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UTILITY MAXIMIZATION IN A LARGE MARKET

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We study the problem of expected utility maximization in a large market, i.e., a market with countably many traded assets. Assuming that agents have von Neumann–Morgenstern preferences with stochastic utility function and that consumption occurs according to a stochastic clock, we obtain the "usual" conclusions of the utility maximization theory. We also give a characterization of the value function in a large market in terms of a sequence of value functions in finite-dimensional models.

KEY WORDS: utility maximization, large markets, incomplete markets, convex duality, optimal investment, stochastic clock.

1. INTRODUCTION

In the mathematical finance literature, the notion of large security market was introduced by Kabanov and Kramkov (1994) as a sequence of probability spaces with corresponding time horizons and semimartingales representing the traded assets.

Investigation of the no-arbitrage conditions in large market settings has naturally attracted the attention of the research community and is done in Klein and Schachermayer (1996a,b), Kabanov and Kramkov (1998), and Klein (2000, 2003, 2006), whereas questions related to completeness are considered in Björk et al. (1997a), Björk, Kabanov, and Runggaldier (1997b), De Donno (2004), De Donno and Pratelli (2004), and Taflin (2005).

In contrast to Kabanov and Kramkov (1994, 1998), Björk and Näslund (1998) assumed that a large market consists of one probability space, but the number of traded assets is countable, and among other contributions developed the arbitrage pricing theory results in such settings. Note that the models with countably many assets embrace the ones with the stochastic dimension of the stock price process (considered, e.g., in Strong 2014). De Donno et al. (2005) extended the formulation in Björk and Näslund (1998) to a model driven by a sequence of semimartingales and established the standard conclusions of the theory for the utility maximization from terminal wealth as well as obtained the dual characterization of the superreplicable claims. Their results are based on the notion of a stochastic integral with respect to a sequence of semimartingales from De Donno and Pratelli (2006). The Merton portfolio problem in the settings with infinitely many traded zero-coupon bonds is investigated in Ekeland and Taflin (2005), and Ringer and Tehranchi (2006). Other applications of large market models in the analysis of fixed

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income securities are considered in Björk et al. (1997a,b), Carmona and Tehranchi (2004, 2006), De Donno and Pratelli (2004), and Taflin (2005).

We consider a market with countably many traded assets driven by a sequence of semimartingales (as in De Donno, Guasoni, and Pratelli, 2005). In such settings, we formulate Merton's portfolio problem for a rational economic agent whose preferences are specified via a stochastic utility of Inada's type defined on the positive real line and whose consumption follows a stochastic clock. We establish the standard existence and uniqueness results for the primal and dual optimization problems under the condition of finiteness of both primal and dual value functions.

We also characterize the primal and dual value functions in terms of the appropriate limits of the sequences of the value functions in the finite-dimensional models.

In particular, we extend the utility maximization results in De Donno et al. (2005) by adding intermediate consumption and assuming randomness of the agent's preferences.

The proof of our results hinges on the dual characterization of the admissible consumption processes given in Proposition 3.1, which allows to link the present model with the abstract theorems of Mostovyi (2015). Note that our formulation of admissible consumption and trading strategies relies on the notion of stochastic integral with respect to a sequence of semimartingales in the sense of De Donno and Pratelli (2006).

We believe that our results provide a convenient set of conditions for analyzing other problems in the settings of the large markets with or without intermediate consumption, such as robust utility maximization, optimal investment with random endowment, utility-based pricing, and existence of equilibria.

The remainder of the paper is organized as follows. Section 2 contains the model formulation and the main results, which are formulated in Theorem 2.2 and Lemma 2.4. Their proofs are given in Section 3.

2. THE MODEL AND THE MAIN RESULT

We consider a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$, where the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ satisfies the usual conditions, $\mathscr{F}_0$ is the completion of the trivial σ-algebra. As in Björk and Näslund (1998) and De Donno et al. (2005), we assume that there is one fixed market which consists of a riskless bond and a sequence of semimartingales $S = (S^n)_{n\geq 1} = ((S^i_t)_{t\in[0,T]})_{i=1}^{\infty}$ that describes the evolution of the stocks. The price of the bond is supposed to be equal to 1 at all times.

The notion of a strategy in the large market relies on the finite-dimensional counterparts, whose definitions we specify first. For $n \in \mathbb{N}$, an *n-elementary strategy* is an $\mathbb{R}^n$-valued, predictable process, which is integrable with respect to $(S^i)_{i\leq n}$. An *elementary strategy* is a strategy which is n-elementary for some n. For $x \geq 0$, an n-elementary strategy H is *x-admissible* if $H \cdot S = \sum_{i\leq n} H^i \cdot S^i$ is uniformly bounded from below by the constant $-x$ $\mathbb{P}$-a.s. Let $\mathscr{H}^n$ denote the set of n-elementary strategies that are also x-admissible for some $x \geq 0$.

