DUALITY FOR OPTIMAL STOPPING IN CONTINUOUS TIME

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Abstract. For the Lagrange-Mayer formulation of the optimal stopping problem, this paper contains the construction of the dual problem, which is of mixed optimal stopping and singular stochastic control type. The idea of the construction and the structure of the proof rely on the results in mathematical finance, particularly on a connection to optimal investment problems with intermediate consumption and discretionary stopping. The results include the dual characterization of the value function and related results, dual representation of the Snell envelope in the Mayer case, and counterexamples that motivate some technical assumptions.

1. Introduction

Methods based on partial differential equations and partial integral differential equations (PDEs), particularly free boundary problems, have become very powerful in studying the optimal stopping problem; see, e.g., the seminal book [PS06]. While the theory of partial differential equations is a very powerful analytical tool in optimal stopping, one usually has to prove probabilistically the verification theorem in order to be able to use such theory. More recently, an approach to optimal stopping based on backward stochastic differential equations (BSDEs), particularly on reflected BSDEs, has been developed, see, e.g., [EKKP97]. It typically relies on strong technical assumptions, for example, the square-integrability of the integrands in the formulation of the optimal stopping problem in [EKKP97].

The proposed approach relies on stochastic analytic techniques. In contrast to the PDE and BSDE-based approaches to optimal stopping, *stochastic analytic methods usually allow the development of the mathematical theory under minimal assumptions.* In particular, no Markovian structure of the underlying problem are needed. Combining stochastic analytic methods with the duality approach allows for multiple developments of the optimal stopping topic, as presented below. In the context of problems in financial mathematics, this approach has been extensively developed for obtaining existence, uniqueness, stability, and asymptotic analysis results in [MM24, Mos15, Mos21, MS20, MS19], which provide foundations for the projects described below.

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The results of this paper open multiple research directions, including stability and asymptotic expansion results via primal-dual techniques and numerical methods for solving optimal stopping problems in Lagrange, Mayer (in the terminology of [PS06]), or Lagrange-Mayer formulations. Besides the novelty in stochastic analysis, applications also include financial mathematics, where optimal stopping with running cost terms arises in the pricing and hedging of financial derivatives with dividends or coupons and a random termination time, and applications to statistics, such as stochastic disorder problems. In the context of the Epstein-Zin problem in financial mathematics, once the duality is established (as in [MX18]), stability analysis can be developed as in [MM24].

The remainder of this paper is organized as follows. Section 2 contains the problem formulation and an intuitive construction of the dual problem. In Section 3, we state and prove the dual characterization of the value function and related results, dual representation of the Snell envelope in the Mayer case, and counterexamples that motivate some technical assumptions. Section 4 contains an auxiliary result needed for the proofs.

2. Problem Formulation

For simplicity of the exposition, we consider a Brownian setting, while extensions to processes with jumps are possible. Let $(W^1, \dots, W^d)$ be a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the augmentation of the filtration generated by this Brownian motion $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_0$ is trivial.

First, for simplicity of the presentation, let us consider a positive local martingale Z starting from 1. Let $V^1(t, x)$ be a function, such that $V^1(t, \cdot)$ is convex on $(0, \infty)$ for every $t \geq 0$ and V^2 is also a convex function on $(0, \infty)$. Denoting by $\mathcal{T}_0$ the set of stopping times, one can formulate the optimal stopping problem in the Lagrange-Mayer form as finding $\tau \in \mathcal{T}_0$ that minimizes the objective

$$\mathbb{E}\left[\int_0^\tau V^1(t, Z_t)dt + V^2(\tau, Z_\tau)\right]$$

over $\tau \in \mathcal{T}_0$. That is, we consider the following optimal stopping problem

$$\inf_{\tau\in\mathcal{T}_0} \mathbb{E}\left[\int_0^\tau V^1(t, Z_t)dt + V^2(\tau, Z_\tau)\right]. \tag{1}$$

2.1. **Discussion of the existing approaches.** Problems of the form (1) can be found in [Shi17] and [EM23], among others. *To the best of the authors' knowledge, the duality for optimal stopping is only developed in the Mayer form*, that is, when $V^1 \equiv 0$, see [DK94] and [PS06] and, for financial applications, [Jam07] and [Rog02], as well as a more recent overview in [Bel13]. In (1), while the running cost term $\int_0^\tau V^1(t, Z_t)dt$ can be absorbed in the terminal reward term $V^2(\tau, Z_\tau)$ by an increase of dimensionality[1], the structure of the running cost term gets hidden, and so, one cannot use the results in [DK94] to obtain the dual characterization

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[1] [FR75, Chapter II.4] contains an illuminating presentation of this method.

as in Theorem 3.7 below. This is somewhat similar to the Hamilton-Jacobi-Belman equations for optimal control problems, where the running cost term enters the differential equation and is represented by the convex conjugates of $V^1(t,\cdot)$'s in the dual equation.

2.2. **Difficulties in the construction of the dual problem. The lack of convexity for the set of stopping times $\mathcal{T}_0$ is the main issue in the construction of the dual problem.** The proposed approach allows to bypass this issue. Additional difficulties contained in finding the sharp conditions for proving the duality theorems and developing the consequences of the duality approach, as explained in section 3 below. The conditions assumptions below are possible, but filling the proof details might result in change of Assumptions 3.1, 3.2, and 3.4.

2.3. **Representation of the objective in** (1)**.** For every $\tau \in \mathcal{T}_0$, let us introduce a stochastic field

$$V^\tau(t,\omega,z) = V^1(t,z)1_{\{t<\tau(\omega)\}} + V^2(t,z)1_{\{t\geq\tau(\omega)\}}, \quad (t,\omega,z) \in [0,\infty)\times\Omega\times(0,\infty),$$

where the symbol ω is omitted for brevity and a stochastic clock κ^τ associated with a given stopping time τ as

$$\kappa^\tau_t = t1_{\{t<\tau\}} + (\tau+1)1_{\{t\geq\tau\}}, \quad t\geq 0. \tag{2}$$

Then, we can restate (1) as

$$\inf_{\tau\in\mathcal{T}_0} \mathbb{E}\left[\int_0^\tau V^\tau(t,Z_t)d\kappa^\tau_t\right].$$

Next, let us introduce the value function that we parametrize by the initial value of the process Z and consider the value function of the form

$$v(z) = \inf_{\tau\in\mathcal{T}_0} \mathbb{E}\left[\int_0^\tau V^\tau(t,zZ_t)d\kappa^\tau_t\right], \quad z>0. \tag{3}$$

This is the Lagrange-Mayer formulation of the optimal stopping problem. By considering $V^1\equiv 0$ or $V^2\equiv 0$, we can recover the Lagrange or Mayer formulation of the optimal stopping problem, respectively, in the terminology of [PS06].

