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ON THE ANALYTICITY OF THE VALUE FUNCTION IN OPTIMAL INVESTMENT AND STOCHASTICALLY DOMINANT MARKETS

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Abstract. We study the analyticity of the value function in optimal investment with expected utility from terminal wealth and the relation to stochastically dominant financial models. We identify both a class of utilities and a class of semimartingale models for which we establish analyticity. Specifically, these utilities have completely monotonic inverse marginals, while the market models have a maximal element in the sense of infinite-order stochastic dominance. We construct two counterexamples, themselves of independent interest, which show that analyticity fails if either the utility or the market model does not belong to the respective special class. We also provide explicit formulas for the derivatives, of all orders, of the value functions as well as their optimizers. Finally, we show that for the set of supermartingale deflators, stochastic dominance of infinite order is equivalent to the apparently stronger dominance of second order.

1. Introduction

One of the most fundamental problems in mathematical finance is to maximize the expected utility of terminal wealth,

$$u(x) := \sup_{X\in\mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right], \quad x>0, \tag{1.1}$$

where $U:(0,\infty)\to\mathbb{R}$ is the agent's utility function and $\mathcal{X}(x)$ the set of admissible wealth processes starting at x. This stochastic optimization problem has been extensively studied in a number of seminal papers and foundational results on, among others, its solvability and the connection between the primal and the dual formulation, have been established for general semimartingale markets (see, among others, [17], [16], [19]).

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The work herein focuses on the *higher-order regularity* and, ultimately, the *analyticity* of the value function u in (1.1). The core of our work is based on a deep connection we develop with *stochastic dominance* in the domain of the dual optimization problem. This interplay between regularity/analyticity of the value function and stochastic dominance of various degrees in the dual domain is, to our knowledge, explored and developed for the first time. It offers new insights on distinct characteristics between the primal and the dual problems. Furthermore, it also led us to derive new results of independent interest regarding stochastic dominance of different orders, including the infinite one, for the set of supermartingale deflators.

For the regularity of u, the authors in [19, 20] showed that, under general market assumptions, if $U \in C^1((0,\infty))$, so is u. For higher-order regularity, there are two types of results. If the utility function U is either power or logarithmic, the value function also inherits this form and it is thus analytic. This is a direct consequence of homotheticity and holds under minimal model assumptions. For utilities beyond homothetic ones, however, only the second-order differentiability of u has been established for twice differentiable utilities and additional model assumptions (see [21]).

To our knowledge, no other regularity and analyticity results exist to date besides the aforementioned limited cases. We are motivated to investigate such questions for various reasons. To begin with, the degree of regularity per se has always been of central interest in stochastic optimization. It is used to establish verification results and also study the regularity and other properties of the optimal control functionals. It is also employed in the analysis of higher-order sensitivities of the value function and the control policies with respect to various model inputs. The question of whether the indirect utility u inherits all the properties of U may appear as a rather abstract task at first that is asking if the map $U \to u$ has some semigroup/invariance properties.

For the expected utility stochastic optimization problem (1.1), in particular, the motivation for studying the higher-order regularity of the value function goes beyond purely mathematical considerations. Indeed, we first recall that the derivatives $U^{(n)}$ of the utility function are directly related to foundational economic indices like risk aversion, risk tolerance, prudence, temperance, the higher-degree Ross indices (see, for example, [8], [13], [7], and [24]). In turn, their counterparts $u^{(n)}$ are also related to analogous indices. Important questions then arise on the connection between the various utility indices with the ones of the value function. Here, the understanding is quite limited, mainly due to the narrow results on the higher-order regularity of the value function.

As the indirect utility, u, may become the *direct* utility of a new optimization problem, for example, in an iterative model or in indifference valuation, preserving the properties of U to u may go beyond describing the higher-order risk. The derivatives of the value function also appear in quantities like certainty equivalents, risk measures, optimized certainty equivalents, and other valuation quantities. Furthermore, they are linked to expansion-based approaches (see, for example, [9] and

[10], who used series expansions to investigate various properties in optimal investments). In a different direction, analyticity plays an important role in the studies of endogenous completeness, see [1] and [12].

Naturally, both the regularity and analyticity of the value function are expected to directly depend on similar properties for their utility analogs. On the other hand, the value function depends crucially on the market environment in which the related utility optimization problems are cast. We are, then, motivated to ask the following question:

Can we identify both a class of semimartingale market models and a class of utility functions such that the value function in (1.1) retains the (highest possible) regularity of the utility function?

Studying this question constitutes the main contribution of our work herein. We outline the main findings. We propose

i) the class of market models which possess a non-zero dual maximal element in the sense of infinite-degree stochastic dominance and ii) the class of utilities whose inverse marginal is a completely monotonic function (and thus analytic). We denote these classes by $\mathcal{SD}(\infty)$ and $\mathcal{CMIM}$, respectively.

We establish the following result: *If the market model is in* $\mathcal{SD}(\infty)$ *and the utility is in* $\mathcal{CMIM}$*, then the value function u is also in* $\mathcal{CMIM}$ *and is, thus, analytic. In other words, we show that, for such market models and such analytic utilities, the associated indirect utility inherits the analyticity and, furthermore, remains in the same* $\mathcal{CMIM}$ class.

We also examine the necessity of these classes of models and utilities. We provide two counterexamples, showing that the results fail outside the family of $\mathcal{SD}(\infty)$ models and/or the utility class $\mathcal{CMIM}$. In the first counterexample, we construct a market model in $\mathcal{SD}(\infty)$ and an analytic, but not in $\mathcal{CMIM}$, utility and show that the value function is not infinitely differentiable (and, thus, not analytic). In the second one, we show that, for any non-homothetic $\mathcal{CMIM}$ utility (actually, the utility being two-times differentiable suffices), there exists a market model outside the $\mathcal{SD}(\infty)$ class, for which the value function is not even twice differentiable. We also derive results when some of the above notions are weakened to finite-degree analogs.

As mentioned above, under minimal model assumptions - well beyond the ones for the $\mathcal{SD}(\infty)$ class - homothetic utilities yield homothetic value functions. Such utilities belong to the $\mathcal{CMIM}$ family and are analytic, and these properties are also inherited by their value functions.

Finally, we show that the class $\mathcal{SD}(\infty)$ is precisely $\mathcal{SD}(2)$. This result is of independent interest. It is based on a delicate simultaneous change of measure and numéraire, combined with an approximation argument relying on a one-point compactification and the monotone class theorem.

The paper is organized as follows: in section 2, we specify the settings for problem (1.1). In section 3, we discuss the background notions on complete monotonicity and stochastic dominance and provide their characterizations. In section 4, we introduce the class of market models and utilities that we propose, followed by the main results on the analyticity of the value function together with the explicit expressions for the primal and dual optimizers and their derivatives of all orders, as well as other regularity results. Section 5 provides a counterexample for non-$\mathcal{CMIM}$ utilities, while section 6 contains a counterexample for non-$\mathcal{SD}(\infty)$ market models.

2. The optimal investment problem

2.1. **The market model.** The market consists of a riskless asset, offering zero interest rate, and d traded stocks, whose price processes form a d-dimensional semi-martingale S on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Here $T \in (0,\infty)$ is the investment horizon.

A trading strategy H is a predictable and S-integrable process. It generates the wealth process $X := x + H \cdot S$, starting at $x > 0$, which, for the utilities considered herein, is taken to be non-negative. Using the notation of [19], we denote the set of admissible wealth processes,

$$\begin{aligned}\mathcal{X}(x) := \{X :\ & X_t = x + H \cdot S_t \geq 0,\ t \in [0,T], \\ & for\ some\ S-integrable\ process\ H\}, \quad x > 0.\end{aligned} \tag{2.1}$$

Following [15], we say that a sequence $(X^n)_{n\in\mathbb{N}} \subset \mathcal{X}(1)$ generates an *unbounded profit with bounded risk (UPBR)*, if the family of the random variables $(X^n_T)_{n\in\mathbb{N}}$ is unbounded in probability, i.e., if

$$\lim_{m\uparrow\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left[X^n_T > m\right] > 0.$$

If no such sequence exists, the condition of *no unbounded profit with bounded risk (NUPBR)* is satisfied. A characterization of *NUPBR* is given via the dual feasible set, $\mathcal{Y}(y)$, introduced in [19],

$$\begin{aligned}\mathcal{Y}(y) := \{Y :\ & Y_0 = y\ and\ XY = (X_tY_t)_{t\in[0,T]}\ is\ a\ supermartingale \\ & for\ every\ X \in \mathcal{X}(1)\}, \quad y > 0.\end{aligned} \tag{2.2}$$

The elements of $\mathcal{Y}(1)$ are called super-martingale deflators, see [15]. It was established in [15] that *NUPBR* is equivalent to the existence of a strictly positive super-martingale deflator, namely,

$$\mathcal{Y}(1)\ contains\ a\ strictly\ positive\ element. \tag{2.3}$$

In [34] and [14], it was later proven that *NUPBR* is equivalent to the existence of a strictly positive local martingale deflator; see, also, [4]. Furthermore, it was shown in [14] that *NUPBR* is equivalent to other no-arbitrage conditions, such as no arbitrage of the first kind and no asymptotic arbitrage of the first kind; we refer the reader to [14, Lemma A.1] for further details.

2.2. **Utility functions.** We recall the standard class of utility functions $U : (0,\infty) \to \mathbb{R}$ which are strictly concave, strictly increasing, continuously differentiable and satisfy the Inada conditions

$$\lim_{x\downarrow 0} U'(x) = \infty \quad and \quad \lim_{x\uparrow\infty} U'(x) = 0. \tag{2.4}$$

To facilitate the upcoming exposition, we will denote the class of all such utility functions by $-\mathcal{C}$, in that

$$U \in -\mathcal{C} \iff -U \in \mathcal{C}.$$

2.3. **Primal problem and the indirect utility.** We recall the optimal investment problem from terminal wealth

$$u(x) := \sup_{X\in\mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right], \quad x>0, \tag{2.5}$$

where $U \in -\mathcal{C}$ and $\mathcal{X}(x)$ as in (2.1).

2.4. **Dual problem and the dual function.** For any $U \in -\mathcal{C}$, its Legendre transform is given by

$$V(y) := \sup_{x>0} \left(U(x) - xy\right), \quad y>0, \tag{2.6}$$

and, by biconjugacy,

$$-\mathcal{C} \ni U \Longleftrightarrow V \in \mathcal{C}.$$

In turn, we recall the dual value function,

$$v(y) := \inf_{Y\in\mathcal{Y}(y)} \mathbb{E}\left[V\left(Y_T\right)\right], \quad y>0. \tag{2.7}$$

with V as in (2.6) and $\mathcal{Y}(y)$ as in (4.10).

It was shown in [15] that condition *NUPBR* is necessary for the non-degeneracy of problem (2.5) in that, if *NUPBR* does not hold, then, for any utility function U, (2.5) has either infinitely many solutions or no solution at all. Specifically, if $U(\infty) = \infty$, then $u(x) = \infty$, $x > 0$. Therefore, either there is no solution (when the supremum is not attained) or there are infinitely many solutions (when the supremum is attained). On the other hand, if $U(\infty) < \infty$, there is no solution.

If condition *NUPBR* holds, problem (2.5) has a solution under the weak assumption that the dual value function v in (2.7) is finite, i.e., $v(y) < \infty$, $y > 0$. In this case, all standard conclusions of the utility maximization theory hold; see, for example, [18] and [27] for details.

3. Complete monotonicity and stochastic dominance

3.1. **Complete monotonicity.** Completely monotonic functions have been well-studied in the literature, see [37] and [33] for the historic overview of the development of the subject therein. A function $f : (0,\infty) \to \mathbb{R}$ is called *completely monotonic*, denoted by $f \in \mathcal{CM}$, if it has derivatives of all orders and

$$(-1)^n f^{(n)}(x) \geq 0,\ x>0 \ \text{ and } \ n = 0,1,2,\ldots$$

Whenever needed, we extend f to $[0,\infty)$ setting $f(0) := \lim_{x\downarrow 0} f(x)$, where $f(0) \leq \infty$.

The celebrated Bernstein theorem (see [33, Theorem 1.4] or [37, Theorem 12b]) gives a characterization of completely monotonic functions, stating that $f \in \mathcal{CM}$ if and only if

$$f(x) = \int_0^\infty e^{-xz} d\mu(z), \quad x \geq 0, \tag{3.1}$$

where μ is a nonnegative sigma-finite measure on $[0,\infty)$ such that the integral converges for every $x > 0$.

Definition 3.1. We define $\mathcal{D}$ to be the class of functions $W : [0,\infty) \to \mathbb{R}$, which satisfy

(1) $-W' \in \mathcal{CM}$,
(2) $W'(\infty) = 0$.

The reader should note that the definition above is related, but not the same, to what is called in literature a Bernstein functions, see, e.g., [33, p. 15]. Bernstein functions would assume bounds on W, but no Inada-type conditions on W'. For a $W \in \mathcal{D}$, we have $W'(y) = -\int\limits_0^\infty e^{-yz} d\mu(z)$ from the Bernstein representation characterization of completely monotonic functions. We then deduce that

$$W'(0+) := \lim_{y \downarrow 0} W'(y) = -\mu([0,\infty)) \quad and \quad W'(\infty) := \lim_{y \uparrow \infty} W'(y) = -\mu(\{0\}).$$

Therefore, the definition of $\mathcal{D}$ dictates that the measure μ has no mass at $z = 0$, to satisfy

$$\mu(\{0\}) = -W'(\infty) = 0. \tag{3.2}$$

We note that the Inada-type condition $W'(0) = -\infty$ holds if and only if $\mu([0,\infty)) = \mu((0,\infty)) = \infty$, not assumed for $W \in \mathcal{D}$.

3.2. **Monotonicity of finite order.** A weaker notion of complete monotonicity is the monotonicity of finite order. We adopt the slightly more restrictive definition of monotonicity of order n in the paper [25] and not the somewhat weaker definitions in the earlier works [38] and [29].

A function $f : (0,\infty) \to \mathbb{R}$ is called monotonic of (finite) order n, denoted by $f \in \mathcal{CM}(n)$, if it has derivatives of order $k = 1, 2, ..., n$ and

$$(-1)^k f^{(k)}(x) \geq 0, \quad x > 0 \quad and \quad k = 0, 1, 2, \dots, n.$$

As in the $\mathcal{CM}$ case, whenever needed, we extend f to $[0,\infty)$ by $f(0) := \lim_{x \downarrow 0} f(x)$, where $f(0) \leq \infty$. In analogy to the class $\mathcal{D}$, we introduce the following definition.

