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OPTIMAL INVESTMENT WITH INTERMEDIATE CONSUMPTION AND RANDOM ENDOWMENT

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We consider an optimal investment problem with intermediate consumption and random endowment, in an incomplete semimartingale model of the financial market. We establish the key assertions of the utility maximization theory, assuming that both primal and dual value functions are finite in the interiors of their domains and that the random endowment at maturity can be dominated by the terminal value of a self-financing wealth process. In order to facilitate the verification of these conditions, we present alternative, but equivalent conditions, under which the conclusions of the theory hold.

KEY WORDS: utility maximization, random endowment, incomplete markets, convex duality, optimal investment, stochastic clock.

1. INTRODUCTION

Existing work. The problem of utility maximization in incomplete markets is of central importance in mathematical finance. The theory was developed, among others, by He and Pearson (1991a, b), Karatzas et al. (1991), Karatzas and Shreve (1998), Kramkov and Schachermayer (1999, 2003), Karatzas and Žitković (2003), and Žitković (2005).

In this paper, we consider the problem of an agent, who in addition to initial wealth receives a random endowment. The agent's goal is to consume and invest so as to maximize expected utility. In complete market settings, this problem is considered by Karatzas and Shreve (1998), Chapter 4. Using a replication argument, the authors were able to reduce the problem to one without endowment. As replication might not be possible in incomplete markets, alternative methods have been used. For example, Cuoco (1997) used martingale techniques to reformulate the dynamic optimization problem as an equivalent static one. In Markovian settings, one possible approach is to use a Hamilton–Jacobi–Bellman equation for the value function, see Duffie and Zariphopoulou (1993) and Duffie et al. (1997). Cvitanić, Schachermayer, and Wang (2001) considered the optimal investment problem of terminal wealth for an agent with random endowment in an incomplete semimartingale market. Using the space $(\mathbf{L}^{\infty})^*$ of finitely additive measures as the domain of the dual problem, they were able to characterize the value function and the optimal terminal wealth in terms of the solution to the dual problem.

In contrast to Cvitanić et al. (2001), Hugonnier and Kramkov (2004) treated both the initial capital and the number of shares of random endowment as the variables of the value function. Although it increased the dimensionality of the problem, such an

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approach permitted the relaxation of some technical assumptions. Stability of this utility maximization problem was investigated by Kardaras and Žitković (2011). Karatzas and Žitković (2003) as well as Žitković (2005) extended the results of Cvitanić et al. (2001) to include intermediate consumption.

Mostovyi (2015) considered the problem of optimal investment with intermediate consumption under the condition that both primal and dual value functions are finite in their domains and has shown that such conditions are both necessary and sufficient for the validity of the "key" conclusions of the theory.

Our contributions. We consider a problem of optimal investment with intermediate consumption and random endowment, in an incomplete semimartingale market with a finite time horizon. This extends the model in Mostovyi (2015) by incorporating a random endowment process (in the finite time horizon settings), and expands the framework of Hugonnier and Kramkov (2004) by adding intermediate consumption. The results, in particular, enable us to reduce the number of technical conditions on the utility in Karatzas and Žitković (2003) and Žitković (2005). In addition, our formulation avoids the use of finitely additive measures in the construction of the dual domain and (as a consequence) gives a unique minimizer to the dual problem (and not just a minimizer, which is unique up to a singular part).

Our approach is based on a formulation of the primal and dual problems that shifts the ideas of the proofs toward multidimensional convex analytic techniques. As in Hugonnier and Kramkov (2004), we consider the number of shares of random endowment to be an additional variable of the value function. Such an increase of dimensionality enables us to obtain existence and uniqueness results assuming that both primal and dual value functions are finite in the interiors of their domains. In order to facilitate the verification of this condition, following Mostovyi (2015), we present an equivalent criterion in terms of the finiteness of the value functions without the endowment. The endowment process at maturity is assumed to be dominated by the terminal value of a nonnegative self-financing wealth process (as in Hugonnier and Kramkov 2004). This condition can also be formulated in several equivalent ways, which we specify as well.

In addition to the usual conclusions of the utility maximization theory, it is possible to establish certain properties of the value functions on the boundaries of their domains, such as upper semi-continuity of the primal value function and lower semi-continuity of the dual value function. Note that, in the setting of optimal investment from terminal wealth, semi-continuity and boundary behavior were studied in Siorpaes (2016).

We would like to stress that the motivations behind this work are the extension of the utility-based pricing theory (see, e.g., Kramkov and Sîrbu 2006) to the stochastic utility framework and the investigation of existence and uniqueness of equilibria in incomplete continuous-time markets. Note that random utilities naturally arise in optimal investment problems under standard transformations, such as changes of numéraire. Example 4.2 illustrates this phenomenon and shows how existence and uniqueness results can be established not only for the "usual" utility maximization problem, but also for a utility maximization under a new (bounded away from zero) numéraire, via a simple application of Theorem 2.4 and Lemma 2.6 below. In Example 4.1, we obtain the standard conclusions of the theory in the setting, where the agent is not allowed to invest in stocks.

The organization of the paper. In Section 2, we describe the mathematical model and state our main results, whose proofs are contained in Section 3. The examples are presented in Section 4.

2. MAIN RESULTS

We consider a financial market model with finite time horizon $[0, T]$ and a zero interest rate. The price process $S = (S^i)_{i=1}^d$ of the stocks is assumed to be a semimartingale on a complete stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, where $\mathcal{F}_0$ is trivial. We assume that there are nontraded contingent claims with payment processes $(F^i)_{i=1}^N = F$. If $(q_i)_{i=1}^N = q$ are the numbers of such claims, then the cumulative payoff of this portfolio is

$$qF \triangleq \sum_{i=1}^N q_i F^i = \left(\sum_{i=1}^N q_i F^i_t\right)_{t\in[0,T]}.$$

The random variable qF_t represents the cumulative amount of endowment received by a holder of q claims during the time interval $[0, t]$. Both processes S and F are given exogenously.

As in Mostovyi (2015), we define a *stochastic clock* as a nondecreasing, càdlàg, adapted process such that

$$\kappa_0 = 0,\ \mathbb{P}[\kappa_T > 0] > 0,\ \text{and}\ \kappa_T \leq A \tag{2.1}$$

for some finite constant A.

Define a portfolio Π as a quadruple (x, q, H, c), where the constant x is the initial value of the portfolio, vector q gives the number of shares of illiquid contingent claims, $H = (H_i)_{i=1}^d$ is a predictable S-integrable process that specifies the amount of each stock in the portfolio, and $c = (c_t)_{t\in[0,T]}$ is the consumption rate, which we assume to be optional and nonnegative.

The *wealth process* $V = (V_t)_{t\in[0,T]}$ generated by the portfolio is

$$V_t = x + \int_0^t H_s dS_s - \int_0^t c_s d\kappa_s + qF_t, \quad t \in [0, T].$$

A portfolio Π with $c \equiv 0$ and $q = 0$ is called *self-financing*. The collection of nonnegative wealth processes generated by self-financing portfolios with initial value $x \geq 0$ is denoted by $\mathcal{X}(x)$, i.e.,

$$\mathcal{X}(x) \triangleq \left\{X \geq 0 : X_t = x + \int_0^t H_s dS_s,\, t \in [0, T]\right\},\, x \geq 0.$$

A probability measure $\mathbb{Q}$ is an *equivalent local martingale measure* if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and every $X \in \mathcal{X}(1)$ is a local martingale under $\mathbb{Q}$. We denote the family of equivalent local martingale measures by $\mathcal{M}$ and assume that

$$\mathcal{M} \neq \emptyset. \tag{2.2}$$

This condition is equivalent to the absence of arbitrage opportunities in the market, see Delbaen and Schachermayer (1994, 1998) as well as Karatzas and Kardaras (2007) for the exact statements and further references.

