particular, [SSZ16], in non-Markovian settings.

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However, the complete analysis of general FPPs goes beyond the scope of the current paper, it is being currently investigated in [MZ19].

Mathematically, the results below hinge on the dual characterization of the indirect utility associated with a possibly suboptimal trading strategy. Both the dynamic formulation and such a suboptimality lead to multiple difficulties related to establishing convex-analytic results functions whose codomain is a space of random variables \mathbb{L}^0 (and not \mathbb{R}), i.e., for random elements. It turns out that even fundamental theorems of convex analysis are harder to establish in such settings, and for example, the classical Fenchel-Moreau (or biconjugation) theorem over a *dual pair of Banach spaces* have been proven only recently, see [DJK]. In the present settings, however, we need not work with a dual pair, but rather with a pair of *polar sets* of stochastic processes, where polarity has to be understood appropriately. We identify precisely such polar sets and use the classical characterizations of the sets of wealth processes from [DS97] combined with changes of numéraire and results from [Mos15] to obtain the biconjugation result.

In the process of proving the biconjugacy, we establish a result, which is closely related to the conditional minimax theorem. Another version of this theorem can be found in [BK10]. Note that in our formulation we also do not require compactness of either domain, but only *boundedness in probability* is needed. Such boundedness often appears in the mathematical finance literature in both primal and dual domains under natural no-arbitrage conditions, for example in [KS99]. Note that minimax without compactness is a classical subject of analysis, see e.g., [Fan79], [Ha81], [LQ91]; a version of the minimax theorem that is helpful in financial applications, in particular below, and that does not require compactness of either domain can be found in [BK17].

For the stability analysis, we introduce a parametrization of perturbations, which allows considering distortions of the drift or volatility of the stock together or separately. Then we identify certain primal and dual feasible elements under such perturbations and prove convergence of the forward performances and complement this with the finding the associated derivatives with respect to a parameter. Dual characterization is key here.

Our construction of the dual domain is consistent with the weak no-arbitrage condition, no unbounded profit with bounded risk (NUPBR) introduced in [KK07], that still allows for the meaningful structure of the underlying problem. Thus, one of the contributions of the present paper is in showing that FPPs on a finite time horizon can be considered under NUPBR and possibly without stronger no free lunch with vanishing risk (NFLVR) condition, which was predominantly used for the investigation of FPPs in the past.

The remainder of the paper is organized as follows: in section 2 we specify the model, section 3 contains the dual characterization, in section 4 we show the stability of the indirect utility process.

2. MODEL

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where the filtration satisfies the usual conditions, \mathcal{F}_0 is trivial. There is a bond, whose price equals to 1 at all times, and a stock. The conditions on the stock will alter in different sections, thus in section 3, it can be considered to be a general multidimensional semimartingale, and for stability analysis in section 4 and below, we will work with a 1-dimensional continuous process.

2.1. Utility field and maximization problem. Let us consider an Inada stochastic utility field $U: [0, \infty) \times \Omega \times [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$, i.e., a stochastic field, which satisfies the following assumption.

Assumption 2.1. For every $(t, \omega) \in [0, \infty) \times \Omega$, $U(t, \omega, \cdot)$ is an Inada utility function, that is, a strictly increasing, strictly concave, differentiable function, which satisfies the Inada conditions:

$$\lim_{x \downarrow 0} U'(t, \omega, x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} U'(t, \omega, x) = 0,$$

where U' denotes the derivative with respect to the last argument. At $x = 0$, we suppose that $U(t, \omega, 0) = \lim_{x \downarrow 0} U(t, \omega, x)$. This value may be $-\infty$. We also suppose that $U(\cdot, \cdot, x)$ is optional for every $x > 0$. As it is common in the probability literature, the symbol ω will usually be omitted.

We refer to [Jar18, Chapter 9] for an overview of utility functions. In this and the following section, we consider only one market, where there is a d -multidimensional stock with a return process R^0 and a bond, whose price equals to 1 at all times. Following [KS99], we denote by $\mathcal{X}(x)$ the set of nonnegative wealth processes:

$$\mathcal{X}(x) = \{X \geq 0 : X = x + H \cdot R^0 \text{ for some } R^0\text{-integrable } H\}, \quad x \geq 0.$$

In this market, we fix an initial wealth $\bar{x} \geq 0$ and a predictable and R^0 -integrable process π , which specifies the proportions of wealth invested in corresponding stocks and such that $X^\pi = \bar{x} \mathcal{E}(\pi \cdot R^0) \geq 0$, where $\mathcal{E}(\cdot)$ denotes the stochastic exponential. The set of wealth processes in $\mathcal{X}(\bar{x})$, which equal to X^π on $[0, t]$, is denoted by $\mathcal{A}(X_t^\pi, t)$, that is

$$(1) \quad \mathcal{A}(X_t^\pi, t) := \left\{ \tilde{X} \in \mathcal{X}(\bar{x}) : \tilde{X}_s = X_s^\pi, \text{ for } s \in [0, t], \mathbb{P}\text{-a.s.} \right\}.$$

In such settings, we define an indirect utility up to T of π as

$$(2) \quad u(X_t^\pi, t, T) := \operatorname{ess\,sup}_{\tilde{X} \in \mathcal{A}(X_t^\pi, t)} \mathbb{E} \left[U \left(T, \tilde{X}_T \right) \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

Note that this definition is closely related to the definition of forward performance processes, see, e.g., [BRT09], [NZ14], and [ASS18] on a finite time horizon. However, (2)

does not require the existence of the optimizer (instead, it is proven below), and the supermartingale structure of $u(X_t^\pi, t, T)$, $t \in [0, T]$, i.e., $\mathbb{E}[u(X_{t_2}^\pi, t_2, T) | \mathcal{F}_{t_1}] \leq u(X_{t_1}^\pi, t_1, T)$ for $0 \leq t_1 \leq t_2 \leq T$, can also be shown.

2.2. Reformulation of (2). For the analysis below, we need to extend the definition of (2) to the closure of the convex solid hull of $\{X_t : X \in \mathcal{X}(x)\}$. This is done in the following two-step procedure. First, we define

$$(3) \quad \mathcal{C}_t(x) \triangleq \{g \in m\mathcal{F}_t : g \leq X_t \text{ for some } X \in \mathcal{X}(x)\}, \quad x \geq 0,$$

$$(4) \quad \mathcal{C}_{tT}(x) := \left\{ g \in m\mathcal{F}_T : g \leq x + \int_{t+}^T H_u dR_u^0, \text{ for some } R^0\text{-integrable } H \right\}, \quad x \geq 0,$$

and we will denote $\mathcal{C}_t(1)$ by \mathcal{C}_t as well as $\mathcal{C}_{tT}(1)$ by \mathcal{C}_{tT} , respectively.

Remark 2.2. The idea behind such definitions is that every stochastic integral of the form $x + \int_0^T H dR^0$ can be represented as $(x + \int_0^t H_u dR_u^0)(1 + \int_{t+}^T \tilde{H}_u dR_u^0)$ for an appropriate \tilde{H} . Also, similarly to the argument in [KS99, Section 4, p. 926] one can show that the sets \mathcal{C}_t and \mathcal{C}_{tT} are closed with respect to the topology of convergence in measure. This, in particular, allows for the following representation of \mathcal{C}_T :

$$\mathcal{C}_T = \mathcal{C}_t \mathcal{C}_{tT} = \{\xi \rho : \xi \in \mathcal{C}_t, \rho \in \mathcal{C}_{tT}\}.$$

Further, as U is increasing, the optimizer to (2) is a maximal element of \mathcal{C}_{tT} , thus by enlarging the domain of (2), we do not lose the structure of the solution to (2). On the other hand, by passing from the set of wealth processes to the closure of the (convex) solid hull of such processes we gain the properties needed, in particular, for the conjugacy characterization below.

Thus, we extend the definition of u in (2) from X^π to \mathcal{C}_t as follows

$$(5) \quad u(\xi, t, T) := \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t[U(T, \xi \rho)], \quad \xi \in \bigcup_{x \geq 0} \mathcal{C}_t(x),$$

with the version of an effective domain of $u(\cdot, t, T)$ being the set

$$(6) \quad \mathcal{E}_u := \left\{ \xi \in \bigcup_{x \geq 0} \mathcal{C}_t(x) : \text{there exists } \rho \in \mathcal{C}_{tT}, \text{ such that } \mathbb{E}[U^-(\xi \rho)] < \infty \right\}.$$

2.3. Technical Assumptions. We will impose the following conditions.

Assumption 2.3. (NUBPR up to T) Let $T > 0$ be fixed. The set $\{X_T : X \in \mathcal{X}(1)\}$ is bounded in probability.

Assumption 2.4. (fin value at T) Let $T > 0$ be fixed. We suppose that

$$u(z, 0, T) > -\infty, \quad z > 0 \quad \text{and} \quad \sup_{x > 0} (u(x, 0, T) - xz) < \infty.$$

These conditions are necessary of the 0-model to be nondegenerate, see e.g., the abstract theorems in [Mos15] and [CCFM17].

Remark 2.5. Assumption 2.4 will imply that the conditional expectations below are well-defined, where we adopt the definition [Shi84, Definition 1, p. 211], which does not require integrability.

3. DUAL CHARACTERIZATION

In this section, we will suppose that the stock prices process is a general multidimensional semimartingale.

3.1. **Existence and uniqueness of a solution to (2).** Let us denote

$$(7) \quad \mathcal{Z}_t := \left\{ (Z_s)_{s \in [t, T]} \geq 0 : \right. \\ \left. Z_t \leq 1 \text{ and } (Z_s X_s)_{s \in [t, T]} \text{ is a supermartingale for every } X \in \mathcal{X}(1) \right\},$$

and recall that the notion of Fatou-convergence of stochastic processes, is introduced in [FK97, Definition 5.2]. The following lemma shows existence of an optimizer to (5).

Lemma 3.1. *Let $T > 0$ be fixed and let us suppose that Assumptions 2.1, 2.3, and 2.4 hold. Then for every $\xi \in \mathcal{E}_u$, where \mathcal{E}_u is defined in (6), there exists $\rho \in \mathcal{C}_{tT}$, such that*

$$u(\xi, t, T) = \mathbb{E}_t [U(T, \xi \rho)].$$

Moreover, if $\xi > 0$, \mathbb{P} -a.s., then such a ρ is unique.