Definition 3.2. For $n \geq 1$, we define $\mathcal{D}(n)$ to be the class of functions $W : [0,\infty) \to \mathbb{R}$, which satisfy

(1) $-W' \in \mathcal{CM}(n-1)$,
(2) $W'(\infty) = 0$.

We note that $W \in \mathcal{D}(n)$ is not necessarily strictly decreasing by definition. While we assume that $W'(\infty) = 0$ (both to make it similar to the $n = \infty$ case and to simplify the upcoming definition of stochastic dominance), we do not impose the condition $W'(0) = \infty$ for $W \in \mathcal{D}(n)$, nor assume that such a W is bounded below.

Proposition 3.3. Fix $W \in \mathcal{D}(n)$, $n \in \{2, 3, \dots\}$. Then,

$$W'(\infty) = W''(\infty) = \cdots = W^{(n-1)}(\infty) = 0 \tag{3.3}$$

and

$$0 \leq -W'(y_1) = \int\limits_{y_1}^{\infty} \cdots \int\limits_{y_{n-1}}^{\infty} (-1)^n W^{(n)}(y_n) dy_n \dots dy_2 < \infty, \quad y_1 > 0.$$

Therefore, any $W \in \mathcal{D}(n)$ has the representation

$$W(y) = W(y_0) + \int_y^{y_0} \int_{y_1}^{\infty} \cdots \int_{y_{n-1}}^{\infty} (-1)^n W^{(n)}(y_n) dy_n \dots dy_2 dy_1, \quad y > 0. \tag{3.4}$$

For each fixed $y_0 > 0$, the above representation holds.

Proof. As $W \in \mathcal{D}(n)$, we have $(-1)^k W^{(k)}(y) \geq 0$ for $k = 1, \dots, n$ and $W'(\infty) = 0$. Assume now that $W^{(k)}(\infty) = 0$ for some $k \leq n-2$. Since $(-1)^{k+2} W^{(k+2)}(y) \geq 0$, we conclude that the function

$$y \to (-1)^{k+1} W^{(k+1)}(y) \geq 0$$

is decreasing. Next, assume that $W^{(k+1)}(\infty) \neq 0$, so

$$(-1)^{k+1} W^{(k+1)}(\infty) > 0.$$

This, however, would contradict the monotonicity of $y \to (-1)^k W^{(k)}(y)$, which is decreasing, and the assumption that $W^{(k)}(\infty) = 0$. An inductive argument completes the proof. □

3.3. **Stochastic dominance of finite order.** Let F and G be two cumulative distribution functions with supports on $\mathbb{R}_+ = [0, \infty)$. We recall that F *stochastically dominates* G *in the first order* if

$$F(y) \leq G(y), \quad y \geq 0.$$

To define stochastic dominance of higher orders, following, for example, [35], we set

$$F_1 = F \quad \text{and} \quad F_i(y) = \int_0^y F_{i-1}(z) dz, \quad i = 2, 3 \dots. \tag{3.5}$$

Since $0 \leq F \leq 1$, the integrals are well defined. The functions G_i are defined similarly. Next, we depart slightly from the definition customary in the literature, e.g., in [35], see also [39] and [36]. On the one hand, we use a somewhat weaker definition, while, on the other, we can treat unbounded supports. More comments follow the definition.

Definition 3.4. For any $n \geq 1$, we say that F stochastically dominates G in the sense of the n-th order, and denote $F \succeq_n G$, if $F_n(y) \leq G_n(y)$, $y \geq 0$. For two random variables $\xi, \eta \geq 0$ we say that $\xi \succeq_n \eta$ if $F_\xi \succeq_n F_\eta$.

Remark 3.5. For $n \geq 3$, it is customary in the literature, in order to define $F \succeq_n G$, to both

(1) assume that F and G are supported on a finite interval $[0, b]$,
(2) have the additional condition $F_k(b) \leq G_k(b)$, $k = 1, \dots, n-1$.

Our definition by-passes both points above since we will only use a restrictive set of "test" functions, namely $\mathcal{D}(n)$. For such test functions, condition (3.3) ensures that we do not (even formally) need the extra assumption. Furthermore, our definition works well for $n \geq 3$ for measures fully supported on the $[0, \infty)$ that we need.

Proposition 3.6. Consider two non-negative random variables ξ and η. Fix $n \geq 2$. Then, the following conditions are equivalent:

(1) $\xi \succeq_n \eta$,
(2) $\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$ for every function $W \in \mathcal{D}(n)$, such that $W(\infty) > -\infty$, (i.e., W is bounded from below),
(3) $\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$ for every function $W \in \mathcal{D}(n)$ such that $\mathbb{E}[W^-(\xi)] < \infty$ and $\mathbb{E}[W^-(\eta)] < \infty$.

Proof. If $W \in \mathcal{D}(n)$ is bounded below, we will suppose that $W(\infty) = 0$, without loss of generality. For $y_0 = \infty$, representation (3.4) becomes

$$\begin{aligned} W(y) &= \int_y^\infty \int_{y_1}^{\infty} \dots \int_{y_{n-1}}^{\infty} (-1)^n W^{(n)}(y_n) dy_n \ \dots dy_2 dy_1 \\ &= \int_{\mathbb{R}_+^n} 1_{\{y \leq y_1 \leq \dots \leq y_n\}} (-1)^n W^{(n)}(y_n) dy_n \ \dots dy_2 dy_1, \quad y > 0. \end{aligned} \tag{3.6}$$

Therefore, Fubini's theorem yields

$$\begin{aligned} \mathbb{E}[W(\xi)] &= \mathbb{E}\left[\int_{\mathbb{R}_+^n} 1_{\{\xi \leq y_1 \leq \dots \leq y_n\}} (-1)^n W^{(n)}(y_n) dy_n \ \dots dy_2 dy_1 \right] \\ &= \int_0^\infty \left(\int_{\mathbb{R}_+^{n-1}} \mathbb{E}\left[1_{\{\xi \leq y_1 \leq \dots \leq y_n\}} \right] dy_1 \ \dots dy_{n-1} \right) (-1)^n W^{(n)}(y_n) dy_n. \end{aligned} \tag{3.7}$$

Fix y_n. Using the cdf F of ξ we can rewrite

$$\begin{aligned} \int_{\mathbb{R}_+^{n-1}} \mathbb{E}\left[1_{\{\xi \leq y_1 \leq \dots \leq y_n\}} \right] dy_1 \ \dots dy_{n-1} &= \int_0^{y_n} \dots \int_0^{y_2} \mathbb{P}\left[\xi \leq y_1 \right] dy_1 \ \dots dy_{n-1} \\ &= \int_0^{y_n} \dots \int_0^{y_2} F(y_1) dy_1 \ \dots dy_{n-1} = F_n(y_n), \end{aligned}$$

where we have used that $\xi \geq 0$. Together with (3.7), we obtain

$$\mathbb{E}[W(\xi)] = \int_0^\infty W(y) dF(y) = \int_0^\infty (-1)^n W^{(n)}(y_n) F_n(y_n) dy_n. \tag{3.8}$$

This shows that (1) and (2) above are equivalent.

To show (2) $\Rightarrow$ (3), for a general $W \in \mathcal{D}(n)$ in (3), first assume that $W(1) = 0$, without any loss of generality. Next, one has to use a smooth cut-off of the n-th derivative of W away from infinity. More precisely, consider an increasing sequence of functions

$$0 \leq f_i(y) \nearrow 1, 0 < y < \infty$$

and such that

$$supp(f_i) \subset (\frac{1}{i}, i).$$

Then we set

$$W_i^{(n)}(y) := f_i(y) W^{(n)}(y), y > 0,$$

and

$$W_i(1) = 0, W'(\infty) = \cdots = W^{(n-1)}(\infty) = 0.$$

We therefore recover all the lower order derivatives, up to the first order, using the computation in Proposition 3.3:

$$0 \le -W_i'(y_1) = \int_{y_1}^{\infty} \cdots \int_{y_{n-1}}^{\infty} (-1)^n f_i(y_n) W^{(n)}(y_n) dy_n \dots dy_2 < \infty, \quad y_1 > 0,$$

so

$$0 \le -W_i'(y_1) \nearrow -W'(y_1), 0 < y_1 < \infty.$$

We integrate the above relation to conclude that

$$\begin{aligned} W_i^+(y) &= 1_{\{y \le 1\}} W_i(y) \\ &= 1_{\{y \le 1\}} \int_y^1 -W_i'(y_1) dy_1 \nearrow 1_{\{y \le 1\}} \int_y^1 -W'(y_1) dy_1 = W^+(y), \end{aligned}$$

and

$$\begin{aligned} W_i^-(y) &= 1_{\{y > 1\}} (-W_i(y)) \\ &= 1_{\{y > 1\}} \int_1^y -W_i'(y_1) dy_1 \nearrow 1_{\{y > 1\}} \int_1^y -W'(y_1) dy_1 = W_+(y). \end{aligned}$$

Using (2) for the bounded test functions $W_i \in \mathcal{D}(n)$, we can pass to the limit separately for the positive and negative parts to conclude that (2) ⇒ (3). □

3.4. **Stochastic dominance of infinite degree.** The infinite-order stochastic dominance is, intuitively, defined by letting $n \uparrow \infty$ in Definition 3.4. This, however, has to be done carefully. We again depart from [35] for our definition.

To provide some intuition, we first note that, for each $z > 0$, the exponential function $W(y) = e^{-zy}$, $y > 0$, is in $\mathcal{D}(n)$, for every $z > 0$ and $n \ge 1$. For every $z > 0$, relation (3.8) reads

$$\mathbb{E}[e^{-z\xi}] = \int_0^{\infty} e^{-zy} dF(y) = \int_0^{\infty} z^n e^{-zy} F_n(y) dy, \quad n \ge 1.$$

Therefore, if for any n, no matter how large, we have $F \succeq_n G$, then the exponential moments of the two distributions compare, for all positive values of z. It thus appears to us that the weakest possible form of dominance, obtaining by letting $n \uparrow \infty$, is the one below.

Definition 3.7. Consider two cumulative distributions F and G on $[0,\infty)$. We say that F dominates G in *infinite degree stochastic dominance*, and denote by $F \succeq_\infty G$, if

$$\int_0^\infty e^{-zy} dF(y) \le \int_0^\infty e^{-zy} dG(y), \quad z > 0.$$

For nonnegative random variables ξ and η, we say that ξ dominates η in *infinite-order stochastic dominance*, and denote $\xi \succeq_\infty \eta$ if $F_\xi \succeq_\infty F_\eta$, that is

$$\mathbb{E}\left[e^{-z\xi}\right] \le \mathbb{E}\left[e^{-z\eta}\right], \quad z > 0.$$

Remark 3.8. To the best of our knowledge, the name of *infinite-order stochastic dominance* first appeared in [35] (but for a somewhat less precise definition), whereas [3] does not use the specific name of infinite-order dominance.

Below, we provide a characterization of infinite-order stochastic dominance.

Proposition 3.9. Consider two non-negative random variables ξ and η. Then, the following conditions are equivalent:

(1) $\xi \succeq_\infty \eta$,
(2) $\mathbb{E}[W(\xi)] \le \mathbb{E}[W(\eta)]$ for every function $W \in \mathcal{D}$, such that $W(\infty) > -\infty$, i.e., W is bounded from below,
(3) $\mathbb{E}[W(\xi)] \le \mathbb{E}[W(\eta)]$ for every function $W \in \mathcal{D}$ such that

$$\mathbb{E}[W^-(\xi)] < \infty \quad \textit{and} \quad \mathbb{E}[W^-(\eta)] < \infty. \tag{3.9}$$

Proof. (2) $\Rightarrow$ (1). Let us suppose that (2) holds and assume by contradiction that there exists $z > 0$, such that

$$\mathbb{E}\left[e^{-z\xi}\right] > \mathbb{E}\left[e^{-z\eta}\right]. \tag{3.10}$$

Let us consider a measure μ such that

$$\mu(\{0\}) = 0, \qquad \mu((0,\infty)) = \infty, \quad \textit{and} \quad \int_0^\infty \int_0^\infty e^{-yt} \mu(dt) dy = \int_0^\infty \frac{1}{t} \mu(dt) < \infty.$$

For example

$$\mu(dt) = \begin{cases} t^{-\frac{1}{2}} dt, & t \ge 1 \\ 0\ dt, & t \in (0,1) \end{cases},$$

works. Next, let us define W as

$$W(y) := \int_y^\infty \int_0^\infty e^{-zt} \mu(dt) dz, \quad y > 0.$$

Then $W \in \mathcal{D}$ and is bounded from below by 0 and from above by

$$\int_0^\infty \int_0^\infty e^{-yt} \mu(dt) dy < \infty.$$

Now, for a constant $c > 0$, we set

$$\nu_c := \mu + c\delta_z,$$

where δ_z is a delta function centered at z. We further define

$$W_c(y) := \int_y^\infty \int_0^\infty e^{-zt}\nu_c(dt)dz, \quad y > 0.$$

One can see that $W_c \in \mathcal{D}$ and is bounded from below by 0. By Tonelli's theorem, we have

$$W_c(y) = \int_0^\infty e^{-yt}\frac{\nu_c(dt)}{t}, \quad y > 0,$$

and thus

$$\begin{aligned}\mathbb{E}\left[W_c(\xi)\right] &= \mathbb{E}\left[\int_0^\infty e^{-\xi t}\frac{\nu_c(dt)}{t}\right] = \int_0^\infty \mathbb{E}\left[e^{-\xi t}\right]\frac{\nu_c(dt)}{t} \\ &= \int_0^\infty \mathbb{E}\left[e^{-\xi t}\right]\frac{\mu(dt)}{t} + \frac{c}{z}\mathbb{E}\left[e^{-\xi z}\right] = \mathbb{E}\left[W(\xi)\right] + \frac{c}{z}\mathbb{E}\left[e^{-\xi z}\right].\end{aligned}$$

Therefore, from (3.10), and for a sufficiently large c, we obtain that

$$\mathbb{E}\left[W_c(\xi)\right] = \mathbb{E}\left[W(\xi)\right] + \frac{c}{z}\mathbb{E}\left[e^{-\xi z}\right] > \mathbb{E}\left[W(\eta)\right] + \frac{c}{z}\mathbb{E}\left[e^{-\eta z}\right] = \mathbb{E}\left[W_c(\eta)\right],$$

which contradicts (2).