In the present setting, specification of the admissible wealth processes and trading strategies is based on integration with respect to a sequence of semimartingales in the sense of De Donno and Pratelli (2006). Thus, we recall several definitions from De Donno and Pratelli (2006), upon which the formulation of the set of admissible consumptions is based. The reader familiar with this construction might proceed to the definition of an x-admissible generalized strategy. Recall that $\mathbb{R}^{\mathbb{N}}$ is the space of all real sequences.

An *unbounded functional* on $\mathbb{R}^{\mathbb{N}}$ is a linear functional F, whose domain $\mathrm{Dom}(F)$ is a subspace of $\mathbb{R}^{\mathbb{N}}$. A *simple integrand* is a finite sum of bounded predictable processes of the form $\sum_{i\leq n} h^i e^i$, where (e^i) is the canonical basis for $\mathbb{R}^{\mathbb{N}}$ and h^i are one-dimensional bounded and predictable processes.

A process H with values in the set of unbounded functionals on $\mathbb{R}^{\mathbb{N}}$ is *predictable* if there exists a sequence of simple integrands (H^n), such that $H = \lim_{n\to\infty} H^n \mathbb{P} - a.s.$, which means that $x \in \mathrm{Dom}(H)$ if the sequence (H^n) converges and $\lim_{n\to\infty} H^n(x) = H(x)$.

A predictable process H with values in the set of unbounded functionals on $\mathbb{R}^{\mathbb{N}}$ is *integrable* with respect to S if there exists a sequence (H^n) of simple integrands, such that (H^n) converges to H and the sequence of semimartingales $(H^n \cdot S)$ converges to a semimartingale Y in the semimartingale topology. In this case, we define the stochastic integral $H \cdot S$ to be Y.

For every $x \geq 0$, a process H is an *x-admissible generalized strategy* if H is integrable with respect to the semimartingale S and there exists an approximating sequence (H^n) of x-admissible elementary strategies, such that $(H^n \cdot S)$ converges to $H \cdot S$ in the semimartingale topology. Note that this is definition 2.5 from De Donno et al. (2005).

Let us define a *portfolio* Π as a triple (x, H, c), where the constant x is an initial value, H is a predictable and admissible S-integrable process (with values in the set of unbounded functionals on $\mathbb{R}^{\mathbb{N}}$) specifying the amount of each asset held in the portfolio, and $c = (c_t)_{t\in[0,T]}$ is a nonnegative and optional process that specifies the consumption rate in units of the bond.

Hereafter, we fix a *stochastic clock* $\kappa = (\kappa_t)_{t\in[0,T]}$, which is a nondecreasing, càdlàg, adapted process such that

$$\kappa_0 = 0, \mathbb{P}[\kappa_T > 0] > 0, \text{ and } \kappa_T \leq A \tag{2.1}$$

for some finite constant A. The stochastic clock represents the notion of time according to which consumption occurs. Note that, in view of the utility maximization problem (2.3) defined, we will only consider consumption processes that are absolutely continuous with respect to $d\kappa$, i.e., of the form $c \cdot \kappa$, as other ones are suboptimal.

For a portfolio (x, H, c), we define the *wealth process* as

$$X \triangleq x + H \cdot S - c \cdot \kappa.$$

Note that the closure of the sets of wealth processes in the semimartingale topology is investigated in De Donno et al. (2005) and Kardaras (2013) (with the definition of a wealth process being different from the one here). For $x \geq 0$, we define the set of x-admissible consumptions as

$$\begin{aligned}\mathscr{A}(x) \triangleq \{c \geq 0 : &\ c \text{ is optional, and there exists}\\ &\ \text{an } x\text{-admissible generalized strategy } H,\\ &\ s.t.\ x + H \cdot S - c \cdot \kappa \geq 0\}.\end{aligned}$$

Thus, a constant strictly positive consumption $c^*_t \triangleq x/A,\ t \in [0, T]$, belongs to $\mathscr{A}(x)$ for every $x > 0$.

For $n \geq 1$, let $\mathscr{Z}^n$ denote the set of càdlàg densities of equivalent local martingale measure for n-elementary strategies, i.e.,

$$\begin{aligned}\mathscr{Z}^n \triangleq \big\{Z > 0 : &\ Z \text{ is a càdlàg martingale, s.t. } Z_0 = 1 \text{and}\\ &\ (1 + H \cdot S)Z \text{ is a local martingale for every } H \in \mathscr{H}^n,\\ &\ H\, is\ 1 - admissible\big\}.\end{aligned}$$

Note that $\mathscr{Z}^{n+1} \subseteq \mathscr{Z}^n$, $n \geq 1$. We also define

$$\mathscr{Z} \triangleq \bigcap_{n \geq 1} \mathscr{Z}^n,$$

and assume that

$$\mathscr{Z} \neq \emptyset, \tag{2.2}$$

which coincides with the no-arbitrage condition in De Donno et al. (2005).

The preferences of an economic agent are modeled via a stochastic utility $U : [0, T] \times \Omega \times [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ that satisfies the conditions below.