Proof. Let us fix $\xi \in \bigcup_{x \geq 0} \mathcal{C}_t(x)$, such that (6) holds. First, we will show that the set

$$(8) \quad \mathcal{U}_t \triangleq \{ \mathbb{E}_t [U(T, \xi \rho)] : \rho \in \mathcal{C}_{tT} \}$$

is closed under pairwise maximization. Let ρ^1 and ρ^2 be some elements of \mathcal{C}_{tT} and let H^1 and H^2 be such that

$$\rho^i \leq 1 + \int_{t+}^T H_u^i dS_u, \quad i = 1, 2.$$

We define

$$A := \{ \mathbb{E}_t [U(T, \xi \rho^1)] > \mathbb{E}_t [U(T, \xi \rho^2)] \} \in \mathcal{F}_t,$$

and set $H := H^1 1_A + H^2 1_{A^c}$. Then we obtain for $\rho := \rho^1 1_A + \rho^2 1_{A^c}$ that

$$(9) \quad \rho \leq 1 + \int_{t+}^T H_u dS_u,$$

as

$$\rho = \begin{cases} \rho^1 \leq 1 + \int_{t+}^T H_u^1 dS_u, & \text{on } A, \\ \rho^2 \leq 1 + \int_{t+}^T H_u^2 dS_u, & \text{on } A^c. \end{cases}$$

Since $1 + \int_{t+}^T H_u dS_u \geq 0$, \mathbb{P} -a.s., we conclude from (9) that $\rho \in \mathcal{C}_{tT}$. Consequently,

$$\mathbb{E}_t [U(T, \xi\rho)] = \max \left(\mathbb{E}_t [U(T, \xi\rho^1)], \mathbb{E}_t [U(T, \xi\rho^2)] \right),$$

i.e., \mathcal{U}_t defined in (8) is closed under pairwise maximization. Applying [KS98, Theorem A.3, p. 324] (or rather an extension of this theorems to extended-real valued random variables, see e.g., [CP15, Proposition 2.6.1]), we deduce that there exists a sequence $(\rho^n)_{n \in \mathbb{N}} \subset \mathcal{C}_{tT}$, such that

$$\lim_{n \rightarrow \infty} \mathbb{E}_t [U(T, \xi\rho^n)] = \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\rho)].$$

By Komlos'-type lemma, see e.g., [DS94, Lemma A1.1], we may find a sequence of convex combinations $\tilde{\rho}^n \in \operatorname{conv}(\rho^n, \rho^{n+1}, \dots)$, $n \in \mathbb{N}$, and a random variable \hat{g} , such that $(\tilde{\rho}_T^n)_{n \in \mathbb{N}}$ converges to \hat{g} , \mathbb{P} -a.s.. By concavity of $U(T, \cdot)$, $(\tilde{\rho}_T^n)_{n \in \mathbb{N}}$ is also a maximizing sequence in the sense that

$$(10) \quad \lim_{n \rightarrow \infty} \mathbb{E}_t [U(T, \xi\tilde{\rho}^n)] = \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\rho)].$$

Similarly to the argument in [KS99, Proof of Proposition 3.1], \mathcal{C}_{tT} is closed in probability. Therefore, $\hat{g} \in \mathcal{C}_{tT}$. Via [Mos15, Lemma 3.5] and the symmetry between primal and dual problems in [Mos15], one can show that $U^+(T, \xi\tilde{\rho}^n)$, $n \in \mathbb{N}$, is a uniformly integrable sequence, so is $\mathbb{E}_t [U^+(T, \xi\tilde{\rho}^n)]$, $n \in \mathbb{N}$, and therefore we have

$$\operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\rho)] = \lim_{n \rightarrow \infty} \mathbb{E}_t [U(T, \xi\tilde{\rho}^n)] \leq \mathbb{E}_t [U(T, \xi\hat{g})], \quad \mathbb{P}\text{-a.s.},$$

and thus \hat{g} is the maximizer to (5). Further, if $\xi > 0$, \mathbb{P} -a.s., the uniqueness of the maximizer follows from the strict concavity of $U(T, \cdot)$. \square

Remark 3.2. If the stock is continuous, after (10) one can apply argument based on a version of the optional decomposition theorem for arbitrary filtrations, see [KK15, Optional Decomposition Theorem 1.4]. Let \tilde{X}^n be nonnegative processes of the form $1 + \int_{t+}^s H_u^n dS_u$, $s \in (t, T]$, and $\tilde{X}_t^n = 1$, for some S -integrable H^n 's, such that

$$\tilde{X}_T^n \geq \tilde{\rho}^n, \quad \mathbb{P}\text{-a.s.}, \quad n \in \mathbb{N}.$$

Following the proof of [Mos15, Lemma 4.2], we pick a strictly positive $Y \in \mathcal{Z}_t$ (whose existence follows from Assumption 2.3) and consider $\tilde{X}^n Y$, $n \in \mathbb{N}$. Let

$$\mathcal{T} := (\mathbb{Q} \cap (t, T)) \cup \{t\} \cup \{T\},$$

where \mathbb{Q} is the set of rational numbers.

Then, passing to convex combinations, we may find a subsequence of convex combinations $\tilde{\tilde{X}}^n$, $n \in \mathbb{N}$, such that $\tilde{\tilde{X}}^n Y$, $n \in \mathbb{N}$, is Fatou convergent to a supermartingale VY

on \mathcal{T} and such that

$$(11) \quad V_t \leq \liminf_{s \downarrow t, s \in \mathcal{T}} \liminf_{n \rightarrow \infty} \mathbb{E} \left[\tilde{X}_s^n \frac{Y_s}{Y_t} \middle| \mathcal{F}_t \right] \leq 1 \quad \text{and} \quad V_T \geq \hat{g}.$$

Similarly to [Mos15, Lemma 4.2], one can show that $(V_s Z_s)_{s \in [t, T]}$ is a supermartingale for every $Z \in \mathcal{Z}_t$. We stress that the supermartingale property is only required on $[t, T]$. One can see that¹ for every supermartingale deflator Z , the process \bar{Z} of the form

$$\bar{Z}_s = \frac{Z_{t+s}}{Z_t}, \quad s \in [0, T-t],$$

is an element of \mathcal{Z}_t . We also set

$$\mathcal{G}_s := \mathcal{F}_{t+s}, \quad s \in [0, T-t].$$

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{G}_s)_{s \in [0, T-t]}$, $(V_s \bar{Z}_s)_{s \in [0, T-t]}$ is a supermartingale for every $\bar{Z} \in \mathcal{Z}_t$. Therefore V satisfies the conditions of (item 1 of) [KK15, Optional Decomposition Theorem 1.4]. As a result, there exists a decomposition of V of the form

$$(12) \quad V_s = V_0 + \int_{0+}^s \bar{H}_u dS_u - A_s, \quad s \in (0, T-t],$$

where V_0 is $\mathcal{G}_0 = \mathcal{F}_t$ -measurable random variable, \bar{H} is predictable and S -integrable and A is a nondecreasing, right-continuous, adapted process, such that $A_0 = 0$.

We denote $\hat{H}_{t+s} := \bar{H}_s$, $s \in [0, T-t]$. Therefore, using (11) and (12), we deduce that

$$\hat{X}_T = 1 + \int_{t+}^T \hat{H}_u dS_u \geq \hat{g}$$

and $\hat{X}_T \in \mathcal{C}_{tT}$. It follows from Assumption 2.4 and [Mos15, Lemma 3.5] that $(U^+(X_T))_{X \in \mathcal{X}(x)}$ is uniformly integrable, therefore, so is $(\mathbb{E}_t[U^+(X_T)])_{X \in \mathcal{X}(x)}$, and using monotonicity of U , we get

$$\operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t[U(T, \xi \rho)] = \lim_{n \rightarrow \infty} \mathbb{E}_t[U(T, \xi \hat{\rho}^n)] \leq \mathbb{E}_t[U(T, \xi \hat{g})] \leq \mathbb{E}_t[U(T, \xi \hat{X}_T)].$$

We deduce that \hat{X}_T is the maximizer to (5). If $\xi > 0$, \mathbb{P} -a.s., the uniqueness of the maximizer follows from the strict concavity of U .

3.2. Structure of the dual process. First, we set

$$V(t, \omega, y) := \sup_{x > 0} (U(t, \omega, x) - xy), \quad (t, \omega, y) \in [0, T] \times \Omega \times [0, \infty).$$

For $t \in [0, T]$, we set

$$(13) \quad \mathcal{D}_t(y) := \{\eta \in m\mathcal{F}_t : \eta \leq z_t \text{ for some } z \in \mathcal{Z}_0\},$$

¹We use the convention $\frac{0}{0} = 0$.

i.e., $\mathcal{D}_t(y)$ is a subset the closure of the convex solid hull of the elements of \mathcal{Z}_0 sampled at time t . We define

$$(14) \quad \mathcal{N}_t := \bigcup_{y \geq 0} \mathcal{D}_t(y).$$

For $t \in [0, T]$, let \mathcal{Z}_t be given by (7) and we set

$$(15) \quad v(\eta, t, T) := \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V(T, \eta z_T)], \quad \eta \in \mathcal{N}_t.$$

Lemma 3.3. *Under the conditions of Lemma 3.1, for every $\eta \in \mathcal{N}_t$, there exists $\widehat{z} \in \mathcal{Z}_t$, such that*

$$(16) \quad v(\eta, t, T) = \mathbb{E} [V(T, \eta \widehat{z}_T) | \mathcal{F}_t] \quad \text{on} \quad \{v(\eta, t, T) < \infty\}.$$

Proof. Let us consider \bar{z}^1 and \bar{z}^2 in \mathcal{Z}_t , let

$$A := \{\omega : \mathbb{E} [V(T, \eta \bar{z}_T^1) | \mathcal{F}_t] (\omega) < \mathbb{E} [V(T, \eta \bar{z}_T^2) | \mathcal{F}_t] (\omega)\} \in \mathcal{F}_t$$

and

$$\bar{z}_{t'} := \bar{z}_{t'}^1 1_A + \bar{z}_{t'}^2 1_{A^c}, \quad t' \in [t, T].$$

Then on $[t, T]$, zX is a supermartingale deflator for every $X \in \mathcal{X}(1)$. Therefore, $\bar{z} \in \mathcal{Z}_t$. By direct computations, we have

$$\mathbb{E} [V(T, \eta \bar{z}_T) | \mathcal{F}_t] = \min (\mathbb{E} [V(T, \eta \bar{z}_T^1) | \mathcal{F}_t], \mathbb{E} [V(T, \eta \bar{z}_T^2) | \mathcal{F}_t]).$$

Therefore, from [Pha09, Theorem A.2.3, p. 215], we deduce that there exists a sequence $(z^n)_{n \in \mathbb{N}}$, such that

$$(17) \quad \lim_{n \rightarrow \infty} \mathbb{E} [V(T, \eta z_T^n) | \mathcal{F}_t] = v(\eta, t, T), \quad \mathbb{P} - a.s.$$

By passing to convex combinations, we obtain a subsequence, which we do not relabel, such that $\lim_{n \rightarrow \infty} z^n = \widehat{z}$, where the limit is considered in the Fatou sense (in the terminology of [FK97, Definition 5.2]) on the set of rational numbers on (t, T) augmented with t and T . Note that by convexity of $V(T, \cdot)$, such a subsequence will also satisfy (17). It follows from the definition of \mathcal{N}_t and [Mos15, Lemma 3.5] that $(V^-(T, \eta z_T^n))_{n \in \mathbb{N}}$ is uniformly integrable. Therefore, we get

$$(18) \quad \lim_{n \rightarrow \infty} \mathbb{E} [V(T, \eta z_T^n) | \mathcal{F}_t] \geq \mathbb{E} [V(T, \eta \widehat{z}_T) | \mathcal{F}_t].$$

By direct computations, we deduce that for every $X \in \mathcal{X}(1)$, $(\widehat{z}_{t'} X_{t'})_{t' \in [t, T]}$ is a nonnegative supermartingale on $[t, T]$, such that $z_t \leq 1$ by properties of Fatou-convergence. Therefore $\widehat{z} \in \mathcal{Z}_t$. Via (17) and (18), we conclude that (16) holds. \square

Let

$$(19) \quad \mathcal{B}_t^+ := \{X \in m\mathcal{F}_t : X \in [0, 1], \mathbb{P}\text{-a.s.}\}.$$

and

$$(20) \quad \mathcal{E}_v := \{\eta \in \mathcal{N}_t : \text{there exists } z \in \mathcal{Z}_t, \text{ such that } \mathbb{E}[V^+(T, \eta z_T)] < \infty\}$$

which corresponds to the effective domain of $v(\cdot, t, T)$.