We observe that with

$$v^\tau(z) := \mathbb{E}\left[\int_0^\tau V^\tau(t,zZ_t)d\kappa^\tau_t\right], \quad z>0, \quad \tau\in\mathcal{T}_0, \tag{4}$$

we can restate (3) as

$$v^\tau(z) = \inf_{\tau\in\mathcal{T}_0} v^\tau(z), \quad z>0, \quad \tau\in\mathcal{T}_0. \tag{5}$$

2.4. **Construction of the dual problem.** Using the martingale representation theorem, with $\mathcal{E}$ denoting the stochastic exponential, heuristically, we can represent Z as $Z_0\mathcal{E}(\alpha\cdot W)$ for some W-integrable d-dimensional process α. Let us consider a d-dimensional stochastic process R, whose dynamics is component-wise given by

$$R^i_t = -\int_0^\cdot \alpha^i_s ds + W^i, \quad i = 1, \dots, d.$$

Next, let us consider a set of nonnegative stochastic integrals with respect to d-dimensional stochastic process R. We can represent them as

$$\mathcal{X}(x) = \left\{X \geq 0 : \ X = x\mathcal{E}\left(\pi\cdot R\right), \ for\ some\ R\text{-}integrable\ process\ \pi\right\}, \quad x \geq 0.$$

For every $\tau \in \mathcal{T}_0$, let us also consider a set of nonnegative optional processes c, whose stochastic integrals with respect to κ^τ can be dominated by elements of $\mathcal{X}(x)$, $x\geq 0$, that is

$$\mathcal{A}^\tau(x) = \left\{c \geq 0 : \int_0^t c_s d\kappa^\tau_s \leq X_t, \ t\geq 0, \ for\ some\ X\in\mathcal{X}(x)\right\}, \quad x\geq 0. \tag{6}$$

Let us also define a convex conjugate of V^τ as

$$U^\tau(t,\omega,x) := \inf_{z>0}\left(V^\tau(t,\omega,z) + zx\right), \quad (t,\omega)\in[0,\infty)\times\Omega. \tag{7}$$

Compared to [Mos15], U^τ is a stochastic utility field in the terminology there. The possible (joint) measurability conditions on U^τ can also be found in [Mos15]. Setting

$$U^1(t,x) = \inf_{z>0}(V^1(t,z)+zx), \ (t,x)\in[0,\infty)\times(0,\infty), \ U^2(x) = \inf_{z>0}(V^2(z)+zx), \ x\in(0,\infty), \tag{8}$$

one can see that

$$U^\tau(t,\omega,x) = U^1(t,x)1_{\{t<\tau(\omega)\}} + U^2(t,x)1_{\{t\geq\tau(\omega)\}}, \quad (t,\omega,x)\in[0,\infty)\times\Omega\times(0,\infty),$$

Now, we can define

$$u^\tau(x) = \sup_{c\in\mathcal{A}^\tau(x)} \mathbb{E}\left[\int_0^\infty U^\tau(t,\omega,c_t)d\kappa^\tau_t\right], \quad x\geq 0, \tag{9}$$

and

$$u(x) = \inf_{\tau\in\mathcal{T}_0} u^\tau(x) = \inf_{\tau\in\mathcal{T}_0}\sup_{c\in\mathcal{A}^\tau(x)} \mathbb{E}\left[\int_0^\infty U^\tau(t,\omega,c_t)d\kappa^\tau_t\right], \quad x>0,$$

By rewriting U^τ in terms of U^1 and U^2 and supressing the symbol ω, we can further restate $u(x)$ as

$$u(x) = \inf_{\tau\in\mathcal{T}_0}\sup_{c\in\mathcal{A}^\tau(x)} \mathbb{E}\left[\int_0^\tau U^1(t,c_t)dt + U^2(\tau,c_\tau)\right], \quad x>0, \tag{10}$$

where, for every $t\geq 0$, $U^1(t,\cdot)$ and U^2 are convex conjugates of $V^1(t,\cdot)$ and V^2, respectively. Problem (10) is of mixed singular stochastic control and optimal stopping type and is closely related to the one in [KW01]. One can show that

$$u(x) \leq v(z) + xz, \quad (x,z)\in(0,\infty)\times(0,\infty). \tag{11}$$

Let us fix $z > 0$ and suppose the existence of the optimal $\tau^*(z)$ for (1). Sufficient conditions for the existence of the optimal stopping time are contained in [KS98, Theorem D.12] and [PS06, Theorem 2.7 and Corollary 2.9] Then, by the results [Mos15], for an appropriate $x^*(z) > 0$, we have

$$u^{\tau^*(z)}(x^*(z)) = v^{\tau^*(z)}(z) + x^*(z)z. \tag{12}$$

Since additionally

$$u^{\tau^*(z)}(x) \leq v^{\tau^*(z)}(z) + xz, \quad \textit{for every} \quad (x, z) \in (0, \infty) \times (0, \infty), \tag{13}$$

(12) and (13) give duality relation between u^{τ^*} and v^{τ^*}. In view of (11), one can call u in (10) the dual value function to v in (1).

2.5. **Intuition behind the construction of the dual problem.** The intuition behind this construction comes from the financial mathematics, where R^i can be thought of as cumulative returns of the risky asset on a *complete market*[2], on which additionally there is one riskless traded asses, whose price process equals to 1 at all times, this corresponds to discounting the risky assets in terms of the units of the riskless asset. Then, $\mathcal{X}(x)$ denotes the set of nonnegative wealth processes starting from the initial wealth x. Such nonnegative wealth processes are often called *admissible*, see [DS06], among many other references. The elements of $\mathcal{X}(x)$ represent the wealth processes obtained by dynamic self-financing trading in d risky and single riskless assets. The convex conjugates of V's in the sense of (7) are utility stochastic fields in the terminology of [KK21, Section 3.4], [KŽ03], and [Mos15], and the sets $\mathcal{A}^\tau(x)$'s, that is, the admissible sets for the dual domain, are the sets of consumption rates that are associated with the stochastic consumption clock κ^τ defined in (2) and that are financeable by the dynamic trading in the market with the initial wealth x. Thus, the formulation of the dual problem in (10) has an interpretation in financial mathematics and, in particular, is closely related to the [Mos15].

2.6. **Extension to the case when Z is not a martingale.** In this section, let us suppose that Z is a positive process on the same probability space with the same filtration as above. Let us suppose the dynamics of Z is given by

$$dZ_t = Z_t \left(\mu_t dt + \sigma_t dW\right), \quad Z_0 = const > 0, \tag{14}$$

where μ_t and σ_t are measurable adapted processes, μ is one-dimensional, and σ is a d-dimensional process whose every component is strictly positive $\mathbb{P}$–a.s., such that the strong solution to (14) exists and is unique.

For the duality approach to work, it is important to ensure that there exists a change of measure, locally in time, under which process Z is a local martingale. Let us suppose that the objective is still given by (1) with such a process Z. With σ^1 denoting the first component

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[2] A financial market is called complete if every bounded contingent claim (bounded random variable) can be replicated by a dynamic trading strategy.

of σ, assuming that $\frac{\mu}{\sigma^1}$ is bounded (or sufficiently integrable), we can define a *new probability measure* $\tilde{\mathbb{P}}$, whose restrictions to $\mathcal{F}_t$'s, $t \geq 0$, are represented by

$$M_t = \frac{d\tilde{\mathbb{P}}_{\mathcal{F}_t}}{d\mathbb{P}_{\mathcal{F}_t}} = \mathcal{E}\left(-\int_0^\cdot \frac{\mu_s}{\sigma^1_s} dW^1_s\right)_t, \quad t \geq 0. \tag{15}$$

Defining $\tilde{\mathbb{P}}$ through restrictions to $\mathcal{F}_t$ is weaker than requiring the existence of $\tilde{\mathbb{P}}$ on the underlying probability space and requiring $\tilde{\mathbb{P}}$ to be equivalent to $\mathbb{P}$. An illuminating example of the case when restrictions to $\mathcal{F}_t$, as in (15) exist, but there is no equivalent probability measure on $[0,\infty)$ is in the Nobel Prize winner in economics, Merton's seminal work [Mer69] (one the infinite! time horizon), in general, and this motivated different notions of no-arbitrage in mathematical finance, see [KK07].

