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Sensitivity analysis of the utility maximisation problem with respect to model perturbations

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Abstract We consider the expected utility maximisation problem and its response to small changes in the market price of risk in a continuous semimartingale setting. Assuming that the preferences of a rational economic agent are modelled by a general utility function, we obtain a second-order expansion of the value function, a first-order approximation of the terminal wealth, and we construct trading strategies that match the indirect utility function up to the second order. The method, which is presented in an abstract version, relies on a simultaneous expansion with respect to both the state variable and the parameter, and convex duality in the direction of the state variable only (as there is no convexity with respect to the parameter). If a risk-tolerance wealth process exists, using it as numéraire and under an appropriate change of measure, we reduce the approximation problem to a Kunita–Watanabe decomposition.

Keywords Sensitivity analysis · Optimal investment · Duality theory · Kunita–Watanabe decomposition

Mathematics Subject Classification (2010) 91G10 · 93E20

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1 Introduction

It is well known, see for example Delbaen and Schachermayer [5] or Hulley and Schweizer [15], that for a continuous (and strictly positive) stock price process, the

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no-arbitrage condition implies that the return J of the stock price has the representation

$$J = M + \lambda \cdot \langle M \rangle,$$

where M is a continuous local martingale and λ a predictable process, i.e., the finite variation part of the return is absolutely continuous with respect to its quadratic variation. In the formulation above, we analyse the effect of perturbations of the market price of risk λ on the utility maximisation problem *with a general utility function*. More precisely, we study the second-order asymptotics of the value function and first-order asymptotics of optimal strategies when the market price of risk is parametrised by a small parameter δ as $\lambda + \delta\nu$, around the "base" model for $\delta = 0$.

In the particular case of *power utility functions*, such a perturbation analysis appears in Larsen et al. [24] (and Chau and Rásonyi [2], but for lower-order asymptotics). Here, we obtain the expansions for *general utility functions*. Besides the mathematical (and arguably more important) motivation in itself, the generalisation to utilities which are not powers provides financial insight in two directions:

1) Going back to Merton [28], it has been clear that utility maximisation analysis has to be understood outside the limited setting of power utilities. There are important situations where the base model can still be solved rather explicitly, *even with a general utility function*, but the approximate model cannot. This is the case for example if the base model is complete (at least from the point of view of utility maximisation, as in Kramkov and Sîrbu [23]), but a perturbation to the market price of risk leads to genuine incompleteness. If one can solve the base model but not the approximate one, an asymptotic result is needed. Example 8.5 along these lines is presented below.

2) Our analysis provides the general structure of the approximation, in the directions of both the parameter and the wealth, giving additional insight on why the constant relative risk aversion case is so particular and therefore more explicit. In the case of power utilities, the optimal wealth process depends trivially on the initial wealth by scaling; so the optimal wealth and the risk-tolerance wealth process coincide (up to a multiplicative constant).

The mathematics, which we consider to be the main contribution, for general utilities is substantially different from Larsen et al. [24]. More precisely:

1) We consider perturbations simultaneously in the direction of the state (initial wealth) and the parameter and thus increase the dimensionality of the value function. As the proofs show, this is a necessary step for the approach to work.

2) In order to understand the general structure of the approximation, we need to formulate auxiliary quadratic stochastic control problems and relate the second-order approximations of both primal and dual value functions to these problems.

3) Finally, if the risk-tolerance wealth process exists, we use it as numéraire and change the measure accordingly, to identify solutions to the general quadratic optimisation problems above in terms of a Kunita–Watanabe decomposition (of a certain martingale) generated by the perturbation process.

As an important part of our contribution, we also have an abstract version of the theorems that can be potentially applied to other stochastic control problems which are convex in the state variable, but not convex with respect to the parameter.

To the best of our knowledge, the closest paper from the mathematical viewpoint is Kramkov and Sîrbu [23], where the authors obtain a second-order expansion of the value function with respect to simultaneous perturbations of the initial wealth and the number of units of random endowment held in the portfolio. We should like to stress that unlike the present setting, the value function in [23] is jointly concave (in both the initial wealth and the number of units of random endowment held in the portfolio), a fact that plays a significant role in the proofs there. Here, we use convexity only in the direction of the state variable. Note that the method of primal–dual simultaneous expansions, used in [23] and the present paper, was introduced in mathematical finance in [11].

As it is a central problem in financial economics, a large body of work is devoted to finding closed-form solutions to utility maximisation problems. Starting with the seminal work of Merton [27], such closed-form solutions have been constructed in Kim and Omberg [18], Zariphopoulou [33], Kraft [19], Liu [26], Guasoni and Robertson [9]. For a non-power utility, the optimal strategies are characterised in Horst et al. [12] and Santacroce and Trivellato [30] using backward stochastic differential equations. For the cases where closed-form solutions do not exist, a first-order asymptotic expansion with respect to perturbations of the market price of risk is obtained in Chau and Rásonyi [2] and Veraguas and Silva [32]. A second-order analysis is performed in Larsen et al. [24] for power utilities with $p < 0$ in a Brownian setting and under multiple integrability conditions. In contrast to [24], where the second-order correction terms are obtained via the martingale representation theorem, the key structural objects that drive the asymptotic expansions in the present paper are quadratic minimisation problems in the spirit of Kramkov and Sîrbu [23] that are closely related to those in Schweizer [31], Gouriéroux et al. [8], Fontana and Schweizer [7] and Czichowsky and Schweizer [4].

In the constant relative risk aversion case, the risk-tolerance wealth process exists and equals the optimal wealth process (up to a multiplicative constant) and thus, the reduction to the Kunita–Watanabe decomposition (as in Theorem 8.3 below) is done under the optimal wealth process (as numéraire) and optimal dual measure accounting for the change of numéraire. This allowed Larsen et al. [24] to jump to the Kunita–Watanabe decomposition, which in view of the Brownian filtration is given via the martingale representation theorem, and to make a direct conjecture (later verified by duality) what the correction to the optimal strategy is. Also, as opposed to the multiple integrability conditions in Larsen et al. [24], the analysis here is performed under one integrability condition, Assumption 2.6. A counterexample in Sect. 6 supports this assumption by showing that in its absence, the expansions need not exist.

The remainder of the paper is organised as follows. In Sect. 2, we present the model, state the technical assumptions and the expansion theorems. Section 3 contains the approximation of optimal trading strategies. Section 4 includes abstract versions of Theorems 2.7, 2.8, 2.10, and 2.12 with proofs, and Sect. 5 contains the proofs of non-abstract theorems and Theorem 3.1, where a construction of corrections to the optimal trading strategies (accurate up to the second order of the value function) is specified. Section 7 contains sufficient conditions for Assumption 2.6. In Sect. 8, we relate the asymptotic expansions from previous sections to the existence of the risk-tolerance wealth process and a Kunita–Watanabe decomposition and present an important example.

We finish this section with some of the notations used in the paper. For a vector $a \in \mathbb{R}^2$ with components a_1 and a_2 and a 2×2 matrix A, we define their norms as

$$|a| := \sqrt{a_1^2 + a_2^2} \quad \text{and} \quad |A| := \sup_{a \in \mathbb{R}^2} \frac{|Aa|}{|a|}.$$

We also often write $\triangle x^2$ for $(\triangle x)^2$ for brevity.

2 Model

2.1 The parametrised family of stock price processes

Let us consider a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, where $T \in (0, \infty)$ is the time horizon, the filtration satisfies the usual conditions and $\mathcal{F}_0$ is a trivial σ-algebra. We assume that there are two traded securities, a bank account with zero interest rate and a stock. Let M be a one-dimensional continuous local martingale and λ a predictable process such that

$$(\lambda^2 \cdot \langle M \rangle)_T < \infty \qquad \mathbb{P}\text{-a.s.} \tag{2.1}$$

The stock price return process for the unperturbed or 0-model is given by

$$J^0 := \lambda \cdot \langle M \rangle + M.$$

Remark 2.1 The decomposition $M + \lambda \cdot \langle M \rangle$ used here for the return process is often used for the stock process itself; see e.g. Hulley and Schweizer [15]. As a consequence, the processes M and λ have slightly different meanings here than e.g. in [15].

Next we consider a parametric family of semimartingales J^δ, $\delta \in \mathbb{R}$, with the same martingale part M and where the market price of risk is perturbed, i.e.,

$$J^\delta := \lambda^\delta \cdot \langle M \rangle + M,$$

where for some predictable process ν such that

$$(\nu^2 \cdot \langle M \rangle)_T < \infty \qquad \mathbb{P}\text{-a.s.}, \tag{2.2}$$

we have

$$\lambda^\delta := \lambda + \delta \nu, \qquad \delta \in \mathbb{R}.$$

2.2 Primal problem

Let U be a utility function that satisfies Assumption 2.2 below.

Assumption 2.2 The utility function U on $(0,\infty)$ is strictly increasing, strictly concave, two times continuously differentiable, and there exist positive constants c_1 and c_2 such that

$$c_1 \le A(x) := -\frac{U''(x)x}{U'(x)} \le c_2, \tag{2.3}$$

i.e., the relative risk aversion of U is uniformly bounded away from zero and infinity.

The family of primal feasible sets is defined, for $(x,\delta) \in (0,\infty)\times\mathbb{R}$, as

$$\mathcal{X}(x,\delta) := \{X \ge 0 : X = x + H\cdot J^{\delta}, H \text{ is predictable and } J^{\delta}\text{-integrable}\},$$

where H represents the amount invested in the stock. The corresponding family of value functions is given by

$$u(x,\delta) := \sup_{X\in\mathcal{X}(x,\delta)} \mathbb{E}[U(X_T)], \qquad (x,\delta)\in(0,\infty)\times\mathbb{R}. \tag{2.4}$$

Remark 2.3 The utility maximisation problem (2.4) without the existence of a local martingale measure (or rather without a local martingale deflator) is investigated in Karatzas et al. [16] in an Itô process setting. The results for the general semimartingale framework follow from the abstract theorems in Kramkov and Schachermayer [20]. However, the results in [20] are stated in a form that is linked to assuming the existence of a local martingale measure. The direct assertions for the no-unbounded-profit-with-bounded-risk setting are given in [17, Theorem 1.4] and [1, Theorem 1]. Also, in certain points below, we refer to the results in Kramkov and Sîrbu [22], for example in (2.11)–(2.13), that are obtained in [22, Theorem 1] under the assumption of NFLVR. We shall show that these formulas also hold in the present setting. In turn, Theorem 8.1 is a version of [23, Theorem 4], which is proved there under NFLVR, but the proof goes through also under the conditions of the present paper. Therefore, we shall not give the proof of Theorem 8.1.

Remark 2.4 There are multiple optimisation problems in this paper. Each of them, under the conditions here and for δ sufficiently close to 0, admits a unique solution. For easier readability, we introduce the notation for the solution right after the corresponding problem.

Let $\widehat{X}(x,\delta)$ be the optimiser to (2.4). Also, we use the convention

$$\mathbb{E}[U(X_T)] := -\infty \qquad \text{if } \mathbb{E}[U^-(X_T)] = \infty,$$

where U^- is the negative part of U.

2.3 Dual problem

The investigation of the primal problem (2.4) is conducted via the dual problem. First, let us define the dual domain, for $(y,\delta)\in(0,\infty)\times\mathbb{R}$, as

$$\begin{aligned}\mathcal{Y}(y,\delta) := \{Y :& Y \text{ is a nonnegative supermartingale such that } Y_0 = y \text{ and}\\ & XY = (X_tY_t)_{t\ge 0} \text{ is a supermartingale for every } X\in\mathcal{X}(1,\delta)\}.\end{aligned}$$

We define the convex conjugate to the utility function U as

$$V(y) := \sup_{x>0} \big(U(x) - xy\big), \qquad y > 0.$$

Note that for $y = U'(x)$, we have

$$V''(y) = -\frac{1}{U''(x)}$$

and

$$B(y) := -\frac{V''(y)y}{V'(y)} = \frac{1}{A(x)}.$$

Therefore, Assumption 2.2 implies that

$$\frac{1}{c_2} \leq B(y) \leq \frac{1}{c_1}, \qquad y > 0.$$

The parametrised family of dual value functions is given by

$$v(y,\delta) := \inf_{Y \in \mathcal{Y}(y,\delta)} \mathbb{E}\left[V(Y_T)\right], \qquad (y,\delta) \in (0,\infty) \times \mathbb{R}. \tag{2.5}$$

Let $\widehat{Y}(y,\delta)$ be the optimiser to (2.5). We use the convention

$$\mathbb{E}\left[V(Y_T)\right] := \infty \qquad \text{if } \mathbb{E}[V^+(Y_T)] = \infty,$$

where V^+ is the positive part of V.

2.4 Technical assumptions

We recall the assumption that M is continuous. The absence of arbitrage opportunities in the 0-model in the sense of no unbounded profit with bounded risk follows from condition (2.1), which implies that $\mathcal{Y}(1,0) \neq \emptyset$. Note that (2.1) and (2.2) imply no unbounded profit with bounded risk for every $\delta \in \mathbb{R}$, thus

$$\mathcal{Y}(1,\delta) \neq \emptyset, \qquad \delta \in \mathbb{R}.$$

In order for the problem (2.4) to be non-degenerate, we also need to assume that

$$u(x,0) < \infty \qquad \text{for some } x > 0. \tag{2.6}$$

Remark 2.5 Conditions (2.1) and (2.6) are necessary for the expected utility maximisation problem to be non-degenerate. Note that we only impose them for $\delta = 0$.

As in the works of Kramkov and Sîrbu [22, 23], an important role is played by the probability measure $\mathbb{R}(x,0)$ given by

$$\frac{d\mathbb{R}(x,0)}{d\mathbb{P}} := \frac{\widehat{X}_T(x,0)\widehat{Y}_T(y,0)}{xy}$$

for $x > 0$ and $y = u_x(x, 0)$. As Example 6.1 below demonstrates, we need to impose an integrability condition. First, let us define

$$\zeta(c, \delta) := \exp\Big(c\big(|(\nu \cdot J^\delta)_T| + \langle \nu \cdot J^\delta \rangle_T\big)\Big), \qquad (c, \delta) \in \mathbb{R}^2. \tag{2.7}$$

Assumption 2.6 Let $x > 0$ be fixed. There exists a $c > 0$ such that

$$\mathbb{E}^{\mathbb{R}(x,0)}\left[\zeta(c, 0)\right] < \infty.$$

We discuss Assumption 2.6 and give sufficient conditions in Sect. 7. We also set

$$L^\delta := \mathcal{E}\big(-(\delta\nu) \cdot J^0\big)_T, \qquad \delta \in \mathbb{R}. \tag{2.8}$$

Here and below, $\mathcal{E}$ denotes the Doléans-Dade exponential. One can see that L^δ is the terminal value of an element of $\mathcal{X}(1, 0)$ for every $\delta \in \mathbb{R}$.

2.5 Expansion theorems

In Theorem 2.7, we prove finiteness of the value functions and first-order derivatives with respect to δ.

Let us set

$$F := (\nu \cdot J^0)_T \qquad \text{and} \qquad G := (\nu^2 \cdot \langle M \rangle)_T. \tag{2.9}$$

Theorem 2.7 *Let $x > 0$ be fixed, assume that (2.1) and (2.6) as well as Assumptions 2.2 and 2.6 hold, and define $y = u_x(x, 0)$ which is well defined by the abstract theorems in [20]. Then there exists a $\delta_0 > 0$ such that for every $\delta \in (-\delta_0, \delta_0)$, we have*

$$u(x, \delta) \in \mathbb{R}, \quad x > 0, \qquad \textit{and} \qquad v(y, \delta) \in \mathbb{R}, \quad y > 0.$$

In addition, u and v are jointly differentiable (and consequently continuous) at $(x, 0)$ and $(y, 0)$, respectively. We also have

$$\nabla u(x, 0) = \begin{pmatrix} y \\ u_\delta(x, 0) \end{pmatrix} \qquad \textit{and} \qquad \nabla v(y, 0) = \begin{pmatrix} -x \\ v_\delta(y, 0) \end{pmatrix}, \tag{2.10}$$

where

$$u_\delta(x, 0) = v_\delta(y, 0) = xy\mathbb{E}^{\mathbb{R}(x,0)}[F].$$

In order to characterise the second-order derivatives of the value functions, we need the following notations. Let $S^{X(x,0)}$ be the price process of the traded securities under the numéraire $\frac{\widehat{X}(x,0)}{x}$, i.e.,

$$S^{X(x,0)} = \left(\frac{x}{\widehat{X}(x, 0)}, \frac{x\mathcal{E}(J^0)}{\widehat{X}(x, 0)}\right).$$

For every $x > 0$, let $\mathbf{H}_0^2(\mathbb{R}(x,0))$ denote the space of square-integrable martingales under $\mathbb{R}(x,0)$ starting from 0. Let us define

$$\mathcal{M}^2(x,0) := \big\{M \in \mathbf{H}_0^2\big(\mathbb{R}(x,0)\big) : M = H \cdot S^{X(x,0)} \text{ for some } S\text{-integrable } H\big\},$$
$$\mathcal{N}^2(y,0) := \big\{N \in \mathbf{H}_0^2\big(\mathbb{R}(x,0)\big) : MN \text{ is an } \mathbb{R}(x,0)\text{-martingale for every } M \in \mathcal{M}^2(x,0)\big\}, \qquad \text{where } y = u_x(x,0).$$

2.6 Auxiliary minimisation problems

As in [22], for $x > 0$, let us consider the quadratic optimisation problems

$$a_{\{x,x\}} := \inf_{M\in\mathcal{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)}\big[A\big(\widehat{X}_T(x,0)\big)(1+M_T)^2\big], \tag{2.11}$$

$$b_{\{y,y\}} := \inf_{N\in\mathcal{N}^2(y,0)} \mathbb{E}^{\mathbb{R}(x,0)}\big[B\big(\widehat{Y}_T(y,0)\big)(1+N_T)^2\big], \qquad y = u_x(x,0), \tag{2.12}$$

where A is the relative risk aversion and B the relative risk tolerance of U. It is shown in [22, Lemmas 2 and 6] that (2.11) and (2.12) admit unique solutions $M^0(x,0)$ and $N^0(y,0)$, respectively, and

$$\begin{aligned}
u_{xx}(x,0) &= -\frac{y}{x}a_{\{x,x\}},\\
v_{yy}(y,0) &= \frac{x}{y}b_{\{y,y\}},\\
a_{\{x,x\}}b_{\{y,y\}} &= 1,\\
A\big(\widehat{X}_T(x,0)\big)\big(1+M_T^0(x,0)\big) &= a_{\{x,x\}}\big(1+N_T^0(y,0)\big).
\end{aligned} \tag{2.13}$$

In order to characterise the derivatives of the value functions with respect to δ, we recall F and G from (2.9) and consider the quadratic minimisation problems

$$a_{\{\delta,\delta\}} := \inf_{M\in\mathcal{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)}\big[A\big(\widehat{X}_T(x,0)\big)(M_T+xF)^2 - 2xFM_T - x^2(F^2+G)\big], \tag{2.14}$$

$$b_{\{\delta,\delta\}} := \inf_{N\in\mathcal{N}^2(y,0)} \mathbb{E}^{\mathbb{R}(x,0)}\big[B\big(\widehat{Y}_T(y,0)\big)(N_T-yF)^2 + 2yFN_T - y^2(F^2-G)\big]. \tag{2.15}$$

Along the lines of [22, Lemma 2], one can show that there exist unique solutions $M^1(x,0)$ and $N^1(y,0)$ to (2.14) and (2.15), respectively. We also set

$$a_{\{x,\delta\}} := \mathbb{E}^{\mathbb{R}(x,0)}\big[A\big(\widehat{X}_T(x,0)\big)\big(1+M_T^0(x,0)\big)\big(xF+M_T^1(x,0)\big) - xF\big(1+M_T^0(x,0)\big)\big], \tag{2.16}$$

$$b_{\{y,\delta\}} := \mathbb{E}^{\mathbb{R}(x,0)}\big[B\big(\widehat{Y}_T(y,0)\big)\big(1+N^0_T(y,0)\big)\big(N^1_T(y,0)-yF\big) + yF\big(1+N^0_T(y,0)\big)\big]. \tag{2.17}$$

Theorems 2.8, 2.10 and 2.12 below contain the second-order expansions of the value functions, derivatives of the optimisers, and properties of such derivatives.