ASSUMPTION 2.1. *For every* $(t, \omega) \in [0, T] \times \Omega$ *the function* $x \to U(t, \omega, x)$ *is strictly concave, increasing, continuously differentiable on* $(0, \infty)$ *and satisfies the Inada conditions:*

$$\lim_{x \downarrow 0} U'(t, \omega, x) = +\infty \quad \text{and} \quad \lim_{x \to \infty} U'(t, \omega, x) = 0,$$

where U' *denotes the partial derivative with respect to the third argument. At* $x = 0$ *we suppose, by continuity,* $U(t, \omega, 0) = \lim_{x \downarrow 0} U(t, \omega, x)$*, this value may be* $-\infty$*. For every* $x \geq 0$*, the stochastic process* $U(\cdot, \cdot, x)$ *is optional.*

The conditions on U coincide with the ones in Mostovyi (2015) (on the finite time horizon). For simplicity of notations for a nonnegative optional process c, the processes with trajectories $(U(t, \omega, c_t(\omega)))_{t \in [0,T]}$, $(U'(t, \omega, c_t(\omega)))_{t \in [0,T]}$, and $(U^-(t, \omega, c_t(\omega)))_{t \in [0,T]}$ (where U^- designates the negative part of U) will be denoted by $U(c)$, $U'(c)$, and $U^-(c)$, respectively.

For a given initial capital $x > 0$, the goal of the agent is to maximize his expected utility. The value function of this problem is denoted by

$$u(x) \triangleq \sup_{c \in \mathscr{A}(x)} \mathbb{E}\left[U(c) \cdot \kappa_T\right], \quad x > 0. \tag{2.3}$$

We use the convention

$$\mathbb{E}\left[U(c) \cdot \kappa_T\right] \triangleq -\infty \quad \text{if} \quad \mathbb{E}\left[U^-(c) \cdot \kappa_T\right] = +\infty. \tag{2.4}$$

To study (2.3), we employ standard duality arguments as in Kramkov and Schachermayer (1999) and Žitković (2005) and define the *conjugate stochastic field* V to U as

$$V(t, \omega, y) \triangleq \sup_{x > 0} \left(U(t, \omega, x) - xy\right), \quad (t, \omega, y) \in [0, T] \times \Omega \times [0, \infty).$$

It is well known that $-V$ satisfies Assumption 2.1. For $y \geq 0$, we also denote

$$\mathscr{Y}(y) \triangleq \mathrm{cl}\left\{Y : Y \text{ is càdlàg adapted and } 0 \leq Y \leq yZ \ (d\kappa \times \mathbb{P}) \text{ a.e. for some } Z \in \mathscr{Z}\right\},$$

where the closure is taken in the topology of convergence in measure $(d\kappa \times \mathbb{P})$ on the space of finite-valued optional processes. We will denote this space $\mathbb{L}^0(d\kappa \times \mathbb{P})$ or $\mathbb{L}^0$ for brevity.

Similar to the composition of U with c, for a nonnegative optional process Y, the stochastic processes, whose realizations are $(V(t, \omega, Y_t(\omega)))_{t \in [0,T]}$ and $(V^+(t, \omega, Y_t(\omega)))_{t \in [0,T]}$ (where V^+ is the positive part of V), will be denoted by $V(Y)$

and $V^+(Y)$, respectively. After these preparations, we define the value function of the dual optimization problem as

$$v(y) \triangleq \inf_{Y \in \mathscr{Y}(y)} \mathbb{E}\left[V(Y) \cdot \kappa_T\right], \quad y > 0, \tag{2.5}$$

where we use the convention:

$$\mathbb{E}\left[V(Y) \cdot \kappa_T\right] \triangleq +\infty \quad \text{if} \quad \mathbb{E}\left[V^+(Y) \cdot \kappa_T\right] = +\infty. \tag{2.6}$$

The following theorem constitutes the main contribution of this paper.

THEOREM 2.2. *Assume that conditions* (2.1) *and* (2.2) *and Assumption 2.1 hold true and suppose*

$$v(y) < \infty \ \ \textit{for all } y > 0 \quad \textit{and} \quad u(x) > -\infty \ \ \textit{for all } x > 0.$$

Then we have

(1) $u(x) < \infty$ *for all* $x > 0$, $v(y) > -\infty$ *for all* $y > 0$. *The functions* u *and* v *are conjugate, i.e.,*

$$v(y) = \sup_{x>0} \left(u(x) - xy\right), \quad y > 0,$$
$$u(x) = \inf_{y>0} \left(v(y) + xy\right), \quad x > 0.$$

The functions u *and* $-v$ *are continuously differentiable on* $(0, \infty)$, *strictly increasing, strictly concave and satisfy the Inada conditions:*

$$u'(0) \triangleq \lim_{x \downarrow 0} u'(x) = +\infty, \quad -v'(0) \triangleq \lim_{y \downarrow 0} -v'(y) = +\infty,$$
$$u'(\infty) \triangleq \lim_{x \to \infty} u'(x) = 0, \quad -v'(\infty) \triangleq \lim_{y \to \infty} -v'(y) = 0.$$