Lemma 3.4. *Let the condition of Lemma 3.1 hold, $\lambda \in \mathcal{B}_t^+$ and η^1 and η^2 are some elements of \mathcal{E}_v , then $\eta := \lambda\eta^1 + (1 - \lambda)\eta^2 \in \mathcal{E}_v$ and we have*

$$(21) \quad v(\eta, t, T) \leq \lambda v(\eta^1, t, T) + (1 - \lambda)v(\eta^2, t, T).$$

Proof. As $\eta^i \in \mathcal{D}_t(y^i)$, for some $y^i > 0$, $i = 1, 2$, one can see that $\eta \leq Y_t$ for some $Y \in \mathcal{Y}(y^1 + y^2, 0)$. Next we will show (21). By Lemma 3.3, we deduce the existence of \hat{z}^1 and \hat{z}^2 , the optimizers to (15) corresponding to η^1 and η^2 , respectively. With

$$z_s := \frac{\lambda\eta^1\hat{z}_s^1 + (1 - \lambda)\eta^2\hat{z}_s^2}{\eta} 1_{\{\eta \neq 0\}} + (\lambda\hat{z}_s^1 + (1 - \lambda)\hat{z}_s^2) 1_{\{\eta=0\}}, \quad s \in [t, T],$$

by direct computations, one can see that $(z_{t'}X_{t'})_{t' \in [t, T]}$ is a supermartingale for every $X \in \mathcal{X}$. By construction, $z_t \leq 1$. We conclude that $z \in \mathcal{Z}_t$. To show that $\eta \in \mathcal{E}_v$, we use convexity of $V^+(T, \cdot)$ and we get

$$\begin{aligned} \mathbb{E}[V^+(T, \eta z_T)] &= \mathbb{E}[V^+(T, \lambda\eta^1\hat{z}_T^1 + (1 - \lambda)\eta^2\hat{z}_T^2)] \\ &\leq \mathbb{E}[\lambda V^+(T, \eta^1\hat{z}_T^1) + (1 - \lambda)V^+(T, \eta^2\hat{z}_T^2)] \leq \mathbb{E}[V^+(T, \eta^1\hat{z}_T^1)] + \mathbb{E}[V^+(T, \eta^2\hat{z}_T^2)] < \infty. \end{aligned}$$

Likewise, using convexity of V , we get

$$(22) \quad \begin{aligned} v(\eta, t) &\leq \mathbb{E}_t[V(T, \eta z_T)] = \mathbb{E}_t[V(T, \lambda\eta^1\hat{z}_T^1 + (1 - \lambda)\eta^2\hat{z}_T^2)] \\ &\leq \lambda\mathbb{E}_t[V(T, \eta^1\hat{z}_T^1)] + (1 - \lambda)\mathbb{E}_t[V(T, \eta^2\hat{z}_T^2)] = \lambda v(\eta^1, t, T) + (1 - \lambda)v(\eta^2, t, T). \end{aligned}$$

Thus, (21) holds. □

An important role in the proofs below will be played by the set

$$(23) \quad \mathcal{G}_t := \{\eta z_T : \eta \in \mathcal{N}_t, z \in \mathcal{Z}_t\}.$$

Lemma 3.5. *Let the condition of Lemma 3.1 hold, $\lambda \in \mathcal{B}_t^+$, and $\eta^i z^i \in \mathcal{G}_t$, $i = 1, 2$. Then with*

$$(24) \quad \eta := \lambda\eta^1 + (1 - \lambda)\eta^2,$$

we have: $\eta \in \mathcal{N}_t$,

$$(25) \quad z_s := \frac{\lambda \eta^1 z_s^1 + (1 - \lambda) \eta^2 z_s^2}{\eta} 1_{\{\eta \neq 0\}} + (\lambda z_s^1 + (1 - \lambda) z_s^2) 1_{\{\eta = 0\}}, \quad s \in [t, T],$$

is an element of \mathcal{Z}_t ,

$$(26) \quad \lambda \eta^1 z_T^1 + (1 - \lambda) \eta^2 z_T^2 = \eta z_T \in \mathcal{G}_t.$$

In particular, (24) holds if λ is a constant taking values in $[0, 1]$, i.e., \mathcal{G}_t is a convex set and if η^i are in \mathcal{E}_v , then $\eta \in \mathcal{E}_v$.

Proof. Let us consider η and z are defined in (24) and (25), respectively. As $\eta \leq \eta^1 + \eta^2$, we deduce (trivially) that $\eta \in \mathcal{N}_t$. It follows from Lemma 3.4 that if, additionally, η^1 and η^2 are some elements of \mathcal{E}_v , then $\eta \in \mathcal{E}_v$. Following the proof of the same lemma, $z \in \mathcal{Z}_t$. The validity of

$$\lambda \eta^1 z_T^1 + (1 - \lambda) \eta^2 z_T^2 = \eta z_T$$

is a consequence of the definitions of η and z . As, by the argument above, $\eta \in \mathcal{N}_t$ and $z \in \mathcal{Z}_t$, we deduce that (26) holds. Thus, in particular, \mathcal{G}_t is convex. \square

3.3. Conjugacy of u and v . We recall that $\mathcal{C}(x)$, $x > 0$, are defined in (3). The goal is to show that

$$u(\xi, t, T) = \operatorname{ess\,inf}_{\eta \in \mathcal{N}_t} (v(\eta, t, T) + \xi \eta), \quad \xi \in \bigcup_{x > 0} \mathcal{C}(x)$$

and

$$v(\eta, t, T) = \operatorname{ess\,sup}_{\xi \in \bigcup_{x > 0} \mathcal{C}_t(x)} (u(\xi, t, T) - \xi \eta), \quad \eta \in \mathcal{N}_t.$$

Lemma 3.6. *Let the condition of Lemma 3.1 hold. Then for every $\eta \in \mathcal{N}_t$ and $\xi \in \bigcup_{x > 0} \mathcal{C}_t(x)$, we have:*

$$u(\xi, t, T) \leq v(\eta, t, T) + \xi \eta,$$

equivalently

$$u(\xi, t, T) \leq \operatorname{ess\,inf}_{\eta \in \mathcal{N}_t} (v(\eta, t, T) + \xi \eta).$$

Proof. Let $\xi \in \bigcup_{x > 0} \mathcal{C}_t(x)$ and $\eta \in \mathcal{N}_t$ be fixed. We need to show that

$$(27) \quad \operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}[U(T, \xi \rho) | \mathcal{F}_t] - \xi \eta \leq \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t].$$

From the definition of the conjugate function, for every $\rho \in \mathcal{C}_{tT}$ and every $z \in \mathcal{Z}_t$, we have

$$(28) \quad \mathbb{E}_t[U(T, \xi \rho)] \leq \mathbb{E}_t[V(T, \eta z_T) + \xi \rho \eta z_T].$$

As $z \in \mathcal{Z}_t$, $\mathbb{E}_t[\rho z_T] \leq 1$ and we have

$$(29) \quad \mathbb{E}_t[\xi \rho \eta z_T] = \xi \eta \mathbb{E}_t[\rho z_T] \leq \xi \eta.$$

Combining (28) and (29), we get

$$\mathbb{E}_t [U(T, \xi\rho)] \leq \mathbb{E}_t [V(T, \eta z_T)] + \eta\xi,$$

which implies (27). This completes the proof of the lemma. \square

Lemma 3.7. *Let the condition of Lemma 3.1 hold and $\eta \in \mathcal{N}_t$. Then we have*

$$(30) \quad \lim_{x \rightarrow \infty} \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \operatorname{ess\,sup}_{\xi \in \mathcal{C}_T(x)} \mathbb{E}_t [(U(T, \xi) - \xi\eta z_T)] \geq \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V(T, \eta z_T)] = v(\eta, t, T).$$

Further, for every $A \in \mathcal{F}_t$, we have

$$(31) \quad \lim_{x \rightarrow \infty} \operatorname{inf}_{z \in \mathcal{Z}_t} \operatorname{sup}_{\xi \in \mathcal{C}_T(x)} \mathbb{E} [(U(T, \xi) - \xi\eta z_T) 1_A] \geq \operatorname{inf}_{z \in \mathcal{Z}_t} \mathbb{E} [V(T, \eta z_T) 1_A].$$

Proof. Step 1. First, we suppose that

$$(32) \quad \mathbb{E} [U(T, 1)] > -\infty.$$

Let us fix $z \in \mathcal{Z}_t$ and set

$$V^n(T, y) := \sup_{x \in (0, n]} (U(T, x) - xy), \quad y \geq 0, \quad n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, as $n \in \mathcal{C}_T(n)$, we get

$$(33) \quad \operatorname{ess\,sup}_{\xi \in \mathcal{C}_T(n)} \mathbb{E}_t [(U(T, \xi) - \xi\eta z_T)] \geq \mathbb{E}_t [V^n(T, \eta z_T)].$$

Next we set

$$v^n(\eta, t, T) := \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V^n(T, \eta z_T)], \quad \eta \in \mathcal{N}_t,$$

and observe that $v^n(\eta, t, T)$, $n \in \mathbb{N}$, in an increasing sequence. From (33), we obtain

$$(34) \quad \lim_{n \rightarrow \infty} \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \operatorname{ess\,sup}_{\xi \in \mathcal{C}_T(n)} \mathbb{E}_t [(U(T, \xi) - \xi\eta z_T)] \geq \lim_{n \rightarrow \infty} v^n(\eta, t, T).$$

As $V^n(T, y) \leq V(T, y)$, $y \geq 0$, similarly to Lemma 3.3, we deduce that there exists $\hat{z}^n \in \mathcal{Z}_t$, such that

$$v^n(\eta, t, T) = \mathbb{E}_t [V^n(T, \eta \hat{z}_T^n)], \quad n \in \mathbb{N}.$$

One can pass to convex combinations, which we denote \tilde{z}^n , $n \in \mathbb{N}$, to obtain a Fatou-limit of \tilde{z}^n 's, which we denote $\hat{z} \in \mathcal{Z}_t$. As

$$V^n(T, y) \geq V^2(T, y) \geq V(T, y) 1_{\{y \geq U'(T, 2)\}} + (U(T, 2) - 2U'(T, 2)) 1_{\{y < U'(T, 2)\}}, \quad y \geq 0, n \geq 2,$$

using convexity of $U(T, \cdot)$, we get $U'(T, 2) \leq U(T, 2) - U(T, 1)$, and thus

$$V^n(T, y) \geq \min(V(T, y), 2U(T, 1) - U(T, 2)), \quad y \geq 0, n \geq 2,$$

and (similarly to [Mos15, Lemma 3.9]) uniform integrability of $(V^n)^-(T, \hat{z}_T^n \eta)$, $n \geq 2$, follows from [Mos15, Lemma 3.5] and (32). As a consequence, using convexity of $V^n(T, \cdot)$'s and Fatou's lemma, we get

(35)

$$\lim_{n \rightarrow \infty} v^n(\eta, t, T) = \lim_{n \rightarrow \infty} \mathbb{E}_t [V^n(T, \hat{z}_T^n \eta)] \geq \lim_{n \rightarrow \infty} \mathbb{E}_t [V^n(T, \tilde{z}_T^n \eta)] \geq \mathbb{E}_t [V(T, \hat{z}_T \eta)] \geq v(\eta, t, T).$$

Combining (34) and (35), we obtain (30). In turn, (31) can be proven similarly.