To rule out doubling strategies in the presence of random endowment, we need to impose additional restrictions. Following Delbaen and Schachermayer (1997a), we say that a nonnegative process in $\mathcal{X}(x)$ is *maximal* if its terminal value cannot be dominated by that of any other process in $\mathcal{X}(x)$.

As in Delbaen and Schachermayer (1997a), we define an *acceptable* process to be a process of the form $X = X' - X''$, where X' is a nonnegative wealth process generated by a self-financing portfolio and X'' is maximal. Following Hugonnier and Kramkov (2004), we denote by $\mathcal{X}(x,q)$ the set of acceptable processes with initial values x, whose terminal values dominate the random payoff $-qF_T$:

$$\mathcal{X}(x,q) \triangleq \{X : X \text{ is acceptable}, \; X_0 = x \; \text{ and } \; X_T + qF_T \geq 0\}. \tag{2.3}$$

The set $\mathcal{X}(x,q)$ may be empty for some $(x,q) \in \mathbb{R}^{N+1}$. We are interested in the values of x and q, for which $\mathcal{X}(x,q) \neq \emptyset$, and define

$$\mathcal{K} \triangleq \operatorname{int}\left\{(x,q) \in \mathbb{R}^{N+1} : \; \mathcal{X}(x,q) \neq \emptyset\right\}. \tag{2.4}$$

Hereafter, we shall impose the following conditions on the endowment process.

ASSUMPTION 2.1. *$(F_T^i)_{i=1,\dots,N}$ are $\mathcal{F}_T$-measurable functions. There exists a maximal nonnegative wealth process X' generated by a self-financing portfolio, such that*

$$X'_T \geq \sum_{i=1}^{N} |F_T^i|. \tag{2.5}$$

Lemma 6 in Hugonnier and Kramkov (2004) shows that

$$\mathrm{cl}\mathcal{K} = \left\{(x,q) \in \mathbb{R}^{N+1} : \; \mathcal{X}(x,q) \neq \emptyset\right\},$$

where $\mathrm{cl}\mathcal{K}$ denotes the closure of the set $\mathcal{K}$ in $\mathbb{R}^{N+1}$.

We restrict our attention to the wealth processes with nonnegative *terminal* values. Thus for each $(x,q) \in \mathrm{cl}\mathcal{K}$ we set

$$\begin{aligned}\mathcal{A}(x,q) \triangleq \Big\{c = (c_t)_{t\in[0,T]} &: c \text{ is nonnegative, optional,} \\ &\text{and there exists } X \in \mathcal{X}(x,q) \text{ s.t. } X_T - \int_0^T c_t d\kappa_t + qF_T \geq 0\Big\}.\end{aligned} \tag{2.6}$$

Note that $c \equiv 0$ belongs to $\mathcal{A}(x,q)$ for every $(x,q) \in \mathrm{cl}\mathcal{K}$.

REMARK 2.2. As the endowment process F in definition (2.6) enters only via its terminal value, it is natural to impose a condition on F_T (and not on the whole F), as in Assumption 2.1. For conditions equivalent to (2.5), see lemma 1 in Hugonnier and Kramkov (2004).

The preferences of an economic agent are modeled with a *utility stochastic field* $U = U(t,\omega,x) : [0,T] \times \Omega \times [0,\infty) \to \mathbb{R} \cup \{-\infty\}$. As in Mostovyi (2015), we assume that U satisfies the conditions below.

ASSUMPTION 2.3. *For every $(t,\omega) \in [0,T] \times \Omega$, the function $x \to U(t,\omega,x)$ is strictly concave, increasing, continuously differentiable on $(0,\infty)$ and satisfies the Inada conditions:*

$$\lim_{x\downarrow 0} U'(t,\omega,x) = +\infty \quad \textit{and} \quad \lim_{x\to\infty} U'(t,\omega,x) = 0, \tag{2.7}$$

where U' denotes the partial derivative with respect to the third argument. At $x = 0$ we suppose, by continuity, $U(t,\omega,0) = \lim_{x\downarrow 0} U(t,\omega,x)$, which may be $-\infty$. For every $x \geq 0$ the stochastic process $U(\cdot,\cdot,x)$ is optional.

The agent can control investment and consumption. The goal is to maximize expected utility. The value function u is defined as:

$$u(x,q) \triangleq \sup_{c\in\mathcal{A}(x,q)} \mathbb{E}\left[\int_0^T U(t,\omega,c_t)d\kappa_t\right], \quad (x,q)\in \mathrm{cl}\mathcal{K}. \tag{2.8}$$

We use the convention

$$\mathbb{E}\left[\int_0^T U(t,\omega,c_t)d\kappa_t\right] \triangleq -\infty \quad \text{if} \quad \mathbb{E}\left[\int_0^T U^-(t,\omega,c_t)d\kappa_t\right] = +\infty.$$

Here and below, W^- and W^+ denote the negative and positive parts of a stochastic field W, respectively.

We are primarily interested in the following questions.

(i) Under what conditions on the market model and on the utility stochastic field U does the maximizer to the problem (2.8) exist for every $(x,q)\in\{u>-\infty\}$?
(ii) What are the properties of the function u?
(iii) What is the corresponding dual problem?

We employ duality techniques to answer these questions and define a convex conjugate stochastic field

$$V(t,\omega,y) \triangleq \sup_{x>0}\left(U(t,\omega,x)-xy\right), \quad (t,\omega,y)\in[0,T]\times\Omega\times[0,\infty).$$

Observe that $-V$ satisfies Assumption 2.3. In order to construct the feasible set of the dual problem, we define the set $\mathcal{L}$ as the relative interior of the polar cone of $-\mathcal{K}$:

$$\mathcal{L} \triangleq \mathrm{ri}\left\{(y,r)\in\mathbb{R}^{N+1} : xy+qr\geq 0 \text{ for all } (x,q)\in\mathcal{K}\right\}. \tag{2.9}$$

It is proven that $\mathcal{L}$ is an open set in $\mathbb{R}^{N+1}$ if and only if for every $q\neq 0$ the random variable qF_T is nonreplicable (see lemma 7 in Hugonnier and Kramkov 2004 for the exact statement).

By $\mathcal{Z}$, we denote the set of càdlàg densities of equivalent local martingale measures:

$$\mathcal{Z} \triangleq \left\{\left(\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}\right)_{t\in[0,T]} : \quad \mathbb{Q}\in\mathcal{M}\right\},$$

and for each $y\geq 0$ we define

$$\begin{aligned}\mathcal{Y}(y) \triangleq \mathrm{cl}\{Y:\ & Y \text{is càdlàg adapted and} \\ & 0\leq Y\leq yZ(d\kappa\times\mathbb{P})\,\text{a.e. for some } Z\in\mathcal{Z}\},\end{aligned} \tag{2.10}$$

where the closure is taken in the topology of convergence in measure $(d\kappa\times\mathbb{P})$ on the space of optional processes. Now we are ready to set the domain of the dual problem:

$$\begin{aligned}\mathcal{Y}(y,r) \triangleq \Big\{ & Y\in\mathcal{Y}(y) : \mathbb{E}\left[\int_0^T c_t Y_t d\kappa_t\right] \leq xy+qr \\ & \text{for every } (x,q)\in\mathrm{cl}\mathcal{K} \text{ and } c\in\mathcal{A}(x,q) \Big\}, \quad (y,r)\in\mathrm{cl}\mathcal{L},\end{aligned} \tag{2.11}$$

and to state the dual optimization problem itself:

$$v(y,r) \triangleq \inf_{Y \in \mathcal{Y}(y,r)} \mathbb{E}\left[\int_0^T V(t,\omega,Y_t)d\kappa_t\right], \ (y,r) \in \mathrm{cl}\mathcal{L}, \tag{2.12}$$

where we use the convention:

$$\mathbb{E}\left[\int_0^T V(t,\omega,Y_t)d\kappa_t\right] \triangleq +\infty \quad \text{if} \quad \mathbb{E}\left[\int_0^T V^+(t,\omega,Y_t)d\kappa_t\right] = +\infty.$$

The following theorem constitutes the main result of this work.