(2) *For every* $x > 0$ *and* $y > 0$, *the optimal solutions* $\hat{c}(x)$ *to* (2.3) *and* $\hat{Y}(y)$ *to* (2.5) *exist and are unique. Moreover, if* $y = u'(x)$ *we have the dual relations*

$$\hat{Y}(y) = U'(\hat{c}(x)), \quad (d\kappa \times \mathbb{P})\ a.e.$$

and

$$\mathbb{E}\left[\left((\hat{c}(x)\hat{Y}(y)) \cdot \kappa\right)_T\right] = xy.$$

(3) *We have,*

$$v(y) = \inf_{Z \in \mathscr{Z}} \mathbb{E}\left[V(yZ) \cdot \kappa_T\right], \ y > 0.$$

2.1. Large Market as a Limit of a Sequence of Finite-Dimensional Markets

Motivated by the question of liquidity, we discuss the convergence of the value functions as the number of available traded securities increases. For this purpose, we need the following definitions. For every $n \geq 1$, we set

$$\mathscr{A}^n(x) \triangleq \{optional\ c \geq 0 :\ there\ exists\ H \in \mathscr{H}^n\ s.t.\ x + H \cdot S_T - c \cdot \kappa_T \geq 0\ \mathbb{P}\text{-a.s.}\},$$

$$u^n(x) \triangleq \sup_{c \in \mathscr{A}^n(x)} \mathbb{E}[U(c) \cdot \kappa_T], \quad x > 0, \tag{2.7}$$

$$\mathscr{Y}^n(y) \triangleq \mathrm{cl}\{Y : Y \text{ is càdlàg adapted and } 0 \leq Y \leq yZ\ (d\kappa \times \mathbb{P}) \text{ a.e. for some } Z \in \mathscr{Z}^n\},$$

where the closure is taken in $\mathbb{L}^0$,

$$v^n(y) \triangleq \inf_{Y \in \mathscr{Y}^n(y)} \mathbb{E}[V(Y) \cdot \kappa_T], \quad y > 0, \tag{2.8}$$

and assume the conventions (2.4) and (2.6). Note that for every $z > 0$, both $(u^n(z))$ and $(v^n(z))$ are increasing sequences. We suppose that

$$\mathscr{A}(1-\varepsilon) \subset \mathrm{cl}\left(\bigcup_{n \geq 1} \mathscr{A}^n(1)\right) \quad \text{for every} \quad \varepsilon \in (0, 1], \tag{2.9}$$

where the closure is taken in $\mathbb{L}^0$.

Let 1_E denotes the indicator function of a set E.

REMARK 2.3. It follows from Proposition 3.1 and Fatou's lemma that $\mathrm{cl}(\bigcup_{n\geq 1} \mathscr{A}^n(1)) \subseteq \mathscr{A}(1)$. Assumption (2.9) gives a weaker version of the reverse inclusion. Note that (2.9) holds if either of the conditions below is valid.

(1) $\kappa_t = 1_T(t)$, $t \in [0, T]$, i.e., if (2.3) defines the problem of optimal investment from terminal wealth. Then, (2.9) follows from lemma 3.4 in De Donno et al. (2005), which also contains the proof of $u^n(x) \to u(x)$.
(2) The process S is (componentwise) continuous. This is the subject of Lemma 3.7.

LEMMA 2.4. *Assume that there exists $n \in \mathbb{N}$, such that*

$$u^n(x) > -\infty \quad \textit{for every} \quad x > 0, \quad v(y) < +\infty \quad \textit{for every} \quad y > 0. \tag{2.10}$$

Then, under conditions (2.1), (2.2), *and* (2.9) *as well as Assumption 2.1, we have*

$$u(x) = \lim_{n\to\infty} u^n(x), \quad x > 0, \quad \textit{and} \quad v(y) = \lim_{n\to\infty} v^n(y), \quad y > 0. \tag{2.11}$$

REMARK 2.5. Condition (2.10) implies finiteness of v, $-u$, v^n, and $-u^n$ (for every n greater than some constant) that are also convex. Theorem 3.1.4 in Hiriart-Urrut and Lemaréchal (2004) ensures that convergence in (2.11) is uniform on compact subsets of $(0, \infty)$. Moreover, theorem 25.7 in Rockafellar (1970) asserts that the derivatives $(v^n)'$ and $(u^n)'$, $n \geq 1$, also converge uniformly on compact intervals in $(0, \infty)$ to v' and u', respectively.

Lemma 2.4 shows that the value function in the market with countably many assets is the limit of the value functions of the finite dimensional models. The following example demonstrates that in general the optimal portfolio and the optimal consumption in the market with infinitely many traded assets are not limits of the optimal portfolios and optimal consumptions in the finite dimensional markets, respectively. The important technical feature in the construction of this example is that in each finite dimensional market *the last stock has the highest expected return*.