3. Main results

Assumption 3.1. Let us suppose that there exists a martingale M such that (15) holds and

$$\int_0^\cdot \left(\int_0^{s-} V^1(t, zZ_t)dt\right) d\left(\frac{1}{M_s}\right)$$

is a uniformly integrable $\tilde{\mathbb{P}}$ martingale, $z > 0$.

Let us introduce

$$\begin{aligned} \tilde{V}^1(t,\omega,z) &:= \frac{V^1(t,z)}{M_t(\omega)}, \quad (t,\omega,z) \in [0,\infty) \times \Omega \times (0,\infty), \quad \textit{and} \\ V^2(t,\omega,z) &:= \frac{V^2(t,z)}{M_t(\omega)}, \quad (t,\omega,z) \in [0,\infty) \times \Omega \times (0,\infty). \end{aligned} \tag{16}$$

Next, for every $\tau \in \mathcal{T}_0$, with

$$\tilde{V}^\tau(t,\omega,z) := \tilde{V}^1(t,z)1_{\{t<\tau(\omega)\}} + \tilde{V}^2(t,z)1_{\{t\geq\tau(\omega)\}}, \quad (t,\omega,z) \in [0,\infty) \times \Omega \times (0,\infty),$$

we can set $\tilde{U}$'s as convex conjugates to $\tilde{V}$'s, similarly to (7), that is as

$$\tilde{U}^\tau(t,\omega,x) := \inf_{z>0}\left(\tilde{V}^\tau(t,\omega,z) + zx\right), \quad (t,\omega,x) \in [0,\infty) \times \Omega \times (0,\infty). \tag{17}$$

We note that $\tilde{U}^\tau$ can be characterized as follows. With $\tilde{V}$'s defined in (16), we can set

$$\begin{aligned} \tilde{U}^1(t,\omega,x) &= \inf_{z>0}\left(\tilde{V}^1(t,\omega,z) + xz\right), \quad (t,\omega,z) \in [0,\infty) \times \Omega \times (0,\infty), \quad \textit{and} \\ \tilde{U}^2(t,\omega,z) &= \inf_{z>0}\left(\tilde{V}^2(t,\omega,z) + xy\right), \quad (t,\omega,z) \in [0,\infty) \times \Omega \times (0,\infty). \end{aligned} \tag{18}$$

Then, one can see that $\tilde{U}^\tau$ from (17) is related to $\tilde{U}^1$ and $\tilde{U}^2$ from (18) via

$$\tilde{U}^\tau(t,\omega,x) = \tilde{U}^1(t,\omega,x)1_{\{t<\tau\}} + \tilde{U}^2(t,\omega,x)1_{\{t\geq\tau\}}, \quad (t,\omega,x) \in [0,\infty) \times \Omega \times (0,\infty). \tag{19}$$

Now, we can state the dual problem entirely similarly to (10), which, in the case when Z is not a local martingale, becomes

$$\begin{aligned} u(x) &= \inf_{\tau\in\mathcal{T}_0} u^\tau(x) \\ &= \inf_{\tau\in\mathcal{T}_0} \sup_{c\in\mathcal{A}^\tau(x)} \tilde{\mathbb{E}}\left[\int_0^\infty \tilde{U}^\tau(t,\omega,c_t)d\kappa^\tau_t\right] \\ &= \inf_{\tau\in\mathcal{T}_0} \sup_{c\in\mathcal{A}^\tau(x)} \tilde{\mathbb{E}}\left[\int_0^\tau \tilde{U}^1(t,c_t)dt + \tilde{U}^2(\tau,c_\tau)\right], \quad x>0. \end{aligned} \tag{20}$$

Conjugacy and duality between u and v can now be obtained similarly to section 2.4. To prove the duality results rigorously, one needs the finiteness of v and u defined in (1) and (10).

Assumption 3.2. Let us suppose that

$$v(y) < \infty \quad \textit{and} \quad u(y) > -\infty, \quad y>0. \tag{21}$$

Remark 3.3. The finiteness of the value functions in a form similar to (21) is central to the standard conclusions of the utility maximization theory; see [Mos15].

For the regularity of the cost functions V^1 and V^2 in (1), we suppose the dual version of the Inada-type condition in the following sense.

Assumption 3.4. We suppose that

$$V^1(t,z) = e^{\delta(t)}\frac{z^{-q}}{q}, \quad \textit{and} \quad V^2(t,z) = c(t)\frac{z^{-q}}{q}, \quad z>0, \quad \textit{for a fixed} \qquad q\in(-1,\infty)\backslash\{0\}.$$

Remark 3.5. Under Assumption 3.4, we have

$$U^1(t,z) = e^{\delta(t)}\frac{x^p}{p}, \quad \textit{and} \quad U^2(z) = c(t)\frac{x^p}{p}, \quad x>0, \quad p=\frac{q}{1+q}.$$

Remark 3.6. The assertions of Theorem 3.7 also hold if change inf to sup in (1), suppose that the functions $V^1(t,\cdot)$, $t>0$, and V^2 are strictly decreasing, strictly concave and differentiable on $(0,\infty)$. With $(V^1)'$ and $(U^1)'$ denote the partial derivative of V^1 and U^1, respectively, with respect to the second argument, we suppose that

$$\begin{aligned} &\lim_{z\downarrow 0}(V^1)'(t,z) = -\infty, \quad \textit{and} \quad \lim_{z\uparrow\infty}(V^1)'(t,z) = 0, \quad t\geq 0, \\ &\lim_{z\downarrow 0}(V^2)'(t,z) = -\infty \quad \textit{and} \quad \lim_{z\uparrow\infty}(V^2)'(t,z) = 0, \quad t\geq 0, \end{aligned} \tag{22}$$

and keep the remaining assumptions unchanged.

One can see that Assumption 3.4 ensures that $-U^1(t,\cdot)$, $t\geq 0$, and $-U^2$ defined in (8) also satisfy Assumption 3.4. With an additional condition on the existence of the optimal stopping time for (1) to be yet determined, the existence of the optimal stopping time for (1) is obtained in multiple particular cases of (1), see, e.g., [EM23] where the optimal solution is obtained via PDE methods, see also [PS06]. In the Mayer formulation of (1), that is, with $V^1\equiv 0$, the existence of optimal stopping rules beyond the Markovian settings in terms of the Snell envelopes contained in [KS98, Theorem D.12].

Theorem 3.7. *Let us suppose that Assumptions 3.1, 3.2, and 3.4 hold and for every $x > 0$, for which there exists an optimal stopping time $\sigma^*(x)$ for* (10). *Then, we have*

(i) both $v(y) \in \mathbb{R}$ and $u(y) \in \mathbb{R}$, $y > 0$. v and $-u$ are strictly concave and continuously differentiable on $(0, \infty)$.