(3) $\Rightarrow$ (2) is trivial. Therefore, it remains to show (1) $\Rightarrow$ (3). Let us consider ξ and η satisfying (1) and $W \in \mathcal{D}$ such that (3.9) holds. Then $-W'$ admits the representation (3.1) for some nonnegative sigma-finite measure μ on $[0,\infty)$, satisfying (3.2), and such that the integral in (3.1) converges for every $x > 0$. Let us also fix $y_0 > 0$. Using the Bernstein representation and Tonelli's theorem, we get

$$\begin{aligned}W(y) - W(y_0) &= \int_y^{y_0} \left(-W'(x)\right)dx = \int_y^{y_0}\int_0^\infty e^{-xt}\mu(dt)dx \\ &= \int_0^\infty\int_y^{y_0} e^{-xt}dx\mu(dt) = \int_0^\infty \left(e^{-yt} - e^{-y_0 t}\right)\frac{\mu(dt)}{t}.\end{aligned} \tag{3.11}$$

Therefore, using (3.9) and Tonelli's theorem, we obtain

$$\begin{aligned}-\infty &< \mathbb{E}\left[\left(W(\xi) - W(y_0)\right)1_{\{\xi\geq y_0\}}\right] \\ &= \mathbb{E}\left[\left(\int_0^\infty \left(e^{-\xi t} - e^{-y_0 t}\right)\frac{\mu(dt)}{t}\right)1_{\{\xi\geq y_0\}}\right] \\ &= \mathbb{E}\left[\int_0^\infty \left(e^{-\xi t} - e^{-y_0 t}\right)1_{\{\xi\geq y_0\}}\frac{\mu(dt)}{t}\right] \\ &= \int_0^\infty \mathbb{E}\left[\left(e^{-\xi t} - e^{-y_0 t}\right)1_{\{\xi\geq y_0\}}\right]\frac{\mu(dt)}{t} \leq 0.\end{aligned}$$

Consequently, using (3.2) and the monotone convergence theorem, we conclude that

$$
\begin{aligned}
-\infty < \mathbb{E}\left[(W(\xi) - W(y_0))\, 1_{\{\xi \geq y_0\}}\right] &= \int_0^\infty \mathbb{E}\left[\left(e^{-\xi t} - e^{-y_0 t}\right) 1_{\{\xi \geq y_0\}}\right] \frac{\mu(dt)}{t} \\
&= \lim_{n\to\infty} \int_{1/n}^{n} \mathbb{E}\left[\left(e^{-\xi t} - e^{-y_0 t}\right) 1_{\{\xi \geq y_0\}}\right] \frac{\mu(dt)}{t} \leq 0.
\end{aligned} \tag{3.12}
$$

Similarly, we obtain

$$
\begin{aligned}
0 \leq & \mathbb{E}\left[(W(\xi) - W(y_0))\, 1_{\{\xi < y_0\}}\right] = \mathbb{E}\left[\int_0^\infty \left(e^{-\xi t} - e^{-y_0 t}\right) 1_{\{\xi < y_0\}} \frac{\mu(dt)}{t}\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} \int_{1/n}^{n} \left(e^{-\xi t} - e^{-y_0 t}\right) 1_{\{\xi < y_0\}} \frac{\mu(dt)}{t}\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\int_{1/n}^{n} \left(e^{-\xi t} - e^{-y_0 t}\right) 1_{\{\xi < y_0\}} \frac{\mu(dt)}{t}\right] \\
&= \lim_{n\to\infty} \int_{1/n}^{n} \mathbb{E}\left[\left(e^{-\xi t} - e^{-y_0 t}\right) 1_{\{\xi < y_0\}}\right] \frac{\mu(dt)}{t},
\end{aligned} \tag{3.13}
$$

where the latter limit might be finite or not. Combining (3.12) and (3.13), we assert that

$$
-\infty < \mathbb{E}\left[W(\xi)\right] = W(y_0) + \lim_{n\to\infty} \int_{1/n}^{n} \left(\mathbb{E}\left[e^{-\xi t}\right] - e^{-y_0 t}\right) \frac{\mu(dt)}{t},
$$

and a similar representation holds for η. Therefore, if $\xi \succeq_\infty \eta$ and (3.9) holds, we have

$$
\begin{aligned}
-\infty < \mathbb{E}\left[W(\xi)\right] &= W(y_0) + \lim_{n\to\infty} \int_{1/n}^{n} \left(\mathbb{E}\left[e^{-\xi t}\right] - e^{-y_0 t}\right) \frac{\mu(dt)}{t} \\
&\leq W(y_0) + \lim_{n\to\infty} \int_{1/n}^{n} \left(\mathbb{E}\left[e^{-\eta t}\right] - e^{-y_0 t}\right) \frac{\mu(dt)}{t} = \mathbb{E}\left[W(\eta)\right],
\end{aligned}
$$

which implies (3). This completes the proof of the proposition. □

4. Main results

4.1. **The $\mathcal{CMIM}$ and $\mathcal{CMIM}\,(n)$ utilities.** As utility-based preferences are invariant under positive linear transformations of the form $U^*(x) = aU(x) + b$, $a > 0$, and in view of the importance of the marginal utility in many problems, it is natural to define a utility function through its derivative. Additionally, it has been observed that the most widely used utility functions have completely monotonic marginals, see [3]. In the present paper, we investigate a class of functions, whose *inverse marginals* are completely monotonic. This is particularly natural in view of the overall importance of the duality approach to the expected utility maximization.

We start with the following definition.

Definition 4.1. We define the $\mathcal{CMIM}$ to be the class of utility functions $U \in -\mathcal{C}$ for which their inverse marginal $(U')^{-1} \in \mathcal{CM}$.

From Bernstein's theorem, we deduce that if $U \in \mathcal{CMIM}$, then we have the representation

$$(U')^{-1}(y) = \int_0^\infty e^{-yz}\mu(dz), \quad y > 0, \tag{4.1}$$

where μ is a nonnegative measure, such that the integral converges for every $y > 0$.

We stress that the Inada conditions (2.4) dictate that the underlying measure μ must satisfy $\mu(\{0\}) = 0$ and $\mu((0,\infty)) = \infty$. Indeed, $U'(0) = \infty$ yields $\mu(\{0\}) = 0$ while $U'(\infty) = 0$ yields $\mu((0,\infty)) = \infty$.

Example 4.2. Here we show that standard utilities are included.

(1) $U(x) = \log x$, $x > 0$. Then, $(U')^{-1}(y) = \frac{1}{y} \in \mathcal{CM}$ and we have

$$\left(U'\right)^{-1}(y) = \int_0^\infty e^{-yz}dz, \quad y > 0.$$

(2) $U(x) = \frac{x^p}{p}$, $x > 0$, $p < 1$, $p \neq 0$. Then, $(U')^{-1}(y) = y^{-\frac{1}{1-p}} \in \mathcal{CM}$ and with $q = -\frac{p}{1-p}$ (i.e., such that $\frac{1}{p} + \frac{1}{q} = 1$), we have

$$\left(U'\right)^{-1}(y) = \frac{1}{\Gamma(1-q)}\int_0^\infty e^{-yz}z^{-q}dz, \quad y > 0,$$

where Γ is the Gamma function, see [33, p. viii, formula (2)].

Assuming less regularity on the utility function but keeping monotonic structure up to finite order leads to the following definition.

Definition 4.3. For $n \in \{2, 3, \dots\}$, we say that a utility function $U \in -\mathcal{C}$ is in the $\mathcal{CMIM}(n)$ class if its inverse marginal is completely monotonic of order $n-1$, that is, $(U')^{-1} \in \mathcal{CM}(n-1)$.

Recalling Definitions 3.1 and 4.1, and denoting by V the convex conjugate of U in the sense of (2.6), we deduce that

$$U \in \mathcal{CMIM} \iff V \in \mathcal{D} \cap \mathcal{C}.$$

Likewise, from Definitions 3.2 and 4.3, we get

$$U \in \mathcal{CMIM}\,(n) \iff V \in \mathcal{D}\,(n) \cap \mathcal{C}.$$

4.2. **The class $\mathcal{SD}(\infty)$ and $\mathcal{SD}(n)$ of market models.** The stochastic dominance had to be formally defined separately for finite and infinite n (see Definitions 3.4 and 3.7). However, for the associated market models, now, we can give a unified definition (for both finite and infinite degree) below.

Definition 4.4. Fix $n \in \{2, 3, \dots\} \cup \{\infty\}$. We say that the financial model satisfies condition $\mathcal{SD}(n)$ if there exists $\widehat{Y} \in \mathcal{Y}(1)$ such that $\widehat{Y}_T \succeq_n Y_T$ for every $Y \in \mathcal{Y}(1)$.

In what follows, we will use the terminologies "market model in $\mathcal{SD}(n)$ class" and "$\mathcal{SD}(n)$-model" interchangeably. In view of Propositions 3.6 and 3.9, we have the following result, for *both* infinite and finite orders.

Lemma 4.5. Fix $n \in \{2, 3, ...\} \cup \{\infty\}$. Assume that the model satisfies condition $\mathcal{SD}(n)$ and that $U \in \mathcal{CMIM}(n)$, thus $V \in \mathcal{D}(n) \cap \mathcal{C}$. Then, the dual value function has the representation

$$v(y) = \mathbb{E}[V(y\widehat{Y}_T)], \quad y > 0. \tag{4.2}$$

This result yields the key property that, up to a multiplicative constant, the dual problem admits the *same optimizer,* $\widehat{Y}$, for *any* initial $y > 0$. Representation (4.2) can be thought of as a *relaxation* of the notion of model *completeness*, in the following sense: while a market model in $\mathcal{SD}(n)$ is, in general, incomplete from the point of view of replication, it does behave like a complete one from the point of view of optimal investment, if the utility function $U \in \mathcal{CMIM}(n)$.

4.3. **Main theorems.** We will assume that

$$v(y) < \infty, \quad y > 0. \tag{4.3}$$

We recall that the above condition is the canonical integrability one on the dual value function that is necessary and sufficient for the standard assertions of the utility maximization theory to hold; see [20] (see, also, [18] for the formulation without *NFLVR* and [27] for the formulation with intermediate consumption and stochastic utility, where (4.3) is combined with the finiteness of the primal value function). We also recall that for $U \in -\mathcal{C}$, under (2.3) and (4.3), for every $x, y > 0$, there exist unique optimizers, $\widehat{X}(x) \in \mathcal{X}(x)$ and $\widehat{Y}(y) \in \mathcal{Y}(y)$, for the primal (2.5) and dual (2.7) problems, respectively. This is a consequence of the abstract theorems in [27].

Theorem 4.6. Consider a financial model for which (2.3) holds, and which is in $\mathcal{SD}(\infty)$. For $U \in \mathcal{CMIM}$ and the measure μ defined by the Bernstein representation (4.1) of $(U')^{-1}$, consider the optimal investment problem (2.5) and assume that (4.3) holds. Then, the following assertions hold:

(1) The value function $u \in \mathcal{CMIM}$ and is thus analytic.

(2) The dual value function $v \in \mathcal{D} \cap \mathcal{C}$. Furthermore, for $n \in \{1, 2, \dots\}$, we have

$$(-1)^n v^{(n)}(y) = (-1)^n \mathbb{E}\left[V^{(n)}(\widehat{Y}_T(y)) \left(\frac{\widehat{Y}_T(y)}{y} \right)^n \right] \tag{4.4}$$
$$= (-1)^n \mathbb{E}\left[V^{(n)}(y\widehat{Y}_T) \left(\widehat{Y}_T \right)^n \right]$$
$$= \mathbb{E}\left[\int_0^\infty e^{-zy\widehat{Y}_T} z^{n-1} \widehat{Y}_T^n \mu(dz) \right] \in (0, \infty), \quad y > 0.$$

(3) The $\mathcal{CM}$ function $-v'$ admits the Bernstein representation

$$v'(y) = -\int_0^\infty e^{-yz} \nu(dz), \quad y > 0, \tag{4.5}$$

for some sigma-finite measure ν supported in $(0, \infty)$, such that

$$\nu(\{0\}) = 0 \quad \textit{and} \quad \nu((0, \infty)) = \infty,$$

and which satisfies

$$\lim_{n \uparrow \infty} \int_{(0,z]} (-1)^{n+1} v^{(n+1)} \left(\frac{n}{\rho} \right) \left(\frac{n}{\rho} \right)^{n+1} d\rho = \frac{\nu((0, z]) + \nu((0, z))}{2}, \quad z > 0. \tag{4.6}$$

(4) For $n \geq 2$ and $f(y) := -\frac{1}{v''(y)}$, $y > 0$, we have

(4.7)
$$u^{(n)}(x) = \sum \frac{(n-2)!}{k_1! 1!^{k_1} \dots k_{n-2}! (n-2)!^{k_{n-2}}} f^{(k_1 + \dots + k_{n-2})}(u'(x)) \prod_{j=1}^{n-2} \left(u^{(j+1)}(x) \right)^{k_j},$$
$$x > 0,$$

where the sum is over all n-tuples of nonnegative integers $(k_1, \dots, k_{n-2})$ satisfying $\sum_{i=1}^{n-2} i k_i = n - 2$.

The following theorem specifies the derivatives of optimizers of all orders for both primal and dual problems.

Theorem 4.7. Under the conditions of Theorem 4.6, the following assertions hold:

(1) The primal and dual optimizers are related via

$$\widehat{X}_T(x) = \int_0^\infty e^{-zu'(x)\widehat{Y}_T} \nu(dz), \quad x > 0. \tag{4.8}$$

(2) Their derivatives are given by

$$\widehat{X}_T^{(1)}(x) := \lim_{h \to 0} \frac{\widehat{X}_T(x+h) - \widehat{X}_T(x)}{h} = -V''(u'(x)\widehat{Y}_T) u''(x) \widehat{Y}_T \tag{4.9}$$
$$= \frac{u''(x)}{U''(\widehat{X}_T(x))} \widehat{Y}_T > 0,$$

(4.10) $$\widehat{Y}_T^{(1)}(y) := \lim_{h\to 0}\frac{\widehat{Y}_T(y+h)-\widehat{Y}_T(y)}{h} = \frac{\widehat{Y}_T(y)}{y} = \widehat{Y}_T.$$

(3) Recursively, for every $n \geq 2$, the higher-order derivatives are given by

$$\widehat{X}_T^{(n)}(x) := \lim_{h\to 0}\frac{\widehat{X}_T^{(n-1)}(x+h)-\widehat{X}_T^{(n-1)}(x)}{h}$$
$$= \sum \frac{n!}{k_1!1!^{k_1}k_2!2!^{k_2}\dots k_n!n!^{k_n}}\left(-V^{(1+k_1+\dots+k_n)}\left(u'(x)\widehat{Y}_T\right)\right)\prod_{j=1}^{n}\left(u^{(j+1)}(x)\widehat{Y}_T\right)^{k_j},$$

where the sum is over all n-tuples of nonnegative integers $(k_1,\dots,k_n)$ satisfying $\sum\limits_{i=1}^{n} ik_i = n$. Trivially,

$$\widehat{Y}_T^{(n)}(y) := \lim_{h\to 0}\frac{\widehat{Y}_T^{(n-1)}(y+h)-\widehat{Y}_T^{(n-1)}(y)}{h} = 0.$$

The limits for the derivatives of the optimizers above are in probability. However, because of the multiplicative structure of the dual $\widehat{Y}(y) = y\widehat{Y}$, the limits above can be understood in the stronger sense: for every sequence $h_k \to 0$, the convergence holds for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Analogous results (and even easier, in many ways) can be stated for the case $n < \infty$.