EXAMPLE 2.6. Let us consider an economic agent, whose preferences are specified by a bounded utility function U defined on the positive real line that is strictly increasing, strictly concave, continuously differentiable, and satisfies the Inada conditions. We consider a one-period model, where there is a riskless bond with $S^0 \equiv 1$, and a sequence of stocks (S^i), such that $S^i_0 = 1$ for every i and (S^i_1) are independent random variables taking values in $\{\frac{1}{2}, s_i\}$, where

$$s_1 = 9, \quad s_{i+1} = (i+2)(2s_i + 1), \quad i \geq 1, \tag{2.12}$$

(we will also denote $s_0 = 1$) with probabilities $1 - p_i$ and p_i, respectively, where (p_i) is an *increasing* sequence such that

$$\frac{1-p_i}{p_i} \leq \min\left(2(s_i - 1)\frac{U'(s_i + s_{i-1} + 1)}{U'\left(\frac{1}{2}\right)}, 15\right), \quad i \geq 1. \tag{2.13}$$

As (p_i) and (s_i) are increasing, we have

$$\max_{k \in \{1,\dots,n\}} \mathbb{E}\left[S^k_1\right] = \mathbb{E}\left[S^n_1\right], \quad n \geq 1,$$

i.e., the last stock of each finite dimensional market has the greatest expected return. Let the stochastic clock κ correspond to the problem of utility maximization of terminal wealth. Note that (2.2) holds, (2.9) is valid by the first item of Remark 2.3, and boundedness of U implies (2.10). Therefore, the assertions of Lemma 2.4 hold. Condition (2.13) results in

$$U(1) = \mathbb{E}\left[U(S^0_1)\right] < \mathbb{E}\left[U(S^1_1)\right]. \tag{2.14}$$

For simplicity of notation, we will assume that the initial wealth of the agent equals to 1. Let h^N_i be the *optimal* number of shares of the ith asset in the market, where N stocks are available for trading, $N \geq 1$. The admissibility condition implies that $h^N_0 \geq -1$. Monotonicity of (s_i) and (p_i) results in the following inequalities:

$$h^N_1 \leq h^N_2 \leq \dots \leq h^N_N, \quad N \geq 1. \tag{2.15}$$

It follows from convexity and monotonicity of U as well as (2.13) that $h^N_i \geq 0$ (if, by contradiction, $h^N_i < 0$, via (2.13) one can show that the portfolio with 0 units of ith stock and $h^N_0 + h^N_i$ units of the riskless asset is admissible, it corresponds to the same initial wealth and gives a higher value of the expected utility). Nonnegativity of h^N_i's and (2.15) give

$$h^N_i \leq \frac{2}{N - i + 1}, \quad i = 1, \dots, N, \quad N \geq 1.$$

Consequently, we get

$$\lim_{N\to\infty} h^N_i = 0, \quad i \geq 1. \tag{2.16}$$

Note that the corresponding wealth processes at time 0 equal $1 = \sum_{i=0}^{N} h_i^N$. Consequently, in the market with countably many stocks, a portfolio that is a limit of the optimal finite dimensional portfolios (i.e., such that (2.16) holds) can have a nontrivial allocation only in the riskless asset. This gives the value of the expected utility $U(1)$. In view of (2.14), such a portfolio is suboptimal.

Moreover, we show below that the sequence of optimizers corresponding to the initial wealth 1, $(\widehat{X}^N)$, *is not a Cauchy sequence in the semimartingale topology*. It follows from (2.13) that $h_N^N \geq h_0^N$ (if by contradiction $h_0^N > h_N^N$, an application of (2.13) implies that the portfolio with h_N^N units of the riskless bond and h_0^N units of the Nth stock gives a higher expected utility). Accordingly, in view of (2.15) and nonnegativity of h_i^N, $i = 1, \dots, N$, we get

$$h_N^N \geq \frac{1}{N+1}, \qquad N \geq 1.$$

Combining this with (2.12), we obtain

$$\widehat{X}_1^N \geq h_N^N s_N + h_0^N \geq 2s_{N-1} \quad \text{with probability } p_N,$$

whereas $\widehat{X}_1^{N-1} \leq 2s^{N-1} - 1$. Consequently,

$$\mathbb{P}\left[|\widehat{X}_1^N - \widehat{X}_1^{N-1}| \geq 1\right] \geq p_N, \qquad N \geq 2.$$

Therefore, $(\widehat{X}^N)$ is not a Cauchy sequence in the semimartingale topology.

3. PROOFS

At the core of the proof of Theorem 2.2 lies the following result.

PROPOSITION 3.1. *Let conditions* (2.1) *and* (2.2) *hold. Then a nonnegative optional process c belongs to* $\mathscr{A}(1)$ *if and only if*

$$\sup_{Z \in \mathscr{Z}} \mathbb{E}\left[((cZ) \cdot \kappa)_T\right] \leq 1. \tag{3.1}$$

The proof of Proposition 3.1 will be given via several lemmas.