Step 2. This case can be reduced to the one above by taking $\hat{\rho} = \arg \max_{\xi \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \frac{1}{2} \xi \rho)]$ and by setting $\rho = \max(\hat{\rho}, \frac{1}{2})$. Then $0 < \rho \in \mathcal{C}_{tT}$ and for

$$\begin{aligned} \tilde{U}(T, x) &:= U(T, \rho x), & \tilde{V}(T, y) &:= V\left(T, \frac{y}{\rho}\right), \\ \tilde{\mathcal{C}}_{tT} &:= \{\tilde{\rho} : \tilde{\rho} \rho \in \mathcal{C}_{tT}\}, & \tilde{\mathcal{Z}}_t &:= \left\{ \tilde{z} : \frac{\tilde{z}}{\rho} \in \mathcal{Z}_t \right\}. \end{aligned}$$

Then we can represent u and v as

$$u(\xi, t, T) = \operatorname{ess\,sup}_{\tilde{\rho} \in \tilde{\mathcal{C}}_{tT}} \mathbb{E}_t [\tilde{U}(T, \tilde{\rho})], \quad v(\eta, t, T) = \operatorname{ess\,inf}_{\tilde{z} \in \tilde{\mathcal{Z}}_t} \mathbb{E}_t [\tilde{V}(T, \eta \tilde{z})]$$

and \tilde{U} satisfies (32). Then, (30) and (31) follow from Step 1. \square

Let us define

$$(36) \quad \mathcal{C}'_{tT} := \left\{ \rho \in \mathcal{C}_{tT} : \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t [\rho z_T] = 1 \right\}$$

and adapt to our settings the notation from [DS97].

$$(37) \quad \mathcal{K}_{tT}^{\max} := \{\text{maximal elements of } \mathcal{X}_{tT}(1)\}.$$

The following lemma extends some characterization of maximal 1-admissible contingent claims from [DS97] to the present settings, mainly to Assumption 2.3.

Lemma 3.8. *Let Assumption 2.3 holds. Then we have*

$$(38) \quad \mathcal{K}_{tT}^{\max} \subseteq \mathcal{C}'_{tT} \subset \mathcal{C}_{tT}.$$

As a consequence, for every $x > 0$ and $\xi \in \mathcal{C}_t(x)$, we have

$$(39) \quad u(\xi, t, T) = \operatorname{ess\,sup}_{\phi \in \mathcal{C}'_{tT}} \mathbb{E}_t [U(T, \xi \phi)] = \operatorname{ess\,sup}_{\phi \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi \phi)],$$

and for every $A \in \mathcal{F}_t$, we have

$$(40) \quad \sup_{\phi \in \mathcal{C}'_{tT}} \mathbb{E} [U(T, \xi \phi) 1_A] = \sup_{\phi \in \mathcal{C}_{tT}} \mathbb{E} [U(T, \xi \phi) 1_A].$$

Proof. The proof is based on a change of numéraire idea. By the results of [KK07], Assumption 2.3 implies the existence of the numéraire portfolio N . Let us assume that for some trading strategy G , $X^{tT} := 1 + \int_{t+}^T G_s dR_s^0$ is a (maximal) element of \mathcal{K}_{tT}^{\max} , then, as we can extend G by 0 on $[0, t]$ to obtain an element of \mathcal{K}^{\max} , one can see that $\frac{X^{tT}}{N}$ is

a maximal element under the numéraire N , and NFLVR holds for $\left(\frac{1}{N}, \frac{R^0}{N}\right)$. As densities of locally equivalent martingale measures under the new numéraire can be represented as $z'zN$, where z' is a supermartingale deflator for S on $[0, t]$ and z is an element of \mathcal{Z}_t , we deduce from [DS97, Theorem 2.5], the existence of $z'z$, such that

$$1 = \mathbb{E} \left[z'_t z_T N_T \frac{X_T^{tT}}{N_T} \right] = \mathbb{E} [z'_t z_T X_T^{tT}] = \mathbb{E} [z'_t \mathbb{E}_t [z_T X_T^{tT}]].$$

As $\mathbb{E}_t [z_T X_T^{tT}] \leq 1$, by construction, it follows that $\mathbb{E}_t [z_T X_T^{tT}] = 1$, \mathbb{P} -a.s., and therefore we have

$$(41) \quad \mathcal{K}_{tT}^{\max} \subseteq \mathcal{C}'_{tT}.$$

Similarly to [KS99, Proposition 3.1], we deduce that

$$\mathcal{C}_{tT} = \{c \in m\mathcal{F}_T : c \leq h, \text{ for some } h \in \mathcal{K}_{tT}^{\max}\} = \left\{ c \in m\mathcal{F}_T : \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t [cz_T] \leq 1 \right\},$$

and thus (38) holds. Moreover, as $U(T, \cdot)$ is nondecreasing, we get

$$\operatorname{ess\,sup}_{\phi \in \mathcal{K}_{tT}^{\max}} \mathbb{E}_t [U(T, \xi\phi)] = \operatorname{ess\,sup}_{\phi \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\phi)].$$

Combining the latter equality with (38), we obtain (39). Finally, (40) can be obtained similarly to (39). \square

Lemma 3.9. *Let the condition of Lemma 3.1 hold and $\eta \in \mathcal{N}_t$ be fixed. Then for every $A \in \mathcal{F}_t$, we have*

$$\begin{aligned} \inf_{z \in \mathcal{Z}_t} \mathbb{E} [V(T, \eta z_T) 1_A] &= \lim_{x \rightarrow \infty} \inf_{z \in \mathcal{Z}_t} \sup_{\xi \rho \in \mathcal{C}_T(x)} \mathbb{E} [(U(T, \xi\rho) - \eta\xi\rho z_T) 1_A] \\ &= \sup_{\substack{\xi \rho \in \bigcup_{x>0} \mathcal{C}_T(x) \\ x>0}} \inf_{z \in \mathcal{Z}_t} \mathbb{E} [(U(T, \xi\rho) - \eta\xi\rho z_T) 1_A]. \end{aligned}$$

Proof. The first equality follows from Lemma 3.7 (see (31)) and the definition of V , whereas the second one is a consequence of the minimax theorem, see [BK17, Theorem B.3]. \square

Remark 3.10. The proof of Lemma 3.9 follows the structure of [Mos15, Lemma 3.9]. However, in view of [BK17, Theorem B.3], one does not need to truncate the domain of u and to invoke the Banach-Alaoglu's theorem.

Remark 3.11 (On the multiplicative decomposition of \mathcal{C}_{tT}). Let us consider

$$\alpha(\rho) := \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t [\rho z_T], \quad \rho \in \mathcal{C}_{tT}.$$

Then, for every $\rho \in \mathcal{C}_{tT}$, $\alpha(\rho)$ takes values in $[0, 1]$, and it follows from Assumption 2.3 that

$$(42) \quad \mathbb{P}[\{\rho > 0\} \cap \{\alpha(\rho) = 0\}] = 0.$$

and we recall that \mathcal{C}'_{tT} and \mathcal{B}_t^+ are defined in (36) and (19), respectively. Then using (42), we get

$$\rho = 1_{\{\alpha(\rho) > 0\}} \alpha(\rho) \frac{\rho}{\alpha(\rho)} + 1_{\{\alpha(\rho) = 0\}} \rho = 1_{\{\alpha(\rho) > 0\}} \alpha(\rho) \frac{\rho}{\alpha(\rho)},$$

i.e., a multiplicative decomposition of ρ into an element of \mathcal{B}_t^+ and an element of \mathcal{C}'_{tT} , which holds on $\{\alpha(\rho) > 0\}$, and which we can extend to $\{\alpha(\rho) = 0\}$ by $\alpha(\rho)$ multiplied by any element of \mathcal{C}'_{tT} (restricted to $\{\alpha(\rho) = 0\}$).

Lemma 3.12. *Let the condition of Lemma 3.1 hold. Then we have*

$$(43) \quad \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E}[(U(T, \xi\rho) - \xi\eta) 1_A] = \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \inf_{z \in \mathcal{Z}_t} \mathbb{E}[(U(T, \xi\rho) - \xi\eta\rho z_T) 1_A].$$

Proof. With \mathcal{B}_t^+ and \mathcal{C}'_{tT} defined in (36) and (19), respectively, and following the argument of Remark 3.11, we can rewrite the right-hand side of (43) as

$$(44) \quad \begin{aligned} & \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \inf_{z \in \mathcal{Z}_t} \mathbb{E}[(U(T, \xi\rho) - \xi\eta\rho z_T) 1_A] \\ &= \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \left(\mathbb{E}[(U(T, \xi\rho) 1_A] - \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\rho z_T 1_A] \right) \\ &= \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\alpha \in \mathcal{B}_t^+} \sup_{\phi \in \mathcal{C}'_{tT}} \left(\mathbb{E}[(U(T, \xi\alpha\phi) 1_A] - \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha\phi z_T 1_A] \right). \end{aligned}$$

Let us consider the latter term, $\sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha\phi z_T 1_A]$, which we can rewrite as

$$(45) \quad \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha\phi z_T 1_A] = \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha 1_A \mathbb{E}_t[\phi z_T]],$$

where by the respective definitions of $\mathcal{C}_t(x)$, \mathcal{N}_t , and \mathcal{B}_t^+ , we deduce that

$$(46) \quad 0 \leq \mathbb{E}[\xi\eta\alpha 1_A] < \infty,$$

and from the definition of \mathcal{C}'_{tT} , for every $\phi \in \mathcal{C}'_{tT}$, we have

$$\text{ess sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t[\phi z_T] = 1, \quad \mathbb{P}\text{-a.s.}$$

Therefore, for every $z \in \mathcal{Z}_t$, we obtain

$$\mathbb{E}[\xi\eta\alpha 1_A \mathbb{E}_t[\phi z_T]] \leq \mathbb{E} \left[\xi\eta\alpha 1_A \text{ess sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t[\phi z_T] \right] = \mathbb{E}[\xi\eta\alpha 1_A].$$

Consequently, for every $\xi \in \mathcal{C}_t(x)$, $\eta \in \mathcal{N}_t$, $\alpha \in \mathcal{B}_t^+$, $\phi \in \mathcal{C}'_{tT}$, we get

$$(47) \quad \sup_{z \in \mathcal{Z}_t} \mathbb{E}[\xi\eta\alpha 1_A \mathbb{E}_t[\phi z_T]] \leq \mathbb{E}[\xi\eta\alpha 1_A].$$

