

Necessary and sufficient conditions in the problem of optimal investment with intermediate consumption

Oleksii Mostovyi

Received: 8 November 2012 / Accepted: 28 February 2014 / Published online: 14 October 2014
© Springer-Verlag Berlin Heidelberg 2014

Abstract We consider the problem of optimal investment with intermediate consumption in the framework of an incomplete semimartingale model of a financial market. We show that a necessary and sufficient condition for the validity of key assertions of the theory is that the value functions of the primal and dual problems are finite.

Keywords Utility maximization · Incomplete markets · Duality theory · Legendre–Fenchel transformation · Stochastic clock

Mathematics Subject Classification (2010) 91G10 · 93E20

JEL Classification G11

1 Introduction

A fundamental problem of mathematical finance is that of an investor who wants to invest and consume in a way that maximizes his expected utility. The first results for continuous-time models were obtained by Merton [28, 29] in a Markovian setting via dynamic programming arguments. An alternative martingale approach was developed among others by Cox and Huang [7, 8], Karatzas, Lehoczky and Shreve [21], and Karatzas and Shreve [19] for complete markets and by Karatzas, Lehoczky, Shreve and Xu [22], He and Pearson [14, 15], Kramkov and Schachermayer [25, 26], Karatzas and Žitković [20] and Žitković [37] in the incomplete case. The main focus

O. Mostovyi (✉)
Department of Mathematics, The University of Texas at Austin, 1 University Station C 1200, Austin,
TX 78712-0257, USA
e-mail: mostovyi@math.utexas.edu

here was to establish conditions under which “key” results hold, such as the existence of primal and dual optimizers.

When the consumption occurs only at maturity and the utility function is deterministic, a necessary and sufficient condition has been obtained in Kramkov and Schachermayer [26]. It is stated as the finiteness of the dual value function. In the case of intermediate consumption and stochastic field utility, the latest sufficient conditions are due to Karatzas and Žitković [20] and Žitković [37]. They are formulated in the form of several regularity assumptions such as a uniform asymptotic elasticity. The results of [20] gave a foundation upon which the problem of robust utility maximization is considered in [38]; robust utility maximization is also studied in [3, 6, 13, 23, 30, 33].

This paper obtains necessary and sufficient conditions in the general framework of an incomplete financial model with a stochastic field utility and intermediate consumption occurring according to some stochastic clock. As in [26], we assume that the dual value function is finite (from above). It turns out that the only other condition we need is the finiteness of the primal value function (from below). In particular, our results do not rely on the asymptotic elasticity of the utility.

Using the notion of a stochastic clock, we treat multiple utility maximization problems in one formulation; the examples of such problems are contained at the end of Sect. 2. The term *stochastic clock* is introduced by Goll and Kallsen [12] and is used in different settings in [2, 4, 20, 37, 38].

In order for our framework to include certain classical problems, such as Merton’s [28] formulation of optimal consumption over an infinite time horizon, we consider an infinite-horizon model and relax the no-arbitrage assumption from the existence of an equivalent local martingale measure (which is used, e.g., in [20, 25, 26, 37] in finite-horizon frameworks) to the existence of an equivalent martingale deflator. This leads to additional technical difficulties in establishing bipolar relations between the primal and dual domains, such as an extension of the dual superreplication characterization of Delbaen and Schachermayer [10, Theorem 5.12] to our setting. This is the subject of Lemma 4.2, whose proof relies on the optional decomposition theorem of Föllmer and Kramkov [11].

The proofs of the main theorems are based on their abstract versions. In turn, consideration of the abstract theorems is built upon the standard machinery of convex duality, in particular the results of Kramkov and Schachermayer [26]. However, since the transition from deterministic to stochastic utility is not always seamless, some crucial steps in the proofs require special treatment.

More specifically, the proof of existence of solutions to the abstract primal and dual problems relies on establishing the uniform integrability of the negative part of the conjugate function composed with the elements of the dual domain. This is the subject of Lemma 3.5. Conjugacy relations between the abstract primal and dual value functions are proved in Lemma 3.9 via the minimax theorem and an application of change of numéraire ideas. In the proofs of these lemmas, the assumption about the finiteness of both value functions is used. The connection of the main theorems with their abstract counterparts is established in Proposition 4.4, where we prove the bipolar relations between the primal and dual domains via the superreplication characterization (Lemma 4.2) and the bipolar theorem of Brannath and Schachermayer [5].

As an application, we demonstrate how existence and uniqueness results can be obtained for a utility maximization problem under a change of numéraire via an application of Theorem 2.3. This constitutes Example 4.5.

We believe that the results of the present paper provide a convenient set of conditions for studying other important problems in mathematical finance, such as robust utility maximization and optimal investment with random endowment, as well as for analyzing utility-based pricing, large markets, and existence of equilibria in incomplete models.

The remainder of the paper is organized as follows. In Sect. 2, we describe the model and state the main results. Their proofs are given in Sect. 4. The abstract versions of the main theorems are presented in Sect. 3.

2 Main results

A model of a security market consists of $d + 1$ assets: one bond and d stocks. We assume that the bond is chosen as a numéraire and denote by $S = (S^i)_{1 \leq i \leq d}$ the discounted price processes of the stocks. We suppose that S is a semimartingale on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ with an infinite time horizon, and \mathcal{F}_0 is the completion of the trivial σ -algebra.

Define a portfolio Π as a triple (x, H, c) , where the constant x is an initial value, $H = (H^i)_{1 \leq i \leq d}$ is a predictable S -integrable process of stocks' quantities, and $c = (c_t)_{t \geq 0}$ is a nonnegative and optional process that specifies the consumption rate in the units of the bond.

Hereafter we fix a *stochastic clock* $\kappa = (\kappa_t)_{t \geq 0}$, which is a nondecreasing, càdlàg, adapted process such that

$$\kappa_0 = 0, \quad \mathbb{P}[\kappa_\infty > 0] > 0 \quad \text{and} \quad \kappa_\infty \leq A \tag{2.1}$$

for some finite constant A . The stochastic clock represents the notion of time according to which consumption occurs.

The discounted value process $V = (V_t)_{t \geq 0}$ of a portfolio Π is defined as

$$V_t := x + \int_0^t H_u dS_u - \int_0^t c_u d\kappa_u, \quad t \geq 0.$$

A portfolio Π with $c \equiv 0$ is called *self-financing*. The collection of nonnegative value processes of self-financing portfolios with initial value 1 is denoted by \mathcal{X} , i.e.,

$$\mathcal{X} := \left\{ X \geq 0 : X_t = 1 + \int_0^t H_u dS_u, t \geq 0 \right\}.$$

A pair (H, c) such that for a given $x > 0$, the corresponding value process V is nonnegative, is called an *x -admissible strategy*. If for a consumption process c we can find a predictable S -integrable process H such that (H, c) is an x -admissible strategy, we say that c is an *x -admissible consumption process*.

The set of x -admissible consumption processes corresponding to the stochastic clock κ is denoted by $\mathcal{A}(x)$, that is,

$$\mathcal{A}(x) := \{c : c \text{ is } x\text{-admissible}\}, \quad x > 0. \tag{2.2}$$

We write $\mathcal{A} := \mathcal{A}(1)$ for brevity. Notice that a constant strictly positive consumption $c_t^* := x/A, t \geq 0$, belongs to $\mathcal{A}(x)$ for every $x > 0$.

The set of *equivalent martingale deflators* is defined as

$$\begin{aligned} \mathcal{Z} := \{Z > 0 : Z \text{ is a càdlàg martingale such that } Z_0 = 1 \text{ and} \\ XZ = (X_t Z_t)_{t \geq 0} \text{ is a local martingale for every } X \in \mathcal{X}\}. \end{aligned}$$

We assume that

$$\mathcal{Z} \neq \emptyset. \tag{2.3}$$

This condition is closely related to the absence of arbitrage opportunities in the sense of [18].

We now introduce an economic agent whose consumption preferences are modeled with a *utility stochastic field* $U = U(t, \omega, x) : [0, \infty) \times \Omega \times [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$ satisfying the conditions below.

Assumption 2.1 For every $(t, \omega) \in [0, \infty) \times \Omega$, the function $x \mapsto U(t, \omega, x)$ is strictly concave, strictly increasing, continuously differentiable on $(0, \infty)$ and satisfies the Inada conditions

$$\lim_{x \downarrow 0} U'(t, \omega, x) = +\infty \quad \text{and} \quad \lim_{x \rightarrow \infty} U'(t, \omega, x) = 0,$$

where U' denotes the partial derivative with respect to the third argument. At $x = 0$, we have by continuity $U(t, \omega, 0) = \lim_{x \downarrow 0} U(t, \omega, x)$; this value may be $-\infty$. For every $x \geq 0$, the stochastic process $U(\cdot, \cdot, x)$ is optional.