(ii) there is a conjugacy between v and u in the sense that

$$v(z) = \sup_{x>0} \left(u(x) - zx\right), \quad z > 0,$$

and, for every $z > 0$, there exists a unique $x^*(z) > 0$, such that

$$v(z) = u(x^*(z)) + x^*(z)z.$$

(iii) for every $z > 0$, the optimal stopping time exists for (3), *$\tau^*(z)$, and is unique. In fact, $\tau^*(z) = \sigma^*(x)$ and does not depend on $z > 0$. For* (10)*, the optimal control $c^*(x)$, exists and is unique and it also does not depend on $x > 0$ up to a multiplicative constant, that is $c^*(x) = xc^*(1)$.*

(iv) for every $z > 0$ and $x^*(z)$ of item (ii), the optimizer $c^*(x^*(z))$ to (10) *is related to Z via*

$$zZ_t = (U^1)'(t, c_t^*(x^*(z)))1_{\{t<\sigma^*\}} + (U^2)'(t, c_t^*(x^*(z)))1_{\{t\geq\sigma^*\}}, \quad t \geq 0,$$

$$c_t^*(x^*(z)) = -(V^1)'(t, zZ_t)1_{\{t<\sigma^*\}} - (V^2)'(t, zZ_t)1_{\{t\geq\sigma^*\}}, \quad t \geq 0,$$

We begin the proof of Theorem 3.7 with the following lemma.

Lemma 3.8. *Under the assumptions of Theorem 3.7, we can restate* (1) *(under $\tilde{\mathbb{P}}$) as*

$$v(z) = \inf_{\tau\in\mathcal{T}_0} \tilde{\mathbb{E}}\left[\int_0^\tau \tilde{V}^1(t, zZ_t)dt + \tilde{V}^2(zZ_\tau)\right], \quad z > 0, \tag{23}$$

where, in the latter expression, the symbol ω is suppressed as it is common in the literature; see, e.g., [KK21, Section 3.4].

Proof. Under Assumption 3.1, $\frac{1}{M}$ is a strictly positive uniformly integrable martingale under $\tilde{\mathbb{P}}$, so that $\tilde{\mathbb{P}}$ can be defined on $[0, \infty)$ without localization, then, for every fixed stopping time $\tau \in \mathcal{T}_0$ and $z > 0$, using integration by parts formula in [JS03, Proposition I.49], and the martingale property of $\frac{1}{M}$ under $\tilde{\mathbb{P}}$, we get

$$\begin{aligned}\mathbb{E}\left[\int_0^\tau V^\tau(zZ_t)d\kappa_t^\tau\right] &= \tilde{\mathbb{E}}\left[\frac{1}{M_\tau}\int_0^\tau V^\tau(zZ_t)d\kappa_t^\tau\right] \\ &= \tilde{\mathbb{E}}\left[\int_0^\tau \left(\int_0^{s-} V^\tau(zZ_t)d\kappa_t^\tau\right) d\left(\frac{1}{M_s}\right) + \int_0^\tau \frac{V^\tau(zZ_t)}{M_t}d\kappa_t^\tau\right].\end{aligned} \tag{24}$$

In the latter expression, by [Pro04, Theorem III.29, p. 128], $\int_0^\cdot \left(\int_0^{s-} V^\tau(Z_t)d\kappa_t^\tau\right) d\left(\frac{1}{M_s}\right)$ is a $\tilde{\mathbb{P}}$ local martingale, Therefore, we can rewrite (24) as

$$\mathbb{E}\left[\int_0^\tau V^\tau(Z_t)d\kappa_t^\tau\right] = \tilde{\mathbb{E}}\left[\int_0^\tau \frac{V^\tau(Z_t)}{M_t}d\kappa_t^\tau\right]. \tag{25}$$

Now, the assertion of the lemma follows from (5), the definition of $\tilde{V}$'s in (16), and (25). □

Lemma 3.9. *Under the assumptions of Theorem 3.7, the value functions are finite-valued, that is*

$$v(z) \in \mathbb{R} \quad and \quad u(z) \in \mathbb{R}, \quad z > 0. \tag{26}$$

We also have

$$v(z) \geq u(x) - xz, \quad for\ every \quad z > 0\ and\ x > 0, \tag{27}$$

and

$$v^\tau(z) \geq u^\tau(x) - zx, \quad for\ every \quad z > 0,\ x > 0\ and\ \tau \in \mathcal{T}_0. \tag{28}$$

Proof. We begin by showing (28). For this, we fix $z > 0$, $x > 0$, and $\tau \in \mathcal{T}_0$. Then, let us consider an arbitrary $c \in \mathcal{A}^\tau(x)$ and denote $C := \int_0^\cdot c_s d\kappa_s^\tau$. Then, there exists $X \in \mathcal{X}(x)$, such that for every stopping time $\sigma \in \mathcal{T}_0$, we have

$$\tilde{\mathbb{E}}\left[Z_\sigma C_\sigma\right] \leq \tilde{\mathbb{E}}\left[Z_\sigma X_\sigma\right], \quad \sigma \in \mathcal{T}_0. \tag{29}$$

Now, by an application of Ito's lemma, one can see that ZX is a local martingale, which is nonnegative by the nonnegativity of X and Z. Therefore, ZX is a nonnegative supermartinale, and

$$Z_0X_0 = x. \tag{30}$$

Therefore, combining the supermartingale property of ZX with (29) and (30), by the Optional Sampling Theorem we deduce that

$$\tilde{\mathbb{E}}\left[Z_\sigma C_\sigma\right] \leq \tilde{\mathbb{E}}\left[Z_\sigma X_\sigma\right] \leq x, \quad \sigma \in \mathcal{T}_0. \tag{31}$$

Next, using [JS03, Proposition I.49], we deduce that

$$ZC = C_- \cdot Z + Z \cdot C.$$

By [Pro04, Theorem III.29], the process $C_- \cdot Z$ is a local martingale. Let σ_m, $m \geq 1$, be a localizing sequence for $C_- \cdot Z$. Then, for every $m \in \mathbb{N}$, by the latter equality, we have

$$Z_{\sigma_m}C_{\sigma_m} = C_- \cdot Z_{\sigma_m} + Z \cdot C_{\sigma_m}, \quad m \in \mathbb{N}.$$

Taking the expectation under $\mathbb{P}$, and using the martingale property of $C_- \cdot Z$ on $[0, \sigma_m]$, we can rewrite the latter expression as

$$\tilde{\mathbb{E}}\left[Z_{\sigma_m}C_{\sigma_m}\right] = \tilde{\mathbb{E}}\left[Z \cdot C_{\sigma_m}\right], \quad m \in \mathbb{N},$$

which, using (31), we can bound by

$$\tilde{\mathbb{E}}\left[Z_{\sigma_m}C_{\sigma_m}\right] = \tilde{\mathbb{E}}\left[Z\cdot C_{\sigma_m}\right] \le x, \quad m\in\mathbb{N}. \tag{32}$$

Taking the limit as $m\to\infty$ and using the monotone convergence theorem, we can obtain from (32) that

$$\tilde{\mathbb{E}}\left[\int_0^\infty Z_s c_s d\kappa^\tau_s\right] = \tilde{\mathbb{E}}\left[Z\cdot C_\infty\right] \le x. \tag{33}$$