Proposition 4.8. Fix $n \in \{2,3,\dots\}$ Consider a financial model for which (2.3) holds, and which is in $\mathcal{SD}(n)$. Let $U \in \mathcal{CMIM}(n)$ so $V = V_U \in \mathcal{D}(n)\cap\mathcal{C}$. Furthermore, assume (4.3) and that V satisfies the inequalities
(4.11)
$$0 < c_k \leq -\frac{y\ V^{(k+1)}(y)}{V^{(k)}(y)} = \frac{(-y)^{k+1}V^{(k+1)}(y)}{(-y)^k V^{(k)}(y)} \leq d_k < \infty,\ y>0,\ \ k=1,\dots,n-1,$$

for some constants c_k, d_k, $k = 1,\dots,n-1$. Then, for the optimal investment problem (2.5), the following assertions hold:

(1) The value function $u \in \mathcal{CMIM}(n)$.
(2) Up to order n, the derivatives of the dual value function v are given by (4.4).
(3) The dual value function v satisfies the bounds (4.11) with respect to the same constants c_k and d_k, for $k = 1,\dots,n-1$.

Stochastic dominance of order n is the natural condition to study n-th order differentiability of value functions, as can be seen from the Proposition above. As defined, $\mathcal{SD}(n)$ for $n \geq 3$ appears a weaker condition than second order stochastic dominance ($n = 2$). However, we show in Proposition 7.2 that, surprisingly, these notions coincide after all, from $n = 2$ to $n = \infty$. Therefore, the class of models considered here is the same as the models with a maximal element in the sense of second order dominance in [22]. In such models, [22, Theorem 5] shows that *the risk tolerance wealth process* $R(x)$ exists, for every $x > 0$.

We briefly recall the definition of $R(x)$ in [22] as the maximal wealth process $R(x) = (R_t(x))_{t\in[0,T]}$, such that $R_T(x) = -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))}$. The process $R(x)$ is equal, up to an initial value, to the first derivative of the primal optimizer:

$$\lim_{h\to 0} \frac{\widehat{X}_T(x+h) - \widehat{X}_T(x)}{h} = \frac{R_T(x)}{R_0(x)},$$

where the limit is in $\mathbb{P}$ probability.

We also note that the existence of the risk tolerance wealth process is connected to asymptotic expansions of second order (see [22], [23]) or [28]). For a formulation and asymptotics with consumption and their relationship to the risk tolerance wealth process, we refer to [11].

Remark 4.9 (On an integrability convention). Recalling a common convention (see, e.g., [27]), we set

$$\mathbb{E}\left[V(Y_T)\right] := \infty \quad if \quad \mathbb{E}\left[V^+(Y_T)\right] = \infty. \tag{4.12}$$

This is done to avoid issues related to $\mathbb{E}\left[V(Y_T)\right]$ not being well-defined.

However, the following argument shows that $\mathbb{E}\left[V(Y_T)\right]$ is well-defined for every $y > 0$ and $Y \in \mathcal{Y}(y)$. To see this, take an arbitrary $y > 0$ and $Y \in \mathcal{Y}(y)$. Then, from the conjugacy between U and V, we get

$$-V(Y_T) \leq Y_T - U(1) \leq Y_T + |U(1)|.$$

Therefore, by the supermartingale property of Y, we obtain

$$\mathbb{E}\left[V^-(Y_T)\right] \leq y + |U(1)| < \infty.$$

Therefore, $\mathbb{E}\left[V(Y_T)\right]$ is well-defined with or without convention (4.12).

In analogy for the primal problem, we set

$$\mathbb{E}\left[U(X_T)\right] := -\infty \quad if \quad \mathbb{E}\left[U^-(X_T)\right] = \infty. \tag{4.13}$$

Assume that

$$v(y) < \infty \quad for\ some\ y > 0, \tag{4.14}$$

which is even weaker than (4.3).

By the argument from the previous paragraph, we know that $\mathbb{E}\left[V(Y_T)\right] \in \mathbb{R}$. For an arbitrary $x > 0$ and $X \in \mathcal{X}(x)$, by conjugacy between U and V, we get

$$U(X_T) \leq V(Y_T) + X_T Y_T \leq V^+(Y_T) + V^-(Y_T) + X_T Y_T.$$

Therefore,

$$U^+(X_T) \leq V^+(Y_T) + V^-(Y_T) + X_T Y_T \ \in \ \mathbb{L}^1(\mathbb{P}).$$

Thus, $\mathbb{E}\left[U(X_T)\right]$ is well-defined for every $x > 0$, and $X \in \mathcal{X}(x)$, with or without the convention (4.13), under the minimal assumption (4.14).

Remark 4.10 (On the positivity of $\widehat{X}_T^{(1)}$). We note that, in general, the derivative of the primal optimizer with respect to the initial wealth does not have to be a positive random variable, see [21, Example 4]. However, for models satisfying the assumptions of Theorem 4.6, (4.9) implies the strict positivity of $\widehat{X}_T^{(1)}$. This complements the results in [22], see Theorem 4 there, which implies the positivity of

$\widehat{X}_T^{(1)}$ in stochastically dominant models in the sense of the second order stochastic dominance.

Proof of Theorem 4.6. As $U \in \mathcal{CMIM}$, its convex conjugate $V \in \mathcal{D} \cap C$. Then, Lemma 4.5 gives that

$$v(y) = \mathbb{E}\left[V(y\widehat{Y}_T)\right], \ y > 0,$$

i.e., the dual minimizer is the same up to the multiplicative constant y: $\widehat{Y}(y) = y\widehat{Y}$, $y > 0$. The expectation above is well-defined, see the discussion in Remark 4.9. Note that (2.3), (4.3) and the structure of the utility function (the Inada conditions, together with the strict monotonicity, concavity, and smoothness) imply the strict concavity and continuous differentiability of both u and $-v$ on $(0,\infty)$; see [20], [27], and [4]. For $n = 1$, (4.4) follows from the standard conclusions of the utility maximization theory, as

$$-v'(y)y = -\mathbb{E}\left[V'(\widehat{Y}_T(y))\widehat{Y}_T(y)\right] = -\mathbb{E}\left[V'(y\widehat{Y}_T)y\widehat{Y}_T\right] \in (0,\infty), \tag{4.15}$$

and therefore, $v'(y) = \mathbb{E}\left[V'(y\widehat{Y}_T)\widehat{Y}_T\right]$, for every $y > 0$.

Next, assume that (4.4) holds for $n = k$, i.e.,

$$v^{(k)}(y) = \mathbb{E}\left[V^{(k)}\left(y\widehat{Y}_T\right)\left(\widehat{Y}_T\right)^k\right] = (-1)^k\mathbb{E}\left[\int_0^\infty e^{-y\widehat{Y}_T z} z^{k-1}\mu(dz)\left(\widehat{Y}_T\right)^k\right], \quad y > 0,$$

where in the second equality we have used (4.1). Let us recall that (4.1) gives

$$V^{(k)}(y) = (-1)^k\int_0^\infty e^{-yz}z^{k-1}\mu(dz), \quad y > 0.$$

Then, let us consider

$$\begin{aligned}\frac{v^{(k)}(y+h) - v^{(k)}(y)}{h} &= \frac{1}{h}\mathbb{E}\left[V^{(k)}((y+h)\widehat{Y}_T)\widehat{Y}_T^k - V^{(k)}(y\widehat{Y}_T)\widehat{Y}_T^k\right] \\ =&(-1)^k\mathbb{E}\left[\int_0^\infty \frac{(\widehat{Y}_T z)^{k-1}\widehat{Y}_T}{h}\left(e^{-(y+h)\widehat{Y}_T z} - e^{-y\widehat{Y}_T z}\right)\mu(dz)\right],\end{aligned} \tag{4.16}$$

Let us fix $y > 0$. As for every $h \neq 0$, we have

$$0 \leq -\frac{1}{h}\left(e^{-(y+h)\widehat{Y}_T z} - e^{-y\widehat{Y}_T z}\right) \leq e^{-(y-|h|)\widehat{Y}_T z} z\widehat{Y}_T,$$

we deduce that, for a constant $h_0 \in (0,y)$, and every $h \in (-h_0,h_0)$, the following inequalities hold

$$0 \leq -\left(\widehat{Y}_T z\right)^{k-1}\frac{1}{h}\left(e^{-(y+h)\widehat{Y}_T z} - e^{-y\widehat{Y}_T z}\right)\widehat{Y}_T \leq (\widehat{Y}_T z)^k e^{-\widehat{Y}_T(y-h_0)z}\widehat{Y}_T. \tag{4.17}$$

Furthermore, as there exists a constant M, such that

$$\bar{z}^k e^{-\bar{z}(y-h_0)} \leq M e^{-\frac{1}{2}\bar{z}(y-h_0)}, \quad \textit{for every } \bar{z} \geq 0, \tag{4.18}$$

and since, by (4.4) for $n = 1$ (see also (4.15)), we have

$$\mathbb{E}\left[\int_0^\infty M\widehat{Y}_T e^{-\frac{1}{2}\widehat{Y}_T(y-h_0)z}\mu(dz)\right] = -Mv'\left(\tfrac{1}{2}(y-h_0)\right) < \infty, \tag{4.19}$$

we deduce from (4.18) and (4.19) that the last expression in (4.17) is $\mathbb{P}\times\mu$ integrable. Therefore, in (4.16), one can pass to the limit as $h \to 0$ to deduce that (4.4) holds for $n = k+1$. We conclude that (4.4) holds for every $n \in \mathbb{N}$. Now, the complete monotonicity of v follows from the complete monotonicity of V and (4.4). In turn, this implies the analyticity of v, see, e.g., [26]. Further, as $(-1)^n V^{(n)}$ do not vanish (see, e.g., [33, Remark 1.5]), we deduce from (4.4) that $(-1)^n v^{(n)}$ are also strictly positive for every $n \in \mathbb{N}$. By [20, Theorem 4], $-v$ satisfies the Inada conditions, which imply that $\nu(\{0\}) = 0$ and $\nu((0,\infty)) = \infty$. Representation (4.5) follows, where (4.6) results from the inversion formula, see [37, Chapter VII, Theorem 7a].

To obtain the properties of u, first, we observe that the biconjugacy relations between the value functions imply that u' exists at every $x > 0$, and it is the inverse of $-v'$. This, and since v' is strictly negative on $(0,\infty)$, imply the analyticity of u. In turn, (4.7) is the consequence of the Faà di Bruno formula, see [30, Section 4.3]. □

Proof of Theorem 4.7. First, we observe that (4.8) is the consequence of (4.1) and standard assertions of the utility maximization theory. In turn, (4.10) follows from the optimality of $y\widehat{Y}$ for every $y > 0$, whereas (4.9) results from the relation

$$\widehat{X}_T(x) = -V'(y\widehat{Y}_T), \quad \text{for} \quad y = u'(x).$$

The higher order derivatives of the dual and primal optimizers follow from the direct computations and an application of the Faà di Bruno formula. □

Proof of Proposition 4.8. From Lemma 4.5, we have

$$v(y) = \mathbb{E}[V(y\widehat{Y}_T)], \quad y > 0.$$

By [20, Theorem 4], v is differentiable and we have

$$v'(y) = \mathbb{E}[\widehat{Y}_T V'(y\widehat{Y}_T)], \quad y > 0.$$

Therefore, it only remains to compute the higher order derivatives of v, recursively, up to order n, as in formula (4.4). The bounds (4.11) for v would follow immediately. In what follows, we show that (4.4) holds up to order n. Assume that, for some $1 \le k \le n-1$, we have

$$(-1)^{k-1}v^{(k-1)}(y) = \mathbb{E}[(-\widehat{Y}_T)^{k-1}V^{(k-1)}(y\widehat{Y}_T)] < \infty, \quad y > 0. \tag{4.20}$$

Using bounds (4.11) and following the proof of [21, Lemma 3] we obtain that, for any $a > 1$, there exist some constants

$$\alpha_k < 1 < \beta_k < \infty$$

for which

$$\alpha_k(-1)^{k-1}V^{(k-1)}(y) \le (-1)^{k-1}V^{(k-1)}(ay) \le \beta_k(-1)^{k-1}V^{(k-1)}(y), \quad y > 0. \tag{4.21}$$

Fix $y > 0$. Then,

$$\begin{aligned}\frac{v^{(k-1)}(y+h) - v^{(k-1)}(y)}{h} &= \mathbb{E}\left[(\widehat{Y}_T)^{k-1}\frac{V^{(k-1)}((y+h)\widehat{Y}_T) - V^{(k-1)}(y\widehat{Y}_T)}{h}\right] \\ &= \mathbb{E}\left[(\widehat{Y}_T)^k V^{(k)}(\xi_h)\right],\end{aligned}$$

where ξ_h is a random variable taking values between $y\widehat{Y}_T$ and $(y+h)\widehat{Y}_T$.