Theorem 2.8 *Let $x>0$ be fixed. Assume all conditions of Theorem 2.7 hold, with $y=u_x(x,0)$. Define*

$$H_u(x,0) := -\frac{y}{x}\begin{pmatrix} a_{\{x,x\}} & a_{\{x,\delta\}} \\ a_{\{x,\delta\}} & a_{\{\delta,\delta\}} \end{pmatrix}, \tag{2.18}$$

where $a_{\{x,x\}}$, $a_{\{\delta,\delta\}}$, $a_{\{x,\delta\}}$ are specified in (2.11), (2.14), (2.16), and

$$H_v(y,0) := \frac{x}{y}\begin{pmatrix} b_{\{y,y\}} & b_{\{y,\delta\}} \\ b_{\{y,\delta\}} & b_{\{\delta,\delta\}} \end{pmatrix}, \tag{2.19}$$

where $b_{\{y,y\}}$, $b_{\{\delta,\delta\}}$, $b_{\{y,\delta\}}$ are specified in (2.12), (2.15), (2.17). Then the value functions u and v admit respective second-order expansions around $(x,0)$ and $(y,0)$ given by

$$u(x+\Delta x,\delta) = u(x,0) + (\Delta x\ \delta)\nabla u(x,0) + \frac{1}{2}(\Delta x\ \delta)H_u(x,0)\begin{pmatrix}\Delta x\\ \delta\end{pmatrix} + o(\Delta x^2+\delta^2) \tag{2.20}$$

and

$$v(y+\Delta y,\delta) = v(y,0) + (\Delta y\ \delta)\nabla v(y,0) + \frac{1}{2}(\Delta y\ \delta)H_v(y,0)\begin{pmatrix}\Delta y\\ \delta\end{pmatrix} + o(\Delta y^2+\delta^2). \tag{2.21}$$

Remark 2.9 In (2.20) and (2.21) above, we only have second-order expansions (and make no claims about the existence of the second-order derivatives $u_{x\delta}, u_{\delta\delta}$, $v_{y\delta}$, $v_{\delta\delta}$). Nevertheless, we may abuse the language and call $H_u(x,0)$ and $H_v(y,0)$ the Hessians of u and v. This causes no confusion; see the discussion e.g. in [25, Sect. 1]. The meaning of the partial derivatives $u_{xx}(x,0)$, $u_{x\delta}(x,0)$ and so on then becomes apparent by identifying entries in these Hessian matrices.

Theorem 2.10 *Let $x>0$ be fixed. Assume all conditions of Theorem 2.7 hold, with $y=u_x(x,0)$. Then we have*

$$\begin{pmatrix} a_{\{x,x\}} & 0 \\ a_{\{x,\delta\}} & -\frac{x}{y} \end{pmatrix}\begin{pmatrix} b_{\{y,y\}} & 0 \\ b_{\{y,\delta\}} & -\frac{y}{x} \end{pmatrix} = I_2, \tag{2.22}$$

where I_2 denotes the 2×2 identity matrix. Moreover,

$$\frac{y}{x} a_{\{\delta,\delta\}} + \frac{x}{y} b_{\{\delta,\delta\}} = a_{\{x,\delta\}} b_{\{y,\delta\}}, \tag{2.23}$$

$$\begin{aligned} &U''\left(\widehat{X}_T(x,0)\right)\widehat{X}_T(x,0)\begin{pmatrix} 1+M_T^0(x,0) \\ xF+M_T^1(x,0)\end{pmatrix} \\ &= -\begin{pmatrix} a_{\{x,x\}} & 0 \\ a_{\{x,\delta\}} & -\frac{x}{y}\end{pmatrix}\widehat{Y}_T(y,0)\begin{pmatrix} 1+N_T^0(y,0) \\ yF-N_T^1(y,0)\end{pmatrix}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} &V''\left(\widehat{Y}_T(y,0)\right)\widehat{Y}_T(y,0)\begin{pmatrix} 1+N_T^0(y,0) \\ -yF+N_T^1(y,0)\end{pmatrix} \\ &= \begin{pmatrix} b_{\{y,y\}} & 0 \\ b_{\{y,\delta\}} & -\frac{y}{x}\end{pmatrix}\widehat{X}_T(x,0)\begin{pmatrix} 1+M_T^0(x,0) \\ xF+M_T^1(x,0)\end{pmatrix}, \end{aligned}$$

and the product of any of $\widehat{X}(x,0)$, $\widehat{X}(x,0)M^0(x,0)$, $\widehat{X}(x,0)M^1(x,0)$ and any of $\widehat{Y}(y,0)$, $\widehat{Y}(y,0)N^0(y,0)$, $\widehat{Y}(y,0)N^1(y,0)$ is a martingale under $\mathbb{P}$, where $M_T^0(x,0)$, $M_T^0(x,0)$, $N_T^0(y,0)$, $N_T^1(y,0)$ are the solutions to (2.11), (2.14), (2.12), (2.15), respectively.

Remark 2.11 Continuing the discussion in Remark 2.9, (2.22) implies that

$$\begin{pmatrix} u_{xx}(x,0) & 0 \\ u_{x\delta}(x,0) & 1\end{pmatrix}\begin{pmatrix} v_{yy}(y,0) & 0 \\ v_{y\delta}(y,0) & -1\end{pmatrix} = -I_2,$$

where

$$\begin{pmatrix} u_{xx}(x,0) & 0 \\ u_{x\delta}(x,0) & 1\end{pmatrix} = -\frac{y}{x}\begin{pmatrix} a_{\{x,x\}} & 0 \\ a_{\{x,\delta\}} & -\frac{x}{y}\end{pmatrix}, \qquad \begin{pmatrix} v_{yy}(y,0) & 0 \\ v_{y\delta}(y,0) & -1\end{pmatrix} = \frac{x}{y}\begin{pmatrix} b_{\{y,y\}} & 0 \\ b_{\{y,\delta\}} & -\frac{y}{x}\end{pmatrix}.$$

Likewise, (2.23) gives

$$-u_{\delta\delta}(x,0) + v_{\delta\delta}(y,0) = -u_{x\delta}(x,0)v_{y\delta}(y,0).$$

Theorem 2.12 *Let $x > 0$ be fixed. Assume all conditions of Theorem 2.7 hold, with $y = u_x(x,0)$. Then the terminal values of the wealth processes $M^0(x,0)$ and $M^1(x,0)$, which are the solutions to (2.11) and (2.14), respectively, satisfy*

$$\begin{aligned} &\lim_{|\Delta x|+|\delta|\to 0}\frac{1}{|\Delta x|+|\delta|}\left|\widehat{X}_T(x+\Delta x,\delta)\right. \\ &\qquad\left. - \frac{\widehat{X}_T(x,0)}{x}\left(x+\Delta x\left(1+M_T^0(x,0)\right)+\delta M_T^1(x,0)\right)\frac{1}{L^\delta}\right| \\ &= 0, \end{aligned} \tag{2.25}$$

where the convergence takes place in $\mathbb{P}$-probability and L^δ is defined in (2.8). Likewise, the terminal values of $N^0(y,0)$ and $N^1(y,0)$, which are the solutions to (2.12) and (2.15), respectively, satisfy

$$\lim_{|\Delta y|+|\delta|\to 0} \frac{1}{|\Delta y|+|\delta|}\Big|\widehat{Y}_T(y+\Delta y,\delta) - \frac{\widehat{Y}_T(y,0)}{y}\Big(y+\Delta y\big(1+N^0_T(y,0)\big)+\delta N^1_T(y,0)\Big)L^\delta\Big| = 0,$$

where the convergence takes place in $\mathbb{P}$-probability.

One can obtain the following corollary.

Corollary 2.13 *Let $x>0$ be fixed. Assume all conditions of Theorem 2.7 hold, with $y=u_x(x,0)$. Then if we define*

$$X'_T(x,0) := \frac{\widehat{X}_T(x,0)}{x}\big(1+M^0_T(x,0)\big), \qquad Y'_T(y,0) := \frac{\widehat{Y}_T(y,0)}{y}\big(1+N^0_T(y,0)\big)$$

and

$$X^\delta_T(x,0) := \frac{\widehat{X}_T(x,0)}{x}\big(M^1_T(x,0)+xF\big), \qquad Y^\delta_T(y,0) := \frac{\widehat{Y}_T(y,0)}{y}\big(N^1_T(y,0)-yF\big),$$

we have

$$\lim_{|\Delta x|+|\delta|\to 0} \frac{|\widehat{X}_T(x+\Delta x,\delta)-\widehat{X}_T(x,0)-\Delta x X'_T(x,0)-\delta X^\delta_T(x,0)|}{|\Delta x|+|\delta|} = 0,$$
$$\lim_{|\Delta y|+|\delta|\to 0} \frac{|\widehat{Y}_T(y+\Delta y,\delta)-\widehat{Y}_T(y,0)-\Delta y Y'_T(y,0)-\delta Y^\delta_T(y,0)|}{|\Delta y|+|\delta|} = 0,$$

where the convergence takes place in $\mathbb{P}$-probability.

Remark 2.14 Even though Corollary 2.13 gives a more explicit form of the derivatives of the terminal wealth, the approximation given in (2.25) turns out to be more useful for example in the construction of optimal trading strategies in Sect. 3.

3 Approximation of the optimal trading strategies

Throughout this section, we suppose that $x>0$ is fixed. Let us denote

$$M^R := J^0 - \widehat{\pi}(x,0)\cdot\langle M\rangle, \tag{3.1}$$

where $\widehat{\pi}(x,0) = (\widehat{\pi}_t(x,0))_{t\in[0,T]}$ is the optimal *proportion* invested in stock corresponding to initial wealth x and $\delta=0$. Note that for every pair of predictable processes G^1 and G^2 such that both integrals $G^1\cdot(\frac{x}{\widehat{X}(x,0)})$ and $G^2\cdot(\frac{x\mathcal{E}(J^0)}{\widehat{X}(x,0)})$ are well defined, we can find by direct computations a process G such that

$$G^1 \cdot \left(\frac{x}{\widehat{X}(x,0)}\right) + G^2 \cdot \left(\frac{x\mathcal{E}(J^0)}{\widehat{X}(x,0)}\right) = G \cdot M^R.$$

Let γ^0 and γ^1 be such that

$$\gamma^0 \cdot M^R = \frac{M^0(x,0)}{x}, \qquad \gamma^1 \cdot M^R = \frac{M^1(x,0)}{x}. \tag{3.2}$$

We define for $\varepsilon > 0$ the families of stopping times

$$\sigma_\varepsilon := \inf\left\{t \in [0,T] : |M^0_t(x,0)| \geq \frac{x}{\varepsilon} \text{ or } \langle M^0(x,0)\rangle_t \geq \frac{x}{\varepsilon}\right\},$$
$$\tau_\varepsilon := \inf\left\{t \in [0,T] : |M^1_t(x,0)| \geq \frac{x}{\varepsilon} \text{ or } \langle M^1(x,0)\rangle_t \geq \frac{x}{\varepsilon}\right\}.$$

We also set, for $\varepsilon > 0$,

$$\gamma^{0,\varepsilon} = \gamma^0 1_{[\![0,\sigma_\varepsilon]\!]}, \qquad \gamma^{1,\varepsilon} = \gamma^1 1_{[\![0,\tau_\varepsilon]\!]}.$$

Theorem 3.1 *Assume that $x > 0$ is fixed and the assumptions of Theorem 2.7 hold. For every $(\Delta x, \delta, \varepsilon) \in (-x, \infty) \times \mathbb{R} \times (0, \infty)$, let us define*

$$X^{\Delta x,\delta,\varepsilon} := (x + \Delta x)\mathcal{E}\Big(\big(\widehat{\pi}(x,0) + \Delta x \gamma^{0,\varepsilon} + \delta(\nu + \gamma^{1,\varepsilon})\big) \cdot J^\delta\Big). \tag{3.3}$$

Then there exists a function $\varepsilon = \varepsilon(\Delta x, \delta) > 0$, $(\Delta x, \delta) \in (-x, \infty) \times \mathbb{R}$, such that

$$\mathbb{E}[U(X_T^{\Delta x,\delta,\varepsilon(\Delta x,\delta)})] = u(x + \Delta x, \delta) - o(\Delta x^2 + \delta^2).$$

Remark 3.2 Theorem 3.1 shows how to correct the optimal proportion in order to match the primal value function up to the second order *jointly in* $(\Delta x, \delta)$.

Remark 3.3 Using the proportions of optimal wealth invested in stock and not the number of shares of stock allows a nicer (or shorter) representation via (3.3) for nearly optimal wealth processes that match the indirect utility up to the second order. The result in Theorem 3.1 complements, in the context of a one-dimensional and continuous stock model, the results in Kramkov and Sîrbu [22] and (in a different additive random endowment framework) those in Kramkov and Sîrbu [23].

4 Abstract version of the expansion theorems

4.1 Abstract version for the 0-model

We begin with the formulation of the abstract version of the expansion theorems for the 0-model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We define $\mathbf{L}^0$ to be the vector space of (equivalence classes of) real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, topologised by the convergence in $\mathbb{P}$-probability; $\mathbf{L}^0_+$ is the positive orthant of $\mathbf{L}^0$. We consider subsets $\mathcal{C}$ and $\mathcal{D}$ of $\mathbf{L}^0_+$ that satisfy the following assumption.

Assumption 4.1 Both $\mathcal{C}$ and $\mathcal{D}$ contain a strictly positive element and

$$\xi \in \mathcal{C} \qquad \text{if and only if} \qquad \mathbb{E}[\xi\eta] \le 1 \text{ for every } \eta \in \mathcal{D},$$

as well as

$$\eta \in \mathcal{D} \qquad \text{if and only if} \qquad \mathbb{E}[\xi\eta] \le 1 \text{ for every } \xi \in \mathcal{C}.$$

Note that Assumption 4.1 is the abstract version of the no-unbounded-profit-with-bounded-risk condition (2.1). We also set $\mathcal{C}(x,0) := x\mathcal{C}$ and $\mathcal{D}(x,0) := x\mathcal{D}$, $x > 0$. Now we can state the abstract primal and dual problems as

$$u(x,0) := \sup_{\xi \in \mathcal{C}(x,0)} \mathbb{E}\left[U(\xi)\right], \qquad x > 0, \tag{4.1}$$

$$v(y,0) := \inf_{\eta \in \mathcal{D}(y,0)} \mathbb{E}[V(\eta)], \qquad y > 0, \tag{4.2}$$

where we follow the standard practice (see e.g. [20]) of denoting the abstract and "concrete" value functions by the same letters. Under finiteness of both primal and dual value functions on $(0,\infty)$, existence and uniqueness of solutions to (4.1) and (4.2) follow from [29, Theorem 3.2].