For a given initial capital $x > 0$, the goal of the agent is to maximize his expected utility. The value function of this problem is denoted by

$$u(x) := \sup_{c \in \mathcal{A}(x)} \mathbb{E} \left[\int_0^\infty U(t, \omega, c_t) d\kappa_t \right], \quad x > 0. \tag{2.4}$$

We use the convention

$$\mathbb{E} \left[\int_0^\infty U(t, \omega, c_t) d\kappa_t \right] := -\infty \quad \text{if} \quad \mathbb{E} \left[\int_0^\infty U^-(t, \omega, c_t) d\kappa_t \right] = +\infty.$$

Here and below, W^- and W^+ denote the negative and positive parts of a stochastic field W , respectively.

Our goal is to find conditions on the financial market and the utility field U under which u satisfies the Inada conditions and the solution $\hat{c}(x) \in \mathcal{A}(x)$ to (2.4) exists.

Remark 2.2 For simplicity of notations, we assume throughout the paper that the argument x in $U(t, \omega, x)$ represents the consumption in discounted units, that is, in the number of bonds. This does not restrict any generality. Indeed, suppose that the investor’s stochastic utility field is given as $\tilde{U} = \tilde{U}(t, \omega, \tilde{x})$, where the consumption \tilde{x} is measured in the number of units of a different asset, whose discounted value is given by a strictly positive semimartingale $A = (A_t)_{t \geq 0}$. Then we arrive to our framework by setting

$$U(t, \omega, x) := \tilde{U}(t, \omega, x/A_t(\omega)).$$

To study (2.4), we employ standard duality arguments as in [25] and [37] and define the *stochastic field V conjugate* to U as

$$V(t, \omega, y) := \sup_{x>0} (U(t, \omega, x) - xy), \quad (t, \omega, y) \in [0, \infty) \times \Omega \times [0, \infty).$$

It is well known that $-V$ satisfies Assumption 2.1. We also denote

$$\mathcal{Y}(y) := \text{cl}\{Y : Y \text{ is càdlàg adapted and } 0 \leq Y \leq yZ \text{ (} d\kappa \times \mathbb{P}\text{)-a.e. for some } Z \in \mathcal{Z}\}, \tag{2.5}$$

where the closure is taken in the topology of convergence in measure ($d\kappa \times \mathbb{P}$) on the space of real-valued optional processes. We write $\mathcal{Y} := \mathcal{Y}(1)$ for brevity.

After these preparations, we define the value function of the dual optimization problem as

$$v(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E} \left[\int_0^\infty V(t, \omega, Y_t) d\kappa_t \right], \quad y > 0, \tag{2.6}$$

where we use the convention

$$\mathbb{E} \left[\int_0^\infty V(t, \omega, Y_t) d\kappa_t \right] := +\infty \quad \text{if } \mathbb{E} \left[\int_0^\infty V^+(t, \omega, Y_t) d\kappa_t \right] = +\infty.$$

Theorems 2.3 and 2.4 constitute our main results.

Theorem 2.3 *Assume that conditions (2.1) and (2.3) and Assumption 2.1 hold true and suppose*

$$v(y) < \infty \text{ for all } y > 0 \quad \text{and} \quad u(x) > -\infty \text{ for all } x > 0. \tag{2.7}$$

Then we have:

1. $u(x) < \infty$ for all $x > 0$, $v(y) > -\infty$ for all $y > 0$. The functions u and v are conjugate, i.e.,

$$v(y) = \sup_{x>0} (u(x) - xy), \quad y > 0,$$

$$u(x) = \inf_{y>0} (v(y) + xy), \quad x > 0.$$

The functions u and $-v$ are continuously differentiable on $(0, \infty)$, strictly increasing, strictly concave and satisfy the Inada conditions

$$u'(0) := \lim_{x \downarrow 0} u'(x) = +\infty, \quad -v'(0) := \lim_{y \downarrow 0} -v'(y) = +\infty,$$

$$u'(\infty) := \lim_{x \rightarrow \infty} u'(x) = 0, \quad -v'(\infty) := \lim_{y \rightarrow \infty} -v'(y) = 0.$$

2. For every $x > 0$ and $y > 0$, the solutions $\hat{c}(x)$ to (2.4) and $\hat{Y}(y)$ to (2.6) exist and are unique. Moreover, if $y = u'(x)$, we have the dual relations

$$\hat{Y}_t(y) = U'(t, \omega, \hat{c}_t(x)), \quad t \geq 0,$$

and

$$\mathbb{E} \left[\int_0^\infty \hat{c}_t(x) \hat{Y}_t(y) d\kappa_t \right] = xy.$$

The finiteness conditions (2.7) are clearly necessary for the conclusions of either claim 1 or 2. Notice that the condition $u(x) > -\infty$ for all $x > 0$ holds trivially if the utility stochastic field U is uniformly bounded from below by a real-valued function, in particular in the settings of Kramkov and Schachermayer [26].

A natural question is whether one can use the set \mathcal{Z} instead of \mathcal{Y} as the dual domain and still obtain the same value function v . Theorem 2.4 below states that the answer is positive, but the minimizer might lie outside of the set \mathcal{Z} in general; see, e.g., Example 5.1 in Kramkov and Schachermayer [25]. Furthermore, due to a certain symmetry between primal and dual problems (that is explored in more detail in Sect. 3), a similar conclusion is valid for the value function u . Let \mathcal{B} be a subset of \mathcal{A} such that

(i) For every $Y \in \mathcal{Y}$, we have

$$\sup_{c \in \mathcal{B}} \mathbb{E} \left[\int_0^\infty c_t Y_t d\kappa_t \right] = \sup_{c \in \mathcal{A}} \mathbb{E} \left[\int_0^\infty c_t Y_t d\kappa_t \right];$$

(ii) The set \mathcal{B} is closed under countable convex combinations, that is, for any sequence $(c^n)_{n \geq 1}$ of optional processes in \mathcal{B} and any sequence of positive numbers $(a^n)_{n \geq 1}$ such that $\sum_{n=1}^\infty a^n = 1$, the process $\sum_{n=1}^\infty a^n c^n$ belongs to \mathcal{B} .

Observe that \mathcal{Z} is closed under countable convex combinations.

Theorem 2.4 Under the conditions of Theorem 2.3, we have

$$v(y) = \inf_{Z \in \mathcal{Z}} \mathbb{E} \left[\int_0^\infty V(t, \omega, yZ_t) d\kappa_t \right], \quad y > 0,$$

$$u(x) = \sup_{c \in \mathcal{B}} \mathbb{E} \left[\int_0^\infty U(t, \omega, xc_t) d\kappa_t \right], \quad x > 0.$$

The proofs of Theorems 2.3 and 2.4 are given in Sect. 4 and rely on Theorems 3.2 and 3.3, which are the “abstract” versions of Theorems 2.3 and 2.4, respectively. We conclude this section with examples of investment problems (see, e.g., Karatzas [17] as well as Karatzas and Shreve [19]) that are included in our formulation. Hereafter, 1_E denotes the indicator function of a set E .

Example 2.5 Maximization of expected utility from consumption:

$$u(x) = \sup_{c \in \mathcal{A}(x)} \mathbb{E} \left[\int_0^T U(t, \omega, c_t) dt \right].$$

Here the clock κ is given by

$$\kappa(t) := \min(t, T), \quad t \geq 0.$$

Example 2.6 Maximization of expected utility from consumption and terminal wealth:

$$u(x) = \sup_{c \in \mathcal{A}(x)} \mathbb{E} \left[\int_0^T U_1(t, \omega, c_t) dt + U_2(\omega, c_T) \right].$$

Here the clock κ is given by

$$\kappa(t) := t 1_{[0, T)}(t) + (T + 1) 1_{[T, \infty)}(t), \quad t \geq 0.$$

Example 2.7 Maximization of expected utility from terminal wealth:

$$u(x) = \sup_{X \in \mathcal{X}} \mathbb{E}[U(\omega, x X_T)]. \tag{2.8}$$

The corresponding clock process is

$$\kappa(t) := 1_{[T, \infty)}(t), \quad t \geq 0.$$

Note that the formulation (2.8) extends the framework of Kramkov and Schachermayer (see [25, 26]) to stochastic utility.

Example 2.8 Maximization of expected utility from consumption over an infinite horizon, that is,

$$u(x) = \sup_{c \in \mathcal{A}(x)} \mathbb{E} \left[\int_0^\infty e^{-\nu t} U(t, \omega, c_t) dt \right], \quad x > 0, \nu > 0,$$

where the clock is defined as

$$\kappa(t) := \int_0^t e^{-\nu s} ds = \frac{1}{\nu}(1 - e^{-\nu t}), \quad t \geq 0.$$

Here ν is an impatience rate.

Example 2.9 Maximization of expected utility from consumption occurring at discrete times (t_1, \dots, t_N) :

$$u(x) = \sup_{c \in \mathcal{A}(x)} \mathbb{E} \left[\sum_{j=1}^N U(t_j, \omega, c_{t_j}) \right], \quad x > 0.$$

Here the clock process is

$$\kappa(t) := \sum_{j=1}^N 1_{[t_j, +\infty)}(t), \quad t \geq 0.$$

3 Abstract versions of the main theorems

Let μ be a finite and positive measure on a measurable space (Ω, \mathcal{F}) . Denote by $\mathbf{L}^0 = \mathbf{L}^0(\Omega, \mathcal{F}, \mu)$ the vector space of (equivalence classes of) real-valued measurable functions on $(\Omega, \mathcal{F}, \mu)$, topologized by convergence in measure μ . Let \mathbf{L}^0_+ denote its positive orthant, i.e.,

$$\mathbf{L}^0_+ = \{\xi \in \mathbf{L}^0(\Omega, \mathcal{F}, \mu) : \xi \geq 0\}.$$

For any ξ and η in \mathbf{L}^0 , we write

$$\langle \xi, \eta \rangle := \int_{\Omega} \xi \eta d\mu.$$

If ξ and η are both nonnegative, the integral is well defined in $[0, \infty]$. Let \mathcal{C}, \mathcal{D} be subsets of \mathbf{L}^0_+ that satisfy the conditions below.

