Noname manuscript No.
(will be inserted by the editor)

A family of non-conforming elements and the analysis of Nitsche's method for a singularly perturbed fourth order problem

Johnny Guzmán · Dmitriy Leykekhman · Michael Neilan

Received: date / Accepted: date

Abstract In this paper we address several issues arising from a singularly perturbed fourth order problem with small parameter ε. First, we introduce a new family of non-conforming elements. We then prove that the corresponding finite element method is robust with respect to the parameter ε and uniformly convergent to order $h^{1/2}$. In addition, we analyze the effect of treating the Neumann boundary condition weakly by Nitsche's method. We show that such treatment is superior when the parameter ε is smaller than the mesh size h and obtain sharper error estimates. Such error analysis is not restricted to the proposed elements and can easily be carried out to other elements as long as the Neumann boundary condition is imposed weakly. Finally, we discuss the local error estimates and the pollution effect of the boundary layers in the interior of the domain.

Mathematics Subject Classification (2000) 65M60, 65N12, 34D15

Keywords nonconforming finite elements, fourth order problem, biharmonic, singular perturbation, Nitsche

1 Introduction

We consider the fourth order problem:

$$\varepsilon^2 \Delta^2 u - \Delta u = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

J. Guzmán
Division of Applied Mathematics, Brown University, Providence, RI 02912
E-mail: johnny_guzman@brown.edu

D. Leykekhman
Department of Mathematics, University of Connecticut, Storrs, CT 06269
E-mail: dmitriy.leykekhman@uconn.edu

M. Neilan
Department of Mathematics and Center for Computation & Technology, Louisiana State University, Baton Rouge, LA 70803
Current address: Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
E-mail: neilan@pitt.edu

where $\Omega \subset \mathbf{R}^d$ $(d = 2, 3)$ is a convex polygonal domain, $f \in L^2(\Omega)$, and n denotes the outward unit normal of $\partial\Omega$. In two dimensions, the boundary value problem (1.1) arises in the context of linear elasticity of thin buckling plates with u representing the displacement of the plate. The dimensionless positive parameter ε, assumed to be small (i.e. $\varepsilon \ll 1$), is defined by

$$\varepsilon = \frac{t^3 E}{12(1-\nu^2)\ell^2 T}.$$

Here, t is the thickness of the plate, E the Young modulus of the elastic material, ν the Poisson ratio, ℓ characteristic diameter of the plate, and T the absolute value of the density of the isotropic stretching force applied at the end of the plate [11]. In three dimensions, problem (1.1) can be considered a gross simplification of the stationary Cahn-Hilliard equation with ε being the length of the transition region of phase separation.

Since problem (1.1) is fourth order, standard conformal finite element methods require function spaces to be subspaces of $H^2(\Omega)$. Such elements require polynomials of high degree and even in two dimensions are not easy to construct. Moreover, the approximation properties of such elements may not even be high on quasi-uniform meshes due to the presence of strong boundary layers that affect global regularity estimates [1,22,23]. Specifically, *a priori* estimates show (cf. Lemma 4)

$$\|u\|_{H^s(\Omega)} \le C\varepsilon^{3/2-s}\|f\|_{L^2(\Omega)}, \quad \text{for } s = 2, 3. \tag{1.2}$$

Therefore, due to their relative simplicity, non-conforming methods are an attractive option.

In this direction, there are several such methods available in the literature. The Morley element is a natural choice since it has the least number of degrees of freedom on each element for fourth order problems, as its basis functions consist of only quadratic polynomials [15,24]. However, since the Morley element is not convergent for second order problems [17,27], either the formulation of the Morley method must be modified or the element itself must be altered in order to obtain robust schemes. In the former category, a modified Morley method was proposed and analyzed in two and three dimensions in [28] and [29]. In these papers, the authors used the original Morley element in conjunction with an enriching operator within their numerical method. In the later category, Nilssen et al. [17] did not change the formulation of the Morley method, but rather enriched second degree polynomials with cubic bubble functions in two dimensions. Tai and Winther later extended this element to three dimensions in [26], and variations of the element were also proposed in [30]; all of these elements are low-order. Another approach to compute the solution to (1.1) is by using C^0 interior penalty methods. For this method, one uses standard continuous Lagrange finite elements while replacing the $C^1(\Omega)$ continuity requirement with penalization techniques. Such methods were proposed in [6,7,10] with $\varepsilon = 1$, and the corresponding simply supported plate problem was constructed and analyzed in [8].

In the three papers [17,28,29] it was shown that the modified Morley methods are convergent uniformly in ε in the energy norm with bounds of order $h^{1/2}$. In fact, in the last remark in [17] the authors argue that this is the best possible ε-independent error estimate for *any* finite element method on quasi-uniform meshes due to the strong boundary layers of the solution. This claim seems plausible if one takes in consideration all possible relations between ε and h. However in some regimes it is not the case. For example, modifying a method by imposing the Neumann boundary conditions weakly, we obtain better error estimates in the case of $\varepsilon < h$, while still retaining the $h^{1/2}$ and ε-uniform error estimate.

As the title suggests, there are several aims of this paper. First, we propose a family of non-conforming finite elements for the singular perturbation problem (1.1) that is robust with respect to the parameter ε. The element is motivated by the element constructed in [17], and although our lowest order element has the same degrees of freedom as the elements in [17,26], the finite element spaces are different. This is because we use face (3D)/edge (2D) bubbles to construct our elements. Our elements also come in a natural hierarchy, and therefore higher order approximations may be possible. We are not aware of any other family of elements with such a hierarchical construction for the biharmonic problem.