LEMMA 3.2. *Let H be a 1-admissible generalized integrand. Under the conditions of Proposition 3.1, $X \triangleq 1 + H \cdot S$ is nonnegative $\mathbb{P}$-a.s. and for every $Z \in \mathscr{Z}$, ZX is a supermartingale.*

As the proof of Lemma 3.2 is straightforward, it is skipped. Note that a discussion of the second assertion of the lemma is presented on p. 2011 of De Donno et al. (2005).

LEMMA 3.3. *Let H be a* 1*-admissible generalized strategy, c be a nonnegative optional process. Under the conditions of Proposition 3.1, the following statements are equivalent:*

(i)
$$c \cdot \kappa_T \leq 1 + H \cdot S_T, \quad \mathbb{P}\text{-}a.s.,$$

(ii)
$$c \cdot \kappa \leq 1 + H \cdot S, \quad \mathbb{P}\text{-}a.s.$$
$$(i.e., \quad c \cdot \kappa_t \leq 1 + H \cdot S_t \quad \text{for every } t \in [0, T], \quad \mathbb{P}\text{-}a.s.).$$

Proof. Let us assume that (i) holds and fix $Z \in \mathscr{Z}$. It follows from Lemma 3.2 that $Z(1 + H \cdot S)$ is a supermartingale. Therefore, using monotonicity of $c \cdot \kappa$, for every $t \leq T$ we have

$$Z_t(c \cdot \kappa_t) = \mathbb{E}\left[Z_T(c \cdot \kappa_t)|\mathscr{F}_t\right] \leq \mathbb{E}\left[Z_T(c \cdot \kappa_T)|\mathscr{F}_t\right] \\ \leq \mathbb{E}\left[Z_T(1 + H \cdot S_T)|\mathscr{F}_t\right] \leq Z_t(1 + H \cdot S_t),$$

which implies (ii). □

Proof of Proposition 3.1. Let $c \in \mathscr{A}(1)$. Fix $Z \in \mathscr{Z}$ and $T > 0$. Then there exists a 1-admissible generalized strategy H, such that

$$1 + H \cdot S_T \geq c \cdot \kappa_T.$$

Multiplying both sides by Z and taking the expectation, we get

$$\mathbb{E}\left[Z_T(1 + H \cdot S_T)\right] \geq \mathbb{E}\left[Z_T(c \cdot \kappa_T)\right], \tag{3.2}$$

where the right-hand side (via monotonicity of $c \cdot \kappa$ and an application of theorem I.4.49 in Jacod and Shiryaev, 1980) can be rewritten as

$$\mathbb{E}\left[Z_T(c \cdot \kappa_T)\right] = \mathbb{E}\left[((Zc) \cdot \kappa)_T\right]. \tag{3.3}$$

By definition of H, there exists a sequence (H^n) of 1-admissible elementary strategies, such that

$$(H^n \cdot S)_{n \geq 1} \quad \text{converges to} \quad H \cdot S \quad \text{in the semimartingale topology.}$$

Consequently, $(H^n \cdot S_T)$ converges to $H \cdot S_T$ in probability, and therefore there exist a subsequence, which we still denote $(H^n \cdot S)$, such that $(H^n \cdot S_T)$ converges to $H \cdot S_T$ $\mathbb{P}$-a.s. Therefore, for every $Z \in \mathscr{Z}$ we obtain from the definition of 1-admissibility and Fatou's lemma

$$1 \geq \liminf_{n \to \infty} \mathbb{E}\left[Z_T(1 + H^n \cdot S_T)\right] \geq \mathbb{E}\left[Z_T(1 + H \cdot S_T)\right].$$

Combining this with (3.2) and (3.3), we conclude that

$$1 \geq \mathbb{E}\left[((Zc) \cdot \kappa)_T\right],$$

which holds for every $Z \in \mathscr{Z}$.

Conversely, let (3.1) holds. Using the same argument as in (3.3), we obtain from (3.1) that

$$1 \geq \sup_{Z \in \mathscr{Z}} \mathbb{E}\left[Z_T(c \cdot \kappa)_T\right].$$

Consequently, the random variable $c \cdot \kappa_T$ satisfies the assumption (i) of theorem 3.1 in De Donno et al. (2005) with $x = 1$. Therefore, we obtain from this theorem that there exists a 1-admissible generalized strategy H such that

$$c \cdot \kappa_T \leq 1 + H \cdot S_T.$$

By Lemma 3.3, this implies that $c \in \mathscr{A}(1)$. This concludes the proof of the proposition. □

Let $\mathbb{L}^0_+$ denote the positive orthant of $\mathbb{L}^0$. We recall that a subset A of $\mathbb{L}^0_+$ is called *solid* if $f \in A$, $g \in \mathbb{L}^0_+$, and $g \leq f$ implies that $g \in A$, a subset $B \subset \mathbb{L}^0_+$ is the *polar* of A, if $B = \{h \in \mathbb{L}^0_+ : \ \mathbb{E}[((hf) \cdot \kappa)_T] \leq 1, \ \text{for every } f \in A\}$, in this case we denote $B = A^o$.