Next, for every $c\in\mathcal{A}^\tau(x)$ and $z>0$, using (32), we get

$$\sup_{c\in\mathcal{A}^\tau(x)} \tilde{\mathbb{E}}\left[\int_0^\infty \tilde{U}^\tau(c_s)d\kappa^\tau_s - \int_0^\infty c_s z Z_s d\kappa^\tau_s\right] \ge \sup_{c\in\mathcal{A}^\tau(x)} \tilde{\mathbb{E}}\left[\int_0^\infty \tilde{U}^\tau(c_s)d\kappa^\tau_s\right] - zx = u^\tau(x) - zx. \tag{34}$$

On the other hand, from the definition of U^τ, we conclude that

$$\sup_{c\in\mathcal{A}^\tau(x)} \tilde{\mathbb{E}}\left[\int_0^\infty \tilde{U}^\tau(c_s)d\kappa^\tau_s - \int_0^\infty c_s z Z_s d\kappa^\tau_s\right] \le \tilde{\mathbb{E}}\left[\int_0^\infty \tilde{V}^\tau(zZ_s)d\kappa^\tau_s\right] = v^\tau(z). \tag{35}$$

Comparing (34) and (35), we assert that for every $x>0$, $z>0$ and $\tau\in\mathcal{T}_0$, we have

$$u^\tau(x) - zx \le v^\tau(z),$$

that is, (28) holds.

Once (28) is established, let us fix $z>0$ and consider a sequence $\tau_n\in\mathcal{T}_0$, $n\in\mathbb{N}$, such that

$$v(z)\ge v^{\tau_n}(z) - \frac{1}{n}, \quad n\in\mathbb{N}.$$

Then, for a fixed $x>0$, using (28) Assumption 3.2, and (28), we get

$$\infty > v(z) \ge v^{\tau_n}(z) - \frac{1}{n} \ge u^{\tau_n}(x) - xz - \frac{1}{n} \ge u(x) - xz - \frac{1}{n} > -\infty. \tag{36}$$

Taking lim sup and lim inf in (36), we conclude that

$$\infty > v(z) \ge \limsup_{n\to\infty} v^{\tau_n}(z) \ge \liminf_{n\to\infty}\left(u^{\tau_n}(x) - xz\right) \ge u(x) - xz > -\infty.$$

As $z>0$ and $x>0$ are arbitrary, (26) and (27) follow.

□

Proof of Theorem 3.7. (i). Finiteness of the value functions follows from Lemma 3.9.

For the conjugacy of u and v, let us consider $x_0>0$, such that the optimal stopping time for (10), $\sigma^*(x_0)$, exists for this x_0. Then, by Assumption 3.4, this stopping time is optimal for every $x>0$, and so, we will denote this stopping time by σ^* for brevity. Then, similarity to the proof of Lemma 3.9, using Asssuption 3.2, for every $x>0$ and $z>0$, we get

$$\infty > v^{\sigma^*}(z) \ge v(z) \ge u(x) - xz = u^{\sigma^*}(x) - xz > -\infty.$$

u^{σ^*} and v^{σ^*} are finite valued. Therefore, the construction of the $\mathcal{A}^{\sigma^*}$ in (6) and u^{σ^*} in (9) together with the finiteness of u^{σ^*} and v^{σ^*} allow to invoke [Mos15, Theorem 3.2], which implies that for every $z > 0$, there exists $x^*(z)$, such that

$$v^{\sigma^*}(z) = u^{\sigma^*}(x^*(z)) - x^*(z)z, \quad z > 0. \tag{37}$$

Combining (37) with the definition of v in (3), we get

$$u(x^*(z)) = u^{\sigma^*}(x^*(z)) = v^{\sigma^*}(z) + x^*(z)z \geq v(z) + x^*(z)z, \tag{38}$$

where, in the latter inequality, we have used (27). On the other hand, from Lemma 3.9, we get

$$v(z) \geq u(x^*(z)) - x^*(z)z. \tag{39}$$

From (38) and (39), we conclude that

$$v^{\sigma^*}(z) = v(z). \tag{40}$$

As additionally $u^{\sigma^*}(x^*(z)) = u(x^*(z))$, from (37) and (40), we obtain that

$$u(x^*(z)) = v(z) + x^*(z)z. \tag{41}$$

As $z > 0$ is arbitrary, the assertions of item *(ii)* follow.

By changing z, particularly by taking limits as $z \to 0$ and $z \to \infty$, one can see that $x^*(z)$ takes values in $(0, \infty)$. Next, as $u = u^{\sigma^*}$ and $v = v^{\sigma^*}$, the finiteness of the value functions established in Lemma 3.9, implies that u^{σ^*} and v^{σ^*} are finite-valued. Therefore, in view of the representation of u given by (20), [Mos15, Theorem 3.2] applies, which asserts that for every $x > 0$, the optimizer to (10) exists and is unique. This implies the assertions of item *(iii)* of this theorem (Theorem 3.7).

For item *(iv)*, we use $u = u^{\sigma^*}$ and $v = v^{\sigma^*}$ established in the proof of item *(ii)* together with the reformulation (20), which allow to invoke again [Mos15, Theorem 3.2]. [Mos15, Theorem 3.2] allows to conclude that the relations between the optimizers of (1) and (10) in item *(iv)* of the statement of this theorem.

□

In the following theorem, we relax Assumption 3.4 to the following one.

Assumption 3.10. We suppose that for every $t \geq 0$, $V^i(t, \cdot)$, $i = 1, 2$, are strictly decreasing strictly convex and satisfy the conditions

$$\lim_{z \downarrow 0} V^i(t, z) = -\infty \quad \textit{and} \quad \lim_{z \uparrow \infty} V^i(t, z) = 0, \quad t \geq 0, \quad i = 1, 2.$$

Theorem 3.11. *Let us suppose that Assumptions 3.1, 3.2, and 3.10 hold and for every $x > 0$, there exists an optimal stopping time $\sigma^*(x)$ for (10). Then, we have*

(i) both $v(y) \in \mathbb{R}$ and $u(y) \in \mathbb{R}$, $y > 0$. v and $-u$ are strictly concave and continuously differentiable on $(0, \infty)$.

(ii) for every $x > 0$, there exists $z(x) > 0$, such that

$$v(z(x)) = u(x) + xz(x).$$

(iii) for every $x > 0$ and $z(x)$ of item (ii), the optimal stopping time for (3) at $z(x)$ exists and is equal to $\sigma^*(x)$, and we have

$$v^{\sigma^*(x_0)}(z^*(x_0)) = v(z^*(x_0)).$$

For (10), and every $x > 0$, the optimal control $c^*(x)$ exists and is unique.

(iv) for every $x > 0$ and $z(x)$ of item (ii), the optimizer $c^*(x)$ to (10) is related to Z via

$$z(x)Z_t = (U^1)'(t, c^*_t(x))1_{\{t<\sigma^*(x)\}} + (U^2)'(t, c^*_t(x))1_{\{t\geq\sigma^*(x)\}}, \quad t \geq 0,$$

$$c^*_t(x) = -(V^1)'(t, z(x)Z_t)1_{\{t<\sigma^*(x)\}} - (V^2)'(t, z(x)Z_t)1_{\{t\geq\sigma^*(x)\}}, \quad t \geq 0,$$

Proof of Theorem 3.11. (i). Finiteness of the value functions follows from Lemma 3.9.