1.

$$\begin{aligned} \xi \in \mathcal{C} &\iff \langle \xi, \eta \rangle \leq 1 \quad \text{for all } \eta \in \mathcal{D}, \\ \eta \in \mathcal{D} &\iff \langle \xi, \eta \rangle \leq 1 \quad \text{for all } \xi \in \mathcal{C}. \end{aligned} \tag{3.1}$$

2. \mathcal{C} and \mathcal{D} contain at least one strictly positive element, i.e.,

$$\text{there are } \xi^* \in \mathcal{C}, \eta^* \in \mathcal{D} \text{ such that } \min(\xi^*, \eta^*) > 0 \quad \mu\text{-a.e.} \tag{3.2}$$

Observe that our construction of the abstract sets \mathcal{C} and \mathcal{D} is similar to the one in [25]; however, we do not require any constant to be an element of \mathcal{C} . This leads to a *symmetry* between the sets \mathcal{C} and \mathcal{D} that plays an important role in the proofs. Also notice that \mathcal{C} and \mathcal{D} are convex and bounded in $\mathbf{L}^0(\mu)$, since they are bounded in $\mathbf{L}^1(\eta^* d\mu)$ and $\mathbf{L}^1(\xi^* d\mu)$, respectively. For $x > 0$ and $y > 0$, we define the sets

$$\begin{aligned} \mathcal{C}(x) &:= x\mathcal{C} := \{x\xi : \xi \in \mathcal{C}\}, \\ \mathcal{D}(y) &:= y\mathcal{D} := \{y\eta : \eta \in \mathcal{D}\}. \end{aligned} \tag{3.3}$$

Consider a *stochastic utility function* $U: \Omega \times [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$, which satisfies the following conditions.

Assumption 3.1 For every $\omega \in \Omega$, the function $x \mapsto U(\omega, x)$ is strictly concave, strictly increasing, continuously differentiable on $(0, \infty)$ and satisfies the Inada conditions

$$\lim_{x \downarrow 0} U'(\omega, x) = +\infty \quad \text{and} \quad \lim_{x \rightarrow \infty} U'(\omega, x) = 0,$$

where $U'(\cdot, \cdot)$ denotes the partial derivative with respect to the second argument. At $x = 0$, we have by continuity $U(\omega, 0) = \lim_{x \downarrow 0} U(\omega, x)$; this value may be $-\infty$. For every $x \geq 0$, the function $U(\cdot, x)$ is measurable.

Define the function V conjugate to U as

$$V(\omega, y) := \sup_{x > 0} (U(\omega, x) - xy), \quad (\omega, y) \in \Omega \times [0, \infty).$$

Observe that $-V$ satisfies Assumption 3.1. For a function W on $\Omega \times [0, \infty)$ and a function $\xi \in \mathbf{L}^0_+$, we write $W(\xi) := W(\omega, \xi(\omega))$. Recall that W^+ and W^- denote the positive and negative parts of W , respectively.

Now we can state the optimization problems; they are

$$u(x) = \sup_{\xi \in \mathcal{C}(x)} \int_{\Omega} U(\xi) d\mu, \quad x > 0, \tag{3.4}$$

$$v(y) = \inf_{\eta \in \mathcal{D}(y)} \int_{\Omega} V(\eta) d\mu, \quad y > 0, \tag{3.5}$$

where we used the convention

$$\int_{\Omega} U(\xi) d\mu := -\infty \quad \text{if} \quad \int_{\Omega} U^-(\xi) d\mu = +\infty,$$

$$\int_{\Omega} V(\eta) d\mu := +\infty \quad \text{if} \quad \int_{\Omega} V^+(\eta) d\mu = +\infty.$$

The following theorem is an abstract version of Theorem 2.3.

Theorem 3.2 Assume that \mathcal{C} and \mathcal{D} satisfy conditions (3.1) and (3.2). Let Assumption 3.1 hold and suppose

$$v(y) < \infty \quad \text{for all } y > 0 \quad \text{and} \quad u(x) > -\infty \quad \text{for all } x > 0.$$

Then we have:

1. $u(x) < \infty$ for all $x > 0$, $v(y) > -\infty$ for all $y > 0$. The functions u and v satisfy the biconjugacy relations, i.e.,

$$v(y) = \sup_{x > 0} (u(x) - xy), \quad y > 0,$$

$$u(x) = \inf_{y > 0} (v(y) + xy), \quad x > 0. \tag{3.6}$$

The functions u and $-v$ are continuously differentiable on $(0, \infty)$, strictly increasing, strictly concave and satisfy the Inada conditions

$$\begin{aligned}
 u'(0) &:= \lim_{x \downarrow 0} u'(x) = +\infty, & -v'(0) &:= \lim_{y \downarrow 0} -v'(y) = +\infty, \\
 u'(\infty) &:= \lim_{x \rightarrow \infty} u'(x) = 0, & -v'(\infty) &:= \lim_{y \rightarrow \infty} -v'(y) = 0.
 \end{aligned}$$

2. For every $x > 0$, the solution $\hat{\xi}(x)$ to (3.4) exists and is unique. For every $y > 0$, the solution $\hat{\eta}(y)$ to (3.5) exists and is unique. If $y = u'(x)$, we have the dual relations

$$\hat{\eta}(y) = U'(\hat{\xi}(x)) \quad \mu\text{-a.e.}$$

and

$$\langle \hat{\xi}(x), \hat{\eta}(y) \rangle = xy.$$

In order to state an abstract version of Theorem 2.4, we need the following definitions. Let $\tilde{\mathcal{D}}$ be a subset of \mathcal{D} such that

- (i) $\tilde{\mathcal{D}}$ is closed under countable convex combinations;
- (ii) for every $\xi \in \mathcal{C}$, we have

$$\sup_{\eta \in \tilde{\mathcal{D}}} \langle \xi, \eta \rangle = \sup_{\eta \in \mathcal{D}} \langle \xi, \eta \rangle. \tag{3.7}$$

Likewise, define $\tilde{\mathcal{C}}$ to be a subset of \mathcal{C} such that

- (iii) $\tilde{\mathcal{C}}$ is closed under countable convex combinations,
- (iv) for every $\eta \in \mathcal{D}$, we have

$$\sup_{\xi \in \tilde{\mathcal{C}}} \langle \xi, \eta \rangle = \sup_{\xi \in \mathcal{C}} \langle \xi, \eta \rangle.$$

Theorem 3.3 Under the conditions of Theorem 3.2, we have

$$\begin{aligned}
 v(y) &= \inf_{\eta \in \tilde{\mathcal{D}}} \int_{\Omega} V(y\eta) \, d\mu, \quad y > 0. \\
 u(x) &= \sup_{\xi \in \tilde{\mathcal{C}}} \int_{\Omega} U(x\xi) \, d\mu, \quad x > 0.
 \end{aligned}$$

The proofs of Theorems 3.2 and 3.3 are given via several lemmas.

Lemma 3.4 Under the conditions of Theorem 3.2, we have

$$v(y) \geq \sup_{x > 0} (u(x) - xy), \quad y > 0.$$

As a result, both u and v are real-valued functions such that

$$\limsup_{x \rightarrow \infty} \frac{u(x)}{x} \leq 0 \quad \text{and} \quad \liminf_{y \rightarrow \infty} \frac{v(y)}{y} \geq 0.$$

Proof Fix $x > 0$ and $y > 0$. We have

$$\sup_{\xi \in \mathcal{C}(x)} \inf_{\eta \in \mathcal{D}(y)} \int_{\Omega} (U(\xi) - \xi\eta) d\mu \leq \inf_{\eta \in \mathcal{D}(y)} \sup_{\xi \in \mathcal{C}(x)} \int_{\Omega} (U(\xi) - \xi\eta) d\mu. \tag{3.8}$$

Using (3.1), we can bound the left-hand side from below by $u(x) - xy$, i.e.,

$$\sup_{\xi \in \mathcal{C}(x)} \inf_{\eta \in \mathcal{D}(y)} \int_{\Omega} (U(\xi) - \xi\eta) d\mu \geq \sup_{\xi \in \mathcal{C}(x)} \left(\int_{\Omega} U(\xi) d\mu - xy \right) = u(x) - xy.$$

Since $V(\eta) \geq U(\xi) - \xi\eta$ for every $\xi \geq 0$ and $\eta \geq 0$, we can bound the right-hand side of (3.8) from above by $v(y)$, i.e.,

$$\inf_{\eta \in \mathcal{D}(y)} \sup_{\xi \in \mathcal{C}(x)} \int_{\Omega} (U(\xi) - \xi\eta) d\mu \leq \inf_{\eta \in \mathcal{D}(y)} \int_{\Omega} V(\eta) d\mu = v(y),$$

and the result follows. □

The techniques in Kramkov and Schachermayer [26] inspired the proof of the following lemma.