LEMMA 3.4. *Under the conditions of Proposition 3.1, we have*

(i) *The sets* $\mathscr{A}(1)$ *and* $\mathscr{Y}(1)$ *are convex, solid, and closed subsets of* $\mathbb{L}^0$.
(ii) $\mathscr{A}(1)$ *and* $\mathscr{Y}(1)$ *satisfy the bipolar relations*

$$\begin{aligned} c \in \mathscr{A}(1) &\Leftrightarrow \mathbb{E}\left[((cY) \cdot \kappa)_T\right] \leq 1, \text{ for every } Y \in \mathscr{Y}(1), \\ Y \in \mathscr{Y}(1) &\Leftrightarrow \mathbb{E}\left[((cY) \cdot \kappa)_T\right] \leq 1, \text{ for every } Y \in \mathscr{A}(1). \end{aligned}$$

(iii) *Both* $\mathscr{A}(1)$ *and* $\mathscr{Y}(1)$ *contain strictly positive elements.*

The proof goes along the lines of the proof of proposition 4.4 in Mostovyi (2015). It is therefore omitted here.

LEMMA 3.5. *Under the conditions of Proposition 3.1, we have*

(i) $\sup_{Z \in \mathscr{Z}} \mathbb{E}[((cZ) \cdot \kappa)_T] = \sup_{Y \in \mathscr{Y}(1)} \mathbb{E}[((cY) \cdot \kappa)_T]$ *for every* $c \in \mathscr{A}(1)$,
(ii) *the set* $\mathscr{Z}$ *is closed under the countable convex combinations, i.e. for every sequence* (Z^m) *in* $\mathscr{Z}$ *and a sequence of positive numbers* (a^m) *such that* $\sum_{m \geq 1} a^m = 1$, *the process* $Z \triangleq \sum_{m \geq 1} a^m Z^m$ *belongs to* $\mathscr{Z}$.

Proof. Let (Z^m) be a sequence in $\mathscr{Z}$ and $Z = \sum_{m \geq 1} a^m Z^m$. By monotone convergence, Z is a càdlàg martingale. Let $X = 1 + H \cdot S$ be a nonnegative wealth process, where $H \in \mathscr{H}^n$ (for some $n \geq 1$),

$$M \triangleq XZ, \quad \text{and} \quad M^N \triangleq X\left(\sum_{m=1}^{N} a^m Z^m\right), \ N \geq 1.$$

As XZ^m is a local martingale for each m, we have that M^N (being a finite sum of local martingales) is a local martingale as well. Moreover, $M^N \geq 0$ for every N and as $a^m \geq 0$ and $XZ^m \geq 0$, for every m, we have that $M^N_t \uparrow M_t$, for each t. By proposition 5.1 in Klein, Lepinette, and Perez-Ostafe (2014), we get that M is a local martingale, and therefore $Z \in \mathscr{Z}$. This implies (ii), whereas (i) results from Fatou's lemma and the definitions of the sets $\mathscr{Z}$ and $\mathscr{Y}(1)$. □

Proof of Theorem 2.2. By Lemma 3.4, the sets $\mathscr{A}(1)$ and $\mathscr{Y}(1)$ satisfy the assumptions of theorem 3.2 in Mostovyi (2015) that implies the assertions (i) and (ii) of Theorem 2.2. The conclusions of item (iii) supervene from Lemma 3.5 and theorem 3.3 in Mostovyi (2015). This completes the proof of Theorem 2.2. □

For the proof of Lemma 2.4, we need the following technical result.

LEMMA 3.6. *Under the conditions of Lemma 2.4, for every* $\varepsilon \in (0, 1)$ *we have*

$$\bigcap_{n \geq 1} \mathscr{Y}^n(1) \subset \mathscr{Y}\left(\frac{1}{1-\varepsilon}\right).$$

Proof. Observe that by proposition 4.4 in Mostovyi (2015), for every $n \geq 1$, the sets $\mathscr{A}^n(1)$ and $\mathscr{Y}^n(1)$ satisfy the bipolar relations, likewise by Lemma 3.4, we have

$\mathscr{A}(1)^o = \mathscr{Y}(1)$. Fix an $\varepsilon \in (0, 1)$. From (2.9) using Fatou's lemma, we obtain

$$\mathscr{A}(1-\varepsilon)^o \supset \left(\bigcup_{n\geq 1} \mathscr{A}^n(1)\right)^o.$$

Therefore, we conclude

$$\mathscr{Y}\left(\frac{1}{1-\varepsilon}\right) = \mathscr{A}(1-\varepsilon)^o \supset \left(\bigcup_{n\geq 1} \mathscr{A}^n(1)\right)^o = \bigcap_{n\geq 1} \mathscr{A}^n(1)^o = \bigcap_{n\geq 1} \mathscr{Y}^n(1).$$

This concludes the proof of the lemma. □

Proof of Lemma 2.4. Without loss of generality, we will assume that $u^1(x) > -\infty$, $x > 0$. We will only show the second assertion, as the proof of the first one is similar. Also, for convenience of notation, we will assume that $y = 1$. Let Z^n be a minimizer to the dual problem (2.8), $n \geq 1$, where the existence of the solutions to (2.8) follows from theorem 2.3 in Mostovyi (2015).