For the conjugacy of u and v, let us consider $x_0 > 0$, such that the optimal stopping time for (10), $\sigma^*(x_0)$, exists for this x_0. Then, similarity to the proof of Lemma 3.9, using Asssuption 3.2, for every $z > 0$, we get

$$\infty > v^{\sigma^*(x_0)}(z) \geq v(z) \geq u(x_0) - x_0 z = u^{\sigma^*(x_0)}(x_0) - x_0 z > -\infty.$$

$v^{\sigma^*(x_0)}$ is finite valued, and so is $u^{\sigma^*(x_0)}$, in view of Lemma 3.9 and (10). Therefore, the construction of the $\mathcal{A}^{\sigma^*(x_0)}$ in (6) and $u^{\sigma^*(x_0)}$ in (9) together with the finiteness of $u^{\sigma^*(x_0)}$ and $v^{\sigma^*(x_0)}$ allow to invoke [Mos15, Theorem 3.2], which implies that there exists $z^*(x_0)$, such that

$$v^{\sigma^*(x_0)}(z^*(x_0)) = u^{\sigma^*(x_0)}(x_0) - x_0 z^*(x_0). \tag{42}$$

Combining (42) with the definition of v in (3), we get

$$u(x_0) = u^{\sigma^*(x_0)}(x_0) = v^{\sigma^*(x_0)}(z^*(x_0)) + x_0 z^*(x_0) \geq v(z^*(x_0)) + x_0 z^*(x_0). \tag{43}$$

On the other hand, from Lemma 3.9, we get

$$v(z^*(x_0)) \geq u(x_0) - x_0 z^*(x_0). \tag{44}$$

From (43) and (44), we conclude that

$$v^{\sigma^*(x_0)}(z^*(x_0)) = v(z^*(x_0)). \tag{45}$$

As additionally $u^{\sigma^*(x_0)}(x_0) = u(x_0)$, from (43) and (45), we obtain that

$$u(x_0) = v(z^*(x_0)) + x_0 z^*(x_0). \tag{46}$$

As $x_0 > 0$ is arbitrary, the assertions of item *(ii)* follow.

Next, in view of the representation of u given by (20) and the existence of the optimal stopping time for (10) at every $x > 0$, [Mos15, Theorem 3.2] applies, which asserts that for every $x > 0$, the optimizer to (10), $c^*(x)$, exists and is unique. In view of (45), $\sigma^*(x_0)$ is optimal for (3) at $z^*(x_0)$, and thus the assertions of item *(iii)* hold.

For item (iv), we use $u(x) = u^{\sigma^*(x_0)}(x_0)$, $x_0 > 0$, and $v(z(x_0)) = v^{\sigma^*(x_0)}(z(x_0))$ (see (45)), where $z(x_0)$ satisfies (42), we invoke again [Mos15, Theorem 3.2]. [Mos15, Theorem 3.2] which allows to conclude that the relations between the optimizers of (1) and (10) in item (iv) of this theorem (Theorem 3.7), where the existence of the optimal stopping time for (3) at $\sigma^*(x)$ and the optimal control $c^*(x)$ for (10) are established in item (iii) .

□

3.1. **Dual representation of the Snell envelope.** The results of this section rely on [Mos21], which is written for the Mayer formulation. Therefore, in this section, we also consider the Mayer formulation of the optimal stopping problem, and suppose that Z is as in Section 2 (a positive local martingale) and $x > 0$ be fixed. In this case, the Snell envelope can be specified as

$$Z^S_t = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[V^2(\tau, Z_\tau)|\mathcal{F}_t\right], \quad t \geq 0. \tag{47}$$

Let us recall some settings from [Mos21]. With η being an $\mathcal{F}_t$-measurable random variable and $\tau \in \mathcal{T}_t$, and

$$\mathcal{Z}_t = \left\{(\tilde{Z}_s)_{s\in[t,\tau]} \geq 0 :\ \tilde{Z}_t \leq 1 \ \mathit{and}\ (\tilde{Z}_s X_s)_{s\in[t,\tau]}\ \mathit{is\ a\ supermartingale\ for\ every}\ X \in \mathcal{X}(1)\right\}.$$

$v^\tau(\eta, t, \tau)$ is given by

$$v^\tau(\eta, t, \tau) = \operatorname*{ess\,inf}_{Z\in\mathcal{Z}_t} \mathbb{E}\left[V^2(\tau, \eta z_\tau)|\mathcal{F}_t\right].$$

The construction of the d-dimensional process R as in Section 2.4 and the assumption that the filtration is generated by the d-dimensional Brownian motion, imply that, in the present settings,

$$\mathcal{Z}_t = \left\{\left(\xi\frac{Z_s}{Z_t}\right)_{s\in[t,\tau]}\right\},$$

where ξ is an $\mathcal{F}_t$-measurable random variable taking values in $[0, 1]$ and Z is process of section 2. The maximal element of $\mathcal{Z}_t$ is given by

$$\mathcal{Z}_t = \left\{\left(\frac{Z_s}{Z_t}\right)_{s\in[t,\tau]}\right\},$$

and, as V^2 is decreasing, v^τ by

$$v^\tau(\eta, t, \tau) = \mathbb{E}\left[V^2\left(\tau, \eta\frac{Z_\tau}{Z_t}\right)|\mathcal{F}_t\right],$$

Therefore, in the present settings, we have the following (primal) representation of the Snell envelope

$$Z^S_t = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_t} v^\tau(Z_t, t, \tau), \quad t \geq 0.$$

The next lemma gives the dual representation of the Snell envelope and of the optimal stopping time. Let us set

$$\mathcal{A}(\xi,t) := \left\{\xi + \int_t^{\cdot} H_s dR_s,\ \mathit{for\ some}\ R-\mathit{integrable}\ H\right\},$$

and

$$u^{\tau}(\xi,t,\tau) = \operatorname*{ess\,sup}_{\tilde{X}\in\mathcal{A}(\xi,t)} \mathbb{E}\left[U^2(\tau,\tilde{X}_\tau)|\mathcal{F}_t\right].$$

Lemma 3.12. *Under the conditions of Theorem 3.11, let* $V^1\equiv 0$ *and* $x>0$ *are fixed. Then,* Z^S *specified in* (47) *can be represented as*

$$Z^S_t = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_t} \operatorname*{ess\,sup}_{X\in\bigcup\limits_{x>0}\mathcal{X}(x)} \left(u^{\tau}(X_t,t,\tau) - X_tZ_t\right),\quad t\geq 0, \tag{48}$$

and $\tau^*(x)$*, the optimal stopping time for* (3)*, can be represented as*

$$\tau^*(x) = \min\left\{t\geq 0:\ \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_t} \operatorname*{ess\,sup}_{X\in\bigcup\limits_{x>0}\mathcal{X}(x)} \left(u^{\tau}(X_t,t,\tau) - X_tZ_t\right) = V^2(t,Z_t)\right\}, \tag{49}$$

is the optimal stopping time.