Our next contribution is that we modify our method to impose the Neumann boundary condition in (1.1b) weakly. The new error estimates show that this approach is superior to that enforcing boundary conditions strongly and may lead to better convergence rates. Although the analysis of such convergence rates is fairly involved, the concept is very natural. If we let $\bar{u}$ be to the solution to the reduced problem

$$-\Delta\bar{u} = f \qquad \text{in } \Omega, \tag{1.3a}$$

$$\bar{u} = 0 \qquad \text{on } \partial\Omega, \tag{1.3b}$$

we see that $\bar{u}$ has no dependence on ε, and therefore no boundary layers. Moreover, since Ω is convex, $\bar{u} \in H^2(\Omega)$ and there holds [12]

$$\|\bar{u}\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{1.4}$$

Furthermore, it can be shown (cf. Lemma 4) that $\|u - \bar{u}\|_{H^1(\Omega)} \le C\varepsilon^{1/2}$; that is for small ε the solution to (1.1) is very close to the solution to the reduces problem (1.3). Thus we naturally want to construct a finite element method that does not only approximate u for all ε, but also approximate $\bar{u}$ well when ε is small. Noting that $\bar{u}$ does not satisfy the second boundary condition in (1.1b), it is then logical to construct a method that does not enforce the normal derivative constraint on the boundary with $\varepsilon = 0$. This naturally leads us to formulation using Nitsche's ideas [18], that is, to impose the boundary condition weakly in the variational formulation using penalization techniques. We note that the use of Nitsche's method for singular problems is not new (e.g. [2,13,14,20,25]). However, as far as we are aware, this is the first time such an approach has been applied for problem (1.1). Moreover, we give a rigorous justification of the advantage of Nitsche's method for (1.1) by proving sharp error estimates in the case ε is small. Similar analysis has been developed by [20] for second order convection-diffusion problems.

The rest of the paper is organized as follows. In the next section we provide the notation that will be used throughout the paper. In Section 3 we introduce a family of elements for the singular perturbation problem in two and three dimensions. Here, we describe the local and global spaces, its associated degrees of freedom, and unisolvency. We also address the approximation properties of the space as well. In Section 4 we define and analyze the finite element method with weakly imposed boundary conditions. We show that the new method satisfies all of the estimates in [17,28,29], but in addition, derive a new estimate which is better for small ε-values. In Section 5, we derive L^2 error estimates. In Section 6 we mention how other popular methods can be modified to include Nitsche's method. Finally, in the last section we discuss the local error estimates and the pollution effect of the boundary layers in the interior of the domain.

2 Notation

We use $H^s(\Omega)$ $(s \ge 0)$ to denote the set of all $L^2(\Omega)$ functions whose distributional derivatives up to order s are in $L^2(\Omega)$, and $H^s_0(\Omega)$ the set of functions whose traces vanish up to order $s-1$ on $\partial\Omega$. We use $(\cdot,\cdot)_D$ to denote the L^2 inner product between two functions over a d-dimensional set D, $\langle\cdot,\cdot\rangle_G$ to denote the L^2 inner product over a set G of dimension less than d, and use the convention $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$.

Let $\mathcal{T}_h$ be a shape regular simplicial triangulation [5,9] of the domain Ω, and let $\mathcal{F}_h$ denote the set of $(d-1)$-dimensional simplices in $\mathcal{T}_h$, i.e., the set of faces (3D) or edges (2D) in $\mathcal{T}_h$. We adopt the following notations:

- $\mathcal{F}_h^i$ = the set of interior $(d-1)$-dimensional simplices in $\mathcal{T}_h$,
- $\mathcal{F}_h^b$ = the set of boundary$(d-1)$-dimensional simplices in $\mathcal{T}_h$,
- h_T = the diameter of T,
- $h = \max_{T\in\mathcal{T}_h} h_T$,
- h_F = the diameter of the $(d-1)$-dimensional simplex F,
- $v^\pm = v\big|_{T^\pm}$, the restriction of the function v to the simplex $T^\pm$,
- $\mathbb{P}^k(G)$ = the space of polynomials of degree less than or equal to k restricted to the set G.

Furthermore, we define the patch of $F \in \mathcal{F}_h$ and $T \in \mathcal{T}_h$ as

$$\mathcal{T}_F = \big\{T \in \mathcal{T}_h :\ F \subset \partial T\big\}, \qquad \mathcal{T}_T = \big\{T' \in \mathcal{T}_h :\ \partial T' \cap \partial T \neq \emptyset\big\},$$

and we use the convention

$$\|v\|^2_{H^m(\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} \|v\|^2_{H^m(T)}, \quad \|v\|^2_{H^m(\mathcal{T}_F)} = \sum_{T\in\mathcal{T}_F} \|v\|^2_{H^m(T)}, \quad \|v\|^2_{H^m(\mathcal{T}_T)} = \sum_{T'\in\mathcal{T}_T} \|v\|^2_{H^m(T')}.$$

For $F \in \mathcal{F}_h^i$, there exist two simplices, $T^+, T^- \in \mathcal{T}_h$ such that $F = \partial T^+ \cap \partial T^-$. We define the jump of the normal derivative of v on F as

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\Big|_F = \frac{\partial v^+}{\partial n_F}\Big|_F - \frac{\partial v^-}{\partial n_F}\Big|_F,$$

where n_F denotes the normal of F pointing from T^+ to T^-. On the boundary $F \in \mathcal{F}_h^b$, we take

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\Big|_F = \frac{\partial v}{\partial n_F}\Big|_F.$$

We also define the average of the normal derivative as

$$\left\{\!\!\left\{\frac{\partial v}{\partial n}\right\}\!\!\right\} = \frac{1}{2}\Big(\frac{\partial v^+}{\partial n_F}\Big|_F + \frac{\partial v^-}{\partial n_F}\Big|_F\Big) \quad \text{if } F \in \mathcal{F}_h^i, \qquad \left\{\!\!\left\{\frac{\partial v}{\partial n}\right\}\!\!\right\} = \frac{\partial v}{\partial n_F}\Big|_F \quad \text{if } F \in \mathcal{F}_h^b.$$

Given $T \in \mathcal{T}_h$, we denote by $\{\lambda_F\}$ the $(d+1)$ barycentric coordinates of T, labeled such that λ_F vanishes on the $(d-1)$-dimensional simplex $F \subset \partial T$. The element bubble and face/edge bubbles are then given by

$$b_T = \prod_F \lambda_F, \qquad b_F = \prod_{G\neq F} \lambda_G,$$

where the product runs over the $(d-1)$-dimensional simplices of T.

We use C to denote a generic constant independent of h, ε, or any penalty parameters that may take different values throughout the paper. In the analysis we will often use the following version of the trace inequality (see e.g., [5, Theorem 1.6.6]).