It follows from (2.1) that the set $\mathscr{Z}^1$ is bounded in $\mathbb{L}^1(d\kappa \times \mathbb{P})$. This in particular implies that $\mathscr{Y}^1(1)$ is bounded in $\mathbb{L}^0(d\kappa \times \mathbb{P})$. Therefore, by lemma A1.1 in Delbaen and Schachermayer (1994), there exists a sequence $\widetilde{Z}^n \in \mathrm{conv}(Z^n, Z^{n+1}, \ldots)$, $n \geq 1$, and an element $Z \in \mathbb{L}^0(d\kappa \times \mathbb{P})$, such that $(\widetilde{Z}^n)$ converges to Z $(d\kappa \times \mathbb{P})$-a.e. We also have

$$Z = \lim_{n\to\infty} \widetilde{Z}^n \in \bigcap_{n\geq 1} \mathscr{Y}^n(1) \subset \mathscr{Y}\left(\frac{1}{1-\varepsilon}\right) \quad \text{for every} \quad \varepsilon \in (0, 1),$$

where the latter inclusion follows from Lemma 3.6. By convexity of V, we get

$$\limsup_{n\to\infty} \mathbb{E}\left[V(\widetilde{Z}^n) \cdot \kappa_T\right] \leq \lim_{n\to\infty} v^n(1). \tag{3.4}$$

Note that $(\widetilde{Z}^n) \subset \mathscr{Y}^1(1)$. Consequently, using lemma 3.5 in Mostovyi (2015), we conclude that $(V^-(\widetilde{Z}^n))$ is uniformly integrable (here V^- denotes the negative part of the stochastic field V). Therefore, from Fatou's lemma and (3.4) we deduce

$$v\left(\frac{1}{1-\varepsilon}\right) \leq \mathbb{E}[V(Z) \cdot \kappa_T] \leq \liminf_{n\to\infty} \mathbb{E}\left[V(\widetilde{Z}^n) \cdot \kappa_T\right] \leq \lim_{n\to\infty} v^n(1)$$

for every $\varepsilon \in (0, 1)$. Taking the limit as $\varepsilon \downarrow 0$ and using the continuity of v (by convexity, see Theorem 2.2), we obtain that

$$v(1) \leq \lim_{n\to\infty} v^n(1).$$

Also, since $\mathscr{Y}(1) \subseteq \mathscr{Y}^n(1)$ for every $n \geq 1$, we have

$$v(1) \geq \lim_{n\to\infty} v^n(1).$$

Thus, $v(1) = \lim_{n\to\infty} v^n(1)$. The proof of the lemma is now complete. □

LEMMA 3.7. *Let S be a continuous process (i.e., every component of S is continuous) that satisfies* (2.2). *Then, under* (2.1), (2.9) *holds.*

Proof. Fix an $\varepsilon \in (0, 1]$ and $c \in \mathscr{A}(1-\varepsilon)$. Let H be a $(1-\varepsilon)$-admissible generalized strategy, such that

$$c \cdot \kappa \leq 1 - \varepsilon + H \cdot S, \quad \mathbb{P}\text{-a.s.}$$

Let (H^n) be a sequence of $(1-\varepsilon)$-admissible elementary strategies, such that $H^n \cdot S$ converges to $H \cdot S$ in the semimartingale topology. Let us define a sequence of stopping times as

$$\tau_n \triangleq \inf \left\{ t \in [0, T] : \ c \cdot \kappa_t > 1 + H^n \cdot S_t \right\} \wedge (T+1).$$

Then, we have

$$\begin{aligned} \mathbb{P}[\tau_n < T+1] &\leq \mathbb{P}\left[\sup_{t \in [0,T]} (c \cdot \kappa_t - 1 + \varepsilon - H^n \cdot S_t) \geq \varepsilon \right] \\ &\leq \mathbb{P}\left[\sup_{t \in [0,T]} (H \cdot S_t - H^n \cdot S_t) \geq \varepsilon \right], \end{aligned}$$

which converges to 0 as $n \to \infty$. Let us define a sequence of consumptions (c^n) as follows:

$$c_t^n \triangleq c_t 1_{[0,\tau_n)}(t), \quad t \in [0, T], \quad n \geq 1.$$

Then, on $\{\tau_n \leq T\}$ we have $c^n \cdot \kappa_{\tau_n -} = c^n \cdot \kappa_{\tau_n}$. By continuity of S, we get

$$c^n \cdot \kappa \leq 1 + H^n \cdot S \quad \text{on } [0, \tau_n \wedge \mathrm{T}] \quad \mathbb{P}\text{-a.s.}, \quad n \geq 1.$$

As $H^n 1_{[0,\tau_n]}$ is a 1-admissible elementary strategy, we deduce that $c^n \in \mathscr{A}^n(1)$, $n \geq 1$. One can also see that (c^n) converges to c in $\mathbb{L}^0$.

This concludes the proof of the lemma. □

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