4.2 Abstract version for the δ-models

For some random variables $G \ge 0$ and F, let us set

$$L^\delta := \exp\bigg(-\Big(\delta F + \frac{1}{2}\delta^2 G\Big)\bigg),$$

$$\mathcal{C}(x,\delta) := \mathcal{C}(x,0)\frac{1}{L^\delta}, \qquad \mathcal{D}(y,\delta) := \mathcal{D}(y,0)L^\delta, \qquad \delta \in \mathbb{R}. \tag{4.3}$$

Now we can state the abstract versions of the perturbed optimisation problems as

$$\begin{aligned} u(x,\delta) &:= \sup_{\xi \in \mathcal{C}(x,\delta)} \mathbb{E}\left[U(\xi)\right] \\ &= \sup_{\xi \in \mathcal{C}(x,0)} \mathbb{E}\bigg[U\Big(\xi\frac{1}{L^\delta}\Big)\bigg], \qquad (x,\delta) \in (0,\infty)\times\mathbb{R}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} v(y,\delta) &:= \inf_{\eta \in \mathcal{D}(y,\delta)} \mathbb{E}[V(\eta)] \\ &= \inf_{\eta \in \mathcal{D}(y,0)} \mathbb{E}[V(\eta L^\delta)], \qquad (y,\delta) \in (0,\infty)\times\mathbb{R}. \end{aligned} \tag{4.5}$$

Under an appropriate integrability assumption specified below, existence and uniqueness of solutions to (4.4) and (4.5) as well as conjugacy relations between $u(\cdot,\delta)$ and $v(\cdot,\delta)$ for every δ sufficiently close to 0 will follow from [29, Theorem 3.2].

4.3 Condition on perturbations

Let $\widehat{\xi}(x,\delta)$ and $\widehat{\eta}(y,\delta)$ denote the solutions to (4.4) and (4.5), respectively, if such solutions exist. By $\mathbb{R}(x,0)$, we denote the probability measure on $(\Omega,\mathcal{F})$ whose Radon–Nikodým derivative with respect to $\mathbb{P}$ is given by

$$\frac{d\mathbb{R}(x,0)}{d\mathbb{P}} := \frac{\widehat{\xi}(x,0)\widehat{\eta}(y,0)}{xy},$$

where $x>0$ and $y=u_x(x,0)$. Note that $\mathbb{R}(x,0)$ is well defined for every $x>0$.

Assumption 4.2 There exists a constant $c>0$ such that

$$\mathbb{E}^{\mathbb{R}(x,0)}\big[\exp\big(c(|F|+G)\big)\big]<\infty.$$

4.4 Auxiliary sets $\mathcal{A}$ and $\mathcal{B}$

Let us now suppose that (4.1) satisfies the standard assertions of the utility maximisation theory (see e.g. [20, Theorems 2.1 and 2.2]) that in turn hold e.g. under the conditions of Theorem 4.4 below. As in Kramkov and Sîrbu [22], we fix $x>0$ and for $\delta=0$, we define

$$\mathcal{A}^\infty(x,0) := \{\alpha\in\mathbf{L}^\infty : \widehat{\xi}(x,0)(1\pm c\alpha)\in\mathcal{C}(x,0) \text{ for some } c=c(\alpha)>0\}.$$

Likewise, for $y>0$ and $\delta=0$, we set

$$\mathcal{B}^\infty(y,0) := \{\beta\in\mathbf{L}^\infty : \widehat{\eta}(y,0)(1\pm c\beta)\in\mathcal{D}(y,0) \text{ for some } c=c(\beta)>0\}.$$

Then $\mathcal{A}^\infty(x,0)$ and $\mathcal{B}^\infty(u_x(x,0),0)$ are orthogonal linear subspaces of

$$\mathbf{L}_0^2\big(\mathbb{R}(x,0)\big) := \big\{\zeta\in\mathbf{L}^0 : \mathbb{E}^{\mathbb{R}(x,0)}[\zeta]=0 \text{ and } \mathbb{E}^{\mathbb{R}(x,0)}[\zeta^2]<\infty\big\},$$

where one can see that the elements of A^∞ and B^∞ have $\mathbb{R}(x,0)$-expectation zero by rewriting such an expectation under the measure $\mathbb{P}$ and recalling the definitions of the primal and dual domains.

Let us denote by $\mathcal{A}^2(x,0)$ and $\mathcal{B}^2(y,0)$ the respective closures of $\mathcal{A}^\infty(x,0)$ and $\mathcal{B}^\infty(y,0)$ in $\mathbf{L}_0^2(\mathbb{R}(x,0))$. One can see that $\mathcal{A}^2(x,0)$ and $\mathcal{B}^2(y,0)$ are closed orthogonal linear subspaces of $\mathbf{L}_0^2(\mathbb{R}(x,0))$. In order to make these sets related to the concrete versions of the expansion theorems, we need the following assumption.

Assumption 4.3 For $\delta=0$ and $x>0$, with $y=u_x(x,0)$, the sets $\mathcal{A}^2(x,0)$ and $\mathcal{B}^2(y,0)$ are complementary linear subspaces in $\mathbf{L}_0^2(\mathbb{R}(x,0))$, i.e.,

$$\alpha\in\mathcal{A}^2(x,0) \quad\text{iff}\quad \alpha\in\mathbf{L}_0^2\big(\mathbb{R}(x,0)\big) \text{ and } \mathbb{E}^{\mathbb{R}(x,0)}[\alpha\beta]=0 \text{ for every } \beta\in\mathcal{B}^2(y,0),$$
$$\beta\in\mathcal{B}^2(y,0) \quad\text{iff}\quad \beta\in\mathbf{L}_0^2\big(\mathbb{R}(x,0)\big) \text{ and } \mathbb{E}^{\mathbb{R}(x,0)}[\alpha\beta]=0 \text{ for every } \alpha\in\mathcal{A}^2(x,0).$$

The following theorem shows joint differentiability and is a consequence of the second-order expansion (as can be seen from Lemma 4.19 below).

Theorem 4.4 *Let $x>0$ be fixed. Suppose that Assumptions 2.2 and 4.1–4.3 hold, $u(x,0)<\infty$ and $y=u_x(x,0)$, which is well defined by the abstract theorems in [20]. Then there exists $\delta_0>0$ such that for every $\delta\in(-\delta_0,\delta_0)$, we have*

$$u(x,\delta)\in\mathbb{R},\quad x>0,\qquad\text{and}\qquad v(y,\delta)\in\mathbb{R},\quad y>0.$$

In addition, u and v are jointly differentiable (and consequently continuous) at $(x,0)$ and $(y,0)$, respectively. We also have

$$\nabla u(x,0)=\begin{pmatrix}y\\ u_\delta(x,0)\end{pmatrix}\qquad\text{and}\qquad\nabla v(y,0)=\begin{pmatrix}-x\\ v_\delta(y,0)\end{pmatrix},\tag{4.6}$$

where

$$u_\delta(x,0)=v_\delta(y,0)=xy\mathbb{E}^{\mathbb{R}(x,0)}[F].$$

Remark 4.5 It is possible to prove Theorem 4.4 without Assumption 4.3. We do not present such a proof for brevity of exposition.

4.5 Auxiliary minimisation problems

As in [22], for $x>0$, let us consider

$$a_{\{x,x\}}:=\inf_{\alpha\in\mathcal{A}^2(x,0)}\mathbb{E}^{\mathbb{R}(x,0)}\big[A\big(\widehat{\xi}(x,0)\big)(1+\alpha)^2\big],\tag{4.7}$$

$$b_{\{y,y\}}:=\inf_{\beta\in\mathcal{B}^2(y,0)}\mathbb{E}^{\mathbb{R}(x,0)}\big[B\big(\widehat{\eta}(y,0)\big)(1+\beta)^2\big],\qquad y=u_x(x,0),\tag{4.8}$$

where A is the relative risk aversion and B the relative risk tolerance of U. It is proved in [22, Lemma 2] that

$$\begin{aligned}u_{xx}(x,0)&=-\frac{y}{x}a_{\{x,x\}},\\ v_{yy}(y,0)&=\frac{x}{y}b_{\{y,y\}},\\ a_{\{x,x\}}b_{\{y,y\}}&=1,\\ A\big(\xi(x,0)\big)\big(1+\alpha(x,0)\big)&=a_{\{x,x\}}\big(1+\beta(y,0)\big),\end{aligned}\tag{4.9}$$

where $\alpha(x,0)$ and $\beta(y,0)$ are the unique solutions to (4.7) and (4.8), respectively. In order to characterise the derivatives of the value functions with respect to δ, we consider the minimisation problems

$$a_{\{\delta,\delta\}}:=\inf_{\alpha\in\mathcal{A}^2(x,0)}\mathbb{E}^{\mathbb{R}(x,0)}\big[A\big(\widehat{\xi}(x,0)\big)(\alpha+xF)^2-2xF\alpha-x^2(F^2+G)\big],\tag{4.10}$$

$$b_{\{\delta,\delta\}}:=\inf_{\beta\in\mathcal{B}^2(y,0)}\mathbb{E}^{\mathbb{R}(x,0)}\big[B\big(\widehat{\eta}(y,0)\big)(\beta-yF)^2+2yF\beta-y^2(F^2-G)\big].\tag{4.11}$$

Denoting by $\alpha_d(x,0)$ and $\beta_d(y,0)$ the unique solutions to (4.10) and (4.11), respectively, we also set

$$a_{\{x,\delta\}} := \mathbb{E}^{\mathbb{R}(x,0)}\big[A\big(\widehat{\xi}(x,0)\big)\big(1+\alpha(x,0)\big)\big(xF+\alpha_d(x,0)\big) - xF\big(1+\alpha(x,0)\big)\big], \tag{4.12}$$

$$b_{\{y,\delta\}} := \mathbb{E}^{\mathbb{R}(x,0)}\big[B\big(\widehat{\eta}(y,0)\big)\big(1+\beta(y,0)\big)\big(-yF+\beta_d(y,0)\big) + yF\big(1+\beta(y,0)\big)\big]. \tag{4.13}$$

4.6 Expansion theorems

We are ready to state the following theorems.

Theorem 4.6 *Let $x>0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y=u_x(x,0)$. Define*

$$H_u(x,0) := -\frac{y}{x}\begin{pmatrix} a_{\{x,x\}} & a_{\{x,\delta\}} \\ a_{\{x,\delta\}} & a_{\{\delta,\delta\}} \end{pmatrix}, \tag{4.14}$$

where $a_{\{x,x\}}$, $a_{\{\delta,\delta\}}$, $a_{\{x,\delta\}}$ are specified in (4.7), (4.10), (4.12), respectively, and

$$H_v(y,0) := \frac{x}{y}\begin{pmatrix} b_{\{y,y\}} & b_{\{y,\delta\}} \\ b_{\{y,\delta\}} & b_{\{\delta,\delta\}} \end{pmatrix}, \tag{4.15}$$

where $b_{\{y,y\}}$, $b_{\{\delta,\delta\}}$, $b_{\{y,\delta\}}$ are specified in (4.8), (4.11), (4.13), respectively. Using the formula (4.6) for the gradients, the second-order expansions of the value functions are given by

$$u(x+\Delta x,\delta) = u(x,0) + (\Delta x\ \delta)\nabla u(x,0) + \frac{1}{2}(\Delta x\ \delta)H_u(x,0)\begin{pmatrix}\Delta x\\ \delta\end{pmatrix} + o(\Delta x^2+\delta^2) \tag{4.16}$$

and

$$v(y+\Delta y,\delta) = v(y,0) + (\Delta y\ \delta)\nabla v(y,0) + \frac{1}{2}(\Delta y\ \delta)H_v(y,0)\begin{pmatrix}\Delta y\\ \delta\end{pmatrix} + o(\Delta y^2+\delta^2). \tag{4.17}$$

Theorem 4.7 *Let $x>0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y=u_x(x,0)$. Let $\xi=\widehat{\xi}(x,0)$ and $\eta=\widehat{\eta}(y,0)$ denote the solutions to (4.1) and (4.2), and $\alpha=\alpha(x,0)$, $\beta=\beta(y,0)$, $\alpha_d=\alpha_d(x,0)$, $\beta_d=\beta_d(y,0)$ the solutions to (4.7), (4.8), (4.10), (4.11), respectively. Then we have*

$$\begin{pmatrix} a_{\{x,x\}} & 0 \\ a_{\{x,\delta\}} & -\frac{x}{y} \end{pmatrix}\begin{pmatrix} b_{\{y,y\}} & 0 \\ b_{\{y,\delta\}} & -\frac{y}{x} \end{pmatrix} = I_2. \tag{4.18}$$

Moreover,

$$\frac{y}{x}a_{\{\delta,\delta\}} + \frac{x}{y}b_{\{\delta,\delta\}} = a_{\{x,\delta\}}b_{\{y,\delta\}} \tag{4.19}$$

and

$$A(\xi)\begin{pmatrix} 1+\alpha \\ xF+\alpha_d \end{pmatrix} = \begin{pmatrix} a_{\{x,x\}} & 0 \\ a_{\{x,\delta\}} & -\frac{x}{y} \end{pmatrix}\begin{pmatrix} 1+\beta \\ -yF+\beta_d \end{pmatrix}, \tag{4.20}$$

or equivalently,

$$B(\eta)\begin{pmatrix} 1+\beta \\ -yF+\beta_d \end{pmatrix} = \begin{pmatrix} b_{\{y,y\}} & 0 \\ b_{\{y,\delta\}} & -\frac{y}{x} \end{pmatrix}\begin{pmatrix} 1+\alpha \\ xF+\alpha_d \end{pmatrix}. \tag{4.21}$$

Theorem 4.8 *Let $x > 0$ be fixed. Assume all conditions of Theorem 4.4 hold. Then the random variables α and α_d which are the solutions to (4.7) and (4.10), respectively, are the partial derivatives of the solution $\widehat{\xi}(x,\delta)$ to (4.4) evaluated at $(x,0)$, that is,*

$$\begin{aligned}\lim_{|\Delta x|+|\delta|\to 0}\frac{1}{|\Delta x|+|\delta|}\Big|&\widehat{\xi}(x+\Delta x,\delta) \\ &- \frac{\widehat{\xi}(x,0)}{x}\Big(x+\Delta x\big(1+\alpha(x,0)\big)+\delta\alpha_d(x,0)\Big)\frac{1}{L^\delta}\Big| \\ = 0,&\end{aligned} \tag{4.22}$$

where the convergence takes place in $\mathbb{P}$-probability. Likewise, β and β_d which are the solutions to (4.8) and (4.11), respectively, are the partial derivatives of the solution $\widehat{\eta}(y,\delta)$ to (4.5) evaluated at $(y,0)$, where $y = u_x(x,0)$, that is,

$$\begin{aligned}\lim_{|\Delta y|+|\delta|\to 0}\frac{1}{|\Delta y|+|\delta|}\Big|&\widehat{\eta}(y+\Delta y,\delta) \\ &- \frac{\widehat{\eta}(y,0)}{y}\Big(y+\Delta y\big(1+\beta(y,0)\big)+\delta\beta_d(y,0)\Big)L^\delta\Big| \\ = 0,&\end{aligned} \tag{4.23}$$

where the convergence takes place in $\mathbb{P}$-probability.

From Theorem 4.8, we obtain the following corollary.

Corollary 4.9 *Under the conditions of Theorem 4.8, (4.22) is equivalent to*

$$\begin{aligned}\lim_{|\Delta x|+|\delta|\to 0}\frac{1}{|\Delta x|+|\delta|}\Big|&\hat{\xi}(x+\Delta x,\delta)-\hat{\xi}(x,0) \\ &- \frac{\hat{\xi}(x,0)}{x}\Big(\Delta x\big(\alpha(x,0)+1\big)+\delta\big(\alpha_d(x,0)+xF\big)\Big)\Big| = 0.\end{aligned}$$

Likewise, (4.23) *holds if and only if*

$$\lim_{|\Delta y|+|\delta|\to 0} \frac{1}{|\Delta y|+|\delta|}\bigg|\widehat{\eta}(y+\Delta y,\delta)-\widehat{\eta}(y,0) \\ -\frac{\widehat{\eta}(y,0)}{y}\Big(\Delta y\big(\beta(y,0)+1\big)+\delta\big(\beta_d(y,0)-yF\big)\Big)\bigg|=0.$$

Both convergences take place in $\mathbb{P}$*-probability.*

4.7 Proofs

We begin the proofs with technical lemmas.

Lemma 4.10 *Let Assumption* 2.2 *hold and* $d\in(\max(\exp(-1/c_2),\exp(-c_1)),1]$. *Then for every* $x>0$, *we have*

$$U'(dx)\le \frac{1}{1+c_2\log d}U'(x),$$
$$-V'(dx)\le \frac{1}{1+\frac{1}{c_1}\log d}\big(-V'(x)\big).$$

Proof Fix $x>0$ and d as above. Then using Assumption 2.2 and the monotonicity of U', we get

$$U'(dx)-U'(x)=\int_d^1\big(-U''(tx)\big)x\,dt=\int_d^1\big(-U''(tx)\big)tx\frac{dt}{t} \\ \le c_2\int_d^1 U'(tx)\frac{dt}{t}\le c_2U'(dx)(-\log d).$$

Therefore, we obtain $U'(dx)(1+c_2\log d)\le U'(x)$, which implies the first assertion of the lemma. The second can be shown analogously. □

Corollary 4.11 *Under the conditions of Lemma* 4.10, *for every* $k\in\mathbb{N}$, *we have*

$$U'(d^kx)\le \frac{1}{(1+c_2\log d)^k}U'(x),$$
$$-V'(d^kx)\le \frac{1}{(1+\frac{1}{c_1}\log d)^k}\big(-V'(x)\big).$$

Below, 1_E denotes the indicator function of a set E.