Lemma 3.5 *Under the conditions of Theorem 3.2, for every $y > 0$, the family $(V^-(h))_{h \in \mathcal{D}(y)}$ is uniformly integrable.*

Proof Fix $y > 0$. Assume by way of contradiction that $(V^-(h))_{h \in \mathcal{D}(y)}$ is not a uniformly integrable family. Then we can find a sequence $(\eta^n)_{n \geq 2} \subset \mathcal{D}(y)$, a sequence $(A^n)_{n \geq 2}$ of disjoint subsets of (Ω, \mathcal{F}) and a constant $\alpha > 0$ such that

$$\int_{\Omega} V^-(\eta^n) 1_{A^n} d\mu \geq \alpha, \quad n \geq 2.$$

Since $v(y) < \infty$, there exists $\eta^1 \in \mathcal{D}(y)$ such that

$$M := \int_{\Omega} V^+(\eta^1) d\mu < \infty.$$

Define the sequence $(\zeta^n)_{n \geq 1}$ as $\zeta^n := \sum_{k=1}^n \eta^k, n \geq 1$. Then by (3.1), for every $\xi \in \mathcal{C}$, we have

$$\langle \zeta^n, \xi \rangle = \sum_{k=1}^n \langle \eta^k, \xi \rangle \leq ny.$$

Thus $\zeta^n \in \mathcal{D}(ny)$, $n \geq 1$. Now, since V^- is nonnegative and nondecreasing, we get

$$\begin{aligned} \int_{\Omega} V^-(\zeta^n) d\mu &\geq \int_{\Omega} \sum_{k=2}^n V^-\left(\sum_{j=1}^n \eta^j\right) 1_{A^k} d\mu \\ &\geq \int_{\Omega} \sum_{k=2}^n V^-(\eta^k) 1_{A^k} d\mu \\ &\geq \alpha(n-1), \quad n \geq 2. \end{aligned}$$

On the other hand, since V^+ is nonincreasing, we obtain

$$\int_{\Omega} V^+(\zeta^n) d\mu \leq \int_{\Omega} V^+(\eta^1) d\mu = M < \infty.$$

Therefore, we deduce that

$$\int_{\Omega} V(\zeta^n) d\mu \leq M - \alpha(n-1), \quad n \geq 2.$$

Consequently,

$$\liminf_{z \rightarrow \infty} \frac{v(z)}{z} \leq \liminf_{n \rightarrow \infty} \frac{\int_{\Omega} V(\zeta^n) d\mu}{ny} \leq \liminf_{n \rightarrow \infty} \frac{M - \alpha(n-1)}{ny} = -\frac{\alpha}{y} < 0,$$

which contradicts the conclusion of Lemma 3.4. □

We need a version of Komlós’ lemma for the set \mathcal{D} . Some other formulations of Komlós’ lemma are proved in [24, 9, 1, 34].

Lemma 3.6 *Assume that the sets \mathcal{C} and \mathcal{D} satisfy (3.1) and (3.2). Let $(\eta^n)_{n \geq 1} \subset \mathcal{D}$. Then there exist a sequence of convex combinations $\zeta^n \in \text{conv}(\eta^n, \eta^{n+1}, \dots)$, $n \geq 1$, and an element $\hat{\eta} \in \mathcal{D}$ such that $(\zeta^n)_{n \geq 1}$ converges μ -a.e. to $\hat{\eta}$.*

Proof Since \mathcal{D} is bounded in \mathbf{L}^0 , using Lemma A1.1 in [9], we can construct a sequence $\zeta^n \in \text{conv}(\eta^n, \eta^{n+1}, \dots)$, $n \geq 1$, such that $(\zeta^n)_{n \geq 1}$ converges μ -a.e. to an element $\hat{\eta}$, where $\hat{\eta}$ is finite μ -a.e. By convexity of the set \mathcal{D} , we obtain that $(\zeta^n)_{n \geq 1}$ is a subset of \mathcal{D} . By Fatou’s lemma, for every $\xi \in \mathcal{C}$, we have

$$\langle \xi, \hat{\eta} \rangle \leq \liminf_{n \rightarrow \infty} \langle \xi, \zeta^n \rangle \leq 1.$$

Hence, $\hat{\eta} \in \mathcal{D}$. □

Lemma 3.7 *Under the conditions of Theorem 3.2, for each $y > 0$, there exists a unique $\hat{\eta}(y) \in \mathcal{D}(y)$ such that*

$$v(y) = \int_{\Omega} V(\hat{\eta}(y)) d\mu. \tag{3.9}$$

As a consequence, v is strictly convex.

Proof Fix $y > 0$. Let $(\eta^n)_{n=1}^\infty \subset \mathcal{D}(y)$ be a minimizing sequence, i.e.,

$$v(y) = \lim_{n \rightarrow \infty} \int_{\Omega} V(\eta^n) d\mu.$$

It follows from Lemma 3.6 that there exist a sequence of convex combinations $\zeta^n \in \text{conv}(\eta^n, \eta^{n+1}, \dots)$, $n \geq 1$, and an element $\hat{\eta}(y) \in \mathcal{D}(y)$ such that $(\zeta^n)_{n=1}^\infty$ converges μ -a.e. to $\hat{\eta}(y)$. Using convexity of V , Lemma 3.5 and Fatou’s lemma, we get

$$v(y) = \liminf_{n \rightarrow \infty} \int_{\Omega} V(\eta^n) d\mu \geq \liminf_{n \rightarrow \infty} \int_{\Omega} V(\zeta^n) d\mu \geq \int_{\Omega} V(\hat{\eta}(y)) d\mu.$$

Therefore, (3.9) holds. Uniqueness of the minimizer to (3.5) as well as strict convexity of v then follow from the strict convexity of V . \square

By the symmetry between the optimization problems (3.4) and (3.5), the following result is a corollary to Lemma 3.7.

Lemma 3.8 *Under the assumptions of Theorem 3.2, for every $x > 0$, there exists a unique maximizer to the primal problem (3.4). As a consequence, u is strictly concave.*

Lemma 3.9 *Under the assumptions of Theorem 3.2, we have*

$$v(y) = \sup_{x > 0} (u(x) - xy), \quad y > 0. \tag{3.10}$$

Proof The two-step proof is based on change of numéraire ideas.

Step 1. Let us show (3.10), assuming that

$$\text{the constant function } 1 \in \mathcal{C} \quad \text{and} \quad \int_{\Omega} U(1) d\mu > -\infty.$$

In this case, $\int_{\Omega} U(x) d\mu$ is finite for any constant $x \geq 1$. Let \mathcal{S}_n be the set of all nonnegative, measurable functions $\xi : \Omega \rightarrow [0, n]$, i.e.,

$$\mathcal{S}_n := \{\xi \in \mathbf{L}^0 : \xi(\omega) \in [0, n] \text{ for all } \omega \in \Omega\}, \quad n > 0.$$

The sets \mathcal{S}_n are $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ -compact. Fix $y > 0$. Since $\mathcal{D}(y)$ is convex and U is concave, the minimax theorem (see [35], Theorem 45.8) gives the equality

$$\sup_{\xi \in \mathcal{S}_n} \inf_{\eta \in \mathcal{D}(y)} \int_{\Omega} (U(\xi) - \xi\eta) d\mu = \inf_{\eta \in \mathcal{D}(y)} \sup_{\xi \in \mathcal{S}_n} \int_{\Omega} (U(\xi) - \xi\eta) d\mu. \tag{3.11}$$

Denote

$$\mathcal{C}'(x) := \left\{ \xi \in \mathcal{C}(x) : \sup_{\eta \in \mathcal{D}(y)} \langle \xi, \eta \rangle = xy \right\}.$$

It follows from (3.3) that $\bigcup_{x>0} \mathcal{C}'(x) \cup \{\xi \equiv 0\} = \bigcup_{x>0} \mathcal{C}(x)$. As a result, since $1 \in \mathcal{C}$, we get

$$\begin{aligned} \sup_{x>0} (u(x) - xy) &= \sup_{x>0} \sup_{\xi \in \mathcal{C}'(x)} \left(\int_{\Omega} U(\xi) d\mu - xy \right) \\ &\geq \lim_{n \rightarrow \infty} \sup_{\xi \in \mathcal{S}_n} \inf_{\eta \in \mathcal{D}(y)} \int_{\Omega} (U(\xi) - \xi \eta) d\mu. \end{aligned} \tag{3.12}$$

In view of (3.11), (3.12) and Lemma 3.4, it suffices to show that

$$v(y) = \lim_{n \rightarrow \infty} \inf_{\eta \in \mathcal{D}(y)} \sup_{\xi \in \mathcal{S}_n} \int_{\Omega} (U(\xi) - \xi \eta) d\mu. \tag{3.13}$$

For each $n \geq 1$, define V^n as

$$V^n(z) := \sup_{0 < x \leq n} (U(x) - xz), \quad z > 0.$$

Then via pointwise maximization, we get

$$\inf_{\eta \in \mathcal{D}(y)} \sup_{\xi \in \mathcal{S}_n} \int_{\Omega} (U(\xi) - \xi \eta) d\mu = \inf_{\eta \in \mathcal{D}(y)} \int_{\Omega} V^n(\eta) d\mu =: v^n(y).$$

Notice that $v^n \leq v$ and $(v^n(y))_{n \geq 1}$ is an increasing sequence. Let $(\eta^n)_{n \geq 1} \subset \mathcal{D}(y)$ be such that

$$\lim_{n \rightarrow \infty} v^n(y) = \lim_{n \rightarrow \infty} \int_{\Omega} V^n(\eta^n) d\mu. \tag{3.14}$$

It follows from Lemma 3.6 that there exists a sequence $\zeta^n \in \text{conv}(\eta^n, \eta^{n+1}, \dots)$, $n \geq 1$, such that $(\zeta^n)_{n \geq 1}$ converges μ -a.e. to a function $\hat{\zeta} \in \mathcal{D}(y)$.