THEOREM 2.4. *Assume that* (2.1), (2.2), *Assumptions 2.1 and 2.3 hold, as well as*

$$\begin{aligned} u(x,q) &> -\infty \quad \textit{for every} \quad (x,q) \in \mathcal{K} \ \textit{and} \\ v(y,r) &< +\infty \quad \textit{for every} \quad (y,r) \in \mathcal{L}. \end{aligned} \tag{2.13}$$

Then we have:

(i) *The functions u and v are finite on $\mathcal{K}$ and $\mathcal{L}$, respectively. u and v satisfy biconjugacy relations:*

$$\begin{aligned} u(x,q) &= \inf_{(y,r)\in \mathrm{cl}\mathcal{L}} (v(y,r) + xy + qr), \ (x,q) \in \mathrm{cl}\mathcal{K}; \\ v(y,r) &= \sup_{(x,q)\in \mathrm{cl}\mathcal{K}} (u(x,q) - xy - qr), \ (y,r) \in \mathrm{cl}\mathcal{L}. \end{aligned} \tag{2.14}$$

(ii) *The function u is upper semi-continuous, $u(x,q) < +\infty$ for every $(x,q) \in \mathrm{cl}\mathcal{K}$. For every $(x,q) \in \{u > -\infty\}$ there exists a unique maximizer to the problem* (2.8). *The function v is lower semi-continuous, $v(y,r) > -\infty$ for every $(y,r) \in \mathrm{cl}\mathcal{L}$. For every $(y,r) \in \{v < +\infty\}$ there exists a unique minimizer to the problem* (2.12).

(iii) *For every $(x,q) \in \mathcal{K}$, the subdifferential of u at (x,q) is nonempty, $(y,r) \in \partial u(x,q)$ if and only if the following conditions hold:*

$$\hat{Y}_t(y,r) = U'(t,\omega,\hat{c}_t(x,q)), \quad (t,\omega) \in [0,T] \times \Omega, \tag{2.15}$$

$$\mathbb{E}\left[\int_0^T \hat{Y}_t(y,r)\hat{c}_t(x,q)d\kappa_t\right] = xy + qr, \tag{2.16}$$

$$|v(y,r)| < +\infty, \tag{2.17}$$

where $\hat{Y}(y,r)$ and $\hat{c}(x,q)$ are optimizers to problems (2.12) *and* (2.8), *respectively.*

Let 1_E denote the indicator function of a set E.

REMARK 2.5. Item (ii) of Theorem 2.4 asserts the existence of the optimal solution to (2.8) at the points (x,q) that might lie on the boundary of $\mathcal{K}$. It turns out that one cannot establish subdifferentiability of u on the boundary of $\mathcal{K}$ in general, as the following example demonstrates.

Consider maximization of the expected utility from terminal wealth, i.e., $\kappa(t) = 1_{[T]}(t)$, $t \in [0,T]$, in the market with no stocks and one contingent claim uniformly distributed on $[-1,1]$, with a power utility of exponent $\alpha \in (-1,0)$, i.e., $U(t,w,x) = \frac{x^\alpha}{\alpha}$. Then $\mathcal{K} = \{(x,q) \in \mathbb{R}^2 : x > |q|\}$, $dom(u) = \mathrm{cl}\mathcal{K}\backslash\{0\}$, but $dom(\partial u) = \mathcal{K}$.

Condition (2.13) might be difficult to verify. The following lemma provides an equivalent criterion in terms of the functions

$$w(x) \triangleq u(x,0) = \sup_{c\in\mathcal{A}(x,0)} \mathbb{E}\left[\int_0^T U(t,\omega,c_t)\,d\kappa_t\right], \quad x>0, \tag{2.18}$$

and

$$\tilde{w}(y) \triangleq \inf_{Y\in\mathcal{Y}(y)} \mathbb{E}\left[\int_0^T V(t,\omega,Y_t)\,d\kappa_t\right], \quad y>0. \tag{2.19}$$

LEMMA 2.6. *Let conditions* (2.1) *and* (2.2) *as well as Assumptions 2.1 and 2.3 hold true. Then condition* (2.13) *holds if and only if*

$$\begin{aligned} &w(x) > -\infty \quad \textit{for every} \quad x>0 \ \textit{and} \\ &\tilde{w}(y) < +\infty \quad \textit{for every} \quad y>0. \end{aligned} \tag{2.20}$$

As pointed out in Mostovyi (2015), $w(x) > -\infty$ for every $x > 0$ if U is uniformly in (t,ω) bounded from below by a finite-valued function. Note that Lemma 2.6 is a generalization of lemma 2 in Hugonnier and Kramkov (2004) to our setting and (2.20) is the condition that was used in Mostovyi (2015) in the statement of the main theorem.

3. PROOFS

The proof of Theorem 2.4 follows Hugonnier and Kramkov (2004). However, our setting and results, which in contrast to Hugonnier and Kramkov (2004) include intermediate consumption, stochastic utility, and properties of the primal and dual value functions on the boundaries of their domains, require special treatment. On the technical side, the proof of Theorem 2.4 relies on the results of Mostovyi (2015). Also, in some proofs below, we will assume that the set $\mathcal{L}$ is open in $\mathbb{R}^{N+1}$. As explained in remark 6 of Hugonnier and Kramkov (2004), we do not lose generality by doing so. We begin with a proposition that gives a useful characterization of the primal and dual domains.

PROPOSITION 3.1. *Under the conditions* (2.1), (2.2)*, and Assumption 2.1, the families* $(\mathcal{A}(x,q))_{(x,q)\in\mathrm{cl}\mathcal{K}}$ *and* $(\mathcal{Y}(y,r))_{(y,r)\in\mathrm{cl}\mathcal{L}}$ *defined in* (2.6) *and* (2.11) *have the following properties:*

(i) For any $(x,q)\in\mathcal{K}$*, the set* $\mathcal{A}(x,q)$ *contains a strictly positive constant process. For every* $(x,q)\in\mathrm{cl}\mathcal{K}$ *a nonnegative optional process* c *belongs to* $\mathcal{A}(x,q)$ *if and only if*

$$\begin{aligned} &\mathbb{E}\left[\int_0^T c_t Y_t d\kappa_t\right] \le xy + qr \\ &\qquad\qquad \textit{for every } (y,r)\in\mathrm{cl}\mathcal{L} \textit{ and } Y\in\mathcal{Y}(y,r). \end{aligned} \tag{3.1}$$

(ii) For every $(y, r) \in \mathcal{L}$, *the set* $\mathcal{Y}(y, r)$ *contains a strictly positive process. For every* $(y, r) \in \mathrm{cl}\mathcal{L}$, *a nonnegative process* Y *belongs to* $\mathcal{Y}(y, r)$ *if and only if*

$$\mathbb{E}\left[\int_0^T c_t Y_t d\kappa_t\right] \leq xy + qr \quad \text{for every } (x, q) \in \mathrm{cl}\mathcal{K} \text{ and } c \in \mathcal{A}(x, q). \tag{3.2}$$

The proof of Proposition 3.1 is based on several lemmas. As in Hugonnier and Kramkov (2004), we define $\mathcal{P}$ to be the set of points at the intersection of $\mathcal{L}$ and the hyperplane $y \equiv 1$, that is,

$$\mathcal{P} \triangleq \{p \in \mathbb{R}^N : (1, p) \in \mathcal{L}\}. \tag{3.3}$$

Note that under (2.2) and Assumption 2.1, it follows from lemma 1 in Hugonnier and Kramkov (2004), that the set $\mathcal{P}$ is bounded.

Let $\mathcal{M}'$ be the set of equivalent local martingale measures $\mathbb{Q}$, such that the process X' (in Assumption 2.1) is a uniformly integrable martingale under $\mathbb{Q}$. According to theorem 5.2 in Delbaen and Schachermayer (1997a), $\mathcal{M}'$ is a nonempty, convex subset of $\mathcal{M}$, which is dense in $\mathcal{M}$ with respect to the variation norm. Note that the results in Delbaen and Schachermayer (1997a) were proven under the assumption that S is locally bounded, however they still hold in our settings, see, e.g., remark 3.4 in Hugonnier, Kramkov, and Schachermayer (2005).