Lemma 4.12 *Let Assumption* 2.2 *hold. Then for every* $z\in(0,1]$ *and* $x>0$, *we have*

$$U'(zx)\le z^{-c_2}U'(x),$$
$$-V'(zx)\le z^{-\frac{1}{c_1}}\big(-V'(x)\big).$$

Proof Fix $d \in (\exp(-1/c_2), 1)$. For every $z \in (0, 1]$ and $x > 0$, we get

$$\begin{aligned} U'(zx) &= \sum_{k=1}^{\infty} U'(zx) 1_{\{z \in (d^k, d^{k-1}]\}} \\ &\le \sum_{k=1}^{\infty} U'(d^k x) 1_{\{z \in (d^k, d^{k-1}]\}} \\ &\le U'(x) \sum_{k=1}^{\infty} \frac{1}{(1 + c_2 \log d)^k} 1_{\{z \in (d^k, d^{k-1}]\}}. \end{aligned} \tag{4.24}$$

Let us set

$$a_1(d) := \frac{1}{1 + c_2 \log d} > 1, \qquad a_2(d) := \frac{\log(1 + c_2 \log d)}{\log d} = -\frac{\log(a_1(d))}{\log d} > 0.$$

As $a_1(d) > 1$ and, for every $k \in \mathbb{N}$,

$$d^k < z \le d^{k-1} \qquad \text{is equivalent to} \qquad \frac{\log z}{\log d} < k \le \frac{\log z}{\log d} + 1,$$

we deduce that for every $z \in (0, 1]$, we have

$$\begin{aligned} \frac{1}{(1 + c_2 \log d)^k} 1_{\{z \in (d^k, d^{k-1}]\}} &\le a_1(d) a_1(d)^{\frac{\log z}{\log d}} 1_{\{z \in (d^k, d^{k-1}]\}} \\ &= a_1(d) \big(a_1(d)^{\frac{1}{\log d}}\big)^{\log z} 1_{\{z \in (d^k, d^{k-1}]\}} \\ &= a_1(d) z^{-a_2(d)} 1_{\{z \in (d^k, d^{k-1}]\}}. \end{aligned} \tag{4.25}$$

Plugging (4.25) in (4.24), we get

$$U'(zx) \le U'(x) \sum_{k=1}^{\infty} a_1(d) z^{-a_2(d)} 1_{\{z \in (d^k, d^{k-1}]\}} = a_1(d) z^{-a_2(d)} U'(x)$$

for every $z \in (0, 1]$ and $x > 0$. As $\lim_{d \uparrow 1} a_1(d) = 1$ and

$$\lim_{d \uparrow 1} a_2(d) = \lim_{d \uparrow 1} \frac{\log(1 + c_2 \log d)}{\log d} = \lim_{y \uparrow 0} \frac{\log(1 + c_2 y)}{y} = \lim_{y \uparrow 0} \frac{c_2}{1 + c_2 y} = c_2,$$

taking the limit in the latter inequality, we obtain that

$$U'(zx) \le \lim_{d \uparrow 1} a_1(d) z^{-a_2(d)} U'(x) = z^{-c_2} U'(x),$$

for every $z \in (0, 1]$ and $x > 0$. The other assertion can be proved similarly. This completes the proof of the lemma. □

Corollary 4.13 *Under Assumption 2.2, for every $z > 0$ and $x > 0$, we have*

$$U'(zx) \leq \max(z^{-c_2}, 1)U'(x) \leq (z^{-c_2} + 1)U'(x),$$

$$-V'(zx) \leq \max(z^{-\frac{1}{c_1}}, 1)\big(-V'(x)\big) \leq (z^{-\frac{1}{c_1}} + 1)\big(-V'(x)\big).$$

4.8 Proof of the second-order expansion

Lemma 4.14 *Let $x > 0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y = u_x(x, 0)$. For arbitrary α^0 and α^1 in $\mathcal{A}^\infty(x, 0)$, let us define, for $(s, t) \in \mathbb{R}^2$,*

$$\psi(s,t) := \frac{1}{x}\big(x + s(1+\alpha^0) + t\alpha^1\big)\frac{1}{L^t},$$
$$w(s,t) := \mathbb{E}\big[U\big(\xi\psi(s,t)\big)\big],$$

where $\xi = \widehat{\xi}(x, 0)$ is the solution to (4.4) corresponding to $x > 0$ and $\delta = 0$. Then w admits at $(0, 0)$ the second-order expansion

$$w(s,t) = w(0,0) + (s\ t)\nabla w(0,0) + \frac{1}{2}(s\ t)H_w\begin{pmatrix} s \\ t\end{pmatrix} + o(s^2+t^2),$$

where

$$w_s(0,0) = u_x(x,0),$$
$$w_t(0,0) = xy\mathbb{E}^{\mathbb{R}(x,0)}[F]$$

and

$$H_w := \begin{pmatrix} w_{ss}(0,0) & w_{st}(0,0) \\ w_{st}(0,0) & w_{tt}(0,0)\end{pmatrix},$$

where the second-order partial derivatives of w at $(0, 0)$ are given by

$$w_{ss}(0,0) = -\frac{y}{x}\mathbb{E}^{\mathbb{R}(x,0)}[A(\xi)(1+\alpha^0)^2],$$
$$w_{st}(0,0) = -\frac{y}{x}\mathbb{E}^{\mathbb{R}(x,0)}[A(\xi)(1+\alpha^0)(xF+\alpha^1) - xF(1+\alpha^0)],$$
$$w_{tt}(0,0) = -\frac{y}{x}\mathbb{E}^{\mathbb{R}(x,0)}[A(\xi)(\alpha^1+xF)^2 - 2xF\alpha^1 - x^2(F^2+G)].$$

Proof As α^0 and α^1 are in $\mathcal{A}^\infty$, there exists a constant $\varepsilon \in (0, 1)$ such that

$$|\alpha^0| + |\alpha^1| \leq \frac{x}{6\varepsilon} - 1 \qquad \mathbb{P}\text{-a.s.} \tag{4.26}$$

Fix an arbitrary $(s, t) \in B_\varepsilon(0, 0)$ and define

$$\widetilde{\psi}(z) := \psi(zs, zt), \qquad z \in (-1, 1).$$

Note that

$$\frac{2}{3} \le \widetilde{\psi}(z) L^{zt} \le \frac{4}{3}, \qquad z \in (-1, 1). \tag{4.27}$$

As

$$\psi_t(s,t) = \frac{\alpha^1}{xL^t} + \psi(s,t)(F + tG) \qquad \text{and} \qquad \psi_s(s,t) = \frac{1+\alpha^0}{xL^t},$$

we get

$$\begin{aligned}\widetilde{\psi}'(z) &= \psi_s(sz, tz)s + \psi_t(sz, tz)t \\ &= \frac{1+\alpha^0}{xL^{zt}} s + \left(\frac{\alpha^1}{xL^{zt}} + \widetilde{\psi}(z)\,(F + ztG)\right) t.\end{aligned} \tag{4.28}$$

Similarly, since $\psi_{ss}(s,t) = 0$ and

$$\begin{aligned}\psi_{tt}(s,t) &= \frac{2\alpha^1}{xL^t}\,(F + tG) + \psi(s,t)\big((F+tG)^2 + G\big), \\ \psi_{st}(s,t) &= \frac{1+\alpha^0}{xL^t}\,(F + tG)\,,\end{aligned}$$

we obtain

$$\begin{aligned}\widetilde{\psi}''(z) &= \psi_{tt}(zs, zt)t^2 + 2\psi_{st}(zs, zt)ts + \psi_{ss}(zs, zt)s^2 \\ &= \left(\frac{2\alpha^1}{xL^{zt}}(F + ztG) + \tilde{\psi}(z)\big((F+ztG)^2 + G\big)\right)t^2 \\ &\quad + 2\frac{1+\alpha^0}{xL^{zt}}\,(F + ztG)\,ts.\end{aligned}$$

Setting $W(z) := U(\xi\widetilde{\psi}(z))$, $z \in (-1, 1)$, we get by direct computations that

$$W'(z) = U'\big(\xi\widetilde{\psi}(z)\big)\xi\widetilde{\psi}'(z), \tag{4.29}$$

$$W''(z) = U''\big(\xi\widetilde{\psi}(z)\big)\big(\xi\widetilde{\psi}'(z)\big)^2 + U'\big(\xi\widetilde{\psi}(z)\big)\xi\widetilde{\psi}''(z). \tag{4.30}$$

Let us define

$$a_2 := 2^{c_2+2} \qquad \text{and} \qquad J := 1 + |F| + G.$$

From (4.28), using (4.26) and (4.27), we get

$$|\widetilde{\psi}'(z)| \le 2J\exp(\varepsilon J), \quad \widetilde{\psi}(z)^{-c_2} + 1 \le 2^{c_2+1}\exp(c_2\varepsilon J), \qquad z \in (-1, 1).$$

Therefore, from (4.29) using Corollary 4.13, we obtain

$$\begin{aligned}\sup_{z\in(-1,1)} |W'(z)| &\le \sup_{z\in(-1,1)} U'(\xi)\xi\left(\big(\widetilde{\psi}(z)\big)^{-c_2} + 1\right)|\widetilde{\psi}'(z)| \\ &\le a_2 U'(\xi)\xi J\exp\big((c_2+1)\varepsilon J\big) \le a_2 U'(\xi)\xi J\exp(a_2\varepsilon J).\end{aligned} \tag{4.31}$$

Similarly, from (4.30), applying Assumption 2.2 and Corollary 4.13, we deduce the existence of a constant $a_3 > 0$ such that

$$\sup_{z\in(-1,1)} |W''(z)| \leq a_3 U'(\xi)\xi J^2 \exp(a_3 \varepsilon J). \tag{4.32}$$

Combining (4.31) and (4.32), we obtain

$$\sup_{z\in(-1,1)} \big(|W'(z)| + |W''(z)|\big) \leq U'(\xi)\xi \big(a_2 J \exp(a_2 \varepsilon J) + a_3 J^2 \exp(a_3 \varepsilon J)\big).$$

Consequently, as $1 \leq J \leq J^2$, by setting $a_1 := \max(a_2, a_3)$, we get for every z_1 and z_2 in $(-1, 1)$ that

$$\left|\frac{W(z_1) - W(z_2)}{z_1 - z_2}\right| + \left|\frac{W'(z_1) - W'(z_2)}{z_1 - z_2}\right| \leq 4a_1 U'(\xi)\xi J^2 \exp(a_1 \varepsilon J). \tag{4.33}$$

By passing to a smaller ε if necessary and by applying Hölder's inequality, we deduce from Assumption 4.2 that the right-hand side of (4.33) integrable. As the right-hand side of (4.33) depends only on ε (and not on (s, t)), the assertion of the lemma follows from the dominated convergence theorem. □

Corollary 4.15 *Let $x > 0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y = u_x(x, 0)$. Then we have*

$$\begin{aligned} u(x + \Delta x, \delta) \geq{}& u(x, 0) + y\Delta x + \delta x y \mathbb{E}^{\mathbb{R}(x,0)}[F] \\ &+ \frac{1}{2}(\Delta x\ \delta) H_u(x, 0) \begin{pmatrix} \Delta x \\ \delta \end{pmatrix} + o(\Delta x^2 + \delta^2), \end{aligned}$$

where $H_u(x, 0)$ is given by (4.14).

Proof The result follows from Lemma 4.14 via an approximation of the solutions to (4.7) and (4.10), which are elements of $\mathcal{A}^2(x, 0)$, by elements of $\mathcal{A}^\infty(x, 0)$. □

Similarly to Lemma 4.14 and Corollary 4.15, we can establish the following results.

Lemma 4.16 *Let $x > 0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y = u_x(x, 0)$. For arbitrary β^0 and β^1 in $\mathcal{B}^\infty(y, 0)$, let us define, for $(s, t) \in \mathbb{R}^2$,*

$$\begin{aligned} \phi(s, t) &:= \frac{1}{y}\left(y + s(1 + \beta^0) + t\beta^1\right) L^t, \\ \bar{w}(s, t) &:= \mathbb{E}\big[V\big(\eta\phi(s, t)\big)\big], \end{aligned}$$

where $\eta = \widehat{\eta}(y, 0)$ is the solution to (4.5) corresponding to $y > 0$ and $\delta = 0$. Then $\bar{w}$ admits at $(0, 0)$ the second-order expansion

$$\bar{w}(s, t) = \bar{w}(0, 0) + (s\ t)\nabla\bar{w}(0, 0) + \frac{1}{2}(s\ t) H_{\bar{w}} \begin{pmatrix} s \\ t \end{pmatrix} + o(s^2 + t^2),$$

where

$$\bar{w}_s(0,0) = v_y(y,0),$$
$$\bar{w}_t(0,0) = xy\mathbb{E}^{\mathbb{R}(x,0)}[F]$$

and

$$H_{\bar{w}} := \begin{pmatrix} \bar{w}_{ss}(0,0) & \bar{w}_{st}(0,0) \\ \bar{w}_{st}(0,0) & \bar{w}_{tt}(0,0) \end{pmatrix},$$

where the second-order partial derivatives of $\bar{w}$ *at* $(0,0)$ *are given by*

$$\bar{w}_{ss}(0,0) = \frac{x}{y}\mathbb{E}^{\mathbb{R}(x,0)}[B(\eta)(1+\beta^0)^2],$$
$$\bar{w}_{st}(0,0) = \frac{x}{y}\mathbb{E}^{\mathbb{R}(x,0)}[B(\eta)(1+\beta^0)(-yF+\beta^1)+yF(1+\beta^0)],$$
$$\bar{w}_{tt}(0,0) = \frac{x}{y}\mathbb{E}^{\mathbb{R}(x,0)}[B(\eta)(\beta^1-yF)^2+2yF\beta^1-y^2(F^2-G)].$$

Lemma 4.17 *Let $x>0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y=u_x(x,0)$. Then we have*

$$v(y+\Delta y,\delta) \le v(y,0) - x\Delta y + \delta xy\mathbb{E}^{\mathbb{R}(x,0)}[F] + \frac{1}{2}(\Delta y\ \delta)H_v(y,0)\begin{pmatrix}\Delta y \\ \delta\end{pmatrix} + o(\Delta y^2+\delta^2),$$

where $H_v(y,0)$ is given by (4.15).

4.9 Closing the duality gap

We begin with the proof of Theorem 4.7.

Proof of Theorem 4.7 It follows from [22, Lemma 2] that

$$A(\xi)(1+\alpha) = a_{\{x,x\}}(1+\beta),$$
$$B(\eta)(1+\beta) = b_{\{y,y\}}(1+\alpha). \tag{4.34}$$

Using standard techniques from calculus of variations, we can show that the solutions to (4.10) and (4.11) satisfy

$$A(\xi)(\alpha_d + xF) - xF = c + \tilde{\beta},$$
$$B(\eta)(\beta_d - yF) + yF = d + \tilde{\alpha}, \tag{4.35}$$

where $\tilde{\beta} \in \mathcal{B}^2(y,0)$, $\tilde{\alpha} \in \mathcal{A}^2(x,0)$, and c and d are some constants. We characterise $\tilde{\beta}$, $\tilde{\alpha}$ and d below. Let us set

$$\tilde{\tilde{\alpha}} := \tilde{\alpha} - d\alpha \in \mathcal{A}^2(x,0). \tag{4.36}$$

It follows from the second equation in (4.35) that

$$\begin{aligned}\beta_d - yF &= A(\xi)(d - yF + \tilde{\alpha}) \\ &= A(\xi)(d + d\alpha - yF + \tilde{\alpha} - d\alpha) \\ &= da_{\{x,x\}}(1+\beta) + A(\xi)(-yF + \tilde{\tilde{\alpha}}),\end{aligned}$$

where we have used (4.34) in the last equality. Multiplying by $-\frac{x}{y}$, we obtain

$$A(\xi)\left(xF - \frac{x}{y}\tilde{\tilde{\alpha}}\right) = -\frac{x}{y}(\beta_d - yF) + \frac{x}{y}da_{\{x,x\}}(1+\beta),$$

and thus

$$A(\xi)\left(xF - \frac{x}{y}\tilde{\tilde{\alpha}}\right) - xF = \tilde{d} + \tilde{\tilde{\beta}},$$

where

$$\tilde{d} = \frac{x}{y}da_{\{x,x\}} \in \mathbb{R} \qquad \text{and} \qquad \tilde{\tilde{\beta}} = \frac{x}{y}da_{\{x,x\}}\beta - \frac{x}{y}\beta_d \in \mathcal{B}^2(y,0).$$

It follows from the characterisation in (4.35) of the unique solution to (4.10) that

$$-\frac{x}{y}\tilde{\tilde{\alpha}} = \alpha_d, \qquad \text{equivalently } \tilde{\tilde{\alpha}} = -\frac{y}{x}\alpha_d.$$

From (4.36), we obtain

$$\tilde{\alpha} = \tilde{\tilde{\alpha}} + d\alpha = -\frac{y}{x}\alpha_d + d\alpha.$$

Plugging this back into the second equality in (4.35), we get

$$B(\eta)(\beta_d - yF) = d(1+\alpha) - \frac{y}{x}(\alpha_d + xF).$$

Multiplying by $\frac{x}{y}A(\xi)$, we obtain

$$A(\xi)(\alpha_d + xF) = \frac{x}{y}da_{\{x,x\}}(1+\beta) - \frac{x}{y}(\beta_d - yF). \tag{4.37}$$

Setting $d' := \frac{x}{y}da_{\{x,x\}}$, we claim that

$$d' = a_{\{x,\delta\}}, \tag{4.38}$$

where $a_{\{x,\delta\}}$ is defined in (4.12). Multiplying both sides of (4.37) by $1+\alpha$, taking expectations under $\mathbb{R}(x,0)$ and using orthogonality of the elements of $\mathcal{A}^2(x,0)$ and $\mathcal{B}^2(y,0)$, we get

$$\begin{aligned}\mathbb{E}^{\mathbb{R}(x,0)}\left[A(\xi)(\alpha_d + xF)(1+\alpha)\right] &= d'\mathbb{E}^{\mathbb{R}(x,0)}\left[(1+\beta)(1+\alpha)\right] \\ &\quad - \frac{x}{y}\mathbb{E}^{\mathbb{R}(x,0)}\left[(\beta_d - yF)(1+\alpha)\right] \\ &= d' + \mathbb{E}^{\mathbb{R}(x,0)}\left[xF(1+\alpha)\right].\end{aligned}$$

Therefore,

$$d' = \mathbb{E}^{\mathbb{R}(x,0)}\left[A(\xi)(\alpha_d + xF)(1+\alpha) - xF(1+\alpha)\right] = a_{\{x,\delta\}},$$

where in the last equality we have used (4.12). Thus, (4.38) holds. Now (4.37) with $\frac{x}{y} da_{\{x,x\}} = a_{\{x,\delta\}}$ and (4.35) prove (4.20). (4.21) can be shown similarly. As $A(\xi) = \frac{1}{B(\eta)}$, (4.20) and (4.21) imply (4.18).