On the other hand, let us fix $\phi \in \mathcal{C}'_{tT}$ and two arbitrary elements of \mathcal{Z}_t , \bar{z}^1 and \bar{z}^2 . With

$$B := \{ \mathbb{E}_t[\phi \bar{z}_T^1] > \mathbb{E}_t[\phi \bar{z}_T^2] \},$$

one can see that

$$\bar{z} := 1_B \bar{z}^1 + 1_{B^c} \bar{z}^2 \in \mathcal{Z}_t,$$

is such that

$$\mathbb{E}_t [\phi \bar{z}_T] = \max \left(\mathbb{E}_t [\phi \bar{z}_T^1], \mathbb{E}_t [\phi \bar{z}_T^2] \right),$$

and therefore by [Pha09, Theorem A.2.3, p. 215], we deduce that there exists a sequence $(z^n)_{n \in \mathbb{N}} \subset \mathcal{Z}_t$, such that

$$(48) \quad \lim_{n \rightarrow \infty} \mathbb{E}_t [\phi z_T^n] = \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t [\phi z_T] = 1, \quad \mathbb{P}\text{-a.s.},$$

where the last equality follows from the definition of \mathcal{C}'_{tT} . Therefore, the left-hand side in (45), can be bounded from below as follows.

$$(49) \quad \sup_{z \in \mathcal{Z}_t} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T]] \geq \lim_{n \rightarrow \infty} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T^n]]$$

As $\mathbb{E}_t [\phi z_T^n] \leq 1$, $n \in \mathbb{N}$, \mathbb{P} -a.s., and in view of (46), an application of the dominated convergence theorem in the right-hand side of (49) gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T^n]] &= \mathbb{E} \left[\xi \eta \alpha 1_A \lim_{n \rightarrow \infty} \mathbb{E}_t [\phi z_T^n] \right] \\ &= \mathbb{E} \left[\xi \eta \alpha 1_A \operatorname{ess\,sup}_{z \in \mathcal{Z}_t} \mathbb{E}_t [\phi z_T] \right] = \mathbb{E} [\xi \eta \alpha 1_A], \end{aligned}$$

where we used (48) in the last equality. Combining these equalities with (49), we get

$$\sup_{z \in \mathcal{Z}_t} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T]] \geq \mathbb{E} [\xi \eta \alpha 1_A],$$

which together with (47) imply that

$$\sup_{z \in \mathcal{Z}_t} \mathbb{E} [\xi \eta \alpha 1_A \mathbb{E}_t [\phi z_T]] = \mathbb{E} [\xi \eta \alpha 1_A].$$

Plugging this equality into (44), we obtain

$$(50) \quad \begin{aligned} &\sup_{\xi \in \mathcal{C}_t(x)} \sup_{\alpha \in \mathcal{B}_t^+} \sup_{\phi \in \mathcal{C}'_{tT}} \left(\mathbb{E} [(U(\xi \alpha \phi) 1_A) - \sup_{z \in \mathcal{Z}_t} \mathbb{E} [\xi \eta \alpha \phi z_T 1_A]] \right) \\ &= \sup_{\xi \in \mathcal{C}_t(x)} \sup_{\alpha \in \mathcal{B}_t^+} \sup_{\phi \in \mathcal{C}'_{tT}} (\mathbb{E} [(U(\xi \alpha \phi) 1_A) - \mathbb{E} [\xi \eta \alpha 1_A]]). \end{aligned}$$

As $\mathcal{C}_t(x) = \mathcal{B}_t^+ \cup \mathcal{C}_t(x)$, it follows from Lemma 3.8 (see (40)), that in (50) the latter equality can be rewritten as

$$\begin{aligned} &\sup_{\xi \in \mathcal{C}_t(x)} \sup_{\alpha \in \mathcal{B}_t^+} \left(\sup_{\phi \in \mathcal{C}'_{tT}} \mathbb{E} [(U(T, \xi \alpha \phi) 1_A) - \mathbb{E} [\xi \eta \alpha 1_A]] \right) \\ &= \sup_{\xi \in \mathcal{C}_t(x)} \left(\sup_{\phi \in \mathcal{C}_{tT}} \mathbb{E} [(U(T, \xi \phi) 1_A) - \mathbb{E} [\xi \eta 1_A]] \right). \end{aligned}$$

Finally, combining the latter equality with (chains of equalities) (44) and (50), we conclude that

$$\sup_{\xi \in \mathcal{C}_t(x)} \sup_{\rho \in \mathcal{C}_{tT}} \inf_{z \in \mathcal{Z}_t} \mathbb{E} [(U(T, \xi \rho) - \xi \eta \rho z_T) 1_A] = \sup_{\xi \in \mathcal{C}_t(x)} \left(\sup_{\phi \in \mathcal{C}_{tT}} \mathbb{E} [(U(T, \xi \phi) 1_A) - \mathbb{E} [\xi \eta 1_A]] \right),$$

i.e., (43) hold. This completes the proof of the lemma. \square

Corollary 3.13. *Let the condition of Lemma 3.1 hold and $\eta \in \mathcal{N}_t$ be fixed. Then for every $A \in \mathcal{F}_t$, we have*

$$\inf_{z \in \mathcal{Z}_t} \mathbb{E} [V(T, \eta z_T) 1_A] = \sup_{\substack{\xi \in \bigcup_{x>0} \mathcal{C}_t(x) \\ \rho \in \mathcal{C}_{tT}}} \mathbb{E} [(U(T, \xi \rho) - \xi \eta) 1_A].$$

Proof. The assertion of the corollary follows from Lemma 3.9 and 3.12. \square

Lemma 3.14. *Let the condition of Lemma 3.1 hold and $\eta \in \mathcal{N}_t$ be fixed. Then, we have*

$$(51) \quad \operatorname{ess\,sup}_{\substack{\xi \in \bigcup_{x>0} \mathcal{C}_t(x) \\ \rho \in \mathcal{C}_{tT}}} (u(\xi, t, T) - \xi \eta) = v(\eta, t, T).$$

Proof. First, it follows from Lemma 3.6 that for every $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$, $\rho \in \mathcal{C}_{tT}$, and $z \in \mathcal{Z}_t$, we have

$$(52) \quad \mathbb{E}_t [U(T, \xi \rho)] - \xi \eta \leq \mathbb{E}_t [V(T, \eta z_T)].$$

Let us fix $m \in \mathbb{N}$ and set

$$A_m := \left\{ \operatorname{ess\,sup}_{\substack{\xi \in \bigcup_{x>0} \mathcal{C}_t(x) \\ \rho \in \mathcal{C}_{tT}}} (\mathbb{E}_t [U(T, \xi \rho)] - \xi \eta) \leq \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V(T, \eta z_T)] - \frac{1}{m} \right\} \in \mathcal{F}_t.$$

Then, in view of (52), for every $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$, $\rho \in \mathcal{C}_{tT}$, and $z \in \mathcal{Z}_t$, we get

$$\mathbb{E}_t [U(T, \xi \rho)] - \xi \eta \leq \mathbb{E}_t [V(T, \eta z_T)] - \frac{1}{m} 1_{A_m}.$$

Multiplying both sides by 1_{A_m} , we obtain

$$\mathbb{E}_t [U(T, \xi \rho) 1_{A_m}] - \xi \eta 1_{A_m} \leq \mathbb{E}_t [V(T, \eta z_T) 1_{A_m}] - \frac{1}{m} 1_{A_m}.$$

Taking the expectation, we deduce that

$$\mathbb{E} [(U(T, \xi \rho) - \xi \eta) 1_{A_m}] \leq \mathbb{E} [V(T, \eta z_T) 1_{A_m}] - \frac{1}{m} \mathbb{P}[A_m].$$

As the above inequality holds for every $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$, $\rho \in \mathcal{C}_{tT}$, and $z \in \mathcal{Z}_t$, we get

$$\sup_{\substack{\xi \in \bigcup_{x>0} \mathcal{C}_t(x) \\ \rho \in \mathcal{C}_{tT}}} \mathbb{E} [(U(T, \xi \rho) - \xi \eta) 1_A] \leq \inf_{z \in \mathcal{Z}_t} \mathbb{E} [V(T, \eta z_T) 1_A] - \frac{1}{m} \mathbb{P}[A_m].$$

Combining the latter inequality with the assertion of Corollary 3.13, we obtain that $\mathbb{P}[A_m] = 0$. As $m \in \mathbb{N}$ is arbitrary, we conclude that

$$\left\{ \operatorname{ess\,sup}_{\substack{\xi \in \bigcup_{x>0} \mathcal{C}_t(x) \\ \rho \in \mathcal{C}_{tT}}} (\mathbb{E}_t [U(T, \xi \rho)] - \xi \eta) < \operatorname{ess\,inf}_{z \in \mathcal{Z}_t} \mathbb{E}_t [V(T, \eta z_T)] \right\} = \bigcup_{m \in \mathbb{N}} A_m$$

has measure 0. Equivalently, we have

$$\operatorname{ess\,sup}_{\xi \in \bigcup_{x>0} \mathcal{C}_t(x)} \left(\operatorname{ess\,sup}_{\rho \in \mathcal{C}_{tT}} \mathbb{E}_t [U(T, \xi\rho)] - \xi\eta \right) = \operatorname{ess\,inf}_{z \in \mathcal{Z}_T} \mathbb{E}_t [V(T, \eta z_T)], \quad \mathbb{P}\text{-a.s.},$$

and thus (51) holds. This completes the proof of the lemma. \square

3.4. The reverse conjugacy.

Remark 3.15. The reverse conjugacy, or biconjugacy, between u and v is a subject closely related to the Fenchel-Moreau theorem. In the present context, this is a delicate topic, as u and $-v$ are defined as essential suprema, and thus they take values in space of \mathcal{F}_t -measurable extended real-valued functions. Therefore, we cannot apply the standard biconjugacy results from convex analysis, e.g., of Rockafellar [Roc70], directly. The topic of the Fenchel-Moreau theorem for $\bar{\mathbf{L}}^0$ -valued functions has been studied recently, see [DJK]. However, the domains of u and v do not form a dual pair of Banach spaces, and thus these domains do not satisfy the assumptions of [DJK]. Therefore, we have to prove biconjugacy by hand.

Lemma 3.16. *Under the conditions of Lemma 3.14, for every $\xi \in \bigcup_{x>0} \mathcal{C}_t(x)$, we have*

$$u(\xi, t, T) = \operatorname{ess\,inf}_{\eta \in \mathcal{N}_t} (v(\eta, t, T) + \xi\eta).$$

Proof. The proof follows the proof of Lemma 3.14 above with some minor modifications. Therefore, we do not present the complete proof and only highlight the differences. First, we need to pass from Z_t to the closure of the convex and solid hull of $\{z_T : z \in \mathcal{Z}_t\}$, which by the Fatou-convergence-type argument above, similarly to the proof of [KS99, Proposition 3.1], can be constructed as

$$(53) \quad \mathcal{D}_{tT} := \{h \in m\mathcal{F}_t : h \leq z_T, \text{ for some } z \in \mathcal{Z}_t\}.$$

Then, (and this is the main step) we need to show that for a given $\xi \in \bigcup_{x>0} \mathcal{C}(x)$ and $A \in \mathcal{F}_t$, we have

$$\limsup_{y \rightarrow \infty} \sup_{\rho \in \mathcal{C}_{tT}} \inf_{\eta z \in \mathcal{D}_T(y)} \mathbb{E} [(V(T, \eta z) + \eta z \xi \rho) 1_A] \leq \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E} [U(T, \xi \rho) 1_A].$$

The latter can be obtained as follows. Let us consider strictly positive elements $\eta \in \mathcal{D}_t$ and $z \in \mathcal{D}_{tT}$ (where \mathcal{D}_t and \mathcal{D}_{tT} are defined in (13) and (53), respectively) such that

$$(54) \quad \mathbb{E} [V(T, \eta z)] < \infty.$$

The existence of such elements follows from Assumption 2.4 (combined with the argument in [KS03, Proposition 1], where it can be proven that the infimum can be reached over the densities of the equivalent martingale measures under NFLVR, this argument

combined with passing to the numéraire portfolio as a numéraire and stochastic utility, or equivalently, by treating the dual problem as in the proof of [Mos15, Theorem 3.3]).