Proposition 1 *For any simply connected domain D with piece-wise smooth boundary ∂D there exists a constant $C > 0$ such that*

$$\|v\|_{L^2(\partial D)} \le C\|v\|^{1/2}_{L^2(D)}\|v\|^{1/2}_{H^1(D)}, \quad \forall v \in H^1(D). \tag{2.1}$$

In particular, by a standard scaling argument we obtain the following estimates on any simplex T

$$\|v\|_{L^2(\partial T)} \le C(h_T^{-1/2}\|v\|_{L^2(T)} + \|v\|_{L^2(T)}^{1/2}\|\nabla v\|_{L^2(T)}^{1/2}), \qquad \forall v \in H^1(T), \tag{2.2a}$$

$$\|v\|_{L^2(\partial T)} \le C(h_T^{-1/2}\|v\|_{L^2(T)} + h_T^{1/2}\|\nabla v\|_{L^2(T)}), \qquad \forall v \in H^1(T). \tag{2.2b}$$

In addition we will also need a standard inverse estimate. Let q *be fixed. Then for all* $v \in \mathbb{P}^q(T)$,

$$\|v\|_{L^2(\partial T)} \le C h_T^{-1/2}\|v\|_{L^2(T)}, \tag{2.3}$$

where C *is independent of* v.

3 A Family of Non-conforming Finite Elements

In this section we introduce a new family of non-conforming elements for the biharmonic problem (1.1). We mainly focus our ideas on the three dimensional case, and briefly discuss the corresponding two dimensional elements below as their construction and properties are similar.

Essentially, we add local function spaces that use bubble functions to Lagrange elements. This strategy was used in [17,26], where the authors developed low-order elements in two and three dimensions. Although our lowest order elements are similar to the non-conforming elements introduced in [17,26] the spaces are different. The key to our element is using face (3D)/edge (2D) bubbles.

First, for $k \ge 2$ we define the local space of the non-conforming element as

$$X^k(T) = \mathbb{P}^k(T) + Q^{k-2}(T), \tag{3.1}$$

with

$$Q^{k-2}(T) = b_T \sum_F b_F Q_F^{k-2}(T), \tag{3.2}$$

where the sum runs over the four faces of T and

$$Q_F^{k-2}(T) = \big\{q \in \mathbb{P}^{k-2}(T):\ (q, b_T b_F w)_T = 0 \text{ for all } w \in \mathbb{P}^{k-3}(T)\big\}. \tag{3.3}$$

That is, $Q_F^{k-2}(T)$ is the space of orthogonal polynomials of degree $\mathbb{P}^{k-2}(T)$ with respect to the inner product $(\cdot, b_T b_F \cdot)_T$. For the case $k = 2$, we set $Q_F^{k-2}(T) = \mathbb{P}^0(T)$.

We define the following degrees of freedom for the local space $X^k(T)$:

$$w(a) \qquad \text{for all vertices } a, \tag{3.4a}$$

$$\langle w, \mu\rangle_e \qquad \text{for all } \mu \in \mathbb{P}^{k-2}(e) \text{ and edges } e \text{ of } T, \tag{3.4b}$$

$$\langle w, \kappa\rangle_F \qquad \text{for all } \kappa \in \mathbb{P}^{k-3}(F) \text{ and faces } F \text{ of } T, \tag{3.4c}$$

$$(w, \rho)_T \qquad \text{for all } \rho \in \mathbb{P}^{k-4}(T), \tag{3.4d}$$

$$\langle \partial w/\partial n_F, \omega\rangle_F \qquad \text{for all } \omega \in \mathbb{P}^{k-2}(F) \text{ and faces } F \text{ of } T. \tag{3.4e}$$

Here we use the convention that if $k = 2$, then the degrees of freedom (3.4c) and (3.4d) are not needed, and if $k = 3$, then the degrees of freedom (3.4d) are not needed. Note that the first four types of degrees of freedom (3.4a)–(3.4d) uniquely determine a polynomial of degree k. Furthermore, by the definition of $Q_F^{k-1}(K)$, we have

$$\dim Q_F^{k-2}(T) = \dim \mathbb{P}^{k-2}(T) - \dim \mathbb{P}^{k-3}(T) = \frac{1}{2}k(k-1) = \dim \mathbb{P}^{k-2}(F). \tag{3.5}$$

Therefore by (3.2) we have $\dim Q^{k-1}(T) \le 4 \dim \mathbb{P}^{k-2}(F)$, and so $\dim X^k(T) \le \dim \mathbb{P}^k(T) + 4 \dim \mathbb{P}^{k-2}(F)$, which is exactly the number of degrees of freedom given in (3.4). The next lemma shows that this last inequality is in fact an equality.

Lemma 1 *There holds*

$$X^k(T) = \mathbb{P}^k(T) \oplus Q^{k-2}(T), \tag{3.6}$$
$$\dim X^k(T) = \dim \mathbb{P}^k(T) + 4\dim \mathbb{P}^{k-2}(F). \tag{3.7}$$

Furthermore, any function $w \in X^k(T)$ is uniquely determined by the degrees of freedom (3.4).

Proof Suppose that $w \in \mathbb{P}^k(T) \cap Q^{k-2}(T)$. Then since b_T is quartic, we have

$$w = b_T q \quad \text{with} \quad q \in \mathbb{P}^{k-4}(T), \quad \text{and} \quad q = \sum_F b_F q_F \quad \text{with} \quad q_F \in Q_F^{k-2}(T).$$

But by the definition of $Q_F^{k-2}(T)$ (3.3), we have

$$(q, b_T q)_T = \sum_F (q_F, b_T b_F q)_T = 0.$$

Therefore, $q \equiv 0$ and (3.6) follows.

To show (3.7), we note that by (3.5) and (3.6), it suffices to show that (3.2) is a direct sum. To this end, we show that if $q = b_K \sum_F b_F q_F = 0$, then $q_F = 0$ for all faces F. First, we note that

$$0 = \frac{\partial q}{\partial n_F}\Big|_F = \frac{\partial b_T}{\partial n_F} b_F q_F\Big|_F = -|\nabla \lambda_F| b_F^2 q_F\Big|_F,$$

where we used that $\frac{\partial b_T}{\partial n_F}\big|_F = -|\nabla \lambda_F| b_F$. Therefore $q_F\big|_F = 0$ for all faces F since b_F^2 is strictly positive on the face F. Thus, we have $q_F = \lambda_F p_F$ for some $p_F \in \mathbb{P}^{k-3}(T)$, and therefore by (3.3), we have

$$0 = (q_F, b_T b_F p_F)_T = (p_F, b_T b_F \lambda_F p_F)_T,$$

which implies $p_F \equiv 0$. Thus, $q_F \equiv 0$, and the dimension count (3.7) immediately follows.