Fix $a > 1$. Using the bounds (4.11) for $V^{(k)}$ in terms of $V^{(k-1)}$, together with (4.21), we conclude that there exists a finite constant C, such that for $|h|$ small enough so that

$$\frac{1}{a} \leq \frac{y-|h|}{y} \leq \frac{y+|h|}{y} \leq a,$$

we have

$$\left|(\widehat{Y}_T)^k V^{(k)}(\xi_h)\right| \leq C\left|(\widehat{Y}_T)^{k-1} V^{(k-1)}(y\widehat{Y}_T)\right|.$$

Since the right-hand side above is integrable, according to (4.20), and

$$\frac{V^{(k-1)}((y+h)\widehat{Y}_T) - V^{(k-1)}(y\widehat{Y}_T)}{h} \to \widehat{Y}_T V^{(k)}(y\widehat{Y}_T), \quad \mathbb{P}-a.s,$$

we can use the Lebesgue dominated convergence theorem to conclude the assertions of part *ii*). The remaining assertions follow. ☐

4.4. **An Example.** The condition that $\mathcal{Y}(1)$ has a maximal element means that, while the market is incomplete with respect to the replication of contingent claims, it behaves like a complete market from the point of view of optimal investment. Our main result, Theorem 4.6, may appear restrictive at first, for this reason. However, the counter-example in Section 6 shows that this is actually the best one can hope. A generic example has to be precisely such a market model with a maximal dual element. This is the case for a multi-dimensional market driven by Brownian motion with a larger dimension, where the coefficients (market prices of risk) are deterministic. For simplicity, we present a one-dimensional stock driven by two Brownian motions.

Let $W^1 = (W^1_t)_{t\in[0,T]}$ and $W^2 = (W^2_t)_{t\in[0,T]}$ be two independent Brownian motions on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by W^1 and W^2. Let us suppose that there are two traded securities: a riskless asset with zero interest rate, $B_t = 1$, $t \in [0,T]$, and a traded stock, whose dynamics is given by

$$S_t = S_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW^1_s, \quad t \in [0,T].$$

for some $S_0 \in \mathbb{R}$, and where μ and σ are *deterministic* measurable functions on $[0,T]$, such that, for some constant $M > 0$, $-M \leq \mu_t \leq M$ and $\frac{1}{M} \leq \sigma_t \leq M$, $t \in [0,T]$.

Next, consider the process $\widehat{Y}$ given by

$$\widehat{Y}_t = \exp\left(-\int_0^t \frac{\mu_s}{\sigma_s} dW_s^1 - \frac{1}{2}\int_0^t \frac{\mu_s^2}{\sigma_s^2} ds\right), \quad t \in [0,T]. \tag{4.22}$$

Following the argument in [22, Example 7, p. 2185], one can show that

$$\widehat{Y}_T \succeq_2 Y_T, \quad \text{for every} \quad Y \in \mathcal{Y}(1), \tag{4.23}$$

in the sense of Definition 3.4, where we recall that the set of supermartingale deflators $\mathcal{Y}(1)$ is given in (2.2).

The reader may note that the market $(B, S, (\mathcal{F}_t^S)_{0\le t\le T})$ is complete, while the original market $(B, S, (\mathcal{F}_t)_{0\le t\le T})$, with all the information available for investment, is incomplete. In such a market (either one), let us consider the optimal investment problem (2.5). If $U \in \mathcal{CMIM}$ and (4.3) holds, then, with $\widehat{Y}$ being given by (4.22), the properties of the primal and dual value functions are given by the assertions of Theorems 4.6, whereas the properties of the optimizers are given by the conclusions of Theorem 4.7. If, instead, one supposes that for a fixed $n \in \{2, 3, \dots\}$, $U \in \mathcal{CMIM}(n)$ and is such that its convex conjugate V satisfies the inequalities (4.11) for some constants c_k, d_k, $k = 1, \dots, n-1$, then, under (4.3), we obtain from Proposition 4.8 that its assertions apply.

5. Counterexample 1: $\mathcal{SD}(\infty)$ market model and $U \notin \mathcal{CMIM}$

We show that the analyticity of the value function may fail if the utility is not $\mathcal{CMIM}$, even if it is analytic, and even if the market model is complete, and thus in the $\mathcal{SD}(n)$ class for every $n \in \{2, 3, ...\} \cup \{\infty\}$. As the construction shows, we will be using completely monotonic functions of finite order. Working with this class allows to tailor the assumptions on the utility function so that differentiability holds up to order n, but fails at order $n+1$, for *any* choice of $n \in \mathbb{N}$.

Proposition 5.1. Fix $n \ge 1$. There exists a complete market model and an analytic utility function $U : (0,\infty) \to \mathbb{R}$ such that

$$U \in \mathcal{CMIM}(n+1) \iff V \in \mathcal{D}(n+1) \cap \mathcal{C},$$

where the dual V satisfies the bounds (4.11) (up to order $k = n-1$, but not up to order $k = n$), and for which the conjugate value functions u and v satisfy

$$u \in \mathcal{CMIM}(n) \iff v \in \mathcal{D}(n) \cap C,$$

together with identical bounds (4.11) up to order $k = n-1$, but with

$$(-1)^{n+1} v^{(n+1)}(1) = \infty.$$

We will need the following Lemma.

Lemma 5.2. There exists an analytic function $f : (0,\infty) \to \mathbb{R}$ with the following properties

(1) $1 \le f \le 2$,
(2) $f' < 0$,
(3) $-f'(i) \ge \frac{i^2}{C}$, $i = 1, 2, \dots$, for some constant $C > 0$.

Proof of Proposition 5.1. Consider the auxiliary dual value function $\bar{V}(y) := y^{-1}$. Consider now the new utility function V defined as

$$V(y) := \int_y^\infty \int_{y_1}^\infty \cdots \int_{y_{n-1}}^\infty (-1)^n f(y_n)\bar{V}^{(n)}(y_n)dy_n \dots dy_2 dy_1 > 0.$$

Note that the intuition behind this definition comes from setting V at the level of the n-th order derivative,

$$V^{(n)}(y) := f(y)\bar{V}^{(n)}(y), \quad y > 0,$$

and then recovering V by integration.

Since $1 \le f \le 2$ and using the integral representations

$$(-1)^k \bar{V}^{(k)}(y_k) = \int_{y_k}^\infty \dots \int_{y_{n-1}}^\infty (-1)^n \bar{V}^{(n)}(y_n)dy_n \ \dots dy_{k+1}, \quad y_k > 0,$$

$$(-1)^k V^{(k)}(y_k) = \int_{y_k}^\infty \dots \int_{y_{n-1}}^\infty (-1)^n f(y_n)\bar{V}^{(n)}(y_n)dy_n \ \dots dy_{k+1}, \quad y_k > 0,$$

we obtain bounds for derivatives of V in terms of derivatives of the same order of $\bar{V}$,

$$(5.1) \qquad (-1)^k \bar{V}^{(k)}(y) \le (-1)^k V^{(k)}(y) \le 2(-1)^k \bar{V}^{(k)}(y) \quad y > 0,\ k = 0, 1, \dots, n.$$

Since

$$\bar{V}''(y) = 2y^{-3}, \bar{V}'''(y) = -6y^{-4}, \dots, \bar{V}^{(n+1)}(y) = (-1)^{n+1} C_{n+1} y^{-n-2}, \quad y > 0,$$

for some explicit positive constants C_k, $\bar{V}$ satisfies some bounds on higher order risk tolerance coefficients of the type (4.11) up to order $k = n-1$. Using the integral representations above, (5.1) yields similar bounds for risk tolerance type coefficients for V: for every $k = 1, \dots, n-1$, there exist $0 < c_k < d_k < \infty$ such that

$$c_k \le -\frac{yV^{(k+1)}(y)}{V^{(k)}(y)} = \frac{(-y)^{k+1}V^{(k+1)}(y)}{(-y)^k V^{(k)}(y)} \le d_k, \quad y > 0,\ k = 1, \dots, n-1.$$

Since $-f'(i) \ge \frac{i^2}{C}$, we choose weights $q_i := \frac{1}{i^3} > 0$, such that

$$s_1 := \sum_{i=1}^\infty i q_i < \infty, \quad \sum_{i=1}^\infty -f'(i) q_i = \infty.$$

Denoting by

$$s_0 := \sum_{i=1}^\infty q_i < \sum_{i=1}^\infty i q_i = s_1 < \infty,$$

we define

$$\varepsilon := \frac{\frac{1}{2}}{\frac{s_1}{s_0} - \frac{1}{2}} \in (0, 1)$$

and consider a random variable Z such that

$$\mathbb{P}(Z = \frac{1}{2}) = 1 - \varepsilon,\ \mathbb{P}(Z = i) = p_i := \varepsilon \frac{q_i}{s_0}, \quad i = 1, 2, 3, \dots.$$

Then, we have

$$\mathbb{E}[Z] = \frac{1}{2}(1-\varepsilon) + \frac{\varepsilon}{s_0}s_1 = \frac{1}{2} + \varepsilon\left(\frac{s_1}{s_0} - \frac{1}{2}\right) = 1.$$

Next, consider any market with the unique martingale measure with density Z. The dual value function is finite since Z is bounded from below. Therefore,

$$v(y) = \mathbb{E}[V(yZ))] < \infty, \quad y > 0, \tag{5.2}$$

i.e., (4.3) holds. Using Proposition 4.8, we obtain that

$$0 < (-1)^k v^{(k)}(y) = (-1)^k \mathbb{E}[V^{(k)}(yZ)Z^k)] < \infty, \quad y > 0,\ k = 1, \dots, n. \tag{5.3}$$

Therefore, $v \in \mathcal{D}(n) \cap \mathcal{C}$ and, in particular, the n-th derivative is given by

$$v^{(n)}(y) = \mathbb{E}[V^{(n)}(yZ)Z^n)], \quad y > 0. \tag{5.4}$$

The $n+1$ derivative of V is

$$V^{(n+1)}(y) = f(y)\bar{V}^{(n+1)}(y) + f'(y)\bar{V}^{(n)}(y).$$

One should note that

$$(-1)^{n+1}V^{(n+1)}(y) = f(y)(-1)^{n+1}\bar{V}^{(n+1)}(y) + (-f'(y))(-1)^n\bar{V}^{(n)}(y) > 0,$$

and

$$(-y)^{(n+1)}V^{(n+1)}(y) > -C_n f'(y).$$

By construction, $\mathbb{E}[-f'(Z)] = \infty$, so $\mathbb{E}[(-Z)^{n+1}V^{(n+1)}(Z)] = \infty$. Finally, from (5.4) we have

$$\begin{aligned}(-1)^{n+1}\frac{v^{(n)}(y) - v^{(n)}(1)}{y-1} &= (-1)^{n+1}\frac{\mathbb{E}[V^{(n)}(yZ)Z^n) - V^{(n)}(Z)Z^n)]}{y-1} \\ &= \mathbb{E}[V^{(n+1)}(\xi)(-Z)^{n+1}],\end{aligned}$$

for some random variable ξ taking values between Z and yZ. Since

$$0 \le V^{(n+1)}(\xi)(-Z)^{n+1} \to V^{(n+1)}(Z)(-Z)^{n+1},$$

we can now apply Fatou's lemma to obtain

$$(-1)^{n+1}v^{(n+1)}(1) = \infty.$$

□

Proof of Lemma 5.2. Consider the Gaussian densities (up to a multiplicative factor) with mean μ and standard deviation $\sigma > 0$, given by

$$g_{\mu,\sigma}(x) := \frac{1}{\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

Then, we have

$$\int_{-\infty}^{\infty} g_{\mu,\sigma}(x)dx = \sqrt{2\pi}, \quad g_{\mu,\sigma^2}(\mu) = \frac{1}{\sigma}.$$

Next, let

$$g(z) := \sum_{i=1}^{\infty}\frac{1}{i^2}g_{i,i^{-4}}(z), \quad z \in \mathbb{C}.$$

While not obvious, it is easy to see that the series converges uniformly on compacts (in the complex plane), so it is analytic (entire). In addition,

$$\int_{-\infty}^{\infty} g(x)dx \leq \sqrt{2\pi}\sum_{i=1}^{\infty}\frac{1}{i^2} =: C < \infty.$$

Furthermore, we have

$$g(i) \geq \frac{1}{i^2}g_{i,i^{-4}}(i) = \frac{1}{i^2}i^4 = i^2.$$

Finally, we set

$$f(y) := 2 - \frac{1}{C}\int_{0}^{y} g(x)dx, \quad y > 0.$$

One can then see that f satisfies the desired properties. □

Remark 5.3. To obtain an example with a utility having positive third derivative $U''' > 0$, we just use Proposition 5.1 for $n = 2$. The fact that $-V''' = (-1)^{(n+1)}V^{(n+1)} > 0$ ensures, by duality, that the corresponding utility

$$U(x) := \inf_{y>0}\left(V(y) + xy\right), \quad x > 0,$$

satisfies the desired condition $U'''(x) > 0$ for all $x > 0$.

6. Counterexample 2: non-stochastically dominant models and lack of differentiability

We show that for *any* non-homothetic utility $U \in -\mathcal{C}$ with $U \in C^2\left((0,\infty)\right)$, we may construct a non-$\mathcal{SD}\left(\infty\right)$ market model such that, at some point $x > 0$, the two-times differentiability of u fails. We recall that standard results in utility maximization theory, in the form of Kramkov and Schachermayer [19], assert the continuous differentiability of the value functions. The result below demonstrates that differentiability might cease to exist at the very next order (even with a $\mathcal{CMIM}$ utility).

We note that, due to the multiplicative structure $\widehat{Y}(y) = y\widehat{Y}$ under the assumptions of Theorem 4.6, we do not make any sigma-boundedness assumption, as in [21]. Our counterexample is somewhat related to the sigma-boundedness counterexample from [21], but it is stronger: we construct a (counterexample) model for *every* Inada utility function with non-constant relative risk aversion.

Let $U \in -\mathcal{C}$ with $U \in C^2\left((0,\infty)\right)$, having non-constant relative risk aversion

$$A(x) := -\frac{U''(x)x}{U'(x)}, x > 0. \tag{6.1}$$

The assumption $U \in C^2\left((0,\infty)\right)$ is without loss of generality. We may also choose $U \in \mathcal{CMIM}$.

Proposition 6.1. For any non-homothetic[1] utility $U \in -\mathcal{C}$, with $U \in C^2\left((0,\infty)\right)$, and thus, $U \in \mathcal{CMIM}(2)$, there exists a non-$\mathcal{SD}\left(\infty\right)$ market model such that the value function is not twice differentiable at some $x > 0$.

[1] I.e., such that $A \neq const$, where A is defined in (6.1).

Proof. We first assume that the risk aversion A satisfies $A(1/m) \neq A(1/k)$, for some m and k in $\mathbb{N}$. As we justify at the end of the proof, this is without loss of generality.

Let us suppose that the sample space $\Omega = \{\omega_0, \omega_1, \dots\}$, and consider a one-period model, where the market consists of a money market account with 0 interest rate and a stock, with $S_0 = 1$ and $S_1(\omega_0) = 2$, $S_1(\omega_n) = \frac{1}{n}$, $n \in \mathbb{N}$.