Further, by Assumption 2.4, we there exist $\xi \in \mathcal{C}_t$ and $\rho \in \mathcal{C}_{tT}$, such that $\mathbb{E}[U(T, \xi\rho)] > -\infty$. Then for such ξ , ρ , η , and z , from the definition of V and since $0 \leq \mathbb{E}[\xi\rho\eta z] < \infty$ (which is a consequence of the respective definitions of \mathcal{D}_t and \mathcal{D}_{tT}), we get

$$(55) \quad -\infty < \mathbb{E}[U(T, \xi\rho)] - y\mathbb{E}[\xi\rho\eta z] \leq \mathbb{E}[V(T, y\eta z)], \quad y > 0,$$

whereas from the monotonicity of $V(T, \cdot)$ and (54), we get

$$(56) \quad \mathbb{E}[V(T, y\eta z)] \leq \mathbb{E}[V(T, \eta z)] < \infty, \quad y \geq 1.$$

Combining (55) and (56), we deduce that (strictly positive elements) $\eta \in \mathcal{N}_t$ and $z \in \mathcal{D}_{tT}$ satisfy

$$-\infty < \mathbb{E}[V(T, y\eta z)] \leq \mathbb{E}[V(T, \eta z)] < \infty, \quad y \geq 1.$$

Next, along the lines of the proof of Lemma 3.7, we define

$$U^n(T, x) := \inf_{0 < y \leq n\eta z} (V(T, y) + xy), \quad (x, \omega) \in (0, \infty) \times \Omega,$$

i.e., from U , we pass to a sequence of truncated stochastic fields (similar to the ones in the proof of [Mos15, Lemma 3.9]). With such U^n 's, as in the proof of Lemma 3.7, we can show that

$$\lim_{n \rightarrow \infty} \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E}[U^n(T, \xi\rho)1_A] = \sup_{\rho \in \mathcal{C}_{tT}} \mathbb{E}[U(T, \xi\rho)1_A].$$

Here the only challenge is to establish the uniform integrability of $(U^n)^+(T, \xi\rho)$, $\rho \in \mathcal{C}_{tT}$. This, however, follows from the following estimates (similar to the ones in [Mos15, Lemma 3.9]): for every $n \geq 2$, one can see that

$$(57) \quad \begin{aligned} U^n(T, x) &\leq U(T, x)1_{\{x > -V'(T, 2\eta z)\}} \\ &\quad + 2(V(T, \eta z) - V(T, 2\eta z))1_{\{x \leq -V'(T, 2\eta z)\}}, \quad (x, \omega) \in (0, \infty) \times \Omega. \end{aligned}$$

(57) implies that for every $n \geq 2$, we have

$$U^n(T, x) \leq \max(U(T, x), 2V(T, \eta z) - V(T, 2\eta z)), \quad (x, \omega) \in (0, \infty) \times \Omega,$$

and the uniform integrability of $(U^n)^+(T, \xi\rho)$, $\rho \in \mathcal{C}_{tT}$, follows from [Mos15, Lemma 3.5]. The remaining parts of the proof of this Lemma are very similar to the proof of Lemma 3.14 and, therefore, they are skipped. \square

Lemma 3.17. *Let the condition of Lemma 3.1 hold, and $\xi \in \mathcal{E}_u$. Then there exists $\hat{\eta}$ in is the closure of \mathcal{N}_t in \mathbb{L}^0 , and $\hat{z} \in \mathcal{Z}_t$, such that*

$$(58) \quad \begin{aligned} u(\xi, t, T) &= \operatorname{ess\,inf}_{\eta \in \mathcal{N}_t} (v(\eta, t, T) + \xi\eta) \\ &= \mathbb{E}[V(T, \hat{\eta}\hat{z}_T) | \mathcal{F}_t] + \hat{\eta}\xi. \end{aligned}$$

Further, let $\hat{\rho}$ be the optimizer to (5) corresponding to ξ . Then we have

$$(59) \quad \mathbb{E}_t[\hat{\rho}\hat{z}_T] = 1, \quad \text{on } \{\xi > 0\}, \quad \mathbb{P} - a.s.$$

and

$$(60) \quad U'(T, \xi\hat{\rho}) = \hat{\eta}\hat{z}_T \quad \text{and} \quad \xi\hat{\rho} = -V'(T, \hat{\eta}\hat{z}_T) \quad \text{on } \{\xi > 0\}, \quad \mathbb{P} - a.s.$$

Proof. Let η^1, η^2 in \mathcal{N}_t and z^1 and z^2 be the corresponding minimizers to (15), then with

$$(61) \quad A := \{\mathbb{E}[V(T, \eta^1 z_T^1) | \mathcal{F}_t] + \eta^1 \xi < \mathbb{E}[V(T, \eta^2 z_T^2) | \mathcal{F}_t] + \eta^2 \xi\} \in \mathcal{F}_t,$$

and via Lemmas 3.4 and 3.5, we get

$$\begin{aligned} \eta &:= \eta^1 1_A + \eta^2 1_{A^c} \in \mathcal{N}_t, \quad \text{as well as} \\ z &:= \frac{1_A \eta^1 z^1 + 1_{A^c} \eta^2 z^2}{\eta} 1_{\{\eta \neq 0\}} + (1_A z^1 + 1_{A^c} z^2) 1_{\{\eta=0\}} \in \mathcal{Z}_t. \end{aligned}$$

In turn, by concavity of V , and using Lemma 3.5, we obtain

$$\begin{aligned} \mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t] + \eta \xi &= (\mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t] + \eta \xi) 1_A + (\mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t] + \eta \xi) 1_{A^c} \\ &= (\mathbb{E}[V(T, \eta z_T 1_A) | \mathcal{F}_t] + \eta^1 \xi) 1_A + (\mathbb{E}[V(T, \eta z_T 1_{A^c}) | \mathcal{F}_t] + \eta^2 \xi) 1_{A^c} \\ &= (\mathbb{E}[V(T, \eta^1 z_T^1) | \mathcal{F}_t] + \eta^1 \xi) 1_A + (\mathbb{E}[V(T, \eta^2 z_T^2) | \mathcal{F}_t] + \eta^2 \xi) 1_{A^c} \\ &= \min(\mathbb{E}[V(T, \eta^1 z_T^1) | \mathcal{F}_t] + \eta^1 \xi, \mathbb{E}[V(T, \eta^2 z_T^2) | \mathcal{F}_t] + \eta^2 \xi), \end{aligned}$$

where in the last equality, we have used (61). Therefore, using [Pha09, Theorem A.2.3, p. 215], we deduce the existence of a sequence $\eta^n z^n$, $n \in \mathbb{N}$, such that

$$(62) \quad \lim_{n \rightarrow \infty} (\mathbb{E}[V(T, \eta^n z_T^n) | \mathcal{F}_t] + \eta^n \xi) = \text{ess inf}_{\eta z \in \mathcal{G}_t} (\mathbb{E}[V(T, \eta z_T) | \mathcal{F}_t] + \eta \xi), \quad \mathbb{P} - a.s.,$$

where we recall that \mathcal{G}_t is defined in (23). By passing to convex combinations, and by applying Lemma 3.5, which asserts that a convex combination of elements of \mathcal{G}_t is an element of \mathcal{G}_t , we may obtain a subsequence of elements of \mathcal{G}_t , which we still denote $\eta^n z^n$, $n \in \mathbb{N}$, and which converges a.s. to a limit, which we denote by ψ .

Let us consider η^n , $n \in \mathbb{N}$. By passing to convex combinations, we may obtain a family of convex weights, $\lambda_k^n \in [0, 1]$, $k \in \{0, \dots, M_n\}$, $n \in \mathbb{N}$, where $M_n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, $\sum_{k=0}^{M_n} \lambda_k^n = 1$, and $\tilde{\eta}^n \triangleq \sum_{k=0}^{M_n} \lambda_k^n \eta^{n+k}$, $n \in \mathbb{N}$, converges along a subsequence to a limit, $\hat{\eta}$, a.s. Applying the same convex weights to $\eta^n z^n$, and passing to the same subsequence², via Lemma 3.5, we get

$$\sum_{k=0}^{M_n} \lambda_k^n \eta^{n+k} z_T^{n+k} = \tilde{\eta}^n z_T^n, \quad n \in \mathbb{N},$$

for some $\tilde{\eta}^n \in \mathcal{N}_t$ and $\tilde{z}^n \in \mathcal{Z}_t$, $n \in \mathbb{N}$. As $\eta^n z_T^n$, $n \in \mathbb{N}$, converges a.s. to ψ , $\tilde{\eta}^n z_T^n$, $n \in \mathbb{N}$, also converges a.s. to the same limit ψ . As both $\tilde{\eta}^n$ and $\tilde{\eta}^n \tilde{z}_T^n$, $n \geq 1$, converge, we

² λ in (24) and (26) (of Lemma 3.5) is the same.

additionally obtain that \tilde{z}_T^n , $n \in \mathbb{N}$, converges to a limit, which we denote \hat{z}_T . Therefore, we have

$$(63) \quad \psi = \lim_{n \rightarrow \infty} \tilde{\eta}^n \tilde{z}_T^n = \hat{\eta} \hat{z}_T.$$

As in the proof of Lemma 3.3, one may show that \hat{z}_T is the terminal value of the element of \mathcal{Z}_t , and thus $\hat{z} \in \mathcal{Z}_t$.