Note that the dimension of $X^k(T)$ is the exact number of degrees of freedom given by (3.4). Thus, to show that the degrees of freedom (3.4) are unisolvent on the space $X^k(T)$, it suffices to show that if the degrees of freedom vanish for $w \in X^k(T)$, then $w \equiv 0$. Write

$$w = w_0 + q \quad \text{with} \quad w_0 \in \mathbb{P}^k(T),\ q \in Q^{k-2}(T),\ \text{and}\ q = b_K \sum_F b_F q_F \quad \text{with} \quad q_F \in Q_F^{k-2}(T).$$

By (3.4a)–(3.4c), we have $w_0 = b_T p$ for some $p \in \mathbb{P}^{k-4}(T)$. But then by (3.4d) and (3.3), we have

$$0 = (w, p)_T = (b_T p, p)_T + \sum_F (q_F, b_T b_F p)_T = (b_T p, p)_T,$$

and therefore $w = q$. Finally by (3.4e) we have for each face F,

$$0 = \big\langle \partial w/\partial n_F, q_F \big\rangle_F = \big\langle b_F q_F (\partial b_T/\partial n_F), q_F \big\rangle_F = -|\nabla \lambda_F| \big\langle b_F^2 q_F, q_F \big\rangle_F, \tag{3.8}$$

where we again used $\frac{\partial b_T}{\partial n_F}\big|_F = -|\nabla \lambda_F| b_F$. It follows from (3.8) that $q_F\big|_F = 0$ for each face F, and therefore, by using the same argument above, we conclude $q_F \equiv 0$.

The degrees of freedom (3.4) naturally lead us to the define the following global spaces:

$$X_h = \big\{w \in H_0^1(\Omega):\ w\big|_T \in X^k(T) \text{ for all } T \in \mathcal{T}_h, \tag{3.9}$$
$$\text{and } \big\langle [\![\partial w/\partial n]\!], \omega\big\rangle_F = 0 \text{ for all } \omega \in \mathbb{P}^{k-2}(F) \text{ and } F \in \mathcal{F}_h^i\big\},$$
$$X_h^0 = \big\{w \in X_h:\ \big\langle \partial w/\partial n_F, \omega\big\rangle_F = 0 \text{ for all } \omega \in \mathbb{P}^{k-2}(F) \text{ and } F \in \mathcal{F}_h^b\big\}. \tag{3.10}$$

The next lemma addresses the approximation properties of the global spaces. The degrees of freedom (3.4) naturally induce an interpolant from $H^2(\Omega) \cap H_0^1(\Omega)$ onto X_h. However, such an interpolant will not be bounded in $H^1(\Omega)$ and hence we need to define a regularized interpolant.

Lemma 2 *There exists an operator $I_h : H_0^1(\Omega) \to X_h$ such that for any $v \in H^s(\Omega) \cap H_0^1(\Omega)$ with $1 \le s \le k+1$, there holds*

$$\|v - I_h v\|_{H^m(\mathcal{T}_h)} \le C h^{s-m} \|v\|_{H^s(\Omega)} \qquad 0 \le m \le s. \tag{3.11}$$

Moreover, there exists $I_h^0 : H_0^1(\Omega) \to X_h^0$ such that

$$\|v - I_h^0 v\|_{H^m(\mathcal{T}_h)} \le C h^{s-m} \|v\|_{H^s(\Omega)} \qquad \text{if } v \in H_0^2(\Omega), \tag{3.12}$$
$$\|v - I_h^0 v\|_{H^m(\mathcal{T}_h)} \le C \left(h^{s-m} \|v\|_{H^s(\Omega)} + h^{1-m}\|v\|_{H^1(\Omega)}\right) \qquad \text{if } v \notin H_0^2(\Omega). \tag{3.13}$$

Proof Let $L_h \subset H_0^1(\Omega)$ be the Lagrange finite element space consisting of globally continuous piecewise polynomials of degree k with vanishing trace. Define the projection $\Pi_h : L_h \to X_h$ locally as

$$\big(\Pi_h w - w\big)(a) = 0 \qquad \text{for all vertices } a, \tag{3.14a}$$
$$\big\langle \Pi_h w - w, \mu\big\rangle_e = 0 \qquad \text{for all } \mu \in \mathbb{P}^{k-1}(e) \text{ and edges } e \text{ of } T, \tag{3.14b}$$
$$\big\langle \Pi_h w - w, \kappa\big\rangle_F = 0 \qquad \text{for all } \kappa \in \mathbb{P}^{k-2}(F) \text{ and faces } F \text{ of } T, \tag{3.14c}$$
$$(\Pi_h w - w, \rho)_T = 0 \qquad \text{for all } \rho \in \mathbb{P}^{k-3}(T), \tag{3.14d}$$
$$\big\langle \partial(\Pi_h w)/\partial n_F - \{\!\{\partial w/\partial n\}\!\}, \omega\big\rangle_F = 0 \qquad \text{for all } \omega \in \mathbb{P}^{k-2}(F) \text{ and faces } F \text{ of } T. \tag{3.14e}$$

By the proof of Lemma 1 and by the definition of X_h, Π_h is well-defined.

Note that for any simplex T that contains F as a face we have

$$\Big|\{\!\{\partial w/\partial n\}\!\}\big|_F - \partial(w\big|_T)/\partial n_F\big|_F\Big| = \begin{cases} \frac{1}{2}\big|[\![\partial w/\partial n]\!]\big| & F \in \mathcal{F}_h^i, \\ 0 & F \in \mathcal{F}_h^b. \end{cases} \tag{3.15}$$

Therefore by (3.14)–(3.15) and a scaling argument, we have

$$\|\Pi_h w - w\|_{L^2(T)} \le C h_T^{3/2} \sum_{F \subset \partial T \cap \mathcal{F}_h^i} \big\|[\![\partial w/\partial n]\!]\big\|_{L^2(F)}. \tag{3.16}$$

We then set $I_h = \Pi_h \Pi_C$, where $\Pi_C : H_0^1(\Omega) \to L_h$ is the Scott-Zhang interpolant [21]. Since the Scott-Zhang interpolant satisfies

$$\|v - \Pi_C v\|_{H^m(T)} \le C h_T^{s-m} \|v\|_{H^s(\mathcal{T}_T)}, \tag{3.17}$$

we have by a scaling argument,

$$h_T^{3/2} \sum_{F \subset \partial T \cap \mathcal{F}_h^i} \big\|[\![\partial(\Pi_C v)/\partial n]\!]\big\|_{L^2(F)} \le C h_T^s \|v\|_{H^s(\mathcal{T}_T)}. \tag{3.18}$$