For every $p \in \mathcal{P}$, we denote

$$\mathcal{M}'(p) \triangleq \{\mathbb{Q} \in \mathcal{M}' : \tilde{\mathbb{E}}_{\mathbb{Q}}[F_T] = p\}.$$

It follows from lemma 8 in Hugonnier and Kramkov (2004) that (under condition (2.2) and Assumption 2.1) $\mathcal{M}'(p)$ is nonempty for every $p \in \mathcal{P}$ and

$$\bigcup_{p \in \mathcal{P}} \mathcal{M}'(p) = \mathcal{M}', \tag{3.4}$$

LEMMA 3.2. *Let the assumptions of Proposition 3.1 hold true and* $p \in \mathcal{P}$. *Then, the càdlàg density process of any* $\mathbb{Q} \in \mathcal{M}'(p)$ *belongs to* $\mathcal{Y}(1, p)$. *For every* $(x, q) \in \mathrm{cl}\mathcal{K}$ *and* $c \in \mathcal{A}(x, q)$, *we have*

$$x + qp \geq \mathbb{E}_{\mathbb{Q}}\left[\int_0^T c_t d\kappa_t\right] = \mathbb{E}\left[\int_0^T \frac{d\mathbb{Q}_t}{d\mathbb{P}_t} c_t d\kappa_t\right]. \tag{3.5}$$

Proof. Fix an arbitrary $(x, q) \in \mathrm{cl}\mathcal{K}$, $c \in \mathcal{A}(x, q)$, and $X \in \mathcal{X}(x, q)$ such that $X_T + qF_T \geq \int_0^T c_t d\kappa_t \geq 0$. By lemma 4 in Hugonnier and Kramkov (2004), X is a supermartingale under $\mathbb{Q}$. Therefore, taking expectation under $\mathbb{Q} \in \mathcal{M}'(p)$ and using localization and proposition I.4.49 in Jacod and Shiryaev (1980), we get (3.5) (note that this part of the proof parallels the proof of proposition 3.1 in Žitković 2005). □

LEMMA 3.3. *Let the assumptions of Proposition 3.1 hold true. Then, for every* $(x, q) \in \mathrm{cl}\mathcal{K}$, *a nonnegative optional process* c *belongs to* $\mathcal{A}(x, q)$ *if and only if*

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T c_t d\kappa_t\right] \leq x + qp \text{ for every } p \in \mathcal{P} \text{ and } \mathbb{Q} \in \mathcal{M}'(p). \tag{3.6}$$

Proof. If $c \in \mathcal{A}(x, q)$ for $(x, q) \in \mathrm{cl}\mathcal{K}$, then the validity of (3.6) is proven in Lemma 3.2. Vice versa, let c be a nonnegative optional process such that (3.6) holds. Let

$$h \triangleq \int_0^T c_t d\kappa_t - qF_T, \quad M \triangleq \max_{1\leq i\leq N} |q_i|.$$

Then $h \geq -MX'_T$ and

$$\begin{aligned}\alpha(h) \triangleq \sup_{\mathbb{Q}\in\mathcal{M}'} \mathbb{E}_{\mathbb{Q}}[h] &= \sup_{p\in\mathcal{P}} \sup_{\mathbb{Q}\in\mathcal{M}'(p)} \mathbb{E}_{\mathbb{Q}}[h] \\ &= \sup_{p\in\mathcal{P}} \sup_{\mathbb{Q}\in\mathcal{M}'(p)} \left(\mathbb{E}_{\mathbb{Q}}\left[\int_0^T c_t d\kappa_t\right] - qp\right) \leq x,\end{aligned}$$

where, in the second equality, we used (3.4). Lemma 5 in Hugonnier and Kramkov (2004) implies the existence of an acceptable process X such that $X_0 = \alpha(h)$ and $X_T \geq h$. It follows that

$$X_T + qF_T \geq \int_0^T c_t d\kappa_t.$$

Therefore $c \in \mathcal{A}(\alpha(h), q) \subseteq \mathcal{A}(x, q)$. □

Proof of Proposition 3.1. We prove item (i) first. Fix $(x, q) \in \mathcal{K}$. As $\mathcal{K}$ is an open set, there exists $\delta > 0$ such that $(x - \delta, q) \in \mathcal{K}$. Take $X \in \mathcal{X}(x - \delta, q)$ then $Z \triangleq X + \delta \in \mathcal{X}(x, q)$. Consequently

$$Z_T + qF_T \geq \delta \geq \int_0^T (\delta/A)\, d\kappa_t,$$

where A is the constant in (2.1). Therefore, the process that takes the constant value δ/A belongs to $\mathcal{A}(x, q)$.

Let c be a nonnegative optional process such that (3.1) holds. For every $p \in \mathcal{P}$, it follows from Lemma 3.2 that the càdlàg density process of any $\mathbb{Q} \in \mathcal{M}'(p)$ is in $\mathcal{Y}(1, p)$. Consequently, c satisfies (3.6). It follows from Lemma 3.3 that $c \in \mathcal{A}(x, q)$. The other direction follows from the definition of the set $\mathcal{Y}(y, r)$. This concludes the proof of item (i).

To prove the assertion of item (ii), let us observe that

$$a\mathcal{Y}(y, r) = \mathcal{Y}(ay, ar) \quad \text{for every } a > 0 \text{ and } (y, r) \in \mathcal{L}.$$

Therefore it suffices to prove the existence of a strictly positive process for $(y, r) = (1, p)$, $p \in \mathcal{P}$. Fix an arbitrary $p \in \mathcal{P}$. By lemma 8 in Hugonnier and Kramkov (2004), we deduce the existence of $\mathbb{Q} \in \mathcal{M}'(p)$. By lemma 3.2, the càdlàg density process $(\frac{d\mathbb{Q}_t}{d\mathbb{P}_t})_{t\in[0,T]}$ is in $\mathcal{Y}(1, p)$. Since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, $(\frac{d\mathbb{Q}_t}{d\mathbb{P}_t})_{t\in[0,T]}$ is strictly positive $\mathbb{P}$ a.s.

Similarly, it suffices to consider $(y, r) \in \mathrm{cl}\mathcal{L}$ with $y = 1$. For every $(1, p) \in \mathrm{cl}\mathcal{L}$, if $Y \in \mathcal{Y}(1, p)$, condition (3.2) follows from the definition of the set $\mathcal{Y}(1, p)$. Conversely, let Y be a nonnegative process such that (3.2) holds for $y = 1$ and $r = p$. Then

$$\mathbb{E}\left[\int_0^T c_t Y_t d\kappa_t\right] \leq 1 \quad \text{for all } c \in \mathcal{A}(1, 0).$$

Note that $\mathcal{A}(1,0)$ is nonempty by lemma 1 in Hugonnier and Kramkov (2004). Therefore, by proposition 4.4 in Mostovyi (2015), Y belongs to the set $\mathcal{Y}(1)$ and is such that (3.2) holds, i.e., $Y \in \mathcal{Y}(1,p)$. □

LEMMA 3.4. *Under the conditions of Theorem 2.4, for every* $(x,q) \in \mathrm{cl}\mathcal{K}$ *and* $(y,r) \in \mathrm{cl}\mathcal{L}$, *we have*

$$u(x,q) \leq v(y,r) + xy + qr. \tag{3.7}$$

As a result, u and v are real-valued functions on $\mathcal{K}$ *and* $\mathcal{L}$, $u < +\infty$ *and* $v > -\infty$ *on* $\mathrm{cl}\mathcal{K}$ *and* $\mathrm{cl}\mathcal{L}$, *respectively.*

Proof. Fix $(x,q) \in \mathrm{cl}\mathcal{K}$ and $(y,r) \in \mathrm{cl}\mathcal{L}$. Take an arbitrary $c \in \mathcal{A}(x,q)$ and $Y \in \mathcal{Y}(y,r)$. It follows from the definition of the set $\mathcal{Y}(y,r)$ and Fenchel's inequality that

$$\begin{aligned}\mathbb{E}\left[\int_0^T U(t,\omega,c_t)d\kappa_t\right] &\leq \mathbb{E}\left[\int_0^T U(t,\omega,c_t)d\kappa_t\right] + xy + qr - \mathbb{E}\left[\int_0^T c_t Y_t d\kappa_t\right] \\ &\leq \mathbb{E}\left[\int_0^T V(t,\omega,Y_t)d\kappa_t\right] + xy + qr.\end{aligned}$$

This implies inequality (3.7). The remaining assertions of the lemma follow from (2.13). □

Let $\mathbf{L}^0 = \mathbf{L}^0(d\kappa \times \mathbb{P})$ be the vector space of optional processes on the stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ equipped with the topology of convergence in measure $(d\kappa \times \mathbb{P})$.