We claim that $(V^n)^-(\zeta^n)$, $n \geq 2$, is a uniformly integrable sequence. Indeed, for $n \geq 2$, we have

$$V^n(\zeta) \geq V^2(\zeta) \geq V(\zeta) 1_{\{\zeta \geq U'(2)\}} + (U(2) - 2U'(2)) 1_{\{\zeta < U'(2)\}}.$$

The concavity of U yields that $U'(2) \leq U(2) - U(1)$. Therefore,

$$V^n(\zeta) \geq \min(V(\zeta), 2U(1) - U(2)), \quad n \geq 2.$$

The uniform integrability of $(V^n)^-(\zeta^n)$, $n \geq 2$, follows now from Lemma 3.5 and the integrability of $U(1)$ and $U(2)$. Therefore, from the convexity of V^n and Fatou's lemma, we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} V^n(\eta^n) d\mu \geq \liminf_{n \rightarrow \infty} \int_{\Omega} V^n(\zeta^n) d\mu \geq \int_{\Omega} V(\hat{\zeta}) d\mu \geq v(y),$$

which in view of (3.14) implies (3.13).

Step 2. Here we show how the general case reduces to the one in Step 1. Let $\hat{\xi} := \arg \max_{\xi \in \mathcal{C}(1/2)} \int_{\Omega} U(\xi) d\mu$ and let ξ_0 be a strictly positive element of $\mathcal{C}(1/2)$. Both $\hat{\xi}$ and ξ_0 exist by Lemma 3.8 and assumption (3.2), respectively. Define

$$\zeta := \max(\hat{\xi}, \xi_0).$$

Then $\zeta \in \mathcal{C}$ and $\int_{\Omega} U(\zeta) d\mu$ is finite. Let

$$\begin{aligned} \tilde{U}(x) &:= U(\zeta x), \\ \tilde{\mathcal{C}}(x) &:= \{\xi : \xi \zeta \in \mathcal{C}(x)\}; \end{aligned}$$

then

$$u(x) = \sup_{\xi \in \tilde{\mathcal{C}}(x)} \int_{\Omega} \tilde{U}(\xi) d\mu, \quad x > 0.$$

Similarly, define

$$\begin{aligned} \tilde{V}(y) &:= V(y/\zeta), \\ \tilde{\mathcal{D}}(y) &:= \{\eta : \eta/\zeta \in \mathcal{D}(y)\}; \end{aligned}$$

then we have

$$v(y) = \inf_{\eta \in \tilde{\mathcal{D}}(y)} \int_{\Omega} \tilde{V}(\eta) d\mu, \quad y > 0.$$

Observe that \tilde{U} satisfies Assumption 3.1, \tilde{V} is the conjugate function to \tilde{U} , and the sets $\tilde{\mathcal{C}}(1)$ and $\tilde{\mathcal{D}}(1)$ satisfy the bipolar relations (3.1) and (3.2). Moreover,

$$1 \in \tilde{\mathcal{C}}(1) \quad \text{and} \quad \int_{\Omega} \tilde{U}(1) d\mu > -\infty.$$

Now (3.10) follows from Step 1. □

Proof of Theorem 3.2 Observe that by Lemmas 3.8 and 3.7, both functions u and $-v$ are strictly concave. Thus the conjugacy relations (3.6) follow from Theorem 12.2 in the book by Rockafellar [32] and Lemma 3.9 (if we extend u by the value $-\infty$ on $(-\infty, 0]$). In turn, the strict concavity of u and $-v$, (3.6) and Theorem 26.3 in [32] imply the differentiability of u and v everywhere in their domains.

Fix $x > 0$ and take $y = u'(x)$. Let $\hat{\eta} \in \mathcal{D}(y)$ be the optimizer to the dual problem (3.5) and $\hat{\xi} \in \mathcal{C}(x)$ the optimizer to the primal problem (3.4). Both $\hat{\eta}$ and $\hat{\xi}$ exist by Lemmas 3.7 and 3.8, respectively. Using the definition of V , (3.1), (3.3) and Theorem 23.5 in [32], we get

$$0 \leq \int_{\Omega} (V(\hat{\eta}) - U(\hat{\xi}) + \hat{\xi} \hat{\eta}) d\mu \leq v(y) - u(x) + xy = 0.$$

Therefore, for μ -a.e. $\omega \in \Omega$, we have

$$V(\hat{\eta}) = U(\hat{\xi}) - \hat{\xi} \hat{\eta}.$$

This implies the remaining assertions of the theorem, i.e.,

$$U'(\hat{\xi}) = \hat{\eta} \quad \mu\text{-a.e.},$$

$$\langle \hat{\xi}, \hat{\eta} \rangle = \int_{\Omega} U(\hat{\xi}) d\mu - \int_{\Omega} V(\hat{\eta}) d\mu = u(x) - v(y) = xy. \quad \square$$

In order to prove Theorem 3.3, we proceed in a similar way as the proof of Proposition 1 in Kramkov and Schachermayer [26]. Define the polar of a set $A \subseteq \mathbf{L}_+^0$ as

$$A^\circ := \{ \xi \in \mathbf{L}_+^0 : \langle \xi, \eta \rangle \leq 1 \text{ for all } \eta \in A \}.$$

A subset A of \mathbf{L}_+^0 is called *solid* if $0 \leq \eta \leq \zeta$ and $\zeta \in A$ implies that $\eta \in A$. Observe that the sets \mathcal{C} and \mathcal{D} satisfy the bipolar relations (3.1). We use a version of the *bipolar theorem* that was proved by Brannath and Schachermayer in [5]: for a subset A of \mathbf{L}_+^0 , the bipolar $A^{\circ\circ}$ is the smallest subset of \mathbf{L}_+^0 containing A which is convex, solid and closed with respect to the topology of convergence in measure.

Lemma 3.10 *Under the conditions of Theorem 3.2, for every fixed $y > 0$, let $\hat{\eta}(y)$ be the minimizer to the dual problem (3.5). Then there exists a sequence $(\zeta^n)_{n \geq 1}$ in $\tilde{\mathcal{D}}$ that μ -a.e. converges to $\hat{\eta}(y)/y$.*

Proof Fix $y > 0$. By assumption, $\tilde{\mathcal{D}}$ is a convex set that satisfies (3.7). Therefore, applying the bipolar theorem (see [5]), we deduce that \mathcal{D} is the smallest convex, closed and solid subset of $\mathbf{L}_+^0(\Omega, \mathcal{F}, \mu)$ containing $\tilde{\mathcal{D}}$. Thus for any $\eta \in \mathcal{D}$, there exists a sequence $(\zeta^n)_{n \geq 1}$ in $\tilde{\mathcal{D}}$ such that $\zeta = \lim_{n \rightarrow \infty} \zeta^n$ exists μ -a.e. and $\zeta \geq \eta$. In particular, such a sequence exists for $\eta = \hat{\eta}(y)/y$. We deduce from optimality of $\hat{\eta}(y)$ that $\eta = \zeta = \lim_{n \rightarrow \infty} \zeta^n$. □

Lemma 3.11 *Under the conditions of Theorem 3.2, for each $y > 0$, we have*

$$\inf_{\eta \in \tilde{\mathcal{D}}} \int_{\Omega} V(y\eta) d\mu < \infty.$$

Proof To simplify notations, we assume that $y = 1$. Let $(a^n)_{n \geq 1}$ be a sequence of strictly positive numbers such that $\sum_{n=1}^{\infty} a^n = 1$. By Lemma 3.7, for each $n \geq 1$, there exists $\hat{\eta}(a^n)$, the minimizer to the dual problem (3.5) when $y = a^n$. One can construct a sequence of strictly positive numbers $(\delta_n)_{n \geq 2}$ that decreases to 0 such that

$$\sum_{n=1}^{\infty} \int_{\Omega} V(\hat{\eta}(a^n)) 1_{A_n} d\mu < \infty, \quad \text{whenever } A_n \in \mathcal{F} \text{ and } \mu(A_n) \leq \delta_n, n \geq 2. \quad (3.15)$$

From Lemma 3.10, we deduce the existence of a sequence $(\eta^n)_{n \geq 1} \subset \tilde{\mathcal{D}}$ such that

$$\mu\left(V(a^n \eta^n) > V(\hat{\eta}(a^n)) + 1 \right) \leq \delta_{n+1}, \quad n \geq 1.$$

Define the sequences of measurable sets $(B_n)_{n \geq 1}$ and $(A_n)_{n \geq 1}$ as

$$B_n := \{V(a^n \eta^n) \leq V(\hat{\eta}(a^n)) + 1\}, \quad n \geq 1,$$

$$A_1 := B_1, \dots, A_n := B_n \setminus \left(\bigcup_{k=1}^{n-1} A_k \right), \dots$$

Then $(A_n)_{n \geq 1}$ is a measurable partition of Ω and $\mu(A_n) \leq \delta_n$ for $n \geq 2$.