Therefore by (3.16)–(3.18) and an inverse estimate, we obtain

$$\begin{aligned}\|v - I_h v\|_{H^m(T)} &\le \|v - \Pi_C v\|_{H^m(T)} + \|\Pi_C v - \Pi_h \Pi_C v\|_{H^m(T)} \\ &\le C h_T^{s-m} \|v\|_{H^s(\mathcal{T}_T)} + C h_T^{3/2-m} \sum_{F\subset \partial T \cap \mathcal{F}_h^i} \big\| [\![\partial(\Pi_C v)/\partial n]\!] \big\|_{L^2(F)} \\ &\le C h_T^{s-m} \|v\|_{H^s(\mathcal{T}_T)}.\end{aligned}$$

To construct an interpolant $I_h^0 v$ in X_h^0, we modify the construction of Π_h by replacing (3.14e) with

$$\begin{aligned}\big\langle \partial(\Pi_h w)/\partial n_F - \{\!\{\partial w/\partial n\}\!\}, \omega\big\rangle_F &= 0 \quad \text{for all } \omega \in \mathbb{P}^{k-2}(F) \text{ and interior faces } F \text{ of } T, \qquad (3.14\text{e}') \\ \big\langle \partial(\Pi_h w)/\partial n_F, \omega\big\rangle_F &= 0 \quad \text{for all } \omega \in \mathbb{P}^{k-2}(F) \text{ and boundary faces } F \text{ of } T.\end{aligned}$$

If we recall (3.15) for interior faces F and use the fact that

$$\big\langle \partial(\Pi_h w - w)/\partial n_F, \omega\big\rangle_F = -\big\langle \partial w/\partial n_F, \omega\big\rangle_F \quad \text{for all } \omega \in \mathbb{P}^{k-2}(F) \text{ and boundary faces } F \text{ of } T.$$

We then have

$$\|\Pi_h w - w\|_{L^2(T)} \le C h_T^{3/2} \sum_{F\subset\partial T} \big\| [\![\partial w/\partial n]\!] \big\|_{L^2(F)}. \tag{3.19}$$

Note that, in contrast to (3.16), the right-hand side of (3.19) may include boundary faces. However, we see that if $v \in H_0^2(\Omega)$, then the estimate (3.18) still holds with the sum taken over all faces of T (and not just interior faces). It then follows by using the same argument as above, that if $v \in H_0^2(\Omega)$, then

$$\|v - I_h^0 v\|_{H^m(T)} \le C h_T^{s-m} \|v\|_{H^s(\mathcal{T}_T)}.$$

However, if $v \notin H_0^2(\Omega)$ then we only obtain the estimate

$$h_T^{3/2} \sum_{F\subset \partial T\cap \mathcal{F}_h^b} \big\| [\![\partial(\Pi_C v)/\partial n]\!] \big\|_{L^2(F)} \le C h_T \|v\|_{H^1(\mathcal{T}_T)}.$$

It then follows that

$$\begin{aligned}\|v - I_h^0 v\|_{H^m(T)} &\le \|v - \Pi_C v\|_{H^m(T)} + \|\Pi_C v - \Pi_h \Pi_C v\|_{H^m(T)} \\ &\le C h_T^{s-m} \|v\|_{H^s(\mathcal{T}_T)} + C h_T^{3/2-m} \sum_{F\subset\partial T} \big\| [\![\partial(\Pi_C v)/\partial n]\!] \big\|_{L^2(F)} \\ &\le C \left(h_T^{s-m} \|v\|_{H^s(\mathcal{T}_T)} + h_T^{1-m} \|v\|_{H^1(\mathcal{T}_T)} \right).\end{aligned}$$

Remark 1 The lackluster estimate (3.13) will play a key role while discussing the advantages of enforcing boundary conditions weakly into the formulation (cf. Remark 3).

3.1 Remarks on the two dimensional elements

The family of two dimensional elements are similar to the three dimensional case, so we only sketch the details. The local space of the two dimensional element takes the same form as (3.1), but the sum in (3.2) now run over all edges. Also note that the definition of b_F in (3.2)–(3.3) are quadratic edge bubbles instead of cubic face bubbles.

The associated degrees of freedom of $X^k(T)$ are defined as follows:

$$w(a) \qquad \text{for all vertices } a \text{ of } T, \tag{3.20a}$$

$$\langle w, \mu \rangle_F \qquad \text{for all } \mu \in \mathbb{P}^{k-2}(F) \text{ and edges } F \text{ of } T, \tag{3.20b}$$

$$(w, \rho)_T \qquad \text{for all } \rho \in \mathbb{P}^{k-3}(T), \tag{3.20c}$$

$$\langle \partial w/\partial n_F, \omega \rangle_F \qquad \text{for all } \omega \in \mathbb{P}^{k-2}(F) \text{ and edges } F \text{ of } T. \tag{3.20d}$$

Lemma 3 *There holds*

$$X^k(T) = \mathbb{P}^k(T) \oplus Q^{k-2}(T), \tag{3.21}$$

$$\dim X^k(T) = \dim \mathbb{P}^k(T) + 3\mathbb{P}^{k-2}(F). \tag{3.22}$$

Furthermore, any function $w \in X^k(T)$ is uniquely determined by the degrees of freedom (3.20)*.*

The proof of Lemma 3 is very similar to that of Lemma 1, so we omit it. Similar to the three dimensional case, we can define the global spaces as (3.9)–(3.10) by substituting faces by edges in the definition. Moreover, following similar arguments to those found in the proof of Lemma 2, there exists interpolants $I_h : H_0^1(\Omega) \to X_h$ and $I_h^0 : H_0^1(\Omega) \to X_h^0$ such that the estimates (3.11)–(3.13) hold. Again, we omit the proof.

4 The finite element method

In this section we define the finite element method for (1.1) and analyze its convergence. First we provide the variational form of the singular biharmonic problem. A function $u \in H_0^2(\Omega)$ is a solution to (1.1) if for all test functions $v \in H_0^2(\Omega)$, there holds

$$A_\varepsilon(u, v) = (f, v), \tag{4.1}$$

where

$$A_\varepsilon(u, v) := \varepsilon^2 a(u, v) + b(u, v), \tag{4.2}$$

with

$$a(u, v) := (D^2 u, D^2 v), \qquad b(u, v) := (\nabla u, \nabla v). \tag{4.3}$$

Here, we have used the notation

$$(D^2 u, D^2 v) = \int_\Omega D^2 u : D^2 v \, dx = \sum_{i,j=1}^d \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \, dx,$$

and

$$(\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v \, dx = \sum_{i=1}^d \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx.$$

Similarly, a function $\bar{u} \in H_0^1(\Omega)$ is defined to be a solution to (1.3) if there holds

$$b(\bar{u}, v) = \int_\Omega f v \, dx \qquad \forall v \in H_0^1(\Omega). \tag{4.4}$$

Before continuing, we first state the following *a priori* estimates and convergence rates to the reduced problem.