It remains to prove (4.19). Let us set

$$\bar{\beta} := \beta + 1, \qquad \bar{\alpha} := \alpha + 1,$$
$$\bar{\beta}_d := \beta_d - yF, \qquad \bar{\alpha}_d := \alpha_d + xF.$$

Then from (4.10) using (4.20), we get

$$\frac{y}{x} a_{\{\delta,\delta\}} = \mathbb{E}^{\mathbb{R}(x,0)}\left[\frac{y}{x} a_{\{x,\delta\}} \bar{\beta}\bar{\alpha}_d - \bar{\beta}_d \bar{\alpha}_d\right] - \frac{y}{x}\mathbb{E}^{\mathbb{R}(x,0)}[2xF\alpha_d] - xy\mathbb{E}^{\mathbb{R}(x,0)}[F^2 + G]. \tag{4.39}$$

Likewise, from (4.11) via (4.39), we obtain

$$\frac{x}{y} b_{\{\delta,\delta\}} = \mathbb{E}^{\mathbb{R}(x,0)}\left[\frac{x}{y} b_{\{y,\delta\}} \bar{\alpha}\bar{\beta}_d - \bar{\beta}_d \bar{\alpha}_d + 2\beta_d xF - xy(F^2 - G)\right]. \tag{4.40}$$

Let us define

$$T_1 := \mathbb{E}^{\mathbb{R}(x,0)}\left[\frac{y}{x} a_{\{x,\delta\}} \bar{\beta}\bar{\alpha}_d + \frac{x}{y} b_{\{y,\delta\}} \bar{\alpha}\bar{\beta}_d\right],$$
$$T_2 := \mathbb{E}^{\mathbb{R}(x,0)}[-2\bar{\beta}_d\bar{\alpha}_d - 2yF\alpha_d + 2xF\beta_d - 2xyF^2].$$

Then adding (4.39) and (4.40), we deduce that

$$\frac{y}{x} a_{\{\delta,\delta\}} + \frac{x}{y} b_{\{\delta,\delta\}} = T_1 + T_2. \tag{4.41}$$

Let us rewrite T_2 as

$$\begin{aligned} T_2 &= \mathbb{E}^{\mathbb{R}(x,0)}[-2\bar{\beta}_d\bar{\alpha}_d - 2yF\alpha_d + 2xF\beta_d - 2xyF^2] \\ &= \mathbb{E}^{\mathbb{R}(x,0)}[-2(\beta_d - yF)(\alpha_d + xF) - 2yF\alpha_d + 2xF\beta_d - 2xyF^2] \\ &= \mathbb{E}^{\mathbb{R}(x,0)}[-2\beta_d\alpha_d - 2\beta_d xF + 2yF\alpha_d + 2xyF^2 - 2yF\alpha_d + 2xF\beta_d - 2xyF^2] \\ &= \mathbb{E}^{\mathbb{R}(x,0)}[-2\beta_d\alpha_d] = 0, \end{aligned} \tag{4.42}$$

as all the terms under the expectation cancel except for $-2\beta_d\alpha_d$, which has still 0 expectation by orthogonality of $\mathcal{A}^2(x,0)$ and $\mathcal{B}^2(y,0)$. Let us consider T_1. First, from (4.18), we get

$$\frac{x}{y} b_{\{y,\delta\}} = \frac{a_{\{x,\delta\}}}{a_{\{x,x\}}} = a_{\{x,\delta\}} b_{\{y,y\}}.$$

Therefore, we can rewrite T_1 as

$$T_1 = \mathbb{E}^{\mathbb{R}(x,0)}\left[\frac{y}{x}a_{\{x,\delta\}}\bar{\beta}\bar{\alpha}_d + a_{\{x,\delta\}}b_{\{y,y\}}\bar{\alpha}\bar{\beta}_d\right]$$
$$= a_{\{x,\delta\}}\mathbb{E}^{\mathbb{R}(x,0)}\left[\frac{y}{x}\bar{\beta}\bar{\alpha}_d + b_{\{y,y\}}\bar{\alpha}\bar{\beta}_d\right]. \tag{4.43}$$

On the other hand, from (4.13), we can express $b_{\{y,\delta\}}$ in terms of $\bar{\beta}$, $\bar{\beta}_d$, $\bar{\alpha}$, $\bar{\alpha}_d$ as

$$b_{\{y,\delta\}} = \mathbb{E}^{\mathbb{R}(x,0)}\left[B(\eta)\,\bar{\beta}_d\bar{\beta} + \frac{y}{x}\bar{\beta}\bar{\alpha}_d\right] = \mathbb{E}^{\mathbb{R}(x,0)}\left[b_{\{y,y\}}\bar{\alpha}\bar{\beta}_d + \frac{y}{x}\bar{\beta}\bar{\alpha}_d\right], \tag{4.44}$$

where in the last equality we have used (4.34). Comparing (4.44) with (4.43), we get

$$T_1 = a_{\{x,\delta\}}b_{\{y,\delta\}}.$$

Plugging this into (4.41) and using (4.42), we deduce that

$$\frac{y}{x}a_{\{\delta,\delta\}} + \frac{x}{y}b_{\{\delta,\delta\}} = a_{\{x,\delta\}}b_{\{y,\delta\}},$$

i.e., (4.19) holds. This completes the proof of the theorem. □

Lemma 4.18 *Let $x > 0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y = u_x(x,0)$. Then for*

$$\Delta y = -\frac{y}{xb_{\{y,y\}}}\left(\frac{x}{y}b_{\{y,\delta\}}\delta + \Delta x\right), \tag{4.45}$$

we have

$$(\Delta y\ \ \delta)\,H_v(y,0)\begin{pmatrix}\Delta y\\ \delta\end{pmatrix} + 2\Delta x\Delta y = (\Delta x\ \ \delta)\,H_u(x,0)\begin{pmatrix}\Delta x\\ \delta\end{pmatrix}. \tag{4.46}$$

Proof First note that $b_{\{y,y\}} > 0$ in (4.45). By direct computations, proving (4.46) is equivalent to establishing the equality

$$-\frac{y}{xb_{\{y,y\}}}\left(\frac{x}{y}b_{\{y,\delta\}}\delta + \Delta x\right)^2 = (\Delta x\ \ \delta)\,H_u(x,0)\begin{pmatrix}\Delta x\\ \delta\end{pmatrix} - \frac{x}{y}b_{\{\delta,\delta\}}\delta^2. \tag{4.47}$$

Now let us consider the right-hand side of (4.47). By direct computations, it can be rewritten as

$$-\frac{y}{x}\Delta x^2 a_{\{x,x\}} + 2\Delta x\delta\left(-\frac{y}{x}a_{\{x,\delta\}}\right) - \delta^2\left(\frac{y}{x}a_{\{\delta,\delta\}} + \frac{x}{y}b_{\{\delta,\delta\}}\right)$$
$$= -\frac{y}{xb_{\{y,y\}}}\Delta x^2 + 2\Delta x\delta\left(-\frac{y}{x}a_{\{x,\delta\}}\right) - \delta^2 a_{\{x,\delta\}}b_{\{y,\delta\}}, \tag{4.48}$$

where the last equality follows from (4.9) and (4.19). We deduce from (4.18) that

$$a_{\{x,\delta\}} = \frac{x}{y}\frac{b_{\{y,\delta\}}}{b_{\{y,y\}}}. \tag{4.49}$$

Plugging (4.49) into (4.48), we can rewrite the right-hand side of (4.48) as

$$-\frac{y}{xb_{\{y,y\}}}\Delta x^2 - 2\Delta x\delta\frac{b_{\{y,\delta\}}}{b_{\{y,y\}}} - \delta^2\frac{x}{y}\frac{(b_{\{y,\delta\}})^2}{b_{\{y,y\}}} = -\frac{y}{xb_{\{y,y\}}}\left(\Delta x + \frac{x}{y}b_{\{y,\delta\}}\delta\right)^2,$$

which is precisely the left-hand side of (4.47). We have just shown that (4.47) holds. By the argument preceding (4.47), this implies that (4.46) is valid as well. This completes the proof of the lemma. □

Lemma 4.19 *Let $x > 0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y = u_x(x,0)$. Then we have*

$$\begin{aligned} u(x+\Delta x,\delta) &= u(x,0) + y\Delta x + \delta xy\mathbb{E}^{\mathbb{R}(x,0)}[F] \\ &\quad + \frac{1}{2}(\Delta x\ \delta)H_u(x,0)\begin{pmatrix}\Delta x\\ \delta\end{pmatrix} + o(\Delta x^2+\delta^2), \end{aligned} \tag{4.50}$$

where $H_u(x,0)$ is given by (4.14). Likewise,

$$\begin{aligned} v(y+\Delta y,\delta) &= v(y,0) - x\Delta y + \delta xy\mathbb{E}^{\mathbb{R}(x,0)}[F] \\ &\quad + \frac{1}{2}(\Delta y\ \delta)H_v(y,0)\begin{pmatrix}\Delta y\\ \delta\end{pmatrix} + o(\Delta y^2+\delta^2), \end{aligned} \tag{4.51}$$

where $H_v(y,0)$ is given by (4.15).

Proof For small Δx and δ and with Δy given by (4.45), we get from conjugacy of u and v (which follows from the abstract theorems in [20]) and Lemma 4.17 that

$$\begin{aligned} u(x+\Delta x,\delta) &\le v(y+\Delta y,\delta) + (x+\Delta x)(y+\Delta y) \\ &\le v(y,0) - x\Delta y + \delta xy\mathbb{E}^{\mathbb{R}(x,0)}[F] + \frac{1}{2}(\Delta y\ \delta)\,H_v(y,0)\begin{pmatrix}\Delta y\\ \delta\end{pmatrix} \\ &\quad + xy + y\Delta x + x\Delta y + \Delta x\Delta y + o(\Delta y^2+\delta^2), \end{aligned} \tag{4.52}$$

where $H_v(y,0)$ is given in (4.15). As $y = u_x(x,0)$ and $x = -v_y(y,0)$, collecting terms on the right-hand side of (4.52), we obtain

$$\begin{aligned} u(x+\Delta x,\delta) &\le u(x,0) + y\Delta x + \delta xy\mathbb{E}^{\mathbb{R}(x,0)}[F] + \Delta x\Delta y \\ &\quad + \frac{1}{2}(\Delta y\ \delta)\,H_v(y,0)\begin{pmatrix}\Delta y\\ \delta\end{pmatrix} + o(\Delta x^2+\delta^2). \end{aligned} \tag{4.53}$$

Likewise, using Corollary 4.15, we get

$$u(x+\Delta x,\delta) \geq u(x,0) + y\Delta x + \delta x y \mathbb{E}^{\mathbb{R}(x,0)}[F] + \frac{1}{2}(\Delta x\ \delta) H_u(x,0) \begin{pmatrix} \Delta x \\ \delta \end{pmatrix} + o(\Delta x^2 + \delta^2). \tag{4.54}$$

By Lemma 4.18, the quadratic terms in (4.53) and (4.54) are equal. Therefore, (4.53) and (4.54) imply that u admits a second-order expansion at $(x,0)$ given by (4.50). Similarly we can prove (4.51). □

Proof of Theorem 4.4 The assertions of Theorem 4.4 follow from Lemma 4.19. □

Proof of Theorem 4.6 The expansions (4.16) and (4.17) follow from Lemma 4.19 and Theorem 4.4. □

4.10 Derivatives of the optimisers

We begin with a technical lemma.

Lemma 4.20 *Let $x > 0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y = u_x(x,0)$, and let $(\delta^n)_{n\in\mathbb{N}}$ be a sequence which converges to 0. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\left[V\left(\widehat{\eta}(y,0)L^{\delta^n}\right)\right] = v(y,0).$$

Proof The proof goes along the lines of the proof of Lemma 4.14; it is therefore skipped. □

Lemma 4.21 *Let $x > 0$ be fixed. Assume all conditions of Theorem 4.4 hold, with $y = u_x(x,0)$, and let $(y^n,\delta^n)_{n\in\mathbb{N}}$ be a sequence which converges to $(y,0)$. Then $\eta^n := \widehat{\eta}(y^n,\delta^n)$, $n\in\mathbb{N}$, converges to $\eta := \widehat{\eta}(y,0)$ in probability, and $V(\eta^n)$, $n\in\mathbb{N}$, converges to $V(\eta)$ in $\mathbf{L}^1(\mathbb{P})$.*

Proof In view of Theorem 4.4, we may assume without loss of generality that $v(y^n,\delta^n)$ is finite for every $n\in\mathbb{N}$. Let us assume by contradiction that $(\eta^n)_{n\in\mathbb{N}}$ does not converge in probability to η. Then there exists $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\left[|\eta^n - \eta| > \varepsilon\right] > \varepsilon.$$

Let us define $\theta^n := \frac{\eta^n}{L^{\delta^n}}$, $n\in\mathbb{N}$, and $\bar{y} := \sup_{n\in\mathbb{N}} y^n$. As $(\theta^n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\bar{y},0)$ and $(L^{\delta^n})_{n\in\mathbb{N}}$ converges to 1 in probability (therefore, in particular, $(L^{\delta^n})_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^0$), we may assume by possibly passing to a smaller ε that

$$\limsup_{n\to\infty} \mathbb{P}\left[|\eta^n - \eta| > \varepsilon, |\theta^n - \eta| L^{\delta^n} \leq \frac{1}{\varepsilon}\right] > \varepsilon.$$

Let us define

$$h^n := \frac{1}{2}(\theta^n + \eta)L^{\delta^n} = \frac{1}{2}(\eta^n + \eta L^{\delta^n}) \in \mathcal{D}\left(\frac{y_n + y}{2}, \delta^n\right), \qquad n \in \mathbb{N}.$$

From convexity of V, we have

$$V(h^n) \le \frac{1}{2}\big(V(\eta^n) + V(\eta L^{\delta^n})\big),$$

and from the strict convexity of V, we deduce the existence of a positive constant ε_0 such that

$$\limsup_{n\to\infty} \mathbb{P}\left[V(h^n) \le \frac{1}{2}\big(V(\eta^n) + V(\eta L^{\delta^n})\big) - \varepsilon_0\right] > \varepsilon_0.$$

Therefore, using Lemma 4.20, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{E}[V(h^n)] &\le \frac{1}{2}\limsup_{n\to\infty} \mathbb{E}[V(\eta^n)] + \frac{1}{2}\limsup_{n\to\infty} \mathbb{E}[V(\eta L^{\delta^n})] - \varepsilon_0^2 \\ &= \frac{1}{2}\limsup_{n\to\infty} v(y^n, \delta^n) + \frac{1}{2} v(y, 0) - \varepsilon_0^2 \\ &= v(y, 0) - \varepsilon_0^2, \end{aligned} \tag{4.55}$$

where in the last equality, we have also used continuity of v at $(y, 0)$, which follows from Theorem 4.4. On the other hand, as $h^n \in \mathcal{D}(\frac{y_n+y}{2}, \delta^n)$, $n \in \mathbb{N}$, we get

$$\limsup_{n\to\infty} v\left(\frac{y_n + y}{2}, \delta^n\right) \le \limsup_{n\to\infty} \mathbb{E}[V(h^n)]. \tag{4.56}$$

Combining (4.55) and (4.56) and using continuity of v at $(y, 0)$ again, we get

$$v(y, 0) = \limsup_{n\to\infty} v\left(\frac{y_n + y}{2}, \delta^n\right) \le \limsup_{n\to\infty} \mathbb{E}[V(h^n)] \le v(y, 0) - \varepsilon_0^2,$$

which is a contradiction as $\varepsilon_0 \neq 0$. Thus $(\eta^n)_{n\in\mathbb{N}}$ converges to η in probability. In turn, this and continuity of v at $(y, 0)$ imply the other assertion of the lemma. □

Proof of Theorem 4.8 We only prove (4.23) as (4.22) can be shown similarly. In view of Theorem 4.4, we assume without loss of generality that for every $n \in \mathbb{N}$, $u(\cdot, \delta^n)$ and $v(\cdot, \delta^n)$ are finite-valued functions. The rest of the proof goes along the lines of the proof of [22, Theorem 2]. Let $(y^n, \delta^n)_{n\in\mathbb{N}}$ be a sequence which converges to $(y, 0)$, where $y = u_x(x, 0) > 0$. Let $\widehat{\eta}^n = \widehat{\eta}(y^n, \delta^n)$, $n \in \mathbb{N}$, denote the corresponding dual optimisers and set

$$\begin{aligned}\phi_1 &:= \frac{1}{2}\min\left(\widehat{\eta}(y, 0), \inf_{n\in\mathbb{N}} \widehat{\eta}^n\right) > 0 \qquad \mathbb{P}\text{-a.s.}, \\ \phi_2 &:= 2\max\left(\widehat{\eta}(y, 0), \sup_{n\in\mathbb{N}} \widehat{\eta}^n\right) < \infty \qquad \mathbb{P}\text{-a.s.}, \\ \theta &:= \inf_{\phi_1 \le t \le \phi_2} V''(t).\end{aligned}$$