Now, for an arbitrary $\tilde{A} \in \mathcal{F}_t$, convexity of V , Lemma 3.16 and (62) imply that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E} [(V(T, \tilde{\eta}^n \tilde{z}^n) + \tilde{\eta}^n \tilde{z}^n \xi \hat{\rho} - U(T, \xi \hat{\rho})) 1_{\tilde{A}}] \\ & \leq \lim_{n \rightarrow \infty} \mathbb{E} [(V(T, \tilde{\eta}^n \tilde{z}^n) + \tilde{\eta}^n \xi - U(T, \xi \hat{\rho})) 1_{\tilde{A}}] = 0. \end{aligned}$$

From the definition of V , for every $n \geq 1$, we have

$$0 \leq V(T, \tilde{\eta}^n \tilde{z}_T^n) + \tilde{\eta}^n \tilde{z}_T^n \xi \hat{\rho} - U(T, \xi \hat{\rho}), \quad \mathbb{P} - a.s.,$$

Therefore, Fatou's lemma and (63) give

$$(64) \quad \mathbb{E} [(V(T, \hat{\eta} \hat{z}_T) + \hat{\eta} \hat{z}_T \xi \hat{\rho} - U(T, \xi \hat{\rho})) 1_{\tilde{A}}] = 0, \quad \tilde{A} \in \mathcal{F}_t.$$

(58) follows. In turn, (64) together with

$$V(T, \hat{\eta} \hat{z}_T) + \hat{\eta} \hat{z}_T \xi \hat{\rho} - U(T, \xi \hat{\rho}) \geq 0, \quad \mathbb{P} - a.s.$$

implies that

$$V(T, \hat{\eta} \hat{z}_T) + \hat{\eta} \hat{z}_T \xi \hat{\rho} = U(T, \xi \hat{\rho}), \quad \mathbb{P} - a.s.,$$

which, via the definition of V , implies (60). In turn, (59) follows from the polar structure of \mathcal{C}_{tT} and \mathcal{Z}_t . \square

Lemma 3.18. *Let the condition of Lemma 3.1 hold, $\xi \in \mathcal{E}_u$. Then, with $\hat{\rho}$, $\hat{\eta}$, and \hat{z}_T being as in Lemma 3.17, we have: both $U(T, \xi \hat{\rho})$ and $V^+(T, \hat{\eta} \hat{z}_T)$ belong to $\mathbb{L}^1(\mathbb{P})$. Further, there exists A_m , $m \in \mathbb{N}$, an \mathcal{F}_t -measurable partition of Ω , such that $\hat{\eta} 1_{A_m} \in \mathcal{N}_t$ and $V(T, \hat{\eta} \hat{z}_T) 1_{A_m} \in \mathbb{L}^1(\mathbb{P})$, $m \in \mathbb{N}$.*

Proof. First, by [Mos15], $U^+(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P})$. By assumption (6), there exists $\rho \in \mathcal{C}_{tT}$, such that $\mathbb{E}[U^-(T, \xi \rho)] < \infty$. From optimality of $\hat{\rho}$, we get

$$(65) \quad \mathbb{E}_t [U^-(T, \xi \hat{\rho})] \leq \mathbb{E}_t [U^+(T, \xi \hat{\rho})] - \mathbb{E}_t [U^+(T, \xi \rho)] + \mathbb{E}_t [U^-(T, \xi \rho)],$$

As $\mathbb{E}[U^+(T, \xi \rho)] < \infty$ (by [Mos15]) and $\mathbb{E}[U^-(T, \xi \rho)] < \infty$ (by (6)), we deduce from (65) that $U^-(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P})$. Therefore, $U(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P})$.

To show that $V(T, \hat{\eta} \hat{z}_T) \in \mathbb{L}^1(\mathbb{P})$, we use the equality

$$(66) \quad V(T, \hat{\eta} \hat{z}_T) + \xi \hat{\rho} \hat{\eta} \hat{z}_T = U(T, \xi \hat{\rho}), \quad \mathbb{P} - a.s.$$

By taking positive part in (66), we obtain

$$V^+(T, \hat{\eta} \hat{z}_T) \leq (V(T, \hat{\eta} \hat{z}_T) + \xi \hat{\rho} \hat{\eta} \hat{z}_T)^+ = U^+(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P}).$$

Likewise

$$(V(T, \hat{\eta} \hat{z}_T) + \xi \hat{\rho} \hat{z}_T)^- = U^-(T, \xi \hat{\rho}) \in \mathbb{L}^1(\mathbb{P}).$$

To show the existence of an \mathcal{F}_t -measurable partition of Ω , such that $\hat{\eta} 1_{A_m} \in \mathcal{N}_t$, $m \geq 1$, first, we observe that it follows from Assumption 2.3, there exists a strictly positive element $\eta_0 \in \mathcal{N}_t$. Let $(\eta_m)_{m \in \mathbb{N}} \subset \mathcal{N}_t$ be a sequence, which converges to $\hat{\eta}$, \mathbb{P} -a.s. Let

$$A_0 := \emptyset \quad \text{and} \quad A_m := \{\hat{\eta} \leq 2\eta_m + \eta_0\} \setminus \bigcup_{k=0}^{m-1} A_k, \quad m \geq 1.$$

Then, by construction on each A_m , $\hat{\eta} 1_{A_m} \in \mathcal{N}_t$, and A_m 's are disjoint subsets \mathcal{F}_t . Finally, from \mathbb{P} -a.s. convergence of η_m , $m \geq 1$, to $\hat{\eta}$, we deduce that $\mathbb{P} \left[\left(\bigcup_{m \geq 1} A_m \right)^c \right] = 0$. Now, $V(T, \hat{\eta} \hat{z}_T) 1_{A_m} \in \mathbb{L}^1(\mathbb{P})$ by the integrability of $U(\xi \hat{\rho})$ and (66). \square

4. STABILITY ANALYSIS

Here we assume that the 0-market consists of a stock and a bond whose price equals to 1 at all times. Let M , a one-dimensional continuous local martingale, that drives the return process of the stock. Then its dynamics is given by

$$(67) \quad R^0 = M + \lambda \cdot \langle M \rangle$$

for some process λ . Thus, the stock price return (from section 2) has the form (67). For the absence of arbitrage in the sense of Assumption 2.3, finite variation part of the return process has to be absolutely continuous with respect to the quadratic variation of its martingale part, see [HS10]. We consider perturbed family of stock returns of the form

$$(68) \quad R^\varepsilon = (1 + \varepsilon\psi) \cdot (M + \lambda(1 + \varepsilon\theta) \cdot \langle M \rangle)$$

and the bond stays unperturbed.

Remark 4.1. Parametrization of perturbations in the form (68) is closely related to the ones that appear in the literature more often:

$$(69) \quad \begin{aligned} R^\varepsilon &= R^0 + (\varepsilon\psi) \cdot M + (\varepsilon\nu) \cdot \langle M \rangle \\ &= (1 + (\varepsilon\psi)) \cdot M + (\lambda + (\varepsilon\nu)) \cdot \langle M \rangle. \end{aligned}$$

Here $\varepsilon\psi$ amount to perturbations of the martingale part (of volatility in the simplest settings) and $\varepsilon\nu$ to perturbations of the finite variation part (or drift) of the stock return. From (69), one can arrive to (68) by assuming that that ν (linearly) depends on ε , and by making the following reparametrization:

$$(70) \quad \nu = \lambda\nu', \quad \nu' = \psi + \nu'', \quad \nu'' = \theta(1 + \varepsilon\psi).$$

The reason for imposing (68) instead of (69) is a simple structure of integrability Assumption 4.4 and no issues related to differentiation with respect to a parameter under

stochastic integration, as in [Mét82] and [HN84]. We give further details on how to get the stability results with (69) in Remarks 4.2 and 4.3 below.

Now we fix a proportion of the total wealth invested in the risky stock³ and investigate the dynamic behavior of the indirect utility under small perturbations of the drift and volatility of the underlying stock as in (68). To make this mathematically precise, we extend the definitions from section 2 in a natural way as follows:

$$\mathcal{X}^\varepsilon(x) \triangleq \{X \geq 0 : X = x + H \cdot R^\varepsilon \text{ for some } R^\varepsilon\text{-integrable } H\}, \quad x \geq 0, \varepsilon \in \mathbb{R}.$$

For every $\varepsilon \in \mathbb{R}$, the initial wealth $\bar{x} \geq 0$ and a predictable and locally bounded process π are the same, but the corresponding family of the wealth processes alters due to different integrators R^ε , i.e., we consider the family

$$X^{\pi, \varepsilon} = \bar{x} \mathcal{E}(\pi \cdot R^\varepsilon), \quad \varepsilon \in \mathbb{R}.$$

Likewise, for every $\varepsilon \in \mathbb{R}$, the set of wealth processes in $\mathcal{X}^\varepsilon(\bar{x})$, which equal to $X^{\pi, \varepsilon}$ on $[0, t]$, is denoted by $\mathcal{A}^\varepsilon(X_t^{\pi, \varepsilon}, t)$, that is

$$(71) \quad \mathcal{A}^\varepsilon(X_t^{\pi, \varepsilon}, t) := \left\{ \tilde{X} \in \mathcal{X}^\varepsilon(\bar{x}) : \tilde{X}_s = X_s^{\pi, \varepsilon}, \text{ for } s \in [0, t], \mathbb{P}\text{-a.s.} \right\}, \quad \varepsilon \in \mathbb{R}.$$

Finally the family of dynamic indirect utilities associated with π up to T is defined as

$$u^\varepsilon(X_t^{\pi, \varepsilon}, t, T) := \operatorname{ess\,sup}_{\tilde{X} \in \mathcal{A}^\varepsilon(X_t^{\pi, \varepsilon}, t)} \mathbb{E} \left[U \left(T, \tilde{X}_T \right) \mid \mathcal{F}_t \right], \quad t \in [0, T], \varepsilon \in \mathbb{R},$$

for brevity, we denote

$$(72) \quad J_t^{\varepsilon, T} := u^\varepsilon(X_t^{\pi, \varepsilon}, t, T), \quad t \in [0, T], \varepsilon \in \mathbb{R}.$$

For every $\varepsilon \in \mathbb{R}$, let us set

$$(73) \quad \eta^\varepsilon = -\varepsilon \lambda \theta \quad \text{and} \quad L^\varepsilon := \mathcal{E}(\eta^\varepsilon \cdot R^0).$$

Note that $L^\varepsilon \in \mathcal{X}^0(1)$ for every ε . The processes L^ε drive the correction terms in Proposition 4.9.

Remark 4.2. If one chooses perturbations (69), the η^ε and L^ε should be defined as

$$\eta^\varepsilon := \left(\lambda - \frac{\lambda + \varepsilon \nu}{1 + \varepsilon \psi} \right) \quad \text{and} \quad L^\varepsilon := \mathcal{E}(\eta^\varepsilon \cdot R^0).$$

This leads to the same heuristic limiting formulas, but stronger integrability conditions (than the one in Assumption 4.4) are needed to complete proofs.

³Fixing the proportion of wealth invested in stock leads to the admissibility of the corresponding wealth process for every $\varepsilon \neq 0$. If one fixes the number of shares of stock in the portfolio, then for $\varepsilon \neq 0$, under both parametrizations (68) and (69), the associated wealth process can be negative with positive probability, in general.

4.1. **Heuristic derivative of L^ε .** We denote

$$(74) \quad \bar{R} = (\lambda\theta) \cdot R^0.$$

Then

$$L^\varepsilon = \mathcal{E}(\eta^\varepsilon \cdot R^0) = \mathcal{E}(-\varepsilon \cdot \bar{R}) = \exp\left(-\varepsilon \bar{R} - \frac{1}{2}\varepsilon^2 \langle \bar{R} \rangle\right).$$

We set $F := -\bar{R}_T$. Therefore, we get

$$\frac{\partial L_T^\varepsilon}{\partial \varepsilon} = L_T^\varepsilon (-\bar{R}_T - \varepsilon \langle \bar{R} \rangle_T), \quad \frac{\partial L_T^\varepsilon}{\partial \varepsilon} \Big|_{\varepsilon=0} = -\bar{R}_T = F.$$

Similarly, we obtain

$$\frac{1}{\partial L_T^\varepsilon} = \frac{1}{L_T^\varepsilon} (\bar{R}_T + \varepsilon \langle \bar{R} \rangle_T), \quad \frac{1}{\partial L_T^\varepsilon} \Big|_{\varepsilon=0} = \bar{R}_T = -F.$$

Remark 4.3. Here we show that under (69), we get the same heuristic formulas for the derivatives of L^ε . We recall that under (69), the corresponding η^ε and L^ε are given by

$$\eta^\varepsilon := \left(\lambda - \frac{\lambda + \varepsilon\nu}{1 + \varepsilon\psi} \right) \quad \text{and} \quad L^\varepsilon := \mathcal{E}(\eta^\varepsilon \cdot R^0).$$

Then, by direct computations and via the reparametrization formulas (70), we obtain

$$\frac{\partial L_T^\varepsilon}{\partial \varepsilon} \Big|_{\varepsilon=0} = F \quad \text{and} \quad \frac{\partial \frac{1}{L_T^\varepsilon}}{\partial \varepsilon} \Big|_{\varepsilon=0} = -F.$$

Thus, the asymptotic behavior of L^ε 's is similar under both parameterizations of perturbations.