Lemma 4 *Let u be the solution to* (4.2)*, and $\bar{u}$ the solution to the reduced problem* (1.3)*. Then $u \in H^3(\Omega)$, and there exists a constant $C > 0$ independent of ε, u, and f such that*

$$\|u\|_{H^s(\Omega)} \le C\varepsilon^{3/2-s}\|f\|_{L^2(\Omega)} \qquad s = 2, 3, \tag{4.5a}$$

$$\|u - \bar{u}\|_{H^1(\Omega)} \le C\varepsilon^{1/2}\|f\|_{L^2(\Omega)}, \tag{4.5b}$$

$$\|u - \bar{u}\|_{L^2(\Omega)} \le C\varepsilon\|f\|_{L^2(\Omega)}. \tag{4.5c}$$

Proof The proofs of the estimates (4.5a)–(4.5b) are given in [17]. To prove (4.5c), we use a duality argument. For arbitrary $v \in L^2(\Omega)$, let ψ be the solution to the following second order problem:

$$-\Delta\psi = v \qquad \text{in } \Omega, \tag{4.6a}$$

$$\psi = 0 \qquad \text{on } \partial\Omega. \tag{4.6b}$$

By (4.6), (1.1), (1.3), and integration by parts, we have

$$(u - \bar{u}, v) = b(u - \bar{u}, \psi) = -\varepsilon^2 a(u, \psi) + \varepsilon^2 \Big\langle \frac{\partial^2 u}{\partial n^2} \frac{\partial \psi}{\partial n} \Big\rangle_{\partial\Omega}.$$

Therefore by the trace inequality (2.1), the estimates (4.5a) and elliptic regularity, we obtain

$$\begin{aligned}
(u - \bar{u}, v) &\le C\varepsilon^2 \Big(\|u\|_{H^2(\Omega)} + \|u\|_{H^2(\Omega)}^{1/2} \|u\|_{H^3(\Omega)}^{1/2} \Big) \|\psi\|_{H^2(\Omega)} \\
&\le C\varepsilon^2 \Big(\varepsilon^{-1/2} + \big(\varepsilon^{-1/2}\big)^{1/2} \big(\varepsilon^{-3/2}\big)^{1/2} \Big) \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\le C\varepsilon \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.
\end{aligned}$$

The estimate (4.5c) then follows from this last inequality.

To define the finite element method, we introduce the bilinear form

$$\begin{aligned}
a_h(v, w) = \sum_{T \in \mathcal{T}_h} (D^2 v, D^2 w)_T &- \sum_{F \in \mathcal{F}_h^b} \Big\langle \frac{\partial^2 v}{\partial n_F^2}, \frac{\partial w}{\partial n_F} \Big\rangle_F \\
&- \sum_{F \in \mathcal{F}_h^b} \Big\langle \frac{\partial v}{\partial n_F}, \frac{\partial^2 w}{\partial n_F^2} \Big\rangle_F + \sigma \sum_{F \in \mathcal{F}_h^b} h_F^{-1} \Big\langle \frac{\partial v}{\partial n_F}, \frac{\partial w}{\partial n_F} \Big\rangle_F,
\end{aligned} \tag{4.7}$$

where σ is a positive penalization parameter. Although, theoretically σ needs to be sufficiently large to ensure the stability of the method (cf. Lemma 5 below), our numerical examples show that the method is robust even for very small values of σ, even for $\sigma = 0$ (cf. Section 8).

The finite element method then reads: Find $u_h \in X_h$ such that

$$A_{\varepsilon,h}(u_h, w) = (f, w) \qquad \forall w \in X_h, \tag{4.8}$$

where

$$A_{\varepsilon,h}(w, v) = \varepsilon^2 a_h(w, v) + b(w, v). \tag{4.9}$$

We define the following norms associated with problem (4.8):

$$\|v\|_{2,h}^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{F\in\mathcal{F}_h^b} h_F^{-1}\|\partial v/\partial n_F\|_{L^2(F)}^2, \tag{4.10}$$

$$\|v\|_{\varepsilon,h}^2 = \varepsilon^2\|v\|_{2,h}^2 + |v|_{H^1(\Omega)}^2. \tag{4.11}$$

The next lemma shows that the finite element method (4.8) is well-posed.

Lemma 5 *There exists a constant $\sigma_0 > 0$ depending only on the shape regularity of $\mathcal{T}_h$, such that for $\sigma \geq \sigma_0$, there holds*

$$\frac{1}{2}\|w\|_{\varepsilon,h}^2 \leq A_{\varepsilon,h}(w,w) \qquad \forall w \in X_h. \tag{4.12}$$

Proof To show (4.12), it suffices to show that the bilinear form $a_h(\cdot,\cdot)$ is coercive with respect to the norm $\|\cdot\|_{2,h}$. By the form's definition (4.7), we have

$$a_h(w,w) = \sum_{T\in\mathcal{T}_h} |w|_{H^2(T)}^2 - 2\sum_{F\in\mathcal{F}_h^b}\left\langle \partial^2 w/\partial n_F^2, \partial w/\partial n_F\right\rangle_F + \sigma\sum_{F\in\mathcal{F}_h^b} h_F^{-1}\|\partial w/\partial n_F\|_{L^2(F)}^2. \tag{4.13}$$

By the Cauchy-Schwarz inequality and a standard scaling argument, there exists a constant $C > 0$ depending only on the shape regularity of the mesh such that

$$\begin{aligned}
2\sum_{F\in\mathcal{F}_h^b}\left\langle \partial^2 w/\partial n_F^2, \partial w/\partial n_F\right\rangle_F &\leq 2\left(\sum_{F\in\mathcal{F}_h^b} h_F\|\partial^2 w/\partial n_F^2\|_{L^2(F)}^2\right)^{1/2}\left(\sum_{F\in\mathcal{F}_h^b} h_F^{-1}\|\partial w/\partial n_F\|_{L^2(F)}^2\right)^{1/2}\\
&\leq C\left(\sum_{F\in\mathcal{F}_h^b} |w|_{H^2(\mathcal{T}_F)}^2\right)^{1/2}\left(\sum_{F\in\mathcal{F}_h^b} h_F^{-1}\|\partial w/\partial n_F\|_{L^2(F)}^2\right)^{1/2}\\
&\leq \frac{1}{2}\sum_{T\in\mathcal{T}_h}|w|_{H^2(T)}^2 + \frac{C^2}{2}\sum_{F\in\mathcal{F}_h^b} h_F^{-1}\|\partial w/\partial n_F\|_{L^2(F)}^2.
\end{aligned}$$