Note that the construction of ϕ_1 and ϕ_2 implies that $\theta > 0$ $\mathbb{P}$-a.s. Let us also fix β^0 and β^1 in $\mathcal{B}^\infty(y,0)$ and define

$$\eta^n := \frac{\widehat{\eta}(y,0)}{y}\big(y + \Delta y^n(\beta^0+1) + \delta^n\beta^1\big)L^{\delta^n} \in \mathcal{D}(y^n,\delta^n), \qquad n \in \mathbb{N},$$

where $\Delta y^n := y^n - y$. As β^0 and β^1 are bounded, we assume without loss of generality that

$$\frac{1}{2}\widehat{\eta}(y,0) \le \eta^n \le 2\widehat{\eta}(y,0), \qquad n \in \mathbb{N},$$

which implies that

$$\phi_1 \le \eta^n \le \phi_2.$$

Using the definition of θ, we get

$$V(\eta^n) - V(\widehat{\eta}^n) \ge V'(\widehat{\eta}^n)(\eta^n - \widehat{\eta}^n) + \theta(\eta^n - \widehat{\eta}^n)^2. \tag{4.57}$$

By [29, Theorem 3.2], $-V'(\widehat{\eta}^n) = \widehat{\xi}(x^n,\delta^n)$ is the solution to (4.4) at the point $x^n = -v_y(y^n,\delta^n)$ so that

$$\mathbb{E}[\widehat{\xi}(x^n,\delta^n)\widehat{\eta}^n] = x^n y^n.$$

Moreover, the bipolar construction of the sets $\mathcal{C}(x^n,\delta^n)$ and $\mathcal{D}(y^n,\delta^n)$ implies that

$$\mathbb{E}[\widehat{\xi}(x^n,\delta^n)\eta^n] \le x^n y^n.$$

Therefore, we obtain

$$\mathbb{E}[V'(\widehat{\eta}^n)(\eta^n - \widehat{\eta}^n)] \ge 0.$$

Combining this with (4.57), we get

$$\mathbb{E}[\theta(\eta^n - \widehat{\eta}^n)^2] \le \mathbb{E}[V(\eta^n)] - v(y^n,\delta^n). \tag{4.58}$$

From Lemma 4.16, we deduce

$$\mathbb{E}[V(\eta^n)] = v(y,0) - x\Delta y^n + v_\delta(y,0)\delta^n + \frac{1}{2}(\Delta y^n\ \delta^n)H_{\bar{w}}\begin{pmatrix}\Delta y^n\\ \delta^n\end{pmatrix} + o\big((\Delta y^n)^2 + (\delta^n)^2\big).$$

Combining this with (4.58) and using the expansion for v from Theorem 4.6, we obtain

$$\limsup_{n\to\infty}\frac{1}{(\Delta y^n)^2 + (\delta^n)^2}\big(\mathbb{E}[\eta^n] - v(y^n,\delta^n)\big) \le \frac{1}{2}|H_{\bar{w}} - H_v(y,0)|. \tag{4.59}$$

In view of Lemma 4.16 (by the choice of β^0 and β^1), we can make the right-hand side of (4.59) arbitrarily small. Combining this with (4.58), we deduce that

$$\limsup_{n\to\infty}\frac{1}{(\Delta y^n)^2 + (\delta^n)^2}\mathbb{E}[\theta(\eta^n - \widehat{\eta}^n)^2]$$

can also be made arbitrarily small. As $\theta > 0$ $\mathbb{P}$-a.s., the assertion of the theorem follows. □

5 Proofs of Theorems 2.7, 2.8, 2.10, 2.12 and 3.1

In order to link the abstract theorems to their concrete counterparts, we first have to establish some structural properties of the perturbed primal and dual admissible sets.

The following lemma gives a useful characterisation of the primal and dual admissible sets after perturbations.

Lemma 5.1 *Under assumption* (2.1), *for every* $\delta \in \mathbb{R}$, *we have*

$$\mathcal{Y}(1,\delta) = \mathcal{Y}(1,0)\mathcal{E}\big(-(\delta\nu)\cdot J^0\big),$$
$$\mathcal{X}(1,\delta) = \mathcal{X}(1,0)\frac{1}{\mathcal{E}(-(\delta\nu)\cdot J^0)}.$$

Proof Fix $\delta \in \mathbb{R}$ and set $X^\delta := \mathcal{E}(\pi \cdot J^\delta)$ for any predictable and J^δ-integrable process π. Then $X^\delta \in \mathcal{X}(1,\delta)$. Let us consider $X^0 := X^\delta \mathcal{E}(-(\delta\nu)\cdot J^0)$. One can see that $X^0 \in \mathcal{X}(1,0)$. The remainder of the proof is straightforward and therefore skipped. □

Proof of Theorem 2.7 Condition (2.1) implies that the respective closures of the convex solid hulls of $\{X_T : X \in \mathcal{X}(1,0)\}$ and $\{Y_T : Y \in \mathcal{Y}(1,0)\}$ satisfy the abstract Assumption 4.1. In view of Lemma 5.1, we have for $\delta \in \mathbb{R}$ that

$$\left\{\frac{X_T}{L^\delta} : X \in \mathcal{X}(1,0)\right\} = \{X_T : X \in \mathcal{X}(1,\delta)\},$$
$$\{Y_T L^\delta : Y \in \mathcal{Y}(1,0)\} = \{Y_T : Y \in \mathcal{Y}(1,\delta)\}.$$

Therefore, the respective closures of the convex solid hulls of $\{X_T : X \in \mathcal{X}(1,\delta)\}$ and $\{Y_T : Y \in \mathcal{Y}(1,\delta)\}$ satisfy the abstract condition (4.3). The relationship between the abstract Assumption 4.2 and Assumption 2.6 is apparent. It remains to show that the sets $\mathcal{M}^2(x)$ and $\mathcal{N}^2(x)$ satisfy the abstract Assumption 4.3. However, this follows from continuity of J^0 and [22, Lemma 6]. Therefore, the assertions of Theorem 2.7 follow from the abstract Theorem 4.4. □

Proof of Theorem 2.8 As in the proof of Theorem 2.7, the assertions of Theorem 2.8 follow from the abstract Theorem 4.6. □

Proof of Theorem 2.10 Similarly to the proof of Theorem 2.8, the assertions of Theorem 2.10 follow from the abstract Theorem 4.7. □

Proof of Theorem 2.12 As above, the affirmations of this theorem follow from the abstract Theorem 4.8. □

For the proof of Theorem 3.1, we need the following technical lemma. First, for $(\delta, \Delta x, \varepsilon) \in \mathbb{R} \times (-x, \infty) \times (0, \infty)$, let us set

$$f(\delta, \Delta x, \varepsilon) := \frac{1}{\Delta x^2 + \delta^2}\Bigg(u(x,0) + (\Delta x\ \delta)\nabla u(x,0) + \frac{1}{2}(\Delta x\ \delta)H_u(x,0)\begin{pmatrix}\Delta x\\ \delta\end{pmatrix} - \mathbb{E}[U(X_T^{\Delta x,\delta,\varepsilon})]\Bigg), \quad (5.1)$$

where $\nabla u(x,0)$, $H_u(x,0)$, $X^{\Delta x,\delta,\varepsilon}$ are defined in (2.10), (2.18), (3.3), respectively.

Lemma 5.2 *Assume that $x > 0$ is fixed and the assumptions of Theorem 2.7 hold. Then for f defined in (5.1), there exists a monotone function g such that*

$$g(\varepsilon) \geq \lim_{|\Delta x|+|\delta|\to 0} f(\delta, \Delta x, \varepsilon), \qquad \varepsilon > 0, \quad (5.2)$$

and

$$\lim_{\varepsilon\to 0} g(\varepsilon) = 0. \quad (5.3)$$

Proof This goes along the lines of the proof of Lemma 4.14. We only outline the main steps for brevity of exposition. For a fixed $\varepsilon > 0$ and $(\Delta x, \delta) \in \mathbb{R}^2$, let us define

$$\psi(\Delta x, \delta) := \frac{x+\Delta x}{x}\exp\Bigg(\big((\Delta x\gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})\cdot M^R\big)_T - \frac{1}{2}\big((\Delta x\gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})^2\cdot\langle M\rangle\big)_T\Bigg)\frac{1}{L^\delta},$$

$$w(\Delta x, \delta) := \mathbb{E}\big[U\big(\widehat{X}_T(x,0)\psi(\Delta x,\delta)\big)\big],$$

where M^R is defined in (3.1). Now first fix $\varepsilon' > 0$, then fix $(\Delta x, \delta) \in B_{\varepsilon'}(0,0)$ and set

$$\widetilde{\psi}(z) := \psi(z\Delta x, z\delta), \qquad z \in (-1, 1).$$

By direct computations, we get

$$\widetilde{\psi}'(z) = \psi_{\Delta x}(z\Delta x, z\delta)\Delta x + \psi_\delta(z\Delta x, z\delta)\delta,$$

where

$$\psi_{\Delta x}(\Delta x, \delta) = \psi(\Delta x, \delta)\Bigg(\frac{1}{x+\Delta x} + \Bigg(\big((\Delta x\gamma^{0,\varepsilon})\cdot M^R\big)_T - \Big(\big((\Delta x\gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})\gamma^{0,\varepsilon}\big)\cdot\langle M\rangle\Big)_T\Bigg)\Bigg),$$

$$\psi_\delta(\Delta x, \delta) = \psi(\Delta x, \delta)\Bigg((\gamma^{1,\varepsilon}\cdot M^R)_T - \Big(\big((\Delta x\gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})\gamma^{1,\varepsilon}\big)\cdot\langle M\rangle\Big)_T + F + \delta G\Bigg),$$

and F and G are defined in (2.9). Similarly, we obtain

$$\widetilde{\psi}''(z) = \psi_{\Delta x\Delta x}(z\Delta x, z\delta)\Delta x^2 + 2\psi_{\Delta x\delta}(z\Delta x, z\delta)\Delta x\delta + \psi_{\delta\delta}(z\Delta x, z\delta)\delta^2,$$

where

$$\begin{aligned}
\psi_{\Delta x\Delta x}(\Delta x, \delta) &= \psi(\Delta x, \delta)\bigg(\frac{1}{x+\Delta x} + \big((\Delta x\gamma^{0,\varepsilon}) \cdot M^R\big)_T \\
&\qquad - \Big(\big((\Delta x\gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})\gamma^{0,\varepsilon}\big) \cdot \langle M\rangle\Big)_T\bigg)^2 \\
&\quad + \psi(\Delta x, \delta)\bigg((\gamma^{0,\varepsilon} \cdot M^R)_T + \big((\gamma^{0,\varepsilon})^2 \cdot \langle M\rangle\big)_T - \frac{1}{(x+\Delta x)^2}\bigg), \\
\psi_{\Delta x\delta}(\Delta x, \delta) &= \psi(\Delta x, \delta)\bigg(\frac{1}{x+\Delta x} + \big((\Delta x\gamma^{0,\varepsilon}) \cdot M^R\big)_T \\
&\qquad - \Big(\big((\Delta x\gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})\gamma^{0,\varepsilon}\big) \cdot \langle M\rangle\Big)_T\bigg) \\
&\quad \times \bigg((\gamma^{1,\varepsilon} \cdot M^R)_T - \Big(\big((\Delta x\gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})\gamma^{1,\varepsilon}\big) \cdot \langle M\rangle\Big)_T \\
&\qquad + F + \delta G\bigg) \\
&\quad + \psi(\Delta x, \delta)\big((\gamma^{1,\varepsilon}\gamma^{0,\varepsilon}) \cdot \langle M\rangle_T\big), \\
\psi_{\delta\delta}(\Delta x, \delta) &= \psi(\Delta x, \delta)\bigg(\big((\gamma^{1,\varepsilon}) \cdot M^R\big)_T \\
&\qquad - \Big(\big((\Delta x\gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})\gamma^{1,\varepsilon}\big) \cdot \langle M\rangle\Big)_T + F + \delta G\bigg)^2 \\
&\quad + \psi(\Delta x, \delta)\Big(\big((\gamma^{1,\varepsilon})^2 \cdot \langle M\rangle\big)_T + G\Big).
\end{aligned}$$

Setting $W(z) := U(\widehat{X}_T(x,0)\widetilde{\psi}(z))$, $z \in (-1, 1)$, we get by direct computations that

$$\begin{aligned}
W'(z) &= U'\left(\widehat{X}_T(x,0)\widetilde{\psi}(z)\right)\widehat{X}_T(x,0)\widetilde{\psi}'(z), \\
W''(z) &= U''\left(\widehat{X}_T(x,0)\widetilde{\psi}(z)\right)\left(\widehat{X}_T(x,0)\widetilde{\psi}'(z)\right)^2 \\
&\quad + U'\left(\widehat{X}_T(x,0)\widetilde{\psi}(z)\right)\widehat{X}_T(x,0)\widetilde{\psi}''(z).
\end{aligned}$$

As in Lemma 4.14, from boundedness of $(\gamma^{0,\varepsilon} \cdot M^R)_T$, $(\gamma^{1,\varepsilon} \cdot M^R)_T$, $((\gamma^{0,\varepsilon})^2 \cdot \langle M\rangle)_T$ and $((\gamma^{1,\varepsilon})^2 \cdot \langle M\rangle)_T$, one can show via Corollary 4.13 and Assumption 4.2 that

$$\left|\frac{W(z_1) - W(z_2)}{z_1 - z_2}\right| + \left|\frac{W'(z_1) - W'(z_2)}{z_1 - z_2}\right| \le \Psi$$

for some random variable Ψ which depends on ε' and is integrable for sufficiently small ε'. By direct computations, the derivatives of W plugged inside the expectation lead to $\nabla u(x,0)$ and a family of Hessians, which converge to $H_u(x,0)$. This results in the existence of a function g satisfying (5.2). Now letting $\varepsilon \to 0$ leads to $H_u^{\varepsilon}(x,0) \to H_u(x,0)$, and therefore we obtain (5.3). Finally, one can choose g to be monotone. □

Proof of Theorem 3.1 First, for f defined in (5.1), we deduce via Lemma 5.2 the existence of a monotone function g such that (5.2) and (5.3) hold. Let us define

$$\phi(\varepsilon) := \{(\delta, \Delta x) : f(t\delta, t\Delta x, \varepsilon) \le 2g(\varepsilon) \text{ for every } t \in [0,1]\}, \qquad \varepsilon > 0,$$

$$r(\varepsilon) := \frac{1}{2}\sup\{r \le \varepsilon : B_r(0,0) \subseteq \phi(\varepsilon)\}, \qquad \varepsilon > 0.$$

Note that $r(\varepsilon) > 0$ for every $\varepsilon > 0$. With

$$\varepsilon(\delta, \Delta x) := \inf\left\{\varepsilon : r(\varepsilon) \ge \sqrt{\Delta x^2 + \delta^2}\right\}, \qquad (\delta, \Delta x) \in \mathbb{R} \times (-x, \infty),$$

we have

$$\lim_{|\Delta x|+|\delta|\to 0} \frac{u(x+\Delta x, \delta) - \mathbb{E}[U(X_T^{\Delta x, \delta, \varepsilon(\delta, \Delta x)})]}{\Delta x^2 + \delta^2} = 0. \qquad \square$$

6 Counterexample

In the following example, we show that even when the 0-model is nice, but Assumption 2.6 fails, we might have

$$u(z,\delta) = v(z,\delta) = \infty \qquad \text{for every } \delta \ne 0 \text{ and } z > 0.$$

Example 6.1 Consider the 0-model where

$$T = 1, \quad M = B, \quad \lambda \equiv 1 \quad \text{and} \quad U(x) = \frac{x^p}{p}, \qquad p \in (0,1).$$

Let us assume that B is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is the usual augmentation of the filtration generated by B. We recall that the convex conjugate for U is $V(y) = \frac{y^{-q}}{q}$, where $q = \frac{p}{1-p}$. Let Z^0 denote the martingale deflator for J^0. Then direct computations yield

$$\mathbb{E}[(Z_1^0)^{-q}] = \exp\left(\frac{1}{2}q(q+1)\right) \in \mathbb{R}.$$

Therefore by the results of Kramkov and Schachermayer [21], the standard conclusions of the utility maximisation theory hold. The primal and dual optimisers are

$$\widehat{X}_1(x,0) = x\exp\left((q+1)B_1 + \frac{1}{2}(1-q^2)\right), \qquad \widehat{Y}_1(y,0) = y\exp\left(-B_1 - \frac{1}{2}\right).$$

Now let us consider a process ν such that

$$(\nu \cdot B)_1 = B_1^3 \qquad \mathbb{P}\text{-a.s.} \tag{6.1}$$

Let us denote $I_t := t$, $t \in [0,1]$. As

$$\frac{d\mathbb{R}(x,0)}{d\mathbb{P}} = \exp\left(-\frac{q(q+1)}{2}\right)\exp\left(qB_1 + q\frac{1}{2}\right) = \exp\left(qB_1 - \frac{q^2}{2}\right), \qquad x > 0,$$

we get, with the notations (2.9), for every $c > 0$ that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{R}(x,0)}\big[\exp\big(c(|F| + G)\big)\big] \\
&= \mathbb{E}\left[\exp\left(qB_1 - \frac{q^2}{2}\right)\exp\big(c|(\nu\cdot B)_1 + (\nu\cdot I)_1| + c(\nu^2\cdot I)_1\big)\right] \\
&= \mathbb{E}\left[\exp\left(qB_1 - \frac{q^2}{2} + c|B_1^3 + (\nu\cdot I)_1| + c(\nu^2\cdot I)_1\right)\right] \\
&\geq \mathbb{E}\left[\exp\left(qB_1 - \frac{q^2}{2} + c|B_1^3| - c(|\nu|\cdot I)_1 + c(\nu^2\cdot I)_1\right)\right] \\
&\geq \exp\left(-\frac{q^2}{2} - \frac{c}{4}\right)\mathbb{E}\left[\exp\left(qB_1 + c|B_1^3| + c\Big(\big(|\nu| - \frac{1}{2}\big)^2\cdot I\Big)_1\right)\right] \\
&\geq \exp\left(-\frac{q^2}{2} - \frac{c}{4}\right)\mathbb{E}[\exp(qB_1 + c|B_1^3|)] \\
&= \exp\left(-\frac{q^2}{2} - \frac{c}{4}\right)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp(qy + c|y^3| - y^2/2)dy = \infty,
\end{aligned}$$

i.e., Assumption 2.6 does not hold.