To finish the proof, let $\eta := \sum_{n=1}^{\infty} a^n \eta^n$. Then $\eta \in \tilde{\mathcal{D}}$ since $\tilde{\mathcal{D}}$ is closed under countable convex combinations. From the construction of $(A_n)_{n \geq 1}$, monotonicity of V and (3.15), we obtain

$$\begin{aligned} \int_{\Omega} V(\eta) d\mu &= \sum_{n=1}^{\infty} \int_{\Omega} V\left(\sum_{j=1}^{\infty} a^j \eta^j\right) 1_{A_n} d\mu \\ &\leq \sum_{n=1}^{\infty} \int_{\Omega} V(a^n \eta^n) 1_{A_n} d\mu \\ &\leq \sum_{n=1}^{\infty} \int_{\Omega} V(\hat{\eta}(a^n)) 1_{A_n} d\mu + \mu(\Omega) \\ &< \infty. \end{aligned}$$

This concludes the proof of the lemma. □

Proof of Theorem 3.3 By symmetry between the primal and dual problems, it suffices to prove that

$$v(y) = \inf_{\eta \in \tilde{\mathcal{D}}} \int_{\Omega} V(y\eta) d\mu, \quad y > 0.$$

Fix $y > 0$ and $\epsilon > 0$. We show that there exists $\eta \in \tilde{\mathcal{D}}$ such that

$$\int_{\Omega} V((y + \epsilon)\eta) d\mu \leq v(y) + \epsilon.$$

Let $\hat{\eta} \in \mathcal{D}(y)$ be the minimizer to the dual problem (3.5) and ζ an element of $\tilde{\mathcal{D}}$ such that

$$\int_{\Omega} V(\epsilon\zeta) d\mu < \infty,$$

whose existence follows from Lemma 3.11. Let $\delta > 0$ be such that

$$\int_{\Omega} (|V(\hat{\eta})| + |V(\epsilon\zeta)|) 1_A d\mu \leq \frac{\epsilon}{2}, \quad \text{whenever } A \in \mathcal{F} \text{ with } \mu(A) \leq \delta.$$

By Lemma 3.10, there exists $\theta \in \tilde{\mathcal{D}}$ such that the set

$$B := \left\{ V(y\theta) > V(\hat{\eta}) + \frac{\epsilon}{2\mu(\Omega)} \right\}$$

has measure $\mu(B) \leq \delta$. Define

$$\eta := \frac{y\theta + \epsilon\zeta}{y + \epsilon}.$$

Since $\tilde{\mathcal{D}}$ is convex, it follows that $\eta \in \tilde{\mathcal{D}}$. By construction of the set B and monotonicity of V , we obtain

$$\begin{aligned} \int_{\Omega} V((y + \epsilon)\eta) d\mu &= \int_{\Omega} V(y\theta + \epsilon\zeta) d\mu \\ &\leq \int_{\Omega} V(y\theta) 1_{B^c} d\mu + \int_{\Omega} V(\epsilon\zeta) 1_B d\mu \\ &\leq \frac{\epsilon}{2} + \int_{\Omega} V(\hat{\eta}) d\mu + \int_{\Omega} (V(\epsilon\zeta) - V(\hat{\eta})) 1_B d\mu \\ &\leq v(y) + \epsilon. \end{aligned} \quad \square$$

4 Proofs of the main theorems

Let us recall the concept of Fatou convergence of stochastic processes; see [11].

Definition 4.1 Let τ be a dense subset of $[0, \infty)$. A sequence of processes $(Y^n)_{n \geq 1}$ is *Fatou-convergent on τ* to a process Y if $(Y^n)_{n \geq 1}$ is uniformly bounded from below and

$$Y_t = \limsup_{\substack{s \downarrow t, \\ s \in \tau}} \limsup_{n \rightarrow \infty} Y_s^n = \liminf_{\substack{s \downarrow t, \\ s \in \tau}} \liminf_{n \rightarrow \infty} Y_s^n$$

almost surely for every $t \geq 0$. If $\tau = [0, \infty)$, then the sequence $(Y^n)_{n \geq 1}$ is called *Fatou-convergent*.

We also recall that a probability measure \mathbb{Q} is called an *equivalent local martingale measure* for \mathcal{X} if \mathbb{Q} is equivalent to \mathbb{P} and every $X \in \mathcal{X}$ is a local martingale under \mathbb{Q} .

The following lemma can be thought of as an extension of Theorem 5.12 in [10] to our setting. Its proof is based on an application of Fatou convergence and the optional decomposition theorem; see [27, 11, 36].

Lemma 4.2 *Let c be a nonnegative optional process and κ a stochastic clock. Under the assumptions (2.1) and (2.3), the following conditions are equivalent:*

- (i) $c \in \mathcal{A}$.
- (ii) $\sup_{Z \in \mathcal{Z}} \mathbb{E}[\int_0^\infty c_t Z_t d\kappa_t] \leq 1$.

Proof Let $c \in \mathcal{A}$. Then there exists a predictable S -integrable process H such that

$$1 + \int_0^t H_u dS_u \geq \int_0^t c_u d\kappa_u \geq 0, \quad t \geq 0.$$

Take an arbitrary $Z \in \mathcal{Z}$. Using the supermartingale property of $Z(1 + \int H dS)$, we obtain for every $T \geq 0$

$$1 \geq \mathbb{E} \left[Z_T \left(1 + \int_0^T H_u dS_u \right) \right] \geq \mathbb{E} \left[Z_T \int_0^T c_u d\kappa_u \right]. \tag{4.1}$$

We claim that

$$\mathbb{E} \left[Z_T \int_0^T c_u d\kappa_u \right] = \mathbb{E} \left[\int_0^T c_u Z_u d\kappa_u \right]. \tag{4.2}$$

Let $C_t := \int_0^t c_u d\kappa_u, t \in [0, T]$. We remark that C is an increasing process that corresponds to the cumulative consumption. For every stopping time $\tau \leq T$, the integration by parts formula gives

$$Z_\tau C_\tau = \int_0^\tau Z_t dC_t + \int_0^\tau C_{t-} dZ_t; \tag{4.3}$$

see also Theorem I.4.49 in Jacod and Shiryaev [16]. By Theorem III.29 in Protter [31], the process $(\int_0^t C_{u-} dZ_u)_{t \in [0, T]}$ is a local martingale. Therefore, there exists an increasing sequence of stopping times $\tau_n \leq T, n \geq 1$, such that $\lim_{n \rightarrow \infty} \mathbb{P}[\tau_n \leq T] = 0$ and $\mathbb{E}[\int_0^{\tau_n} C_{t-} dZ_t] = 0$ for every $n \geq 1$. Since Z is a martingale, using the monotone convergence theorem, we get from (4.1) and (4.3) that

$$\begin{aligned} 1 &\geq \mathbb{E}[Z_T C_T] = \lim_{n \rightarrow \infty} \mathbb{E}[Z_T C_{\tau_n}] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[Z_{\tau_n} C_{\tau_n}] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau_n} Z_t dC_t \right] = \mathbb{E} \left[\int_0^T Z_t dC_t \right], \end{aligned}$$

i.e., (4.2) holds. Combining (4.1) and (4.2) (or equivalently, considering the latter expression), taking $T \rightarrow \infty$, and using monotone convergence, we get (ii).