Therefore by (4.13) we obtain

$$a_h(w,w) \geq \frac{1}{2}\sum_{T\in\mathcal{T}_h}|w|_{H^2(T)}^2 + \big(\sigma - \frac{C^2}{2}\big)\sum_{F\in\mathcal{F}_h^i} h_F^{-1}\|\partial w/\partial n_F\|_{L^2(F)}^2.$$

Choosing $\sigma_0 = \frac{1}{2}(C^2+1)$, we obtain (4.12).

Before deriving our main results, we need a few more technical lemmas. The first measures the interpolation error in the norm given by (4.11).

Lemma 6 *Let $u \in H^s(\Omega)$ with $3 \leq s \leq k+1$ be the solution to* (1.1), *and let I_h be the interpolant from Lemma* 2. *Then there holds*

$$\|u - I_h u\|_{\varepsilon,h} \leq C\big(h^{s-1} + \varepsilon h^{s-2}\big)\|u\|_{H^s(\Omega)}, \tag{4.14a}$$

$$\|u - I_h u\|_{\varepsilon,h} \leq C h^{1/2}\|f\|_{L^2(\Omega)}. \tag{4.14b}$$

Moreover, if the solution to the reduced problem satisfies $\bar{u} \in H^m(\Omega)$ for some $2 \leq m \leq k+1$, then

$$\|u - I_h u\|_{\varepsilon,h} \leq C\big(\varepsilon^{1/2}\|f\|_{L^2(\Omega)} + h^{m-1}\|\bar{u}\|_{H^m(\Omega)}\big). \tag{4.14c}$$

Proof By (3.11) and (4.5a), we have

$$\varepsilon^2 \sum_{T\in\mathcal{T}_h} |u - I_h u|^2_{H^2(T)} \le C\varepsilon^2 h^{2s-4} \|u\|^2_{H^s(\Omega)}, \tag{4.15a}$$

$$\varepsilon^2 \sum_{T\in\mathcal{T}_h} |u - I_h u|^2_{H^2(T)} \le C\varepsilon^2 h \|u\|_{H^2(\Omega)} \|u\|_{H^3(\Omega)} \le Ch\|f\|^2_{L^2(\Omega)}, \tag{4.15b}$$

and

$$\varepsilon^2 \sum_{T\in\mathcal{T}_h} |u - I_h u|^2_{H^2(T)} \le C\varepsilon^2 \|u\|^2_{H^2(\Omega)} \le C\varepsilon\|f\|^2_{L^2(\Omega)}. \tag{4.15c}$$

Similarly, we have by (3.11), (4.5a), and the trace inequality (2.2b),

$$\begin{aligned}\varepsilon^2 \sum_{F\in\mathcal{F}_h^b} h_F^{-1} \big\|\partial(u - I_h u)/\partial n_F\big\|^2_{L^2(F)} &\le C\varepsilon^2 \sum_{T\in\mathcal{T}_h} \Big(h_T^{-2}\|u - I_h u\|^2_{H^1(T)} + \|u - I_h u\|^2_{H^2(T)}\Big) \qquad (4.16a)\\ &\le C\varepsilon^2 h^{2s-4}\|u\|^2_{H^s(\Omega)},\\ \varepsilon^2 \sum_{F\in\mathcal{F}_h^b} h_F^{-1} \big\|\partial(u - I_h u)/\partial n_F\big\|^2_{L^2(F)} &\le C\varepsilon^2 \sum_{T\in\mathcal{T}_h} \Big(h_T^{-2}\|u - I_h u\|^2_{H^1(T)} + \|u - I_h u\|^2_{H^2(T)}\Big) \qquad (4.16b)\\ &\le C\varepsilon^2 h\|u\|_{H^2(\Omega)}\|u\|_{H^3(\Omega)} \le Ch\|f\|^2_{L^2(\Omega)},\end{aligned}$$

and

$$\begin{aligned}\varepsilon^2 \sum_{F\in\mathcal{F}_h^b} h_F^{-1} \big\|\partial(u - I_h u)/\partial n_F\big\|^2_{L^2(F)} &\le C\varepsilon^2 \sum_{T\in\mathcal{T}_h} \Big(h_T^{-2}\|u - I_h u\|^2_{H^1(T)} + \|u - I_h u\|^2_{H^2(T)}\Big) \qquad (4.16c)\\ &\le C\varepsilon^2\|u\|^2_{H^2(\Omega)} \le C\varepsilon\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

Moreover, we have by (3.11), (4.5a), (4.5b), and (1.4),

$$\begin{aligned}|u - I_h u|^2_{H^1(\Omega)} &\le Ch^{2s-2}\|u\|^2_{H^s(\Omega)}, \qquad (4.17a)\\ |u - I_h u|^2_{H^1(\Omega)} &\le C\big(|u - \bar{u} - I_h(u - \bar{u})|^2_{H^1(\Omega)} + |\bar{u} - I_h\bar{u}|^2_{H^1(\Omega)}\big) \qquad (4.17b)\\ &\le C\big(h\|u - \bar{u}\|_{H^1(\Omega)}\|u - \bar{u}\|_{H^2(\Omega)} + h^2\|\bar{u}\|^2_{H^2(\Omega)}\big) \le Ch\|f\|^2_{L^2(\Omega)}\end{aligned}$$

and

$$\begin{aligned}|u - I_h u|^2_{H^1(\Omega)} &\le C\big(|u - \bar{u} - I_h(u - \bar{u})|^2_{H^1(\Omega)} + |\bar{u} - I_h\bar{u}|^2_{H^1(\Omega)}\big) \qquad (4.17c)\\ &\le C\big(\|u - \bar{u}\|^2_{H^1(\Omega)} + h^{2m-2}\|\bar{u}\|^2_{H^m(\Omega)}\big)\\ &\le C\big(\varepsilon\|f\|^2_{L^2(\Omega)} + h^{2m-2}\|\bar{u}\|^2_{H^m(\Omega)}\big).\end{aligned}$$

The first estimate (4.14a) then follows from (4.11), (4.15a), (4.16a) and (4.17a); the second estimate (4.14b) follows from (4.11), (4.15b), (4.16b) and (4.17b); and the third estimate (4.14c) follows from (4.11), (4.15c), (4.16c) and (4.17c).