The following lemma establishes semi-continuity of the value functions. Note that, in the setting of optimal investment from terminal wealth, upper semi-continuity of the primal value function is proven in theorem 6.3 of Siorpaes (2016).

LEMMA 3.5. *Let the conditions of Theorem 2.4 hold true. Then the function u is upper semi-continuous. For every* $(x,q) \in \{u > -\infty\}$, *there exists a unique maximizer to the problem* (2.8). *Likewise, the function v is lower semi-continuous. For every* $(y,r) \in \{v < +\infty\}$, *there exists a unique minimizer to the problem* (2.12).

Proof. Let $(y^n, r^n)_{n\geq 1}$ be a sequence in $\mathcal{L}$ converging to a point $(y,r) \in \{v < +\infty\}$. Without loss of generality, assume that $v(y^n,r^n) < \infty$, $n \geq 1$. Let

$$L \triangleq \liminf_{n\to\infty} v(y^n,r^n). \tag{3.8}$$

Passing, if necessary, to a subsequence, which we still denote $(y^n,r^n)_{n\geq 1}$, we can assume that

$$L = \lim_{n\to\infty} v(y^n,r^n). \tag{3.9}$$

Let $Y^n \in \mathcal{Y}(y^n,r^n)$ be such that

$$\mathbb{E}\left[\int_0^T V\left(t,\omega,Y^n_t\right)d\kappa_t\right] \leq v(y^n,r^n) + \frac{1}{n}, \quad n \geq 1.$$

By lemma A1.1 in Delbaen and Schachermayer (1994), there exists a sequence of convex combinations $\tilde{Y}^n \in \mathrm{conv}(Y^n, Y^{n+1}, \dots)$, $n \geq 1$, and an element $\hat{Y} \in \mathbf{L}^0$, such that $\left(\tilde{Y}^n\right)_{n\geq 1}$ converges to $\hat{Y}$ $(d\kappa \times \mathbb{P})$ a.e. From the convexity of V, we have

$$\sup_{n\geq m} \mathbb{E}\left[\int_0^T V(t, \omega, Y_t^n) d\kappa_t\right] \geq \sup_{n\geq m} \mathbb{E}\left[\int_0^T V(t, \omega, \tilde{Y}_t^n) d\kappa_t\right]. \tag{3.10}$$

For every $(x, q) \in \mathrm{cl}\mathcal{K}$ and $c \in \mathcal{A}(x, q)$, using Fatou's lemma, we get:

$$\mathbb{E}\left[\int_0^T \hat{Y}_t c_t d\kappa_t\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[\int_0^T c_t \tilde{Y}_t^n d\kappa_t\right] \leq xy + qr.$$

Consequently, using Proposition 3.1, we deduce that $\hat{Y} \in \mathcal{Y}(y, r)$.

By Lemma 2.6, the functions w and $\tilde{w}$ satisfy the assumptions of theorem 3.2 in Mostovyi (2015). Let $\bar{y} \triangleq \sup_{n\geq 1} |y^n|$, then $\left(\tilde{Y}^n\right)_{n\geq 1} \subseteq \mathcal{Y}(\bar{y})$. Therefore, from lemma 3.5 in Mostovyi (2015), we deduce that the sequence $(V^-(t, \omega, \tilde{Y}_t^n))_{n\geq 1}$ is uniformly integrable. Consequently, from (3.9) and (3.10), using Fatou's lemma, we obtain

$$\begin{aligned} v(y, r) &\leq \mathbb{E}\left[\int_0^T V(t, \omega, \hat{Y}_t) d\kappa_t\right] \\ &\leq \liminf_{n\to\infty} \mathbb{E}\left[\int_0^T V(t, \omega, \tilde{Y}_t^n) d\kappa_t\right] \\ &\leq \limsup_{n\to\infty} \mathbb{E}\left[\int_0^T V(t, \omega, Y_t^n) d\kappa_t\right] \\ &\leq L, \end{aligned} \tag{3.11}$$

which, in view of (3.8), implies lower semi-continuity of v. Note that $v > -\infty$ holds everywhere in its domain, by Lemma 3.4. Now for every $(y, r) \in \{v < +\infty\}$, taking $(y^n, r^n) = (y, r)$, $n \geq 1$, we deduce from (3.11) the existence of a minimizer to the dual problem (2.12), whose uniqueness follows from the strict convexity of V. The proof of the corresponding assertions for the function u is similar. □

Proof of Theorem 2.4. (i) Concavity of u follows from the concavity of U. Fix $(y, r) \in \mathcal{L}$ and define the following sets:

$$A(y, r) \triangleq \{(x, q) \in \mathrm{cl}\mathcal{K} : xy + qr \leq 1\}, \tag{3.12}$$

$$\mathcal{C}(y, r) \triangleq \bigcup_{(x,q)\in A(y,r)} \mathcal{A}(x, q). \tag{3.13}$$

Note that $A(y, r)$ is closed and bounded in $\mathbb{R}^{N+1}$ (note that the proof of the boundedness of $A(y, r)$ in similar settings can be found in lemma 5.2 in Siorpaes 2016), $\mathcal{C}(y, r)$ is closed with respect to the topology of convergence in measure $(d\kappa \times \mathbb{P})$.

For every $z > 0$, let us set

$$\bar{u}(z) \triangleq \sup_{c\in\mathcal{C}(y,r)} \mathbb{E}\left[\int_0^T U(t, \omega, zc_t) d\kappa_t\right] = \sup_{(x,q)\in zA(y,r)} u(x, q) > -\infty, \tag{3.14}$$

where the latter inequality follows from (2.13). Using Proposition 3.1, we get

$$Y \in \mathcal{Y}(y,r) \Leftrightarrow \mathbb{E}\left[\int_0^T c_t Y_t d\kappa_t\right] \leq 1 \text{ for all } c \in \mathcal{C}(y,r).$$

We deduce that the sets $\mathcal{C}(y,r)$ and $\mathcal{Y}(y,r)$ satisfy the assumption of Theorem 3.2 in Mostovyi (2015). In view of (2.13) and (3.14), the same theorem implies that

$$v(y,r) = \sup_{z>0}(\bar{u}(z) - z) = \sup_{(x,q)\in \mathrm{cl}\mathcal{K}}(u(x,q) - xy - qr).$$

As $-u$ and v are proper closed convex functions (by Lemma 3.5), the latter equality implies the biconjugacy relations (2.14), see Rockafellar (1970), section 12.

(ii) The assertions of this item follow from Lemma 3.5.

(iii) By item (ii), u is a proper concave function, which is finite on $\mathcal{K}$. Therefore, by theorem 23.4 in Rockafellar (1970), $\partial u(x,q)$ is nonempty for every $(x,q) \in \mathcal{K}$. The conjugacy relations (2.14) imply (by corollary 23.5.1 in Rockafellar 1970) that $\partial u(x,q) \subseteq \mathrm{cl}\mathcal{L}$.