Proof. Let us fix a stopping time $\tau\in\mathcal{T}_t$. Then, as Z is a local martingale, NUPBR holds for R on $[t,\tau)$. Therefore, in view of Assumptions 3.2 and 3.10, the conditions of [Mos21, Lemma 3.14] hold in the present settings. With

$$\mathcal{C}_t(x) := \left\{g\in m\mathcal{F}_t:\ g\leq X_t\ \mathit{for\ some}\ X_t\in\mathcal{X}(x)\right\},$$

[Mos21, Lemma 3.14] asserts that

$$v^{\tau}(Z_t,t,\tau) = \operatorname*{ess\,sup}_{\xi\in\bigcup\limits_{x>0}\mathcal{C}_t(x)} \left(u^{\tau}(\xi,t,T) - \xi Z_t\right).$$

Therefore, in view of the defintion of $\mathcal{C}_t(x)$, we obtain

$$Z^S_t = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_t} \operatorname*{ess\,sup}_{\xi\in\bigcup\limits_{x>0}\mathcal{C}_t(x)} \left(u^{\tau}(\xi,t,T) - \xi Z_t\right). \tag{50}$$

Thus, (50) gives the representation of the Snell envelope (50).

As the optimal stopping time for (3) in the Mayer formulation can be specified as

$$\tau^*(x) = \min\left\{t\geq 0: Z^S_t = V^2(t,Z_t)\right\},$$

we conclude from the last equality that (50) implies (49).

□

3.2. **Non-existence of the optimal stopping time.** The first counterexample shows that, in general, the existence of the optimal stopping time in Theorem 3.7 must be imposed, and it does not need to hold under the remaining assumptions of Theorem 3.7.

Example 3.13. In the Mayer formulation of the optimal stopping problem, that is, with $V^1 \equiv 0$, one can build the following counterexample. Let us fix $q \in (-1, 0)$, so that

$$V^2(t, z) = \frac{z^{-q}}{q}, \quad z > 0,$$

and consider

$$Z_t = \exp\left(-\frac{W_t}{q} + \frac{1}{2q}t + \frac{1}{q(t+1)}\right), \quad t \geq 0,$$

So that

$$V^2(t, Z_t) = \frac{1}{q}\exp\left(W_t - \frac{1}{2}t - \frac{1}{t+1}\right) < 0, \quad t \geq 0.$$

Then, as V^2 is negative-valued, it follows that $v(z) < \infty$, where v is defined in (3).

In this case,

$$U^2(t, x) = \frac{x^p}{p}, \quad x > 0, \quad p = \frac{q}{1+q} < 0.$$

One can see that for every $x > 0$, we have

$$X \equiv x \in \mathcal{X}(x),$$

and for every stopping time $\tau \in \mathcal{T}_0$, we have

$$c = \left(x1_{\{t=\tau\}}\right)_{t\geq 0} \in \mathcal{A}^\tau(x), \quad x > 0.$$

Then, for u^τ defined in (9), we have

$$u^\tau(x) \geq U^2(t, x) = \frac{x^p}{p}, \quad \tau \in \mathcal{T}_0, \tag{51}$$

So that

$$u(x) = \inf_{\tau\in\mathcal{T}_0} u^\tau(x) \geq \inf_{\tau\in\mathcal{T}_0} \frac{x^p}{p} > -\infty, \quad x > 0,$$

and thus, u defined in (10) is finite-valued.

We conclude that

$$v(z) < \infty \quad \textit{and} \quad u(z) > -\infty, \quad z > 0,$$

that is, Assumption 3.2 holds.

Next, let us observe that for every $\tau \in \mathcal{T}$, by [Mos15, Theorem 3.2], we have

$$u^\tau(x) = \mathbb{E}\left[U\left(\tau, I(z^\tau Z_\tau)\right)\right], \tag{52}$$

where

$$I(z) = z^{\frac{1}{p-1}} \quad \textit{and} \quad z^\tau = (u^\tau)'(x),$$

which in the case of (51) become

$$z^\tau = \frac{p}{x}u^\tau(x).$$

Let us fix $\tau \in \mathcal{T}$, and suppose that

$$\mathbb{P}[\tau < \infty] > 0. \tag{53}$$

This without loss of generality as $\tau = \infty$ is suboptimal to (3) is the settings of this example. Then, (52) gives

$$u^{\tau}(x) = \frac{\mathbb{E}\left[\left(\frac{p}{x}u^{\tau}(x)Z_{\tau}\right)^{-q}\right]}{p}, \quad x > 0. \tag{54}$$

For every $x > 0$, solving (54) for $u^{\tau}(x)$, we obtain that

$$-\left(-u^{\tau}(x)\right)^{1+q} = \left(\frac{-p}{x}\right)^{-q}\frac{\mathbb{E}\left[Z_{\tau}^{-q}\right]}{p}. \tag{55}$$

Now, let $\sigma := \tau + 1 \in \mathcal{T}$ and $A = \{\tau < \infty\}$. By (53), we have that $\mathbb{P}[A] > 0$, and we have

$$\begin{aligned}\mathbb{E}\left[(Z_{\sigma})^{-q}\right] &= \mathbb{E}\left[\exp\left(W_{\sigma} - \frac{\sigma}{2} - \frac{1}{\sigma+1}\right)\right]\\ &= \mathbb{E}\left[\exp\left(W_{\tau+1} - \frac{\tau+1}{2} - \frac{1}{\tau+2}\right)(1_A + 1_{A^c})\right]\\ &= \mathbb{E}\left[\exp\left(W_{\tau+1} - \frac{\tau+1}{2} - \frac{1}{\tau+2}\right)1_A\right],\end{aligned} \tag{56}$$

as $\exp\left(W_{\tau+1} - \frac{\tau+1}{2} - \frac{1}{\tau+2}\right) = 0$ on $\tau = \infty$. Next, as $\exp(-\frac{1}{\tau+2}) \geq \exp(-\frac{1}{\tau+1})$, in view of (53), we can bound the latter expression in (56) as follows

$$\begin{aligned}\mathbb{E}\left[\exp\left(W_{\tau+1} - \frac{\tau+1}{2} - \frac{1}{\tau+2}\right)1_A\right] &> \mathbb{E}\left[\exp\left(W_{\tau+1} - \frac{\tau+1}{2} - \frac{1}{\tau+1}\right)1_A\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(W_{\tau+1} - \frac{\tau+1}{2} - \frac{1}{\tau+1}\right)1_A|\mathcal{F}_{\tau}\right]\right],\end{aligned} \tag{57}$$

which we can further rewrite as

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}\left[\exp\left(W_{\tau+1} - \frac{\tau+1}{2} - \frac{1}{\tau+1}\right)1_A|\mathcal{F}_{\tau}\right]\right]\\ &= \mathbb{E}\left[\exp\left(W_{\tau} - \frac{\tau}{2} - \frac{1}{\tau+1}\right)1_A\mathbb{E}\left[\exp\left(W_{\tau+1} - W_{\tau} - \frac{1}{2}\right)|\mathcal{F}_{\tau}\right]\right]\\ &= \mathbb{E}\left[\exp\left(W_{\tau} - \frac{\tau}{2} - \frac{1}{\tau+1}\right)1_A\right]\\ &= \mathbb{E}\left[\exp\left(W_{\tau} - \frac{\tau}{2} - \frac{1}{\tau+1}\right)(1_A + 1_{A^c})\right]\\ &= \mathbb{E}\left[(Z_{\tau})^{-q}\right].\end{aligned} \tag{58}$$

Combining (56), (57), and (58), we conclude that

$$\mathbb{E}\left[(Z_{\sigma})^{-q}\right] > \mathbb{E}\left[(Z_{\tau})^{-q}\right]. \tag{59}$$

Therefore, from (60) and recalling that $p < 0$, we obtain

$$- \left(-u^{\sigma}(x)\right)^{1+q} = \left(\frac{-p}{x}\right)^{-q} \frac{\mathbb{E}\left[Z_{\sigma}^{-q}\right]}{p} < \left(\frac{-p}{x}\right)^{-q} \frac{\mathbb{E}\left[Z_{\tau}^{-q}\right]}{p} = - \left(-u^{\tau}(x)\right)^{1+q}. \tag{60}$$

We conclude that

$$u^{\sigma}(x) = u^{\tau+1}(x) < u^{\tau}(x), \quad x > 0, \tag{61}$$

for every τ satisfying (60), and thus, the optimal stopping time for the dual problem (10) does not exist.