4.2. **Rigorous derivation.** For $t \in [0, T]$, and $\hat{\rho}$ and \hat{z}_T being associated with $X_t^{\pi,0}$ (via Lemma 3.17), let us define the probability measure \mathbb{R}^t as

$$(75) \quad \frac{d\mathbb{R}^t}{d\mathbb{P}} \Big|_{\mathcal{F}_s} := 1_{\{s \in [0, t]\}} + \mathbb{E}_s[\hat{\rho} \hat{z}_T] 1_{\{s > t\}}, \quad s \in [0, T].$$

Note that Lemma 3.17 (via (59)) ensures that \mathbb{R}^t is a probability measure. Finally, we suppose that the perturbations are sufficiently bounded in the following sense.

Assumption 4.4. The process $|\psi|$ is uniformly bounded from above and there exists a constant $\bar{c} > 0$, such that

$$\exp(\bar{c}(|\bar{R}_T| + \langle \bar{R} \rangle_T)) \in \mathbb{L}^1(\mathbb{R}^t), \quad t \in [0, T],$$

where the probability measure \mathbb{R}^t and the process \bar{R} are defined in (75) and (74), respectively.

Remark 4.5. If the relative risk aversion is greater than $1 + \varepsilon''$, uniformly in ω , for some $\varepsilon'' > 0$, Assumption 4.4 holds if $|\bar{R}_T| + \langle \bar{R} \rangle_T$ have some exponential moment under \mathbb{P} . This can be shown following the [MS19, Remark 3.5], it is therefore skipped.

Remark 4.6. Under assumption 4.4, one can show that there exists $\delta_0 > 0$, such that for every $\delta \in (-\delta_0, \delta_0)$, both Assumptions 2.3 and 2.4 hold. Therefore, characterizations of both u and v obtained above apply.

We also need to strengthen the assumptions on U .

Assumption 4.7. There exist positive constants γ_1, γ_2 , and δ_0 , such that for every $x > 0$ and $z \in (0, \delta_0]$, we have

$$U'(T, zx) \leq z^{-\gamma_1} U'(T, x) \quad \text{and} \quad U'(T, x/z) \leq z^{\gamma_2} U'(T, x).$$

Remark 4.8. Assumption 4.7 holds if either relative risk aversion or relative risk tolerance are bounded away from 0 and ∞ , see e.g., [MS19, Lemma 5.12].

Theorem 4.9. Let $T > 0$ be fixed and suppose that Assumptions 2.3, 2.4, 4.4, and 4.7 hold as well as $\mathbb{E}[U^-(T, X_T^{\pi,0})] < \infty$. Then, we have

$$(76) \quad \mathbb{P} - \lim_{\varepsilon \rightarrow 0} J_t^{\varepsilon, T} = J_t^{0, T}, \quad t \in [0, T].$$

Further, for each $t \in [0, T]$, with $\hat{\eta}_t$ being associated to $X_t^{\pi,0}$ via (60) and with

$$(77) \quad M^R := R^0 - \pi \cdot \langle R^0 \rangle,$$

we have

$$(78) \quad \mathbb{P} - \lim_{\varepsilon \rightarrow 0} \frac{J_t^{\varepsilon, T} - J_t^{0, T}}{\varepsilon} = X_t^{\pi,0} \hat{\eta}_t \left((\psi\pi - \lambda\theta) \cdot M_t^R + \mathbb{E}_t^{\mathbb{R}^t} [(\lambda\theta) \cdot R_T^0] \right).$$

Proof. Let us fix $t \in [0, T]$. Via a direct application of Ito's formula, one can show that

$$\begin{aligned} X_t^{\pi, \varepsilon} &= x \mathcal{E}(\pi \cdot R^\varepsilon)_t = x \mathcal{E}(\pi \cdot R^0)_t \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t}{\mathcal{E}(-\varepsilon(\lambda\theta) \cdot R^0)_t} \\ &= X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_t^\varepsilon}. \end{aligned}$$

This implies that

$$u^\varepsilon(X_t^{\pi, \varepsilon}, t, T) \geq \mathbb{E}_t \left[U \left(T, X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{L_T^\varepsilon} \right) \right], \quad \mathbb{P} - a.s.,$$

where $\hat{\rho}_t$ is the optimizer to (5) corresponding to $\xi = X_t^{\pi,0}$ and the base model of the stock price. For $\varepsilon > 0$, let us consider

$$\begin{aligned} & \frac{1}{\varepsilon} (u^\varepsilon(X_t^{\pi, \varepsilon}, t, T) - u^0(X_t^{\pi,0}, t, T)) \\ & \geq \frac{1}{\varepsilon} \left(\mathbb{E}_t \left[U \left(T, X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{L_T^\varepsilon} \right) \right] - \mathbb{E}_t [U(T, X_t^{\pi,0} \hat{\rho}_t)] \right) \\ & = \frac{1}{\varepsilon} \left(\mathbb{E}_t \left[\int_0^\varepsilon U' \left(T, X_t^{\pi,0} \hat{\rho}_t \frac{\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_T^\beta} \right) X_t^{\pi,0} \hat{\rho}_t \frac{\partial}{\partial \beta} \left(\frac{\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_T^\beta} \right) d\beta \right] \right). \end{aligned}$$

From Assumption 4.7, we get

$$(79) \quad \begin{aligned} & U' \left(T, X_t^{\pi,0} \hat{\rho}_t \frac{\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_T^\beta} \right) \\ & \leq U'(T, X_t^{\pi,0} \hat{\rho}_t) \max \left(\left(\frac{\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t}{L_T^\beta} \right)^{-\gamma_1}, \delta_0^{-\gamma_1} \right). \end{aligned}$$

We recall that, in general, see e.g., [Shi84, Definition 1, p. 211], the definition of conditional expectation does not require integrability. This, in particular, allows to circumvent any integrability conditions on $\mathcal{E}(\beta(\psi\pi - \lambda\theta) \cdot M^R)_t$. Therefore, from (79), following [MS19, Lemma 5.14], and using Assumption 4.4, we obtain

$$(80) \quad \begin{aligned} & \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (u^\varepsilon(X_t^{\pi,\varepsilon}, t, T) - u^0(X_t^{\pi,0}, t, T)) \\ & \geq \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left(\mathbb{E}_t \left[U \left(T, X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{\mathcal{E}(-\varepsilon(\lambda\theta) \cdot R^0)_T} \right) \right] - \mathbb{E}_t [U(T, X_t^{\pi,0} \hat{\rho}_t)] \right) \\ & = \mathbb{E}_t [U'(T, X_t^{\pi,0} \hat{\rho}_t) X_t^{\pi,0} \hat{\rho}_t ((\psi\pi - \lambda\theta) \cdot M_t^R + (\lambda\theta) \cdot R_T^0)] \\ & = \mathbb{E}_t [X_t^{\pi,0} \hat{\rho}_t \hat{\eta}_t \hat{z}_{tT} ((\psi\pi - \lambda\theta) \cdot M_t^R + (\lambda\theta) \cdot R_T^0)] \\ & = X_t^{\pi,0} \hat{\eta}_t \left((\psi\pi - \lambda\theta) \cdot M_t^R + \mathbb{E}_t^{\mathbb{R}^t} [(\lambda\theta) \cdot R_T^0] \right), \end{aligned}$$

where $\hat{\eta}_t \hat{z}_{tT}$ are given via Lemma 3.17.

Fom the definition of the conjugate field V , we have

$$U \left(T, X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{\mathcal{E}(-\varepsilon(\lambda\theta) \cdot R^0)_T} \right) \leq V(T, \hat{\eta}_t \hat{z}_{tT} L^\varepsilon) + X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{\mathcal{E}(-\varepsilon(\lambda\theta) \cdot R^0)_T} \hat{\eta}_t \hat{z}_{tT} L^\varepsilon,$$

and thus combining with the result of Lemma 3.17, we get

$$(81) \quad \begin{aligned} & U \left(T, X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{\mathcal{E}(-\varepsilon(\lambda\theta) \cdot R^0)_T} \right) - U(T, X_t^{\pi,0} \hat{\rho}_t) \\ & \leq V(T, \hat{\eta}_t \hat{z}_{tT} L^\varepsilon) - V(T, \hat{\eta}_t \hat{z}_{tT}) + X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{\mathcal{E}(-\varepsilon(\lambda\theta) \cdot R^0)_T} \hat{\eta}_t \hat{z}_{tT} L^\varepsilon - X_t^{\pi,0} \hat{\rho}_t \hat{\eta}_t \hat{z}_{tT}, \end{aligned}$$

Using Assumptions 4.4 and 4.7, and by passing into an \mathcal{F}_t -measurable partition of Ω as is Lemma 3.18, we get

$$(82) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E}_t [V(T, \hat{\eta}_t \hat{z}_{tT} L^\varepsilon) - V(T, \hat{\eta}_t \hat{z}_{tT})] = X_t^{\pi,0} \hat{\eta}_t \mathbb{E}_t^{\mathbb{R}^t} [(\lambda\theta) \cdot R_T^0],$$

and

$$(83) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left(\mathbb{E}_t \left[X_t^{\pi,0} \frac{\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t \hat{\rho}_t}{\mathcal{E}(-\varepsilon(\lambda\theta) \cdot R^0)_T} \hat{\eta}_t \hat{z}_{tT} L^\varepsilon - X_t^{\pi,0} \hat{\rho}_t \hat{\eta}_t \hat{z}_{tT} \right] \right) \\ & = X_t^{\pi,0} \hat{\eta}_t \lim_{\varepsilon \rightarrow 0} \left(\mathbb{E}_t^{\mathbb{R}^t} [\mathcal{E}(\varepsilon(\psi\pi - \lambda\theta) \cdot M^R)_t - 1] \right) = X_t^{\pi,0} \hat{\eta}_t (\psi\pi - \lambda\theta) \cdot M_t^R. \end{aligned}$$

From (80) and (81) using (82) and (83), we deduce that

$$\mathbb{P} - \lim_{\varepsilon \downarrow 0} \frac{u^\varepsilon(X_t^{\pi, \varepsilon}, t, T) - u^0(X_t^{\pi, 0}, t, T)}{\varepsilon} = X_t^{\pi, 0} \hat{\eta} \left((\psi\pi - \lambda\theta) \cdot M_t^R + \mathbb{E}_t^{\mathbb{R}^t} [(\lambda\theta) \cdot R_T^0] \right).$$

Similarly, we can show that

$$\mathbb{P} - \lim_{\varepsilon \uparrow 0} \frac{u^\varepsilon(X_t^{\pi, \varepsilon}, t, T) - u^0(X_t^{\pi, 0}, t, T)}{\varepsilon} = X_t^{\pi, 0} \hat{\eta} \left((\psi\pi - \lambda\theta) \cdot M_t^R + \mathbb{E}_t^{\mathbb{R}^t} [(\lambda\theta) \cdot R_T^0] \right).$$

(76) and (78) follow. \square

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