Let $(x,q) \in \mathcal{K}$ and $(y,r) \in \mathrm{cl}\mathcal{L}$ be such that (2.15), (2.16), and (2.17) hold, where $\hat{c}(x,q)$ and $\hat{Y}(y,r)$ are the optimizers to (2.8) and (2.12), respectively, whose existence follows from item (ii). Using conjugacy of U and V we get:

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_0^T (V(t,\omega,\hat{Y}_t(y,r)) - U(t,\omega,\hat{c}_t(x,q)) + \hat{c}_t(x,q)\hat{Y}_t(y,r))d\kappa_t\right] \\ &= v(y,r) - u(x,q) + xy + qr. \end{aligned}$$

Therefore by theorem 23.5 in Rockafellar (1970), the biconjugacy relations (2.14) imply that $(y,r) \in \partial u(x,q)$.

Conversely, fix $(x,q) \in \mathcal{K}$, and let $(y,r) \in \partial u(x,q)$. As $-u$ and v are closed convex functions (by item (ii)) that satisfy (2.14) (by item (i)), using Theorem 23.5 in Rockafellar (1970), we get

$$-u(x,q) + v(y,r) + xy + qr \leq 0. \tag{3.15}$$

Lemma 3.4 gives finiteness of $u(x,q)$, which via (3.15) and another application of Lemma 3.4 implies finiteness of $v(y,r)$ and thus (by item (ii)) the existence of $\hat{Y}(y,r)$, a unique minimizer to the problem (2.12). Analogously, we deduce that there exists $\hat{c}(x,q)$, a unique maximizer to the problem (2.8). Using Proposition 3.1, we obtain from (3.15):

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \left|V(t,\omega,\hat{Y}_t(y,r)) + \hat{c}_t(x,q)\hat{Y}_t(y,r) - U(t,\omega,\hat{c}_t(x,q))\right| d\kappa_t\right] \\ &= \mathbb{E}\left[\int_0^T \left(V\left(t,\omega,\hat{Y}_t(y,r)\right) + \hat{c}_t(x,q)\hat{Y}_t(y,r) - U(t,\omega,\hat{c}_t(x,q))\right) d\kappa_t\right] \\ &\leq v(y,r) + xy + qr - u(x,q) \leq 0, \end{aligned}$$

which gives (2.15) and (2.16). □

Proof of Lemma 2.6. Assume that (2.20) holds. Fix $(x,q) \in \mathcal{K}$. It follows from Assumption 2.1 and lemma 1 in Hugonnier and Kramkov (2004) that $(x,0) \in \mathcal{K}$ for every $x > 0$. As $\mathcal{K}$ is an open convex cone, there exists a point $(x_1,q_1) \in \mathcal{K}$, such that

$$(x, q) = \lambda(x_1, q_1) + (1-\lambda)(x_2, 0)$$

for some $\lambda \in (0, 1)$ and $x_2 > 0$.Take $c \in \mathcal{A}(x_2, 0)$, such that

$$\mathbb{E}\left[\int_0^T U(t, \omega, (1-\lambda)c_t)d\kappa_t\right] > -\infty. \tag{16}$$

Note that process c exists by assumption (2.20). Fix $g \in \mathcal{A}(x_1, q_1)$. Then

$$\lambda g + (1-\lambda)c \in \mathcal{A}(x, q).$$

As U is increasing, we obtain from (16):

$$u(x, q) \geq \mathbb{E}\left[\int_0^T U(t, \omega, \lambda g_t + (1-\lambda)c_t)d\kappa_t\right] \geq$$
$$\geq \mathbb{E}\left[\int_0^T U(t, \omega, (1-\lambda)c_t)d\kappa_t\right] > -\infty.$$

In order to prove that v is finite on $\mathcal{L}$, define the set

$$\mathcal{E} \triangleq \{(y, r) \in \mathrm{cl}\mathcal{L} : \ v(y, r) < +\infty\}.$$

First, we show that $\mathcal{E}$ is nonempty and establish some properties of $\mathcal{E}$. Let

$$\mathcal{B} \triangleq \{(y, r) \in \mathrm{cl}\mathcal{L} : y \leq 1\},$$
$$\mathcal{D} \triangleq \bigcup_{(y,r)\in\mathcal{B}} \mathcal{Y}(y, r).$$

Notice that $\mathcal{D}$ is convex, solid (in the terminology of Brannath and Schachermayer 1999), and closed in $\mathbf{L}^0$, and we have

$$\mathcal{D} \subseteq \mathcal{Y}(1). \tag{3.17}$$

We claim that $\mathcal{D} = \mathcal{Y}(1)$. By theorem 5.2 in Delbaen and Schachermayer (1997a), $\mathcal{M}'$ is dense in $\mathcal{M}$ in the total-variation norm. Therefore, by Lemma 3.2 and Proposition 3.1, we obtain that $\mathcal{Z} \subset \mathcal{D}$. In turn, by the bipolar theorem of Brannath and Schachermayer, this implies that

$$Z^{oo} \subseteq \mathcal{D}. \tag{3.18}$$

On the other hand, from lemma 4.2 and proposition 4.4 in Mostovyi (2015), we conclude

$$Z^{oo} = \mathcal{Y}(1),$$

which produces (via (3.17) and (3.18)) $\mathcal{D} = \mathcal{Y}(1)$.

Using proposition 4.4 in Mostovyi (2015), one can also see that the sets $\mathcal{A}(1, 0)$ and $\mathcal{D}$ satisfy the assumptions of theorem 3.2 in Mostovyi (2015), which in particular assert that for every $x > 0$ there exists $\hat{c}(x)$, the unique maximizer to (2.18). Thus, for every $x > 0$, we define

$$Y_t(x) \triangleq U'(t, \omega, \hat{c}_t(x)), \quad t \in [0, T].$$

It follows from the same theorem that w is a continuously differentiable function that satisfies the Inada conditions and

$$Y(x) \in w'(x)\mathcal{D}.$$

Therefore, there exists $(y, r) \in w'(x)\mathrm{cl}\mathcal{B}$, such that $Y(x) \in \mathcal{Y}(y, r)$. As

$$\mathbb{E}\left[\int_0^T V(t, \omega, Y_t(x))\, d\kappa_t\right] < +\infty$$

(by the same theorem), we conclude that $\mathcal{E} \neq \emptyset$. Moreover, as x can be taken arbitrarily large and w satisfies the Inada conditions, we deduce that the closure of $\mathcal{E}$ contains origin. One can also see that the set $\mathcal{E}$ is convex and

$$\mathcal{E} \supseteq \bigcup_{\lambda \geq 1} \lambda \mathcal{E}.$$

Second, we prove that $\mathcal{L} \subseteq \mathcal{E}$. Fix an arbitrary $(y, r) \in \mathcal{L}$ and let $\delta > 0$ be such that $B_\delta(y, r) \subset \mathcal{L}$, where $B_\delta(y, r)$ denotes the ball in $\mathbb{R}^{N+1}$ of radius δ centered at (y, r). Because the origin is in the closure of $\mathcal{E}$, there exists $(\tilde{y}_2, \tilde{r}_2) \in \mathcal{E} \cap B_{\delta/2}(0)$. Let

$$(\tilde{y}_1, \tilde{r}_1) \triangleq (y - \tilde{y}_2, r - \tilde{r}_2).$$

Then $(\tilde{y}_1, \tilde{r}_1) \in B_{\delta/2}(y, r)$. One can choose $\lambda \in (0, 1)$ such that

$$(y_1, r_1) \triangleq \frac{1}{\lambda}(\tilde{y}_1, \tilde{r}_1) \in B_\delta(y, r).$$