Using [Mos15, Theorem 3.2], from (61), we conclude that the primal value functions satisfy

$$v^{\sigma}(z) = v^{\tau+1}(z) < v^{\tau}(z), \quad z > 0, \tag{62}$$

and thus the optimal stopping time for (3) also does not exist.

Moreover, one can show that the primal problem does not have an optimal stopping rule as

$$\tau^* = \inf \left\{ t \geq 0 : \; V^2(t, Z_t) = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[V^2(\tau, Z_\tau) | \mathcal{F}_t\right] \right\}, \tag{63}$$

satisfies

$$\mathbb{P}\left[\tau^* < \infty\right] = 0.$$

Example 3.14. Counterexample in Lagrange formulation. Let us consider a deterministic function α on $(0, \infty)$ such that

$$\int_0^\infty \alpha_t^2 dt < \infty. \tag{64}$$

We remark that (64) implies that

$$\int_0^\infty \exp\left(\int_0^t \alpha_s^2 ds\right) \alpha_t^2 dt < \infty, \tag{65}$$

which will be used below.

Next, for a constant $q \in (-1, 0)$, let us consider a process Z of the form

$$Z_t = \exp\left(-\frac{1}{q}\int_0^t \alpha_s dW_s + \frac{1}{2q}\int_0^t \alpha_s^2 ds + \frac{1}{q(t+1)}\right), \quad t \geq 0. \tag{66}$$

and let us set

$$V^1(t, z) = \frac{z^{-q}}{q} \frac{1}{(t+1)^2} \quad \textit{and} \quad V^2(t, z) \equiv 0, \quad (t, z) \in [0, \infty) \times (0, \infty).$$

With these preliminaries, the objective in (1) becomes

$$v(z) = \inf_{\tau \in \mathcal{T}_0} \frac{z^{-q}}{q} \mathbb{E}\left[\int_0^\tau \exp\left(\int_0^t \alpha_s dW_s - \frac{1}{q}\int_0^t \alpha_s^2 ds\right) \exp\left(-\frac{1}{t+1}\right) \frac{1}{(t+1)^2} dt\right].$$

Let us consider the following equivalent problem

$$\sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[\int_0^\tau \exp\left(\int_0^t \alpha_s dW_s - \frac{1}{q}\int_0^t \alpha_s^2 ds\right) \exp\left(-\frac{1}{t+1}\right) \frac{1}{(t+1)^2} dt\right]. \tag{67}$$

With

$$M := \mathcal{E}\left(\int_0^{\cdot} \alpha_s dW_s\right) \quad \textit{and} \quad A_t := \exp\left(-\frac{1}{t+1}\right), \quad t \geq 0, \tag{68}$$

we can restate (67) as

$$\sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[\int_0^{\tau} M_t dA_t\right].$$

Now, for every $\tau \in \mathcal{T}_0$, using [JS03, Proposition I.4.49], we get

$$\int_0^{\tau} M_t dA_t = M_\tau A_\tau - M_0 A_0 - \int_0^{\tau} A_{t-} dM_t.$$

It follows from (65) that

$$\mathbb{E}\left[\langle A_- \cdot M\rangle_\infty\right] < \infty. \tag{69}$$

Therefore, for every $\tau \in \mathcal{T}_0$, we have

$$\mathbb{E}\left[\int_0^{\tau} A_{t-} dM_t\right] = 0.$$

This allows us to rewrite the objective in (67) as

$$\sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[M_\tau A_\tau\right] = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[\mathcal{E}\left(\int_0^{\cdot} \alpha_s dW_s\right)_\tau \exp\left(-\frac{1}{\tau+1}\right)\right]. \tag{70}$$

Now, similarly to Example 3.13, we can show that (70) problem with M and A given by (3.13) (and where α satisfies (65)) does not admit an optimal stopping time.

Example 3.15. Here we present an example (1), which does not admit an optimal solution in the Lagrange-Mayer formulation.

With the process Z being as in (66) of the previuous Example 3.14, let us consdier

$$V^1(t,z) = \frac{z^{-q}}{q}\frac{1}{(t+1)^2} \quad \textit{and} \quad V^2(t,z) = \frac{z^{-q}}{q}, \quad (t,z) \in [0,\infty) \times (0,\infty).$$

Then, similarly to the computations in Example 3.14, we can restate the objective as

$$\inf_{\tau \in \mathcal{T}_0} \mathbb{E}\left[\int_0^{\tau} V^1(t,Z_t)dt + V^2(\tau,Z_\tau)\right] = 2\inf_{\tau \in \mathcal{T}_0} \mathbb{E}\left[\mathcal{E}\left(\int_0^{\cdot} \alpha_s dW_s\right)_\tau \exp\left(-\frac{1}{\tau+1}\right)\right], \tag{71}$$

which does not admit an optimal stopping time by the argument in Example 3.14.

4. AN ADDITIONAL RESULT NEEDED

For the proof of Theorem 3.11 under even weaker conditions, we need the following theorem, which is closely related to [Mos15, Theorem 3.2] under weaker integrability condition.

Theorem 4.1. *Let us fix $\tau \in \mathcal{T}_0$ and suppose that Assumptions 3.1, 3.10, hold*

$$u^\tau(x) > -\infty, \quad x > 0,$$

and there exists $x > 0$, such that

$$u^\tau(x) < \infty.$$

Then we have:

- *(i)* $u^\tau(x) < \infty$ *for all* $x > 0$*, and there exists* $z_0 > 0$*, such that* $v^\tau(z)$ *is finitely valued for* $z > z_0$*. The value functions* u^τ *and* v^τ *are biconjugate, that is*

$$v^\tau(z) = \sup_{x>0} \left(u^\tau(x) - xz\right), \quad z > z_0,$$
$$u^\tau(x) = \inf_{z>z_0} \left(v^\tau(z) + xz\right), \quad x > 0.$$

- *(ii) The function* u^τ *is continuously differentiable on* $(0,\infty)$ *and the function* v^τ *is strictly convex on* $\{v^\tau < \infty\}$*. The functions* u^τ *and* v^τ *satisfy*

$$\lim_{x\to 0} \left(u^\tau\right)'(x) = \infty, \quad \textit{and} \quad \lim_{y\to\infty} \left(v^\tau\right)'(y) = 0.$$

We remark that, under the assumptions of Theorem 4.1, we have

$$z \in \partial u^\tau(x) \quad if\ and\ only\ if \quad u^\tau(x) - xz = v^\tau(z).$$

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