We are going to construct probabilities $p_n := \mathbb{P}[\omega_n] > 0$, $n \geq 0$, satisfying the following three properties

$$\mathbb{E}\left[-U''(S_1)\right] < \infty, \tag{6.2}$$

$$\mathbb{E}\left[U'(S_1)S_1\right] = \mathbb{E}\left[U'(S_1)\right] < \infty, \tag{6.3}$$

and

$$\mathbb{E}\left[U'(S_1)(1 - S_1)A(S_1)\right] \neq 0, \tag{6.4}$$

where A is defined in (6.1). Note that, relations (6.3) and (6.4) can hold together only if the function A is non-constant.

Direct computations show that (6.4) holds if and only if

$$\widehat{\Delta} := -\frac{\mathbb{E}\left[U''(S_1)(S_1 - 1)\right]}{\mathbb{E}\left[U''(S_1)(S_1 - 1)^2\right]} \neq 1. \tag{6.5}$$

Furthermore, note that $\widehat{\Delta} \in (-1, 2)$.

In addition to (6.2), (6.3), and (6.4), we will show that there exists $\bar{\varepsilon} \in \left(0, \frac{1}{2}\right]$, such that for a random variable defined as

$$\begin{aligned} G(\omega) := &\min_{\varepsilon \in [0,\bar{\varepsilon}]} U''\left(S_1(\omega) + \varepsilon(1 + \widehat{\Delta}(S_1(\omega) - 1))\right) 1_{\{\widehat{\Delta} < 1\}} \\ &+ \min_{\varepsilon \in [-\bar{\varepsilon},0]} U''\left(S_1(\omega) + \varepsilon(1 + \widehat{\Delta}(S_1(\omega) - 1))\right) 1_{\{\widehat{\Delta} > 1\}}, \quad \omega \in \Omega, \end{aligned}$$

satisfies

$$G \in \mathbb{L}^1(\mathbb{P}). \tag{6.6}$$

Assuming for now that such probabilities indeed exist, we show that under (6.3) and stock as above, $\mathbb{P}$ is not a martingale measure for S, and we have

$$2 > \mathbb{E}\left[S_1\right] > 1. \tag{6.7}$$

Indeed, the monotonicity of U' yields

$$\begin{aligned} \mathbb{E}\left[1 - S_1\right] &= \mathbb{E}\left[(1 - S_1)1_{\{S_1 < 2\}}\right] - \mathbb{P}[\{S_1 = 2\}] \\ &< \mathbb{E}\left[\frac{U'(S_1)}{U'(2)}(1 - S_1)1_{\{S_1 < 2\}}\right] - \mathbb{P}[\{S_1 = 2\}] \\ &= \frac{1}{U'(2)}\left(\mathbb{E}\left[U'(S_1)(1 - S_1)1_{\{S_1 < 2\}}\right] + U'(2)\mathbb{E}[(1 - S_1)1_{\{S_1 = 2\}}]\right) \\ &= \frac{1}{U'(2)}\mathbb{E}\left[U'(S_1)(1 - S_1)\right] = 0, \end{aligned}$$

where in the last equality we used (6.3). This implies (6.7), where the upper bound is also strict as $p_n > 0$, for every $n \geq 0$. Thus, $\mathbb{P}$ is not a martingale measure for S.

Therefore, the constant-valued process $Z \equiv 1$ is not an element of $\mathcal{Y}(1)$, and thus it is not the dual minimizer for $y = 1$.

Furthermore, we claim that (4.3) holds. This is rather clear by observing that, for n_0 large enough, one can choose a martingale measure $\mathbb{Q}$ that changes the probabilities only for $\omega_0, \omega_1, \dots, \omega_{n_0}$ for some n_0, but keeps the same probabilities for $\omega_n, n > n_0$. As the density $Z \in \mathcal{Y}(1)$ of such a martingale measure is bounded below away from 0, (4.3) holds.

Next, we construct appropriate probabilities p_n's (such that (6.3), (6.6), and (6.5) hold; note that one might need to perturb finitely many of these p_n's later such that (6.12) below holds too). For this, we set

$$p_n := \frac{1}{2^{n+1}} \frac{\min(1, U'(2))}{\max\left(1, U'\left(\frac{1}{n}\right) - \min\limits_{s \in \left[\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}\right]} U''(s)\right)}, \quad n \geq 2. \tag{6.8}$$

Note that

$$\min_{s \in \left[\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}\right]} U''(s) \leq \min_{z \in [0,1/3]} \min_{\Delta \in [-1,1]} U''(1/n + z(1 + \Delta(1/n - 1)))$$

and

$$\min_{s \in \left[\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}\right]} U''(s) \leq \min_{z \in [-1/3,0]} \min_{\Delta \in [1,2]} U''(1/n + z(1 + \Delta(1/n - 1))).$$

The intuition behind the exact form of the intervals above comes from taking $\bar{\varepsilon} = 1/3$ in the construction of G satisfying (6.6) when $\widehat{\Delta}$ is not fixed yet.

Then, as $S_1 > 1$ only for ω_0, we have

$$\begin{aligned} 0 \leq \mathbb{E}\left[U'(S_1)S_1\right] &\leq U'(2) + \mathbb{E}\left[U'(S_1)\right] \leq 2U'(2) + U'(1) \\ &+ \sum_{n \geq 2} \frac{1}{2^{n+1}} \frac{U'\left(\frac{1}{n}\right)}{\max\left(U'\left(\frac{1}{n}\right) - \min\limits_{s \in \left[\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}\right]} U''(s), 1\right)} < \infty, \end{aligned}$$

and, the finiteness in (6.3) holds (regardless of the choice of p_0 and p_1).

Now, with p_n, $n \geq 2$, given by (6.8), we show that we can simultaneously have (6.3) and (6.5). We define

$$\begin{aligned} p_0 :=& \frac{1}{U'(2)} \sum_{n \geq 2} p_n U'\left(\tfrac{1}{n}\right)\left(1 - \tfrac{1}{n}\right) \\ =& \frac{\min(U'(2), 1)}{U'(2)} \sum_{n \geq 2} \frac{1}{2^{n+1}} \frac{U'\left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)}{\max\left(U'\left(\frac{1}{n}\right) - \min\limits_{s \in \left[\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}\right]} U''(s), 1\right)}. \end{aligned} \tag{6.9}$$

Then, using the above, we rewrite

$$2p_0 U'(2) + \sum_{n \geq 1} p_n U'\left(\tfrac{1}{n}\right) \frac{1}{n} = p_0 U'(2) + \sum_{n \geq 1} p_n U'\left(\tfrac{1}{n}\right),$$

and (6.3) follows. Thus, (6.3) holds with p_n, $n \geq 2$, given by (6.8) and p_0 specified by (6.9). Note that $p_0 \leq 1/4$, $\sum_{n \geq 2} p_n \leq 1/4$, and, therefore, $p_1 := 1 - (p_0 + \sum_{n \geq 2} p_n) \geq 1/2$.

To show (6.6), we observe that there exist constants a and a', such that $0 < a < a'$ and, for an appropriate $\bar{\varepsilon}$, we have

$$0 \geq \mathbb{E}\left[G\right] \geq 2 \min_{s\in[a,a']} U''\left(s\right) + \sum_{n\geq 2} \frac{1}{2^{n+1}} \frac{\min\limits_{s\in\left[\frac{2}{3n},\frac{2}{3}+\frac{2}{3n}\right]} U''\left(s\right)}{\max\left(U'\left(\frac{1}{n}\right) - \min\limits_{s\in\left[\frac{2}{3n},\frac{2}{3}+\frac{2}{3n}\right]} U''\left(s\right), 1\right)} > -\infty.$$

Next, we show that (6.4) holds for the choice of p_n's or for a slightly perturbed choice of p_n's, where the distortion is such that the remaining assumptions of the example do not change. To this end, we rewrite (6.5) as

$$-U''(2)p_0 + \sum_{n\geq 2} U''(\tfrac{1}{n})(1-\tfrac{1}{n})p_n \neq U''(2)p_0 + \sum_{n\geq 2} U''(\tfrac{1}{n})(1-\tfrac{1}{n})^2 p_n. \tag{6.10}$$

Collecting terms and plugging the expression for p_0 from (6.9), we can rewrite (6.10) as

$$0 \neq -\frac{2U''(2)}{U'(2)} \sum_{n\geq 2} p_n U'\left(\tfrac{1}{n}\right)\left(1-\tfrac{1}{n}\right) + \sum_{n\geq 2} U''(\tfrac{1}{n})(1-\tfrac{1}{n})\tfrac{1}{n}p_n$$

or, in turn,

$$0 \neq \sum_{n\geq 2} p_n U'(\tfrac{1}{n})\left(1-\tfrac{1}{n}\right)\left(A(2) - A(\tfrac{1}{n})\right). \tag{6.11}$$

We note that, if $x \to A(x)$, $x > 0$, is strictly monotone[2], (6.5) holds, as all terms under the sum in (6.11) are non-zero and of the same sign.

When the relative risk aversion is not monotone, but also non-constant, and if (6.11) does not hold, it is enough to perturb finitely many of the p_n's in a way to get simultaneously

$$\begin{aligned} &\sum_{n\geq 0} p_n = 1, \\ &\sum_{n\geq 0} p_n U'(s_n)(1-s_n) = 0, \\ &\sum_{n\geq 0} p_n U'(s_n)(1-s_n)A(s_n) \neq 0, \end{aligned} \tag{6.12}$$

while preserving the positivity of p_n's (here $s_0 = 2$ and $s_n = 1/n$, $n \in \mathbb{N}$). As $A(1/m) \neq A(1/k)$, for some m and k, such a distortion of p_n's exists. As we have only perturbed finitely many p_n's, (6.6) still holds. This results in the choice of a probability measure, such that (6.3), (6.5), and (6.6) hold.

[2]An example of an Inada utility function of class $\mathcal{CMIM}$, where the relative risk aversion is strictly monotone, is given via $-V'(y) = y^{-k}\frac{1}{y+1}$, $y > 0$, for some constant $k > 0$. Here $-V'$ is completely monotonic as a product of the completely monotonic functions $y \to y^{-k}$ and $y \to \frac{1}{y+1}$, $y > 0$, see [33, Corollary 1.6]. Then, the relative risk tolerance at $x = -V'(y)$ is given by $B(y) = -\frac{V''(y)y}{V'(y)} = k + \frac{y}{y+1}$, which is a strictly monotone function of y on $(0,\infty)$. As $A(x)B(y) = 1$ for $y = U'(x)$, we deduce that A is also a strictly monotone function on $(0,\infty)$.

We show that $u''(1)$ does not exist. First, we will assume that in (6.5), the left-hand side is strictly less than 1, i.e.,

$$-\frac{\mathbb{E}\left[U''(S_1)(S_1-1)\right]}{\mathbb{E}\left[U''(S_1)(S_1-1)^2\right]} < 1. \tag{6.13}$$

As for every $x \geq 0$, $\{x + \Delta(S_1 - 1) : \ \Delta \in [-x, x]\}$ is the set of terminal values of the elements of $\mathcal{X}(x)$, we observe that buying the portfolio consisting of one share of stock is admissible for $x = 1$. Then, by conjugacy, we have

$$\mathbb{E}\left[U(S_1)\right] = \mathbb{E}\left[V(U'(S_1)) + U'(S_1)S_1\right]. \tag{6.14}$$

Using (6.3) and with an arbitrary $\Delta \in [-1, 1]$, we can rewrite the latter expression as

$$\begin{aligned} &\mathbb{E}\left[V(U'(S_1)) + U'(S_1)S_1\right] = \mathbb{E}\left[V(U'(S_1)) + U'(S_1)\right] \\ &\quad = \mathbb{E}\left[V(U'(S_1)) + U'(S_1)(1 + \Delta(S_1 - 1))\right] \geq \mathbb{E}\left[U(1 + \Delta(S_1 - 1))\right]. \end{aligned} \tag{6.15}$$

Combining (6.14) and (6.15), we get

$$\mathbb{E}\left[U(S_1)\right] \geq \mathbb{E}\left[U(1 + \Delta(S_1 - 1))\right], \quad \Delta \in [-1, 1],$$

which yields that $\widehat{X}(1) = S$.

In turn, by the relations between the primal and dual optimizers, we get

$$u'(1) = \mathbb{E}\left[S_1 U'(S_1)\right] = \mathbb{E}\left[U'(S_1)\right], \tag{6.16}$$

where the second equality follows from (6.3).