Conversely, assume that $\sup_{Z \in \mathcal{Z}} \mathbb{E}[\int_0^\infty c_t Z_t d\kappa_t] \leq 1$. Then as in (4.2), we get

$$\mathbb{E} \left[Z_n \int_0^n c_u d\kappa_u \right] = \mathbb{E} \left[\int_0^n c_u Z_u d\kappa_u \right], \quad n \geq 0.$$

One can see that $\{(Z_t)_{t \in [0, n]} : Z \in \mathcal{Z}\}$ coincides with the set of càdlàg densities of equivalent local martingale measures for \mathcal{X} on (Ω, \mathcal{F}_n) . Let us denote the set of such measures by $\mathcal{M}_n^\mathbb{Q}$. Then by Proposition 4.2 in [27], there exists a càdlàg process V^n on $[0, n]$, given by

$$V_t^n = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{M}_n^\mathbb{Q}} \mathbb{E}^\mathbb{Q} \left[\int_0^n c_u d\kappa_u \mid \mathcal{F}_t \right], \quad t \in [0, n],$$

which is a supermartingale under every $\mathbb{Q} \in \mathcal{M}_n^\mathbb{Q}$. Notice that $V_t^n \geq \int_0^t c_u d\kappa_u, t \in [0, n]$, and $V_0^n \leq 1$. Now, applying Theorem 4.1 in [11], we can write V^n as

$$V_t^n = V_0^n + \int_0^t H_u^n dS_u - A_t^n, \quad t \in [0, n],$$

where H^n is predictable S -integrable and A^n optional increasing with $A^n_0 = 0$. Extend H^n to $[0, \infty)$ by setting $H^n_t := 0$ for $t > n$. Using Lemma 5.2 in [11], we can construct a sequence of stochastic processes $Y^n \in \text{conv}(1 + \int H^n dS, 1 + \int H^{n+1} dS, \dots)$, $n \geq 1$, and a process Y such that $(ZY^n)_{n \geq 1}$ is Fatou-convergent on the set of positive rational numbers to a supermartingale ZY for every $Z \in \mathcal{Z}$. Then, we have $Y_t \geq \int_0^t c_u d\kappa_u, t \geq 0$, and $Y_0 \leq 1$. Now using Theorem 4.1 in [11] on $[0, n]$, we get

$$Y_t = Y_0 + \int_0^t G^n_u dS_u - B^n_t, \quad t \in [0, n],$$

where G^n is predictable S -integrable and B^n optional increasing with $B^n_0 = 0$. Let us set $G^n_t := 0$ for $t > n$. Denoting

$$n(t) := \min \{n \in \mathbb{N} : n > t\}, \quad t \geq 0,$$

we deduce that the process

$$\tilde{G}_t := \sum_{k=1}^{n(t)} (G^k_t - G^k_{t-1}), \quad t \geq 0,$$

is such that $1 + \int_0^t \tilde{G}_u dS_u \geq \int_0^t c_u d\kappa_u, t \geq 0$. Thus, $c \in \mathcal{A}$. □

Lemma 4.3 *Let κ be a stochastic clock. Under the assumptions (2.1) and (2.3), for every $c \in \mathcal{A}$, we have*

$$\sup_{Z \in \mathcal{Z}} \mathbb{E} \left[\int_0^\infty c_t Z_t d\kappa_t \right] = \sup_{Y \in \mathcal{Y}} \mathbb{E} \left[\int_0^\infty c_t Y_t d\kappa_t \right] \leq 1.$$

Proof By the definition (2.5), for an arbitrary $Y \in \mathcal{Y}$, we can find a sequence $(Y^n)_{n \geq 1}$ in the solid hull of \mathcal{Z} (i.e., such that $Y^n \leq Z^n$ ($d\kappa \times \mathbb{P}$)-a.e. for some $Z^n \in \mathcal{Z}$) such that $(Y^n)_{n \geq 1}$ converges ($d\kappa \times \mathbb{P}$)-a.e. to Y . Using Fatou’s lemma and Lemma 4.2, we get

$$\mathbb{E} \left[\int_0^\infty c_t Y_t d\kappa_t \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\int_0^\infty c_t Y^n_t d\kappa_t \right] \leq \sup_{Z \in \mathcal{Z}} \mathbb{E} \left[\int_0^\infty c_t Z_t d\kappa_t \right] \leq 1. \quad \square$$

Denote by $\mathbf{L}^0 = \mathbf{L}^0(d\kappa \times \mathbb{P})$ the linear space of (equivalence classes of) real-valued optional processes on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ which we equip with the topology of convergence in measure ($d\kappa \times \mathbb{P}$). Let \mathbf{L}^0_+ be the positive orthant of \mathbf{L}^0 . Recall that the polar of a set $A \subseteq \mathbf{L}^0_+$ is defined as

$$A^\circ := \left\{ Y \in \mathbf{L}^0_+ : \mathbb{E} \left[\int_0^\infty c_t Y_t d\kappa_t \right] \leq 1 \text{ for all } c \in A \right\}.$$

In view of Theorems 3.2 and 3.3, in order to complete the proofs of Theorems 2.3 and 2.4, it suffices to establish the following proposition. Note that the sets \mathcal{C}, \mathcal{D} and measure μ correspond to the sets \mathcal{A}, \mathcal{Y} and measure $(d\kappa \times \mathbb{P})$, and the sets $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ accord with the sets \mathcal{B} and \mathcal{Z} , respectively.

Proposition 4.4 *Assume that an \mathbb{R}^d -valued semimartingale S satisfies (2.3). Under the condition (2.1), the sets \mathcal{A} and \mathcal{Y} defined in (2.2) and (2.5), respectively, have the following properties:*

- (i) \mathcal{A} and \mathcal{Y} are subsets of L^0_+ that are convex, solid and closed in the topology of convergence in measure ($d\kappa \times \mathbb{P}$).
- (ii) The sets \mathcal{A} and \mathcal{Y} satisfy the bipolar relations

$$c \in \mathcal{A} \iff \mathbb{E} \left[\int_0^\infty c_t Y_t d\kappa_t \right] \leq 1 \quad \text{for all } Y \in \mathcal{Y},$$

$$Y \in \mathcal{Y} \iff \mathbb{E} \left[\int_0^\infty c_t Y_t d\kappa_t \right] \leq 1 \quad \text{for all } c \in \mathcal{A}.$$

- (iii) There exists $c \in \mathcal{A}$ such that $c > 0$, and there exists $Y \in \mathcal{Y}$ such that $Y > 0$.

Proof (i) It is enough to show closedness of \mathcal{A} . Let $(c^n)_{n \geq 1}$ be a sequence in \mathcal{A} that $(d\kappa \times \mathbb{P})$ -a.e. converges to c . For an arbitrary $Z \in \mathcal{Z}$, using Fatou’s lemma and Lemma 4.2, we get

$$\mathbb{E} \left[\int_0^\infty c_t Z_t d\kappa_t \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\int_0^\infty c^n_t Z_t d\kappa_t \right] \leq 1.$$

Therefore, by Lemma 4.2, $c \in \mathcal{A}$, and thus \mathcal{A} is closed.

- (ii) It follows from Lemma 4.2 that

$$\mathcal{A} = \mathcal{Z}^o,$$

whereas from Lemma 4.3, we deduce

$$\mathcal{Y} \subseteq \mathcal{A}^o = \mathcal{Z}^{oo}. \tag{4.4}$$

Since \mathcal{Y} is closed, convex and solid and $\mathcal{Z} \subset \mathcal{Y}$, it follows from the bipolar theorem of Brannath and Schachermayer [5] that $\mathcal{Z}^{oo} \subseteq \mathcal{Y}$. Combining this with (4.4), we conclude that

$$\mathcal{Y} = \mathcal{A}^o. \tag{4.5}$$

On the other hand, it follows from part (i) that \mathcal{A} is also convex, closed and solid. Thus $\mathcal{A} = \mathcal{A}^{oo}$ by the bipolar theorem. Therefore, from (4.5), we get

$$\mathcal{A} = \mathcal{Y}^o.$$

- (iii) Since \mathcal{X} contains the constant function $\mathbf{1} = (1)_{t \geq 0}$, the existence of $c \in \mathcal{A}$ such that $c > 0$ follows from the definition of the set \mathcal{A} . The existence of $Y \in \mathcal{Y}$ with $Y > 0$ follows from assumption (2.3). This completes the proof of Proposition 4.4. \square

The following example shows an application of our results to a utility maximization problem under a change of numéraire. Essentially, we start from the framework of Kramkov and Schachermayer [26] and show how more conclusions to their Theorem 2 can be added, namely existence and uniqueness results for a utility maximization problem under a new numéraire, via an application of Theorem 2.3.

Example 4.5 Consider the problem of optimal investment from terminal wealth (where $\kappa(t) = 1_{[T, \infty)}(t)$, $t \geq 0$), where T is a positive constant, S is a d -dimensional semimartingale and the price process of the bond is constant, U is a deterministic utility, i.e., $U(t, \omega, x) = \bar{U}(x)$, $(t, \omega, x) \in [0, \infty) \times \Omega \times [0, \infty)$, where \bar{U} is an *Inada utility*, i.e., a strictly increasing, strictly concave, continuously differentiable function that satisfies the Inada conditions and is extended at 0 by continuity. Notice that Assumption 2.1 holds for U . Let us assume that

$$v(y) < \infty, \quad y > 0,$$

and that condition (2.3) holds. Then the assumptions of Theorem 2.3 are satisfied (also notice the relationship with Theorem 2 in [26]). Therefore, existence and uniqueness hold for the solutions of the primal and dual problems as well as conjugacy relationships between them, where the wealth process is given *in the units of the bond*.

We now show how Theorem 2.3 can be used to establish the standard conclusions of the theory for the problem of utility maximization under a *new* numéraire, which we designate by N .