For every $\delta \in \mathbb{R}$, we can express the local martingale deflator Z^δ as

$$Z_t^\delta = \exp\left(-\big((\lambda + \delta\nu)\cdot B\big)_t - \frac{1}{2}\big((\lambda+\delta\nu)^2\cdot I\big)_t\right), \qquad t \in [0,1].$$

For $p \in (0,1)$, as $q > p > 0$, we have

$$\begin{aligned}
\mathbb{E}[(Z_1^\delta)^{-q}] &= \mathbb{E}\left[\exp\left(q\big((\lambda+\delta\nu)\cdot B\big)_1 + \frac{q}{2}\big((\lambda+\delta\nu)^2\cdot I\big)_1\right)\right] \\
&\geq \mathbb{E}\left[\exp\left(q\big((\lambda+\delta\nu)\cdot B\big)_1\right)\right].
\end{aligned}$$

Therefore, using (6.1), we get

$$\begin{aligned}
\mathbb{E}[(Z_1^\delta)^{-q}] &\geq \mathbb{E}\left[\exp\left(q\big((\lambda+\delta\nu)\cdot B\big)_1\right)\right] \\
&= \mathbb{E}\big[\exp\big(q(B_1 + \delta B_1^3)\big)\big] \\
&= \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2} + q(y + \delta y^3)\right)dy = \infty
\end{aligned}$$

for every $\delta \neq 0$. Consequently, $v(1,\delta)=\infty$ for every $\delta \neq 0$. Moreover, one can find a constant $D>0$ such that

$$\begin{aligned} u(1,\delta) &\geq \mathbb{E}\left[U\left(\widehat{X}^0_1(1,0)\exp\left(\delta F+\frac{1}{2}\delta^2 G\right)\right)\right] \\ &= D\mathbb{E}\left[\exp\left(qB_1+\frac{q}{2}\right)\exp\left(p\delta(\nu\cdot B)_1+p\delta(\nu\cdot I)_1+\frac{p}{2}\delta^2(\nu^2\cdot I)_1\right)\right] \\ &= D\mathbb{E}\left[\exp\left(qB_1+p\delta B_1^3+\frac{q-p}{2}+\frac{p}{2}\big((\delta\nu+1)^2\cdot I\big)_1\right)\right]. \end{aligned}$$

As $q-p$ and $\frac{p}{2}((\delta\nu+1)^2\cdot I)_1$ are nonnegative, we get

$$u(1,\delta)\geq D\mathbb{E}[\exp(qB_1+p\delta B_1^3)] = D\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp(qy+p\delta y^3-y^2/2)dy=\infty$$

for every $\delta\neq 0$.

7 On sufficient conditions for Assumption 2.6

We now discuss Assumption 2.6. Recall that $\zeta(c,\delta)$ was defined in (2.7).

Remark 7.1 The stronger condition

$$\sup_{(x',\delta)\in B_\varepsilon(x,0)} \mathbb{E}^{\mathbb{R}(x',\delta)}[\zeta(c,\delta)]<\infty, \tag{7.1}$$

for some $\varepsilon>0$ and $c>0$, where $B_\varepsilon(x,0)$ denotes the ball in $\mathbb{R}^2$ of radius ε centred at $(x,0)$, implies *local semiconcavity* of the value function $u(x,\delta)$. Consequently, in the quadratic expansions of u and v given by (2.20) and (2.21), the matrices $H_u(x,0)$ and $H_v(y,0)$ defined in (2.18) and (2.19), respectively, are Hessian matrices, i.e., are derivatives of gradients. This follows from Lemma 4.14. However, the very restrictive condition (7.1) is an assumption that depends on the solutions for $\delta\neq 0$ and is thus usually impossible to check.

Remark 7.2 A stronger condition that implies Assumption 2.6 is the existence of a wealth process $\widetilde{X}$ under the numéraire $\widehat{X}(x,0)$ and a constant $c>0$ such that

$$\exp\left(c\big(|(\nu\cdot J^0)_T|+(\nu^2\cdot\langle M\rangle)_T\big)\right)\leq \widetilde{X}_T \qquad \text{a.s.}$$

This can be seen by taking the expectation under $\mathbb{R}(x,0)$ and by representing the integrand under the measure $\mathbb{P}$.

Remark 7.3 Let us assume that $c_1>1$ in (2.3), i.e., that the relative risk aversion of U is strictly greater than 1. (For example this holds if $U(x)=\frac{x^p}{p}$ with $p<0$.) In this case, a sufficient condition for Assumption 2.6 to hold is the existence of some positive exponential moments under $\mathbb{P}$ of $|(\nu \cdot J^0)_T|$ and $(\nu^2 \cdot \langle M\rangle)_T$. This can be shown as follows. Let us set

$$q_i := -\left(1 - \frac{1}{c_i}\right), \qquad i = 1, 2.$$

As $c_2 \geq c_1 > 1$, we deduce that $q_i \in (-1, 0)$, $i = 1, 2$. Using Lemma 4.12, one can find a constant $C > 0$ such that

$$-V'(y)y \leq C(y^{-q_1} + y^{-q_2}), \qquad y > 0. \tag{7.2}$$

To prove (7.2), observe that from Lemma 4.12, we get for every $z \in (0, 1]$ that

$$\begin{aligned} U'(z) &\leq z^{-c_2} U'(1), \\ -V'(z) &\leq z^{-\frac{1}{c_1}}\big(-V'(1)\big). \end{aligned} \tag{7.3}$$

As $(U')^{-1} = -V'$, the first inequality implies that there exists z_0 such that

$$-V'(z) \leq \big(U'(1)\big)^{\frac{1}{c_2}} z^{-\frac{1}{c_2}} \qquad \text{for every } z \geq z_0.$$

Combining this inequality with (7.3) and since

$$\sup_{z \in [\min(z_0,1), \max(z_0,1)]} |-V'(z)z| < \infty,$$

we obtain (7.2). Thus if some positive exponential moments of $|\nu \cdot J^0_T|$ and $\nu^2 \cdot \langle M\rangle_T$ exist under $\mathbb{P}$, using Hölder's inequality, one can find a positive constant a such that

$$\mathbb{E}[\zeta(a, 0)] < \infty, \tag{7.4}$$

where $\zeta(a, 0)$ is defined in (2.7). Let us set

$$c := a(1 + q_2)$$

and note that $\frac{c}{1+q_1} = a\frac{1+q_2}{1+q_1} \leq a$. With $y = u_x(x, 0)$, using again Hölder's inequality (note that $\frac{1}{1+q_i}$ are the Hölder conjugates of $\frac{1}{-q_i}$, $i = 1, 2$) and (7.2), we get

$$\begin{aligned} xy\mathbb{E}^{\mathbb{R}(x,0)}[\zeta(c, 0)] &\leq C\mathbb{E}\Big[\Big(\big(\widehat{Y}_T(y, 0)\big)^{-q_1} + \big(\widehat{Y}_T(y, 0)\big)^{-q_2}\Big)\zeta(c, 0)\Big] \\ &\leq C\mathbb{E}[\widehat{Y}_T(y, 0)]^{-q_1}\mathbb{E}\left[\zeta\left(\frac{c}{1+q_1}, 0\right)\right]^{1+q_1} \\ &\quad + C\mathbb{E}[\widehat{Y}_T(y, 0)]^{-q_2}\mathbb{E}\left[\zeta\left(\frac{c}{1+q_2}, 0\right)\right]^{1+q_2} \\ &\leq Cy^{-q_1}\mathbb{E}[\zeta(a, 0)]^{1+q_1} + Cy^{-q_2}\mathbb{E}[\zeta(a, 0)]^{1+q_2} < \infty, \end{aligned}$$

where the last inequality follows from the supermartingale property of $\widehat{Y}(y, 0)$ and (7.4). Thus Assumption 2.6 holds.

Remark 7.4 Assumption 2.6 is related to the condition [23, Assumption 4] on the random endowment via the following argument. Assume that for some $x' > 0$ and $c > 0$, there exists a wealth process $X \in \mathcal{X}(x', 0)$ such that

$$\zeta(c,0) \leq \frac{X_T}{\widehat{X}_T(x,0)}, \tag{7.5}$$

where $\widehat{X}(x,0)$ is the solution to (2.4). By rewriting the expectation under $\mathbb{P}$, one can see that Assumption 2.6 is satisfied. The wealth process $\frac{X}{\widehat{X}(x,0)}$ under the numéraire $\widehat{X}(x,0)$ in condition (7.5) is a local martingale under $\mathbb{R}(x,0)$. In [23, Assumption 4], it is assumed that $\frac{X}{\widehat{X}(x,0)}$ is a square-integrable martingale under $\mathbb{R}(x,0)$.

8 Relationship to the risk-tolerance wealth process

Following Kramkov and Sîrbu [23], we recall that for an initial wealth $x > 0$ and $\delta = 0$, the *risk-tolerance wealth process* is a maximal wealth process $R(x,0)$ such that

$$R_T(x,0) = -\frac{U'(\widehat{X}_T(x,0))}{U''(\widehat{X}_T(x,0))}, \tag{8.1}$$

i.e., it is a replication process for the random payoff given by the right-hand side of (8.1). In general, the risk-tolerance wealth process $R(x,0)$ need not exist. It is shown in [23, Theorems 8 and 9] that the existence of the risk-tolerance wealth process is closely related to some important properties of the marginal utility-based prices and to the validity of second-order expansions of the value functions under the presence of a random endowment. Below we establish a relationship between the existence of $R(x,0)$ and the second-order expansions of the value functions in the present context.

The following theorem is a version of [23, Theorem 4], which we present without a proof as the one from [23, Theorem 4] goes through; see the discussion in Remark 2.3 above.

Theorem 8.1 *Let $x > 0$ be fixed, assume that* (2.1), (2.6) *and Assumption* 2.2 *hold, and denote $y = u_x(x,0)$. Then the following assertions are equivalent*:

1) *The risk-tolerance wealth process $R(x,0)$ exists.*
2) *The value function $u(x,\delta)$ admits the expansion* (2.20) *at the point $(x,0)$, and $u_{xx}(x,0) = -\frac{y}{x}a_{\{x,x\}}$ satisfies*

$$\frac{(u_x(x,0))^2}{u_{xx}(x,0)} = \mathbb{E}\left[\frac{(U'(\widehat{X}_T(x,0)))^2}{U''(\widehat{X}_T(x,0))}\right],$$

$$u_{xx}(x,0) = \mathbb{E}\left[U''\big(\widehat{X}_T(x,0)\big)\left(\frac{R_T(x,0)}{R_0(x,0)}\right)^2\right].$$

3) *The value function $v(y,\delta)$ admits the expansion* (2.21) *at the point $(y,0)$, and $v_{yy}(y,0) = \frac{x}{y}b_{\{y,y\}}$ satisfies*

$$y^2 v_{yy}(y,0) = \mathbb{E}\big[\big(\widehat{Y}_T(y,0)\big)^2 V''\big(\widehat{Y}_T(y,0)\big)\big] = xy\mathbb{E}^{\mathbb{R}(x,0)}\big[B\big(\widehat{Y}_T(y,0)\big)\big].$$

In addition, if these assertions are valid, then the initial value of $R(x,0)$ *is given by*

$$R_0(x,0) = -\frac{u_x(x,0)}{u_{xx}(x,0)} = \frac{x}{a_{\{x,x\}}},$$

the product $R(x,0)Y(y,0)$ *is a uniformly integrable martingale and*

$$\lim_{\Delta x \to 0} \frac{\widehat{X}_T(x+\Delta x,0) - \widehat{X}_T(x,0)}{\Delta x} = \frac{R_T(x,0)}{R_0(x,0)}, \tag{8.2}$$

$$\lim_{\Delta y \to 0} \frac{\widehat{Y}_T(y+\Delta y,0) - \widehat{Y}_T(y,0)}{\Delta y} = \frac{\widehat{Y}_T(y,0)}{y}, \tag{8.3}$$

where the limits in (8.2) *and* (8.3) *take place in* $\mathbb{P}$*-probability.*

As in [23], for $x > 0$ and with $y = u_x(x,0)$, let us define

$$\frac{d\widetilde{\mathbb{R}}(x,0)}{d\mathbb{P}} := \frac{R_T(x,0)\widehat{Y}_T(y,0)}{R_0(x,0)y}$$

and choose $\frac{R(x,0)}{R_0(x,0)}$ as a numéraire, i.e., let us set

$$S^{R(x,0)} := \left(\frac{R_0(x,0)}{R(x,0)}, \frac{R_0(x,0)\mathcal{E}(J^0)}{R(x,0)}\right).$$

We define the space of martingales

$$\widetilde{\mathcal{M}}^2(x,0) := \big\{M \in \mathbf{H}_0^2\big(\widetilde{\mathbb{R}}(x,0)\big) : M = H \cdot S^{R(x,0)}$$
$$\text{for some } S^{R(x,0)}\text{-integrable process } H\big\},$$

and $\widetilde{\mathcal{N}}^2(y,0)$ as its orthogonal complement in $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x,0))$. We start with a simple lemma (stated without a proof) relating the change of numéraire to the structure of martingales.

Lemma 8.2 *Let* $x > 0$ *be fixed. Assume all conditions of Theorem* 8.1 *hold, with* $y = u_x(x,0)$*. Then we have*

$$M \in \mathcal{M}^2(x,0) \qquad \text{if and only if} \qquad M\frac{\widehat{X}_T(x,0)}{R_T(x,0)} \in \widetilde{\mathcal{M}}^2(x,0),$$

and

$$N \in \mathcal{N}^2(y,0) \qquad \text{if and only if} \qquad N \in \widetilde{\mathcal{N}}^2(y,0).$$

The following theorem describes the structural properties of the approximations in Theorems 2.8, 2.10 and 2.12 under the assumption that the risk-tolerance wealth process exists. In words, *the second-order approximation of the value function amounts to a Kunita–Watanabe decomposition* under the changes of measure and numéraire described above.

Theorem 8.3 *Let $x > 0$ be fixed. Assume all conditions of Theorem 8.1 hold, with $y = u_x(x, 0)$. Let us also assume that the risk-tolerance wealth process $R(x, 0)$ exists. Consider the Kunita–Watanabe decomposition of the square-integrable martingale*

$$P_t := \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)}\Big[\Big(A\big(\widehat{X}_T(x,0)\big) - 1\Big)xF\Big|\mathcal{F}_t\Big], \qquad t \in [0,T],$$

given by

$$P = P_0 - \widetilde{M}^1 - \widetilde{N}^1, \qquad \text{where } \widetilde{M}^1 \in \widetilde{\mathcal{M}}^2(x,0),\ \widetilde{N}^1 \in \widetilde{\mathcal{N}}^2(y,0),\ P_0 \in \mathbb{R}. \quad (8.4)$$

Then the solutions $M^1(x, 0)$ and $N^1(y, 0)$ of the quadratic optimisation problems (2.14) and (2.15) can be obtained from (8.4) by reverting to the original numéraire, according to Lemma 8.2, through the identities

$$\widetilde{M}^1_t = \frac{\widehat{X}_t(x,0)}{R_t(x,0)}M^1_t(x,0), \quad \widetilde{N}^1_t = \frac{x}{y}N^1_t(y,0), \qquad t \in [0,T]. \quad (8.5)$$

In addition, the Hessian terms in the quadratic expansions of u, v can be identified as

$$\begin{aligned} a_{\{\delta,\delta\}} &= \frac{R_0(x,0)}{x}\inf_{\widetilde{M}\in\widetilde{\mathcal{M}}^2(x,0)}\mathbb{E}^{\widetilde{\mathbb{R}}(x,0)}\Big[\Big(\widetilde{M}_T + xF\Big(A\big(\widehat{X}_T(x,0)\big) - 1\Big)\Big)^2\Big] + C_a \\ &= \frac{R_0(x,0)}{x}\mathbb{E}^{\widetilde{\mathbb{R}}(x,0)}[(\widetilde{N}^1_T)^2] + \frac{R_0(x,0)}{x}P_0^2 + C_a, \end{aligned} \quad (8.6)$$

where

$$C_a := x^2\mathbb{E}^{\mathbb{R}(x,0)}\Big[F^2\frac{A(\widehat{X}_T(x,0)) - 1}{A(\widehat{X}_T(x,0))} - G\Big], \quad (8.7)$$

and

$$\begin{aligned} b_{\{\delta,\delta\}} &= \frac{R_0(x,0)}{x}\inf_{\widetilde{N}\in\mathcal{N}^2(y,0)}\mathbb{E}^{\widetilde{\mathbb{R}}(y,0)}\Big[\Big(\widetilde{N}_T + yF\Big(A\big(\widehat{X}_T(x,0)\big) - 1\Big)\Big)^2\Big] + C_b \\ &= \frac{R_0(x,0)}{x}\Big(\frac{y}{x}\Big)^2\mathbb{E}^{\widetilde{\mathbb{R}}(y,0)}[(\widetilde{M}^1_T)^2] + \frac{R_0(x,0)}{x}\Big(\frac{y}{x}\Big)^2P_0^2 + C_b, \end{aligned}$$

where

$$C_b := y^2\mathbb{E}^{\mathbb{R}(x,0)}\Big[G + F^2\Big(1 - A\big(\widehat{X}_T(x,0)\big)\Big)\Big]. \quad (8.8)$$

The terms $a_{x,\delta}$ from (2.16) and $b_{y,\delta}$ from (2.17) can be represented as

$$a_{\{x,\delta\}} = P_0,$$

$$b_{\{y,\delta\}} = \frac{y}{x}\frac{P_0}{a_{\{x,x\}}}.$$

With these identifications, all the conclusions of Theorem 2.8 and Corollary 2.13 hold true.