Set $(y_2, r_2) \triangleq \frac{1}{1-\lambda}(\tilde{y}_2, \tilde{r}_2)$, then

$$(y, r) = \lambda(y_1, r_1) + (1 - \lambda)(y_2, r_2).$$

Fix a process $Y' \in \mathcal{Y}(y_1, r_1)$. As $(\tilde{y}_2, \tilde{r}_2) \in \mathcal{E}$, there exists a process $Y'' \in \mathcal{Y}(y_2, r_2)$, such that

$$\mathbb{E}\left[\int_0^T V(t, \omega, (1 - \lambda) Y''_t) d\kappa_t\right] < +\infty.$$

As V is decreasing and $(\lambda Y' + (1 - \lambda) Y'') \in \mathcal{Y}(y, r)$, we deduce

$$\begin{aligned} v(y, r) &\leq \mathbb{E}\left[\int_0^T V(t, \omega, \lambda Y'_t + (1 - \lambda) Y''_t) d\kappa_t\right] \\ &\leq \mathbb{E}\left[\int_0^T V(t, \omega, (1 - \lambda) Y''_t) d\kappa_t\right] < +\infty. \end{aligned}$$

Conversely, if (2.13) holds then for every $p \in \mathcal{P}$, as $\mathcal{Y}(y, yp)$ is a subset of $\mathcal{Y}(y)$, we have

$$\tilde{w}(y) \leq v(y, yp) < +\infty, \quad y > 0.$$

Let us recall that $w(x) = u(x, 0)$. Therefore, the other assertion of (2.20) follows trivially. □

4. EXAMPLES

The following example is a generalization of example 4.2 in Karatzas and Žitković (2003). We would like to stress the simplicity with which we prove existence and uniqueness for problems (4.1) and (4.4) below by application of Lemma 2.6. Note that verification of the assumptions of Lemma 2.6 is essentially reduced to checking (4.2) and (4.3).

EXAMPLE 4.1 (Optimal consumption from random endowment). Consgider a market with no stocks and an agent who receives a cumulative endowment F satisfying Assumption 2.1, which in the absence of stocks implies boundedness of F_T. Let us assume that the objective has the form

$$u(x, q) \triangleq \sup_{c \in \mathcal{A}(x,q)} \mathbb{E}\left[\int_0^T e^{-\nu t}\bar{U}(c_t)dt\right], \quad (x, q) \in \mathrm{cl}\mathcal{K}, \tag{4.1}$$

where $\nu \geq 0$ is an impatience rate and $\bar{U}$ is an Inada utility, i.e., a strictly increasing, strictly concave, continuously differentiable function that satisfies the Inada conditions and such that $\bar{U}(0) = \lim\limits_{x \to 0} \bar{U}(x)$. Notice that we do not require $\bar{U}$ to satisfy the asymptotic elasticity condition (e.g., $\bar{U}$ can be of the form $\bar{U}(x) = \frac{x}{\log x}$ for large values of x and such that $\bar{U}$ is an Inada utility function). Problem (4.1) is a particular case of (2.8) if we let the stochastic clock be $\kappa_t = t,\ t \in [0, T]$, so that κ satisfies (2.1) with $A = T$. The set $\mathcal{K}$ in our case is given by

$$\mathcal{K} = \mathrm{int}\left\{(x, q) : x + qF_T \geq 0\right\}.$$

For problem (4.1), verification of existence and uniqueness for every $(x, q) \in \mathcal{K}$ becomes an application of Lemma 2.6. Let

$$C \triangleq \begin{cases} \frac{1-e^{-\nu T}}{\nu}, & \text{if } \nu > 0, \\ T, & \text{if } \nu = 0. \end{cases}$$

Then, for every $x > 0$, as the constant consumption $c \equiv (x/T)_{t \in [0,T]}$ is in $\mathcal{A}(x, 0)$, we have

$$w(x) = \sup_{c \in \mathcal{A}(x,0)} \mathbb{E}\left[\int_0^T e^{-\nu t}\bar{U}(c_t)dt\right] \geq C\bar{U}(x/T) > -\infty. \tag{4.2}$$

As for the dual problem, because there are no stocks, every probability measure $\mathbb{Q} \sim \mathbb{P}$ is an equivalent local martingale measure. In particular, the constant process $1 \in \mathcal{Y}(1)$. Therefore condition (2.2) holds, and we get for every $y > 0$

$$\tilde{w}(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[\int_0^T e^{-\nu t}\bar{V}(Y_t)dt\right] \leq C\bar{V}(y) < +\infty, \tag{4.3}$$

where $\bar{V}$ is the conjugate to $\bar{U}$. Conditions (4.2) and (4.3) (together with (2.1) and (2.2) as well as Assumptions 2.1 and 2.3) imply via Lemma 2.6 that the assumptions of Theorem

2.4 are satisfied, and therefore that the conclusions of Theorem 2.4 are valid for u and v, where the dual value function is

$$v(y, r) \triangleq \inf_{Y \in \mathcal{Y}(x,q)} \mathbb{E}\left[\int_0^T e^{-\nu t} \bar{V}(Y_t)dt\right], \quad (y, r) \in \mathrm{cl}\mathcal{L}. \tag{4.4}$$

If, in addition, $\bar{U}$ is bounded from below, i.e.,

$$\bar{U}(0) > -\infty,$$

then we can extend the existence of a solution to (4.1) to the closure of $\mathcal{K}$, because for every $(x, q) \in \mathrm{cl}\mathcal{K}$ the consumption $c \equiv 0$ is admissible, i.e., belongs to $\mathcal{A}(x, q)$, and

$$u(x, q) \geq \mathbb{E}\left[\int_0^T e^{-\nu t} \bar{U}(0)dt\right] = C\bar{U}(0) > -\infty.$$

Likewise, if $\bar{U}$ is bounded from above, i.e.,

$$\lim_{x\to\infty} \bar{U}(x) = \bar{V}(0) < +\infty,$$

then, because for every $(y, r) \in \mathrm{cl}\mathcal{L}$ the process $Y \equiv 0$ is in $\mathcal{Y}(y, r)$, we deduce

$$v(y, r) \leq \mathbb{E}\left[\int_0^T e^{-\nu t} \bar{V}(0)dt\right] = C\bar{V}(0) < +\infty, g$$

which implies the existence of a solution to (4.4) for every $(y, r) \in \mathrm{cl}\mathcal{L}$.

One of the advantages of working with a stochastic utility is the flexibility that this framework exhibits with respect to some standard transformations in mathematical finance, such as a change of numéraire. In the following example, we show how Theorem 2.4 and Lemma 2.6 provide existence and uniqueness results for a utility maximization problem under a different numéraire.

EXAMPLE 4.2. Consider the problem of optimal investment from terminal wealth ($\kappa(t) = 1_{[T]}(t),\ t \in [0, T]$), where S is a locally bounded semimartingale, U is a constant with respect to (t, ω) utility, i.e.,

$$U(t, \omega, x) = \bar{U}(x), \quad (t, \omega, x) \in [0, T] \times \Omega \times [0, \infty),$$

for some Inada utility function $\bar{U}$ (i.e., $\bar{U}$ is the same as in Example 4.1). Assume that (2.2) and Assumption 2.1 hold for an endowment process F and

$$\widetilde{w}(y) < +\infty, \quad y > 0, \tag{4.5}$$

where $\widetilde{w}$ is defined in (2.19). Then, the conditions of Lemma 2.6 are satisfied and therefore the conclusions of Theorem 2.4 hold for u and v. Observe the relationship with lemma 2 and theorem 2 in Hugonnier and Kramkov (2004). Let

$$N \text{ be a maximal process in } \mathcal{X}(1), \text{ such that } N_T \geq \frac{1}{M} \tag{4.6}$$

for some constant $M > 1$. We will show that the conclusions of Theorem 2.4 can be established for a (different) utility maximization problem under the numéraire N.