For ε being small, Taylor's expansion yields

$$U(\widehat{X}_1(1+\varepsilon)) - U(S_1) = \varepsilon U'(S_1)\frac{\widehat{X}_1(1+\varepsilon) - S_1}{\varepsilon} + \frac{\varepsilon^2}{2}U''(\eta(\varepsilon))\frac{\left(\widehat{X}_1(1+\varepsilon) - S_1\right)^2}{\varepsilon^2}, \tag{6.17}$$

where $\eta(\varepsilon)$ is a random variable taking values between S_1 and $\widehat{X}_1(1+\varepsilon)$. Therefore, from (6.16) and (6.17), we obtain

$$\begin{aligned} \tfrac{2}{\varepsilon^2}\left(u(1+\varepsilon) - u(1) - \varepsilon u'(1)\right) &= \frac{2}{\varepsilon}\mathbb{E}\left[U'(S_1)\left(\frac{\widehat{X}_1(1+\varepsilon) - S_1}{\varepsilon} - 1\right)\right] \\ &\quad + \mathbb{E}\left[U''(\eta(\varepsilon))\frac{\left(\widehat{X}_1(1+\varepsilon) - S_1\right)^2}{\varepsilon^2}\right]. \end{aligned} \tag{6.18}$$

Let us consider the first term in the right-hand side of (6.18),

$$\frac{2}{\varepsilon}\mathbb{E}\left[U'(S_1)\left(\frac{\widehat{X}_1(1+\varepsilon) - S_1}{\varepsilon} - 1\right)\right]. \tag{6.19}$$

As $\widehat{X}_1(1+\varepsilon) = 1+\varepsilon+\widehat{\Delta}(1+\varepsilon)(S_1-1)$, for some (fixed and nonrandom) $\widehat{\Delta}(1+\varepsilon) \in [-1-\varepsilon, 1+\varepsilon]$, we can rewrite (6.19) as

$$\begin{aligned}
&\frac{2}{\varepsilon}\mathbb{E}\left[U'(S_1)\left(\frac{1+\varepsilon+\widehat{\Delta}(1+\varepsilon)(S_1-1)-S_1}{\varepsilon}-1\right)\right]\\
&= \frac{2}{\varepsilon}\mathbb{E}\left[U'(S_1)\left(\frac{1+\varepsilon-S_1-\varepsilon}{\varepsilon}\right)\right] + \frac{2}{\varepsilon^2}\widehat{\Delta}(1+\varepsilon)\mathbb{E}\left[U'(S_1)\left(S_1-1\right)\right]\\
&= \frac{2}{\varepsilon^2}\mathbb{E}\left[U'(S_1)\left(1-S_1\right)\right] + \frac{2}{\varepsilon^2}\widehat{\Delta}(1+\varepsilon)\mathbb{E}\left[U'(S_1)\left(S_1-1\right)\right]\\
&= 0,
\end{aligned} \tag{6.20}$$

where in the last equality we used (6.3). Therefore, (6.18) becomes

$$\tfrac{2}{\varepsilon^2}\left(u(1+\varepsilon)-u(1)-\varepsilon u'(1)\right) = \mathbb{E}\left[U''(\eta(\varepsilon))\frac{\left(\widehat{X}_1(1+\varepsilon)-S_1\right)^2}{\varepsilon^2}\right]. \tag{6.21}$$

Next, we look at

$$\limsup_{\varepsilon\uparrow 0}\mathbb{E}\left[U''(\eta(\varepsilon))\frac{\left(\widehat{X}_1(1+\varepsilon)-S_1\right)^2}{\varepsilon^2}\right]. \tag{6.22}$$

Using that $u(1) < \infty$, [19, Lemma 3.6] together with the symmetry between the primal and dual problems (see the abstract theorems from [27]) allow to conclude that $\widehat{X}_1(1+\varepsilon) \to \widehat{X}_1(1) = S_1$, as $\varepsilon \to 0$, in probability. Consequently, since $\eta(\varepsilon)$ takes values between $\widehat{X}_1(1)$ and $\widehat{X}_1(1+\varepsilon)$, we deduce that $\eta(\varepsilon) \to S_1$ in probability. Passing to the limit along a subsequence in (6.22), and applying Fatou's lemma, we get

$$\limsup_{\varepsilon\uparrow 0}\mathbb{E}\left[U''(\eta(\varepsilon))\frac{\left(\widehat{X}_1(1+\varepsilon)-S_1\right)^2}{\varepsilon^2}\right] \leq \sup_{\widetilde{\Delta}\geq 1}\mathbb{E}\left[U''(S_1)\left(\widetilde{\Delta}S_1+1-\widetilde{\Delta}\right)^2\right], \tag{6.23}$$

where we used the representation $\frac{\widehat{X}_1(1+\varepsilon)-S_1}{\varepsilon} = \widetilde{\Delta}S_1+1-\widetilde{\Delta}$, for some constant $\widetilde{\Delta}$, which, for $\varepsilon \in (-1,0)$, is bounded from below by 1, and the observation that, on $\widetilde{\Delta} > 2$, $U''(\eta(\varepsilon))\left(\widetilde{\Delta}S_1+1-\widetilde{\Delta}\right)^2$ is monotone in $\widetilde{\Delta}$, for every sufficiently small and negative ε and for every ω.

Combining (6.21) and (6.23), we deduce that

$$\limsup_{\varepsilon\uparrow 0}\tfrac{2}{\varepsilon^2}\left(u(1+\varepsilon)-u(1)-\varepsilon u'(1)\right) \leq \sup_{\widetilde{\Delta}\geq 1}\mathbb{E}\left[U''(S_1)\left(\widetilde{\Delta}S_1+1-\widetilde{\Delta}\right)^2\right]. \tag{6.24}$$

On the other hand, there exists a constant $\varepsilon_0' > 0$, such that for every $\bar{\Delta} \in [-1-2/\varepsilon_0', 1]$, we have that

$$\mathcal{X}(1+\varepsilon) \ni X^{\varepsilon,\bar{\Delta}} := 1+\varepsilon+(1+\bar{\Delta}\varepsilon)(S-1), \quad \textit{for every} \quad \varepsilon \in (0,\varepsilon_0']. \tag{6.25}$$

In particular, for every $\bar{\Delta} < 1$, we can choose ε_0' such that (6.25) holds. We then obtain

$$\begin{aligned} &\liminf_{\varepsilon\downarrow 0} \tfrac{2}{\varepsilon^2}\left(u(1+\varepsilon) - u(1) - \varepsilon u'(1)\right) \\ &\geq \liminf_{\varepsilon\downarrow 0} \tfrac{2}{\varepsilon^2}\left(\mathbb{E}\left[U\left(X_1^{\varepsilon,\bar{\Delta}}\right)\right] - u(1) - \varepsilon u'(1)\right), \end{aligned} \tag{6.26}$$

where $\mathbb{E}\left[U\left(X_1^{\varepsilon,\bar{\Delta}}\right)\right]$ is well-defined see the justification in Remark 4.9. Since

$$X^{\varepsilon,\bar{\Delta}} = S + (\bar{\Delta}\varepsilon)S + \varepsilon(1-\bar{\Delta}),$$

applying Taylor's expansion once more in (6.26) gives

$$\begin{aligned} \frac{2}{\varepsilon^2}\left(\mathbb{E}\left[U\left(X_1^{\varepsilon,\bar{\Delta}}\right)\right] - u(1) - \varepsilon u'(1)\right) =& \frac{2}{\varepsilon^2}\Big(\varepsilon(\bar{\Delta}-1)\mathbb{E}\left[U'(S_1)(S_1-1)\right] \\ &+\frac{1}{2}\mathbb{E}\left[U''(\widetilde{\eta}(\varepsilon))(\bar{\Delta}\varepsilon S_1 + \varepsilon(1-\bar{\Delta}))^2\right]\Big) \\ =& \frac{1}{\varepsilon^2}\mathbb{E}\left[U''(\widetilde{\eta}(\varepsilon))(\bar{\Delta}\varepsilon S_1 + \varepsilon(1-\bar{\Delta}))^2\right], \end{aligned} \tag{6.27}$$

for some random variable $\widetilde{\eta}(\varepsilon)$ taking values between S_1 and $X^{\varepsilon,\bar{\Delta}}$, and where in the last equality, we have used (6.3).

In particular, for $\bar{\Delta} = \widehat{\Delta} = -\frac{\mathbb{E}[U''(S_1)(S_1-1)]}{\mathbb{E}[U''(S_1)(S_1-1)^2]}$, where by assumption (6.13), $\widehat{\Delta} < 1$, we can rewrite the latter expression in (6.27) as

$$\mathbb{E}\left[U''(\widetilde{\eta}(\varepsilon))(\widehat{\Delta}S_1 + (1-\widehat{\Delta}))^2\right].$$

Note that the function $f(\bar{\Delta}) := \mathbb{E}\left[U''(S_1)(\bar{\Delta}S_1 + (1-\bar{\Delta}))^2\right]$, $\bar{\Delta} \in \mathbb{R}$, reaches its strict global maximum at $\widehat{\Delta}$ defined above. Also, from (6.25), we deduce that $X^{\varepsilon,\bar{\Delta}} \to S_1$ as $\varepsilon \downarrow 0$. Combining (6.26) and (6.27) and using (6.6), for $\bar{\Delta} = \widehat{\Delta}$, we deduce that

$$\liminf_{\varepsilon\downarrow 0} \tfrac{2}{\varepsilon^2}\left(u(1+\varepsilon) - u(1) - \varepsilon u'(1)\right) \geq \mathbb{E}\left[U''(S_1)(\widehat{\Delta}S_1 + 1 - \widehat{\Delta})^2\right]. \tag{6.28}$$

Therefore, from (6.23) and (6.28), we conclude that

$$\begin{aligned} &\liminf_{\varepsilon\downarrow 0} \tfrac{2}{\varepsilon^2}\left(u(1+\varepsilon) - u(1) - \varepsilon u'(1)\right) \geq \mathbb{E}\left[U''(S_1)(\widehat{\Delta}S_1 + 1 - \widehat{\Delta})^2\right] \\ &= \sup_{\bar{\Delta}\in\mathbb{R}} \mathbb{E}\left[U''(S_1)\left(\bar{\Delta}S_1 + 1 - \bar{\Delta}\right)^2\right] > \sup_{\bar{\Delta}\geq 1} \mathbb{E}\left[U''(S_1)\left(\bar{\Delta}S_1 + 1 - \bar{\Delta}\right)^2\right] \\ &\geq \limsup_{\varepsilon\uparrow 0} \tfrac{2}{\varepsilon^2}\left(u(1+\varepsilon) - u(1) - \varepsilon u'(1)\right), \end{aligned}$$

which shows that $u''(1)$ does not exist in the case when $\widehat{\Delta} < 1$, where $\widehat{\Delta}$ is defined in (6.5). The case when $\widehat{\Delta} > 1$ can be handled similarly.

We conclude justifying why we may assume that, without loss of generality, A satisfies $A(1/m) \neq A(1/k)$, for some m and k in $\mathbb{N}$. Indeed, for a given utility function U, let us consider the family $U_\lambda := U(\lambda\cdot)$, $\lambda > 0$. Then, for a given $\lambda > 0$ and every $x > 0$, we have

$$A_\lambda(x) := -\frac{U''_\lambda(x)x}{U'_\lambda(x)} = -\frac{\lambda^2 U''(\lambda x)x}{\lambda U'(\lambda x)} = A(\lambda x). \tag{6.29}$$

If $A(a) \neq A(b)$ for some $0 < a < b$, then, we have, by (6.29), that $A_\lambda(a/\lambda) \neq A_\lambda(b/\lambda)$. Therefore, by the choice of λ, we may assume that

$$A_\lambda(a/\lambda) \neq A_\lambda(1/m),$$

for some $m \in \mathbb{N}$ and where $a/\lambda \in (0, 1/m)$. If we add a/λ to the range of S_1, and assign to this state a positive but small probability, the arguments above still go through, and imply that $u''_\lambda(1)$ does not exist, where

$$u_\lambda(x) := \sup_{X\in\mathcal{X}(1)} \mathbb{E}\left[U(\lambda x X_1)\right] = u(\lambda x).$$

Therefore, non-existence of $u''_\lambda(1)$ would imply that $u''(\lambda)$ does not exist either. □

7. $\mathcal{SD}(2) = \mathcal{SD}(\infty)$

While condition $\mathcal{SD}(n)$, $n = 2, \dots, \infty$ is the natural condition for differentiability of order n, or analyticity, for $n = \infty$, it turns out that, when applied to the dual domain, having a maximal element is the same for every n. In addition, we have a characterization of such a maximal element. This is related to [32, Proposition 3.10], but unlike [32, Proposition 3.10], we do not assume that $\widehat{Y}$ is a density of a probability measure, as no free lunch with vanishing risk (NFLVR) is not supposed here. Even if NFLVR is assumed, if $\widehat{Y}$ is maximal in the second-order stochastic dominance, one can conclude that it is a measure.

We recall that a probability measure $\mathbb{Q} \sim \mathbb{P}$ is an *equivalent local martingale measure* for S if every $X \in \mathcal{X}(1)$ is a local martingale under $\mathbb{Q}$. We will denote the family of equivalent local martingale measures by $\mathcal{M}^e(S)$. By [5, Theorem 1.1], the celebrated no-free lunch with vanishing risk (NFLVR) condition for S is equivalent to

$$\mathcal{M}^e_\sigma(S) := \{\mathbb{Q} \sim \mathbb{P} : \ S \text{ is a } \sigma-\text{martingale under } \mathbb{Q}\} \neq \emptyset, \tag{7.1}$$

that is to non-emptiness of the set of *equivalent sigma-martingale measures* for S. Following [5], let us also recall that an *equivalent separating measure* for S is defined as $\mathbb{Q} \sim \mathbb{P}$ such that every $X \in \mathcal{X}(1)$ is a supermartingale under $\mathbb{Q}$. $\mathcal{M}^e_s(S)$ denotes the family of the equivalent separating measures for S.

Remark 7.1. We recall that, using the Ansel and Stricker Theorem, [2, Corollary 3.5], one can show that

$$\mathcal{M}^e_s(S) \supseteq \mathcal{M}^e(S) \supseteq \mathcal{M}^e_\sigma(S)$$

If $\mathcal{M}^e_s(S) \neq \emptyset$, then S satisfies NFLVR, and thus by [5, Theorem 1.1], we have that

$$M^e_\sigma(S) \neq \emptyset.$$

Further, by [5, Proposition 4.7], the density of $\mathcal{M}^e_\sigma(S)$ in $\mathcal{M}^e_s(S)$ implies that, for every non-negative random variable g, we have

$$\sup_{\mathbb{Q}\in\mathcal{M}^e_s(S)} \mathbb{E}^{\mathbb{Q}}[g] = \sup_{\mathbb{Q}\in\mathcal{M}^e_\sigma(S)} \mathbb{E}^{\mathbb{Q}}[g],$$

see the proof of [5, Theorem 5.12].

Proposition 7.2. Let us consider a financial model for which (2.3) holds[3]. Then the following conditions are equivalent:

1. $\mathcal{SD}(\infty):\ \widehat{Y}_T \succeq_\infty Y_T$, for every $Y \in \mathcal{Y}(1)$;
2. $\widehat{Y}_T \geq \mathbb{E}\left[Y_T|\sigma(\widehat{Y}_T)\right]$, for every $Y \in \mathcal{Y}(1)$;
3. $\mathcal{SD}(2):\ \widehat{Y}_T \succeq_2 Y_T$, for every $Y \in \mathcal{Y}(1)$.

In addition if NFLVR holds, then: $\mathbb{E}\left[\widehat{Y}_T\right] = 1$ and the probability measure $\widehat{\mathbb{Q}}$ defined through its derivative as $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} := \widehat{Y}_T$ is a separating measure in the terminology of Delbaen and Schachermayer [5], that is $\widehat{\mathbb{Q}} \in \mathcal{M}^e_s(S)$.

Proof. The implication $3 \Rightarrow 1$ is trivial. Likewise, $2 \Rightarrow 3$ follows from Jensen's inequality. For the additional statement, under NFLVR there exists $Y_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$, $\mathbb{Q} \in \mathcal{M}^e_\sigma(S)$, such that $\mathbb{E}[Y_T] = 1$, and thus from 2, we have

$$\widehat{Y}_T \geq \mathbb{E}\left[Y_T|\sigma(\widehat{Y}_T)\right].$$

Therefore $\mathbb{E}\left[\widehat{Y}_T\right] \geq 1$, and the proof is complete.