Proof Let us prove (8.5) first. Completing the square in (2.14), we get

$$a_{\{\delta,\delta\}} = \inf_{M\in\mathcal{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)}\left[A\big(\widehat{X}_T(x,0)\big)\left(M_T + xF\Big(1-\frac{1}{A(\widehat{X}_T(x,0))}\Big)\right)^2\right] + C_a,$$

where C_a is defined in (8.7). As

$$\frac{d\mathbb{R}(x,0)}{d\widetilde{\mathbb{R}}(x,0)} = \frac{A(\widehat{X}_T(x,0))R_0(x,0)}{x} = \frac{\widehat{X}_T(x,0)R_0(x,0)}{R_T(x,0)x},$$

using Lemma 8.2, we can reformulate (8.6) as

$$\begin{aligned} a_{\{\delta,\delta\}} &= \inf_{M\in\mathcal{M}^2(x,0)} \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)}\left[\left(M_T\frac{\widehat{X}_T(x,0)}{R_T(x,0)} + xF\Big(A\big(\widehat{X}_T(x,0)\big)-1\Big)\right)^2\right] \\ &\qquad \times \frac{R_0(x,0)}{x} + C_a \\ &= \frac{R_0(x,0)}{x}\inf_{\widetilde{M}\in\widetilde{\mathcal{M}}^2(x,0)} \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)}\left[\left(\widetilde{M}_T + xF\Big(A\big(\widehat{X}_T(x,0)\big)-1\Big)\right)^2\right] + C_a. \end{aligned} \tag{8.9}$$

Likewise, completing the square in (2.15), we obtain

$$\begin{aligned} b_{\{\delta,\delta\}} &= \inf_{N\in\mathcal{N}^2(y,0)} \mathbb{E}^{\mathbb{R}(y,0)}\left[B\big(\widehat{Y}_T(y,0)\big)\left(N_T + yF\frac{1-B(\widehat{Y}_T(y,0))}{B(\widehat{Y}_T(y,0)}\right)^2\right] + C_b \\ &= \frac{R_0(x,0)}{x}\inf_{N\in\mathcal{N}^2(y,0)} \mathbb{E}^{\widetilde{\mathbb{R}}(y,0)}\left[\left(N_T + yF\Big(A\big(\widehat{X}_T(x,0)\big)-1\Big)\right)^2\right] + C_b, \end{aligned} \tag{8.10}$$

where C_b is defined in (8.8). Now the decomposition (8.5) (where the constant P_0 is still to be determined) results from (8.9), (8.10) and optimality of $M^1(x,0)$ and $N^1(y,0)$ for (2.14) and (2.15), respectively. As $A(\widehat{X}_T(x,0)) = \frac{\widehat{X}_T(x,0)}{R_T(x,0)}$, taking the expectation in (2.24) under $\widetilde{\mathbb{R}}(x,0)$, we deduce that $P_0 = a_{\{x,\delta\}}$. Therefore, using (2.22), we deduce that $b_{\{y,\delta\}} = \frac{y}{x}\frac{P_0}{a_{\{x,x\}}}$. □

Remark 8.4 Applying Itô's formula, one can find expressions for the corrections of the optimal proportions of the total wealth invested in the stock in terms of the Kunita–Watanabe decomposition when using the risk-tolerance wealth process as numéraire, in the spirit of (3.3) in Theorem 3.1. However, in the general case when we have $\frac{R(x,0)}{R_0(x,0)} = X'(x,0) \neq \widehat{X}(x,0)/x$, such a correction to the proportions also contains the terms $\widehat{X}(x,0)/R(x,0)$ and $\widetilde{M}^1(x,0)$.

Example 8.5 We suppose that B and W are independent Brownian motions on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$, where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by B and W (made right-continuous and complete). Let us assume that the family of returns is given by

$$dJ_t^\delta = (\mu + \delta\nu_t)dt + \sigma dB_t, \qquad \delta \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constant and the perturbation process ν is predictable and such that (2.2) and Assumption 2.6 hold. In other words, the base model $\delta = 0$ has constant volatility and market prices of risk and is driven by one Brownian motion B, but the perturbed model is genuinely incomplete, as the market price of risk depends non-trivially on another source of randomness W.

We additionally suppose that the agent is trying to maximise his/her expected utility, where U satisfies Assumption 2.2. Fix $x > 0$. Using duality, one can see that the solution to the base model does not take into account the information provided by the additional Brownian motion W. More precisely, although the filtration sees W, all the optimal dual elements are given by $\widehat{Y}(y,0) = yY$, $y > 0$, for

$$dY_t = -\frac{\mu}{\sigma} Y_t dB_t, \qquad Y_0 = 1.$$

The reader may note that this is the density of the minimal martingale measure introduced in [6] for the base model considered with the large filtration generated by B and W.

First, the dual value function is

$$v(y,0) = \int_{\mathbb{R}} V\left(y \exp\left(-\frac{\mu}{\sigma}\sqrt{T}z - \frac{1}{2}\frac{\mu^2}{\sigma^2}T\right)\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz, \qquad y > 0,$$

and $u(\cdot, 0)$ can be represented in a similar manner.

As the optimal investment strategy and the risk-tolerance wealth process depend exclusively on J^0 (or B), we obtain their corresponding representations below. It appears more convenient to express $\widehat{X}(x,0)$ and $R(x,0)$ via replication of claims of the type $f(J_T^0)$, for an appropriate function f. The reader may note that Y_T is given by a deterministic function of J_T^0, so one can use the Black–Scholes pricing and hedging argument, rather than the dynamic programming approach relying on the feedback representation.

Using the replication approach, the optimal investment strategy and the risk-tolerance wealth process be represented as follows. With

$$K(z,t) := \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left(-\frac{z^2}{2\sigma^2 t}\right),$$
$$\bar{g}(z) := -V'\left(u_x(x,0)\exp\left(\frac{\mu(\mu T - 2z)}{2\sigma^2}\right)\right), \qquad z \in \mathbb{R},$$

the optimal proportion for $\delta = 0$ is given by

$$\widehat{\pi}_t(x,0) = \frac{\int_{\mathbb{R}} K_x(J_t^0 - y, T - t)\bar{g}(y)dy}{\widehat{X}_t(x,0)} 1_{[0,T)}(t), \qquad t \in [0,T],$$

where K_x denotes the derivative of K with respect to the first argument. Moreover, for $x > 0$, the risk-tolerance wealth process exists and evolves as

$$dR_t(x,0) = \left(\int_{\mathbb{R}} K_x(J_t^0 - y, T-t)\bar{h}(y)dy \right) dJ_t^0,$$

where

$$\bar{h}(z) := u_x(x,0)\exp\left(\frac{\mu(\mu T - 2z)}{2\sigma^2}\right) V''\left(u_x(x,0)\exp\left(\frac{\mu(\mu T - 2z)}{2\sigma^2}\right)\right), \quad z \in \mathbb{R}.$$

For the characterisation of the corrections to the optimal trading strategies below, it is convenient to represent $R(x,0)$ in terms of proportions, i.e.,

$$dR_t(x,0) = R_t(x,0)\rho_t dJ_t^0,$$

where the initial condition is given via the assertion of Theorem 8.1 and the proportion ρ invested in the stock is

$$\rho_t = \frac{\int_{\mathbb{R}} K_x(J_t^0 - y, T-t)\bar{h}(y)dy}{R_t(x,0)} 1_{[0,T)}(t), \qquad t \in [0,T].$$

Keeping the original numéraire, we switch to the minimal martingale measure $\widehat{\mathbb{Q}}$ given by $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = Y_T$. Under this measure, $\frac{1}{\sigma}J^0$ and W are independent Brownian motions. Consider the $\widehat{\mathbb{Q}}$-martingale L defined as

$$\bar{L}_t := \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(\widehat{X}_T(x,0) - R_T(x,0)\right)x(\nu \cdot J^0)_T \middle| \mathcal{F}_t\right], \qquad t \in [0,T].$$

It can be decomposed as

$$\bar{L} = \bar{L}_0 + \bar{\alpha} \cdot J^0 + \bar{\beta} \cdot W. \tag{8.11}$$

This is the Föllmer–Schweizer decomposition of $\bar{L}_T$ (see Föllmer and Schweizer [6] and the discussion in Choulli et al. [3]), but it can also be thought of as the replication of $\bar{L}_T$ in the market which is fictitiously completed by a security with the return process W.

Now we change numéraire to the risk-tolerance wealth process and adjust the measure accordingly. First, the process P defined in Theorem 8.3 is the "price process" $\bar{L}$ changed to the numéraire $R(x,0)$, i.e., $P = \bar{L}/R(x,0)$. Second, let us set $M^{\widetilde{R}} := J^0 - \int_0^{\cdot} \rho_s ds$. We note that $\frac{1}{\sigma}M^{\widetilde{R}}$ and W are independent Brownian motions under the measure $\widetilde{\mathbb{R}}(x,0)$. The martingale $M^{\widetilde{R}}$ drives the return of the traded security under the numéraire $R(x,0)$. The decomposition (8.4) can be written as

$$P = P_0 - \varphi \cdot M^{\widetilde{R}} - \beta \cdot W \tag{8.12}$$

for some processes φ, β. We first identify $\widetilde{M}^1 = \varphi \cdot M^{\widetilde{R}}$ and then, by direct computations, show that using (8.5), we conclude that γ^1 in (3.2) equals

$$\gamma_t^1 = \frac{R_t(x,0)}{\widehat{X}_t(x,0)}\left(\widetilde{M}_t^1(\rho_t - \pi_t) + \varphi_t\right)\frac{1}{x}, \qquad t \in [0,T], \tag{8.13}$$

where φ in both (8.12) and (8.13) is given by

$$\varphi_t = P_t \rho_t - \frac{\bar{\alpha}_t}{R_t(x,0)}, \qquad t \in [0,T],$$

for $\bar{\alpha}$ coming from the representation (8.11). To make an even more explicit representation of γ^1, one needs the perturbation process ν to satisfy extra conditions. In the power utility setting with negative power, if ν is of a certain (very specific) form, the perturbations to Kim and Omberg [18] and Kraft [19] are characterised in [24, Sect. 5] (where in Example 5.2, both base and perturbed models are from Kim and Omberg [18], and in Example 5.3, the base model corresponds to Kraft [19] and the perturbed models are of the form considered in Guasoni and Robertson [10]).

In turn, using (2.25) and (8.2) and direct computations, we deduce that γ^0 from (3.2) is given by

$$\gamma^0_t = \frac{R_t(x,0)}{\widehat{X}_t(x,0)R_0(x,0)}(\rho_t - \pi_t), \qquad t \in [0,T].$$

Remark 8.6 Theorem 8.3 gives an interpretation of $a_{\{x,\delta\}}$ as a utility-based price. Let us start by observing that

$$\begin{aligned} a_{\{x,\delta\}} &= \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)}\Big[\Big(A\big(\widehat{X}_T(x,0)\big) - 1\Big)xF\Big] \\ &= \mathbb{E}\Bigg[\frac{(\widehat{X}_T(x,0) - R_T(x,0))}{R_0(x,0)}xF\frac{\widehat{Y}_T(y,0)}{y}\Bigg]. \end{aligned}$$

If there exists a wealth process X' such that

$$X'_T \geq \big|\big(\widehat{X}_T(x,0) - R_T(x,0)\big)F\big| \tag{8.14}$$

and $X'\widehat{Y}$ is a uniformly integrable martingale,[1] then according to Hugonnier and Kramkov [13] and Hugonnier et al. [14], $a_{\{x,\delta\}}$ represents the marginal utility-based price of the "random endowment"

$$\frac{(\widehat{X}_T(x,0) - R_T(x,0))}{R_0(x,0)}xF.$$

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[1] In particular, such a process X' satisfying both conditions exists if $|F| \leq C$ a.s. for some constant $C > 0$. In this case, $X' = C(R(x,0) + \widehat{X}(x,0))$ satisfies (8.14) and $X'\widehat{Y}(y,0)$ is a $\mathbb{P}$-martingale by Theorem 8.1.

References

1. Chau, H., Cosso, A., Fontana, C., Mostovyi, O.: Optimal investment with intermediate consumption under no unbounded profit with bounded risk. J. Appl. Probab. **54**, 710–719 (2017)
2. Chau, H., Rásonyi, M.: On optimal investment with processes of long or negative memory. Stoch. Process. Appl. **128**, 1095–1113 (2018)
3. Choulli, T., Vandaele, N., Vanmaele, M.: The Föllmer–Schweizer decomposition: comparison and description. Stoch. Process. Appl. **120**, 853–872 (2010)
4. Czichowsky, C., Schweizer, M.: Convex duality in mean–variance hedging under convex trading constraints. Adv. Appl. Probab. **44**, 1048–1112 (2012)
5. Delbaen, F., Schachermayer, W.: The Mathematics of Arbitrage. Springer, Berlin (2006)
6. Föllmer, H., Schweizer, M.: Hedging of contingent claims under incomplete information. In: Davis, M.H.A., Elliott, R.J. (eds.) Applied Stochastic Analysis, Stochastics Monographs, vol. 5, pp. 389–414. Gordon and Breach, London (1991)
7. Fontana, C., Schweizer, M.: Simplified mean–variance portfolio optimization. Math. Financ. Econ. **6**, 125–152 (2012)
8. Gouriéroux, C., Laurent, J.P., Pham, H.: Mean–variance hedging and numéraire. Math. Finance **8**, 179–200 (1998)
9. Guasoni, P., Robertson, S.: Portfolios and risk premia for the long run. Ann. Appl. Probab. **22**, 239–284 (2012)
10. Guasoni, P., Robertson, S.: Static fund separation of long-term investments. Math. Finance **25**, 789–826 (2015)
11. Henderson, V.: Valuation of claims on non-traded assets using utility maximization. Math. Finance **12**, 351–373 (2002)
12. Horst, U., Hu, Y., Imkeller, P., Réveillac, A., Zhang, J.: Forward–backward systems for expected utility maximization. Stoch. Process. Appl. **124**, 1813–1848 (2014)
13. Hugonnier, J., Kramkov, D.: Optimal investment with random endowment in incomplete markets. Ann. Appl. Probab. **14**, 845–864 (2004)
14. Hugonnier, J., Kramkov, D., Schachermayer, W.: On utility-based pricing of contingent claims in incomplete markets. Math. Finance **15**, 203–212 (2005)
15. Hulley, H., Schweizer, M.: On minimal market models and minimal martingale measures. In: Chiarella, C., Novikov, A. (eds.) Contemporary Quantitative Finance. Essays in Honour of Eckhard Platen, pp. 35–51. Springer, Berlin (2010)
16. Karatzas, I., Lehoczky, J.P., Shreve, S.E., Xu, G.L.: Martingale and duality methods for utility maximization in an incomplete market. SIAM J. Control Optim. **29**, 702–730 (1991)
17. Kardaras, K.: On the closure in the Emery topology of semimartingale wealth-process sets. Ann. Appl. Probab. **4**, 1355–1376 (2013)
18. Kim, T.S., Omberg, E.: Dynamic nonmyopic portfolio behavior. Rev. Financ. Stud. **9**, 141–161 (1996)
19. Kraft, H.: Optimal portfolios and Heston's stochastic volatility model: an explicit solution for power utility. Quant. Finance **5**, 303–313 (2005)
20. Kramkov, D., Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Probab. **9**, 904–950 (1999)
21. Kramkov, D., Schachermayer, W.: Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab. **13**, 1504–1516 (2003)
22. Kramkov, D., Sîrbu, M.: On the two-times differentiability of the value functions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab. **16**, 1352–1384 (2006)
23. Kramkov, D., Sîrbu, M.: Sensitivity analysis of utility-based prices and risk-tolerance wealth process. Ann. Appl. Probab. **16**, 2140–2194 (2006)
24. Larsen, K., Mostovyi, O., Žitković, G.: An expansion in the model space in the context of utility maximization. Finance Stoch. **22**, 297–326 (2018)
25. Lewis, A., Sendov, H.: Quadratic expansions of spectral functions. Linear Algebra Appl. **340**, 97–121 (2002)
26. Liu, J.: Portfolio selection in stochastic environments. Rev. Financ. Stud. **20**, 1–39 (2007)
27. Merton, R.C.: Lifetime portfolio selection under uncertainty: the continuous-time case. Rev. Econ. Stat. **51**, 247–257 (1969)
28. Merton, R.C.: Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory **3**, 373–413 (1971)

29. Mostovyi, O.: Necessary and sufficient conditions in the problem of optimal investment with intermediate consumption. Finance Stoch. **19**, 135–159 (2015)
30. Santacroce, M., Trivellato, B.: Forward backward semimartingale systems for utility maximization. SIAM J. Control Optim. **52**, 3517–3537 (2014)
31. Schweizer, M.: Mean-variance hedging for general claims. Ann. Appl. Probab. **2**, 171–179 (1992)
32. Veraguas, J.B., Silva, F.: Sensitivity analysis for expected utility maximization in incomplete Brownian market models. Math. Financ. Econ. **12**, 387–411 (2018)
33. Zariphopoulou, T.: Optimal investment and consumption models with non-linear stock dynamics. Math. Methods Oper. Res. **50**, 271–296 (1999)