Let $\widetilde{S} \triangleq \left(\frac{S}{N}, \frac{1}{N}\right)$, $\widetilde{F} = \frac{F}{N}$ and define the sets $\widetilde{\mathcal{X}}(x)$ and $\widetilde{\mathcal{X}}(x,q)$ analogously to $\mathcal{X}(x)$ and $\mathcal{X}(x,q)$, respectively, for the processes $\widetilde{S}$ and $\widetilde{F}$ instead of S and F. Using corollary 14 in Delbaen and Schachermayer (1997b), one can show that

$$\left\{X_T : X \in \widetilde{\mathcal{X}}(x,q)\right\} = \left\{\frac{X_T}{N_T} : X \in \mathcal{X}(x,q)\right\}, \quad (x,q) \in \mathrm{cl}\mathcal{K}.$$

In particular, we have

$$\mathrm{int}\left\{(x,q) \in \mathbb{R}^{N+1} : \widetilde{\mathcal{X}}(x,q) \neq \emptyset\right\} = \mathcal{K}.$$

The value function under the numéraire N is defined as

$$\widetilde{u}(x,q) \triangleq \sup_{X \in \widetilde{\mathcal{X}}(x,q)} \mathbb{E}\left[\bar{U}(X_T + q\widetilde{F}_T)\right], \quad (x,q) \in \mathrm{cl}\mathcal{K}. \tag{4.7}$$

Let us introduce

$$\widetilde{U}(\omega, x) \triangleq \bar{U}\left(\frac{x}{N_T(\omega)}\right), \quad x \geq 0,\ \omega \in \Omega.$$

Then, the optimization problem (4.7) can be rewritten in the following form:

$$\widetilde{u}(x,q) = \sup_{X \in \mathcal{X}(x,q)} \mathbb{E}[\widetilde{U}(X_T + qF_T)], \quad (x,q) \in \mathrm{cl}\mathcal{K}. \tag{4.8}$$

REMARK 4.3. One can extend $\bar{U}$ and $\widetilde{U}$ to become utility stochastic fields satisfying Assumption 2.3. However, due to the form of the stochastic clock, only the values of the utility at time T matter in this example. With slight abuses of notation we will say that Assumption 2.3 holds for $\bar{U}$ and $\widetilde{U}$.

Let $\bar{V}$ and $\widetilde{V}$ denote the conjugates to $\bar{U}$ and $\widetilde{U}$, respectively. In particular, this implies that

$$\widetilde{V}(\omega, y) = \bar{V}(yN_T(\omega)), \quad y \geq 0,\ \omega \in \Omega.$$

Now, we can formulate the dual problem as follows:

$$\widetilde{v}(y,r) \triangleq \inf_{Y \in \mathcal{Y}(y,r)} \mathbb{E}\left[\widetilde{V}(Y_T)\right], \quad (y,r) \in \mathrm{cl}\mathcal{L}. \tag{4.9}$$

LEMMA 4.4. *Let $\bar{U}$ satisfy Assumption 2.3, and let* (2.2), (4.5), (4.6), *and Assumption 2.1 hold. Then, the conclusions of Theorem 2.4 are valid for the value functions defined in* (4.7) *and* (4.9).

Proof. Consider the primal problem for the numéraire N in the form (4.8).
For every $x > 0$, because $xN \in \mathcal{X}(x)$, we have

$$\widetilde{u}(x,0) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}\left[\bar{U}\left(\frac{X_T}{N_T}\right)\right] \geq \bar{U}(x) > -\infty.$$

Likewise, for every $y > 0$, we deduce

$$\inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[\widetilde{V}(Y_T)] = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[\bar{V}(Y_T N_T)] \leq \widetilde{w}\left(\frac{y}{M}\right) < +\infty.$$

Now the conclusions of Theorem 2.4 follow from Lemma 2.6. □

REFERENCES

BRANNATH, W., and W. SCHACHERMAYER (1999): A Bipolar Theorem for Subsets of $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$, *Séminaire de Probabilités XXXIII, Springer Lecture Notes in Mathematics XXXIII*, 349–354.

CUOCO, D. (1997): Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income, *J. Econom. Theory* 72, 33–73.

CVITANIĆ, J., W. SCHACHERMAYER, and H. WANG (2001): Utility Maximization in Incomplete Markets with Random Endowment, *Finance Stoch*. 5, 259–272.

DELBAEN, F., and W. SCHACHERMAYER (1994): A General Version of the Fundamental Theorem of Asset Pricing, *Math. Ann*. 300, 463–520.

DELBAEN, F., and W. SCHACHERMAYER (1997a): The Banach Space of Workable Contingent Claims in Arbitrage Theory, *Ann. Inst. H. Pincaré Statist. Probab*. 33, 113–144.

DELBAEN, F., and W. SCHACHERMAYER (1997b): The No-Arbitrage Property Under a Change of Numéraire, *Stoch. Stoch. Rep*. 53, 213–226.

DELBAEN, F., and W. SCHACHERMAYER (1998): The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes, *Math. Ann*. 312, 215–250.

DUFFIE, D., and T. ZARIPHOPOULOU (1993): Optimal Investment with Undiversifiable Income Risk, *Math. Finance* 3, 135–148.

DUFFIE, D., W. FLEMING, M. SONER, and T. ZARIPHOPOULOU (1997): Heding in Incomplete Markets with HARA Utility, *J. Econ. Dymaics Control* 21, 753–782.

HE, H., and N. D. PEARSON (1991a): Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constraints: The Finite-Dimensional Case, *Math. Finance* 1, 1–10.

HE, H., and N. D. PEARSON (1991b): Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constraints: The Infinite-Dimensional Case, *J. Econom. Theory* 54, 259–304.

HUGONNIER, J., and D. KRAMKOV (2004): Optimal Investment with Random Endowment in Incomplete Markets, *Ann. Appl. Probab*. 14, 845–864.

HUGONNIER, J., D. KRAMKOV, and W. SCHACHERMAYER (2005): On Utility-Based Pricing of Contingent Claims in Incomplete Markets, *Math. Finance* 15, 203–212.

JACOD, J., and A. N. SHIRYAEV (1980): *Limit Theorems for Stochastic Processes*, Berlin, Heidelberg, New York: Springer.

KARATZAS, I., and K. KARDARAS (2007): The Numéraire Portfolio in Semimartingale Financial models, *Finance Stoch*. 11, 447–493.

KARATZAS, I., and S. SHREVE (1998): *Methods of Mathematical Finance*, New York: Springer-Verlag.

KARATZAS, I., and G. ŽITKOVIĆ (2003): Optimal Consumption from Investment and Random Endowment in Incomplete Semi-Martingale Markets, *Ann. Probab*. 31, 1821–1858.

KARATZAS, I., J. P. LEHOCZKY, S. E. SHREVE, and G. L. XU (1991): Martingale and Duality Methods for Utility Maximization in an Incomplete Market, *SIAM J. Control Optim*. 29, 702–730.

KARDARAS, K., and G. ŽITKOVIĆ (2011): Stability of the Utility Maximization Problem with Random Endowment in Incomplete Markets, *Math. Finance* 21, 313–333.

KRAMKOV, D., and W. SCHACHERMAYER (1999): The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets, *Ann. Appl. Probab*. 9, 904–950.

KRAMKOV, D., and W. SCHACHERMAYER (2003): Necessary and Sufficient Conditions in the Problem of Optimal Investment in Incomplete Markets, *Ann. Appl. Probab*. 13, 1504–1516.

KRAMKOV, D., and M. SÎRBU (2006): Sensitivity Analysis of Utility-Based Prices and Risk-Tolerance Wealth Processes, *Ann. Appl. Probab*. 16(4), 2140–2194.

MOSTOVYI, O. (2015): Necessary and Sufficient Conditions in the Problem of Optimal Investment with Intermediate Consumption, *Finance Stoch.* 19(1), 135–159.

ROCKAFELLAR, R. T. (1970): *Convex Analysis,* Princeton, NJ: Princeton University Press.

SIORPAES, P. (2016): Do Arbitrage-Free Prices Come from Utility Maximization? *Math. Finance* 26(3), 602–616.

ŽITKOVIĆ, G. (2005): Utility Maximization with a Stochastic Clock and an Unbounded Random Endowment, *Ann. Appl. Probab*. 15, 748–777.