It remains to prove $1 \Rightarrow 2$. Assume $\mathcal{SD}(\infty)$ and let $\widehat{Y}$ be the maximal element in $\mathcal{Y}(1)$ in the sense of the infinite order stochastic dominance. Let us consider the dual function V such that the following properties hold:

(i) $-V'$ is $\mathcal{CM}$;
(ii) the Inada conditions (2.4) hold for $-V$;
(iii) V is bounded.

Remark 7.3. One way of constructing such a V is though a measure μ such that

$$\mu(\{0\}) = 0, \qquad \mu((0,\infty)) = \infty, \quad \textit{and} \quad \int_0^\infty \int_0^\infty e^{-yt}\mu(dt) < \infty. \tag{7.2}$$

Then V can be defined (through its derivative) as

$$V(y) := \int_y^\infty \int_0^\infty e^{-zt}\mu(dt)dz, \quad y > 0. \tag{7.3}$$

Then

$$-V'(y) = \int_0^\infty e^{-yt}\mu(dt), \quad y > 0,$$

[3]In particular, we do not suppose (7.1).

is demonstratively $\mathcal{CM}$. The Inada conditions (2.4) and finiteness will hold for $-V$ by (7.2). Thus properties (i)-(iii) hold. Note that, for the finiteness of V, we have

$$\int_0^\infty \left(\int_0^\infty e^{-yt} \mu(dt) \right) dy = \int_0^\infty \left(\int_0^\infty e^{-yt} dy \right) \mu(dt) = \int_0^\infty \frac{1}{t} \mu(dt).$$

This allows for an explicit choice of μ. For example, we can pick μ given by

$$\mu(dt) = \begin{cases} t^{-\frac{1}{2}} dt, & t \geq 1 \\ 0 dt, & t \in (0,1) \end{cases}.$$

Another way to construct V satisfying (i)-(iii) is the following: for some constants $a \in (-1, 0)$ and $b \in (-\infty, 1)$, we can set

$$V(y) := \int_y^\infty z^a (z+1)^b dz, \quad y > 0.$$

Then, $-V'(y) = y^a (y+1)^b$, $y > 0$, is $\mathcal{CM}$ as a product of $\mathcal{CM}$ functions, see [33, Corollary 1.6]. As

$$0 \leq V(y) \leq \lim_{y \downarrow 0} V(y) = \int_0^\infty y^a (y+1)^b dz < \infty,$$

and the Inada conditions (2.4) hold for $-V'$, we deduce that properties (i)-(iii) hold.

Now, with V as above for the utility function $U(x) = \inf_{y>0} (V(y) + xy)$, $x > 0$, let us consider (2.5). Under (2.3), its dual is (2.7), where (4.3) holds by the boundedness of V. Therefore, the abstract theorems in [27] apply, and we deduce the existence of $\widehat{x} = -v'(1)$, such that $\widehat{X} \in \mathcal{X}(\widehat{x})$ and $\widehat{X}\widehat{Y}$ is a true martingale and

$$\widehat{X}_T = -V'(\widehat{Y}_T). \tag{7.4}$$

Let us fix a constant $\lambda > 0$ and consider a new dual function defined (on the level of the derivative) as

$$V_\lambda(y) := -\int_y^\infty V'(z) \left(1 + e^{-\lambda z}\right) dz, \quad y > 0;$$

that is

$$-V_\lambda'(y) = -V'(y) \left(1 + e^{-\lambda y}\right), \quad y > 0.$$

As $-V'(y) \leq -V_\lambda'(y) \leq -2V'(y)$, $y > 0$, we deduce that the Inada conditions (2.4) hold for $-V$ and that V is bounded. Further, $-V_\lambda'$ is $\mathcal{CM}$ by [33, Corollary 1.6]. Thus V_λ satisfies properties (i)-(iii) above, and therefore for $U_\lambda(x) = \inf_{y>0} (V_\lambda(y) + xy)$, $x > 0$, if we consider the utility maximization problem (2.5) and it dual (2.7), we deduce from [27, Theorem 3.2] that $x_\lambda := -v_\lambda'(1)$ is well-defined and that there exists $X^\lambda \in \mathcal{X}(x_\lambda)$ such that $X^\lambda \widehat{Y}$ is a $\mathbb{P}$ martingale and

$$X_T^\lambda = -V_\lambda'(\widehat{Y}_T) = -V'(\widehat{Y}_T) \left(1 + e^{-\lambda \widehat{Y}_T}\right) = \widehat{X}_T \left(1 + e^{-\lambda \widehat{Y}_T}\right),$$

is the optimizer to (2.5) for $x = x_\lambda$ and the utility function U_λ.

Now, one can see that

$$x_\lambda - \widehat{x} > 0 \quad and \quad X^\lambda - \widehat{X} \in \mathcal{X}(x_\lambda - \widehat{x}).$$

Further, $(X^\lambda - \widehat{X})\widehat{Y}$ is a true $\mathbb{P}$-martingale and

$$\frac{X^\lambda_T - \widehat{X}_T}{\widehat{X}_T} = e^{-\lambda \widehat{Y}_T}.$$

Let us consider

$$M := \frac{X^\lambda - \widehat{X}}{\widehat{X}},$$

and change numéraire to $\frac{\widehat{X}}{\widehat{x}}$ and the probability measure to $\mathbb{R}$ defined as

$$\frac{d\mathbb{R}}{d\mathbb{P}} = \frac{\widehat{X}_T \widehat{Y}_T}{\widehat{x}}.$$

One can see that under the numéraire $\frac{\widehat{X}}{\widehat{x}}$ and measure $\mathbb{R}$, the sets of the nonnegative wealth processes and supermartingale deflators are given by

$$\widehat{\mathcal{X}}(x) := \frac{\mathcal{X}(x)}{\widehat{X}} x = \left\{ \frac{X}{\widehat{X}} \widehat{x} = \left(\frac{X_t}{\widehat{X}_t} \widehat{x} \right)_{t \in [0,T]} \ : \ X \in \mathcal{X}(x) \right\}, \quad x > 0;$$

$$\widehat{\mathcal{Y}}(y) := \frac{\mathcal{Y}(y)}{\widehat{Y}} = \left\{ \frac{Y}{\widehat{Y}} = \left(\frac{Y_t}{\widehat{Y}_t} \right)_{t \in [0,T]} \ : \ Y \in \mathcal{Y}(y) \right\}, \quad y > 0.$$

Let us also denote

$$\widehat{S} := \left(\frac{\widehat{x}}{\widehat{X}}, \frac{\widehat{x} S}{\widehat{X}} \right).$$

One can see that

$$M \in \widehat{\mathcal{X}} \left(\frac{x_\lambda - \widehat{x}}{\widehat{x}} \right)$$

and that M is a true $\mathbb{R}$-martingale such that

$$M_T = e^{-\lambda \widehat{Y}_T}.$$

Therefore, M is a bounded replication process for $e^{-\lambda \widehat{Y}_T}$ under the numéraire $\frac{\widehat{X}}{\widehat{x}}$ (and measure $\mathbb{R}$).

We deduce that for every constant $\lambda > 0$, the option

$$0 \leq f_\lambda(\widehat{Y}_T) := e^{-\lambda \widehat{Y}_T} \leq 1$$

is *replicable* by a bounded stochastic integral under the numéraire $\frac{\widehat{X}}{\widehat{x}}$. Let us work under the measure $\mathbb{R}$ and numéraire $\frac{\widehat{X}}{\widehat{x}}$. One can see that $\mathbb{R} \in \mathcal{M}^e_s(\widehat{S})$. In particular (under the numéraire $\frac{\widehat{X}}{\widehat{x}}$), we obtain that

$$\mathcal{M}^e_s(\widehat{S}) \neq \emptyset;$$

and we have

$$\mathbb{E}^{\mathbb{R}} \left[f_\lambda(\widehat{Y}_T) \right] = \mathbb{E}^{\mathbb{Q}} \left[f_\lambda(\widehat{Y}_T) \right], \quad for\ every \quad \lambda > 0 \quad and \quad \mathbb{Q} \in \mathcal{M}^e_s(\widehat{S}). \tag{7.5}$$

Next, from every f_λ of the form $f_\lambda(y) = e^{-\lambda y}$, $y > 0$, we will extend (7.5) to any

$$h : [0,\infty) \to \mathbb{R}, \quad h \text{ is continuous} \quad \text{and} \quad \lim_{y\uparrow\infty} h(y) = 0. \tag{7.6}$$

The latter property of h allows for the Alexandroff extension to $[0,\infty]$. Equivalently, one can consider

$$g(y) := \begin{cases} h(-\log(y)), & y \in (0,1] \\ 0, & y = 0 \end{cases},$$

and observe that g is continuous on $[0,1]$. Therefore, we can uniformly approximate g by the polynomials P_n of the form

$$P_n(y) = a_0 + a_1 y + \cdots + a_n y^n, \quad y \in [0,1].$$

Since $g(0) = 0$, we further conclude that $a_0 = 0$. Going back to h, we deduce that h can be approximated uniformly on $[0,\infty)$ by the functions of the form

$$a_1 e^{-y} + \cdots + a_n e^{-ny} = P_n(e^{-y}), \quad y \geq 0.$$

Consequently, for such an h and any $\mathbb{Q} \in \mathcal{M}^e_s(\widehat{S})$, we get

$$\begin{aligned}\mathbb{E}^{\mathbb{R}}\left[h(\widehat{Y}_T)\right] &= \mathbb{E}^{\mathbb{R}}\left[\lim_{n\uparrow\infty} P_n\left(e^{-\widehat{Y}_T}\right)\right] = \lim_{n\uparrow\infty}\mathbb{E}^{\mathbb{R}}\left[P_n\left(e^{-\widehat{Y}_T}\right)\right] = \\ &= \lim_{n\uparrow\infty}\mathbb{E}^{\mathbb{Q}}\left[P_n\left(e^{-\widehat{Y}_T}\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[\lim_{n\uparrow\infty} P_n\left(e^{-\widehat{Y}_T}\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[h\left(\widehat{Y}_T\right)\right],\end{aligned} \tag{7.7}$$

where we have used the dominated convergence theorem and (7.5).

To recapitulate, we have shown that for any function h satisfying (7.6), we have

$$\mathbb{E}^{\mathbb{R}}\left[h(\widehat{Y}_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[h(\widehat{Y}_T)\right], \quad \textit{for every } \mathbb{Q} \in \mathcal{M}^e_s(\widehat{S}). \tag{7.8}$$

Next, we observe that (7.8) holds for any function $h : [0,\infty) \to \mathbb{R}$, which is *smooth and has a compact support* in $(0,\infty)$, as every such function satisfies (7.6). Further, using truncation and regularization by convolution, one can approximate (in the topology of uniform convergence on compact subsets of $(0,\infty)$) any *bounded and continuous* function $(0,\infty) \to \mathbb{R}$ by a uniformly bounded sequence of smooth functions with compact support in $(0,\infty)$. Therefore, similarly to the computations in (7.7), we deduce that (7.8) holds for every *bounded continuous* function h: $(0,\infty) \to \mathbb{R}$.

Let $\mathcal{H}$ be the set of *bounded Borel-measurable* functions h: $(0,\infty) \to \mathbb{R}$, such that (7.8) holds. One can see that $\mathcal{H}$ is a monotone class. By $\mathcal{C}$ let us denote the set of *bounded continuous* functions $(0,\infty) \to \mathbb{R}$. As $\mathcal{C}$ is closed under the pointwise multiplication, and $\mathcal{C} \subseteq \mathcal{H}$, we deduce from a version of the monotone class theorem, see e.g., [31, Theorem I.8] or [6, Theorem 21, p. 14], that $\mathcal{H}$ contains all bounded $\sigma(\mathcal{C})$-measurable functions. As $\sigma(\mathcal{C})$ is the Borel sigma-field on $(0,\infty)$, we conclude that, for every bounded Borel-measurable function h: $(0,\infty) \to \mathbb{R}$, (7.8) holds. Therefore, *every bounded option* $h(\widehat{Y}_T)$ *is replicable* under the numéraire $\frac{\widehat{X}}{\widehat{x}}$.

Now, let us fix a bounded function h: $(0,\infty) \to [0,\infty)$. Then, there exists $x \geq 0$, and $X \in \mathcal{X}(x)$, such that

$$\widetilde{X} = \frac{X}{\widehat{X}}\widehat{x} \in \widehat{\mathcal{X}}(x) \quad \textit{is bounded} \quad \textit{and} \quad \widetilde{X}_T = h(\widehat{Y}_T). \tag{7.9}$$

We also have

$$x = \mathbb{E}^{\mathbb{R}}\left[h(\widehat{Y}_T)\right] = \mathbb{E}\left[\frac{\widehat{X}_T}{\widehat{x}}\widehat{Y}_T h(\widehat{Y}_T)\right]. \tag{7.10}$$

Now, let us get back to the original numéraire and measure: by (7.9), we have a process $X \in \mathcal{X}(x)$, such that

$$X_T = \frac{\widehat{X}_T}{\widehat{x}} h(\widehat{Y}_T).$$

Therefore, for any $Y \in \mathcal{Y}(1)$, we have

$$x \geq \mathbb{E}\left[X_T Y_T\right] = \mathbb{E}\left[\frac{\widehat{X}_T}{\widehat{x}} h(\widehat{Y}_T) Y_T\right] \tag{7.11}$$

Comparing (7.10) and (7.11), we deduce, for every $Y \in \mathcal{Y}(1)$ and every bounded $h \geq 0$, that

$$\mathbb{E}\left[\widehat{X}_T\widehat{Y}_T h(\widehat{Y}_T)\right] \geq \mathbb{E}\left[\widehat{X}_T Y_T h(\widehat{Y}_T)\right].$$

Consequently, we have

$$\widehat{X}_T\widehat{Y}_T \geq \mathbb{E}\left[\widehat{X}_T Y_T | \sigma(\widehat{Y}_T)\right].$$

We recall that by (7.4), $\widehat{X}_T = -V'(\widehat{Y}_T)$, and thus $\widehat{X}_T$ is $\sigma(\widehat{Y}_T)$-measurable. We conclude that

$$\widehat{Y}_T \geq \mathbb{E}\left[Y_T | \sigma(\widehat{Y}_T)\right],$$

for every $Y \in \mathcal{Y}(1)$. This completes the proof. □

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