Let us denote $\tilde{S} := (\frac{S}{N}, \frac{1}{N})$ and set

$$\tilde{\mathcal{X}} := \left\{ X \geq 0 : X_t = 1 + \int_0^t H_u d\tilde{S}_u \geq 0, \quad t \in [0, T] \right\}.$$

Now we can define the primal value function under the numéraire N as

$$\tilde{u}(x) := \sup_{X \in \tilde{\mathcal{X}}} \mathbb{E}[\bar{U}(x X_T)], \quad x > 0.$$

Setting

$$\tilde{U}(\omega, x) := \bar{U}\left(\frac{x}{N_T(\omega)}\right), \quad x \geq 0,$$

and since $\{X_T : X \in \tilde{\mathcal{X}}\} = \{\frac{X_T}{N_T} : X \in \mathcal{X}\}$, we have

$$\tilde{u}(x) = \sup_{X \in \mathcal{X}} \mathbb{E}[\tilde{U}(x X_T)], \quad x > 0. \tag{4.6}$$

Remark 4.6 Denoting $\tilde{U}(t, \omega, x) := \bar{U}(\frac{x}{N_t(\omega)})$, $(t, \omega, x) \in [0, \infty) \times \Omega \times [0, \infty)$, one can see that \tilde{U} satisfies Assumption 2.1. With a slight abuse of notations, we say that \tilde{U} satisfies Assumption 2.1.

As for the dual problem, first let us observe that

$$\tilde{V}(\omega, y) := \bar{V}(y N_T(\omega)), \quad y \geq 0,$$

is the conjugate to \tilde{U} , where \bar{V} is the conjugate to \bar{U} . Then the value function dual to \tilde{u} can be defined as

$$\tilde{v}(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[\tilde{V}(Y_T)], \quad y > 0.$$

Lemma 4.7 *Let S be a semimartingale, N a process in \mathcal{X} , and \bar{U} an Inada utility. Let (2.3) hold, $v(y) < \infty$ for every $y > 0$, and*

$$\tilde{v}(y) < \infty, \quad y > 0. \tag{4.7}$$

Then the conclusions of Theorem 2.3 hold for \tilde{u} and \tilde{v} .

Remark 4.8 For $N \in \mathcal{X}$, (4.7) holds if either one of the following conditions is valid:

- The utility function \bar{U} is bounded from above, i.e.,

$$\lim_{x \rightarrow \infty} \bar{U}(x) = \bar{V}(0) < \infty.$$

Then we obtain

$$\tilde{v}(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[\bar{V}(Y_T N_T)] \leq \bar{V}(0) < \infty, \quad y > 0.$$

- N_T is bounded away from 0, i.e., there exists a constant $\varepsilon > 0$ such that

$$N_T \geq \varepsilon.$$

In this case, for every $y > 0$, we have

$$\tilde{v}(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[\bar{V}(N_T Y_T)] \leq \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[\bar{V}(\varepsilon Y_T)] = v(\varepsilon y) < \infty.$$

Proof of Lemma 4.7 Let us consider the formulation (4.6). Since \tilde{U} satisfies Assumption 2.1 (see Remark 4.6) and taking into account (4.7), to finish the proof, it suffices to show that

$$\tilde{u}(x) > -\infty \quad \text{for every } x > 0.$$

Fix $x > 0$. Since $N \in \mathcal{X}$, we have

$$\tilde{u}(x) = \sup_{X \in \mathcal{X}} \mathbb{E} \left[\bar{U} \left(\frac{x X_T}{N_T} \right) \right] \geq \bar{U}(x) > -\infty.$$

This concludes the proof of the lemma. □

Acknowledgements This work is part of the author’s PhD thesis. I would like to thank Dmitry Kramkov for being a supportive advisor, whose suggestions and ideas helped in writing this paper. I would also like to thank Giovanni Leoni, Scott Robertson and Mihai Sîrbu for remarks and discussions.

References

1. Beiglböck, M., Schachermayer, W., Veliyev, B.: A short proof of the Doob–Meyer theorem. *Stoch. Process. Appl.* **122**, 1204–1209 (2012)
2. Blanchet-Scalliet, C., El Karoui, N. Nicole, Jeanblanc, M., Martellini, L.: Optimal investment decisions when time-horizon is uncertain. *J. Math. Econ.* **44**, 1100–1113 (2008)

3. Bordigoni, G., Matoussi, A., Schweizer, M.: A stochastic control approach to a robust utility maximization problem. *Stoch. Anal. Appl.* **2**, 125–151 (2007)
4. Bouchard, B., Pham, H.: Wealth-path dependent utility maximization in incomplete markets. *Finance Stoch.* **8**, 579–603 (2004)
5. Brannath, W., Schachermayer, W.: A bipolar theorem for subsets of $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$. In: *Séminaire de Probabilités XXXIII*. Springer Lecture Notes in Mathematics, vol. XXXIII, pp. 349–354 (1999)
6. Burgert, C., Rüschemdorf, L.: Optimal consumption strategies under model uncertainty. *Stat. Decis.* **23**, 1–14 (2005)
7. Cox, J.C., Huang, C.F.: Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Econ. Theory* **49**, 33–83 (1989)
8. Cox, J.C., Huang, C.F.: A variational problem arising in financial economics. *J. Math. Econ.* **20**, 465–487 (1991)
9. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300**, 463–520 (1994)
10. Delbaen, F., Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.* **312**, 215–250 (1998)
11. Föllmer, H., Kramkov, D.: Optional decompositions under constraints. *Probab. Theory Relat. Fields* **109**, 1–25 (1997)
12. Goll, T., Kallsen, J.: A complete explicit solution to the log-optimal portfolio problem. *Ann. Appl. Probab.* **13**, 774–799 (2003)
13. Gundel, A.: Robust utility maximization for complete and incomplete market models. *Finance Stoch.* **9**, 151–176 (2005)
14. He, H., Pearson, N.D.: Consumption and portfolio policies with incomplete markets and short-sale constraints: the finite-dimensional case. *Math. Finance* **1**, 1–10 (1991)
15. He, H., Pearson, N.D.: Consumption and portfolio policies with incomplete markets and short-sale constraints: the infinite-dimensional case. *J. Econ. Theory* **54**, 259–304 (1991)
16. Jacod, J., Shiryaev, A.N.: *Limit Theorems for Stochastic Processes*. Springer, Berlin (1987)
17. Karatzas, I.: Optimization problems in the theory of continuous trading. *SIAM J. Control Optim.* **27**, 1221–1259 (1989)
18. Karatzas, I., Kardaras, K.: The numéraire portfolio in semimartingale financial models. *Finance Stoch.* **11**, 447–493 (2007)
19. Karatzas, I., Shreve, S.: *Methods of Mathematical Finance*. Springer, Berlin (1998)
20. Karatzas, I., Žitković, G.: Optimal consumption from investment and random endowment in incomplete semimartingale markets. *Ann. Probab.* **31**, 1821–1858 (2003)
21. Karatzas, I., Lehoczky, J.P., Shreve, S.E.: Optimal portfolio and consumption decisions for a small investor on a finite horizon. *SIAM J. Control Optim.* **25**, 1557–1586 (1987)
22. Karatzas, I., Lehoczky, J.P., Shreve, S.E., Xu, G.L.: Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control Optim.* **29**, 702–730 (1991)
23. Knispel, T.: Asymptotics of robust utility maximization. *Ann. Appl. Probab.* **22**, 172–212 (2012)
24. Komlós, J.: A generalization of a problem of Steinhaus. *Acta Math. Acad. Sci. Hung.* **18**, 217–229 (1967)
25. Kramkov, D., Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.* **9**, 904–950 (1999)
26. Kramkov, D., Schachermayer, W.: Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab.* **13**, 1504–1516 (2003)
27. Kramkov, D.O.: Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probab. Theory Relat. Fields* **105**, 459–479 (1994)
28. Merton, R.C.: Lifetime portfolio selection under uncertainty: the continuous-time case. *Rev. Econ. Stat.* **51**, 247–257 (1969)
29. Merton, R.C.: Optimum consumption and portfolio rules in a continuous-time model. *J. Econ. Theory* **3**, 373–413 (1971)
30. Owari, K.: Robust utility maximization with unbounded random endowment. In: *Adv. Math. Econ.*, vol. 14, pp. 147–181 (2011)
31. Protter, P.: *Stochastic Integration and Differential Equations*. Springer, Berlin (2004)
32. Rockafellar, R.T.: *Convex Analysis*. Princeton University Press, Princeton (1970)
33. Schied, A., Wu, C.-T.: Duality theory for optimal investments under model uncertainty. *Stat. Decis.* **23**, 199–217 (2005)
34. Schwartz, M.: New proofs of a theorem by Komlós. *Acta Math. Hung.* **47**, 181–185 (1986)

35. Strasser, H.: *Mathematical Theory of Statistics: Statistical Experiments and Asymptotic Decision Theory*. de Gruyter, Berlin (1985)
36. Stricker, C., Yan, J.-A.: Some remarks on the optional decomposition theorem. In: *Séminaire de Probabilités XXXII. Lecture Notes in Mathematics*, vol. 1686, pp. 56–66. Springer, New York (1998)
37. Žitković, G.: Utility maximization with a stochastic clock and an unbounded random endowment. *Ann. Appl. Probab.* **15**, 748–777 (2005)
38. Wittmüss, W.: Robust optimization of consumption with random endowment. *Stoch. Int. J. Probab. Stoch. Process.* **80**, 459–475 (2008)