The next lemma is needed to analyze the error equation. It provides bounds for the interpolation error in the bilinear form restricted to the subspace.

Lemma 7 *Under the same hypotheses of Lemma 6, for any $w \in X_h$ there holds,*

$$A_{\varepsilon,h}(u - I_h u, w) \le C(1+\sigma)\big(h^{s-1} + \varepsilon h^{s-2}\big)\|u\|_{H^s(\Omega)}\|w\|_{\varepsilon,h}, \tag{4.18a}$$
$$A_{\varepsilon,h}(u - I_h u, w) \le C(1+\sigma)h^{1/2}\|f\|_{L^2(\Omega)}\|w\|_{\varepsilon,h}, \tag{4.18b}$$
$$A_{\varepsilon,h}(u - I_h u, w) \le C(1+\sigma)\big(\varepsilon^{1/2}\|f\|_{L^2(\Omega)} + h^{m-1}\|\bar{u}\|_{H^m(\Omega)}\big)\|w\|_{\varepsilon,h}. \tag{4.18c}$$

Proof By (4.9), we have

$$\begin{aligned} A_{\varepsilon,h}(u - I_h u, w) = {}& \varepsilon^2 \sum_{T\in\mathcal{T}_h} (D^2(u - I_h u), D^2 w)_T + (\nabla(u - I_h u), \nabla w) \\ & - \varepsilon^2 \sum_{F\in\mathcal{F}_h^b} \Big\langle \frac{\partial^2(u - I_h u)}{\partial n_F^2} \frac{\partial w}{\partial n_F} \Big\rangle_F - \varepsilon^2 \sum_{F\in\mathcal{F}_h^b} \Big\langle \frac{\partial(u - I_h u)}{\partial n_F}, \frac{\partial^2 w}{\partial n_F^2} \Big\rangle_F \\ & + \varepsilon^2 \sigma \sum_{F\in\mathcal{F}_h^b} h_F^{-1} \Big\langle \frac{\partial(u - I_h u)}{\partial n_F}, \frac{\partial w}{\partial n_F} \Big\rangle_F \\ =: {}& J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned} \tag{4.19}$$

By the Cauchy-Schwarz inequality, (2.3) and (4.11), we can show

$$J_1, J_2, J_4, J_5 \le C(1+\sigma)\|u - I_h u\|_{\varepsilon,h}\|w\|_{\varepsilon,h}.$$

Using (4.14) we have

$$J_1, J_2, J_4, J_5 \le C(1+\sigma)(\varepsilon h^{s-2} + h^{s-1})\|u\|_{H^s(\Omega)}\|w\|_{\varepsilon,h}, \tag{4.20a}$$
$$J_1, J_2, J_4, J_5 \le C(1+\sigma)h^{1/2}\|f\|_{L^2(\Omega)}\|w\|_{\varepsilon,h}, \tag{4.20b}$$

and

$$J_1, J_2, J_4, J_5 \le C(1+\sigma)\big(\varepsilon^{1/2}\|f\|_{L^2(\Omega)} + h^{m-1}\|\bar{u}\|_{H^m(\Omega)}\big)\|w\|_{\varepsilon,h}. \tag{4.20c}$$

Next, by the Cauchy-Schwarz inequality, the trace inequality (2.2b), (4.11) and (3.11), we have

$$\begin{aligned} J_3 &\le C\varepsilon^2 \left(\sum_{F\in\mathcal{F}_h^b} h_F \|\partial^2(u - I_h u)/\partial n_F^2\|_{L^2(F)}^2 \right)^{1/2} \left(\sum_{F\in\mathcal{F}_h^b} h_F^{-1} \|\partial w/\partial n_F\|_{L^2(F)}^2 \right)^{1/2} \\ &\le C\varepsilon \left(\sum_{T\in\mathcal{T}_h} \Big(\|u - I_h u\|_{H^2(T)}^2 + h_T^2 \|u - I_h u\|_{H^3(T)}^2 \Big) \right)^{1/2} \|w\|_{\varepsilon,h} \\ &\le C\varepsilon h^{s-2}\|u\|_{H^s(\Omega)}\|w\|_{\varepsilon,h}. \end{aligned} \tag{4.21a}$$

Furthermore, by the trace inequality (2.2b), (3.11), the inverse estimate (2.3), and (4.5a), we have

$$\begin{aligned} J_3 &\le C\varepsilon^2 \left(\sum_{T\in\mathcal{T}_h} \Big(h_T^{-2} \|u - I_h u\|_{H^2(T)}^2 + \|u - I_h u\|_{H^3(T)}^2 \Big) \right)^{1/2} \|w\|_{H^1(T)} \\ &\le C\varepsilon^2 \|u\|_{H^3(\Omega)}\|w\|_{\varepsilon,h} \le C\varepsilon^{1/2}\|f\|_{L^2(\Omega)}\|w\|_{\varepsilon,h}. \end{aligned} \tag{4.20c}$$

Note that if $\varepsilon \le h$, then this last estimate also implies

$$J_3 \le Ch^{1/2}\|f\|_{L^2(\Omega)}\|w\|_{\varepsilon,h}. \tag{4.20b$'$}$$

On the other hand, if $h \le \varepsilon$ we can use (4.21a) with $s = 3$ and (4.5a) to obtain

$$J_3 \le C\varepsilon h\|u\|_{H^3(\Omega)}\|w\|_{\varepsilon,h} \le Ch^{1/2}\|f\|_{L^2(\Omega)}\|w\|_{\varepsilon,h}. \tag{4.20b$''$}$$

Finally, the estimate (4.18a) follows from (4.19), (4.20a) and (4.21a); the second estimate (4.18b) follows from (4.19), (4.20b) and (4.20b); and the third estimate (4.18c) follows from (4.19), (4.20c) and (4.20c).

We now are now in position to state the prove the main result of this section.

Theorem 1 *Suppose that $u \in H^s(\Omega)$ with $3 \le s \le k+1$. Then there exists a constant C independent of ε and h such that*

$$\|u - u_h\|_{\varepsilon,h} \le C(\varepsilon h^{s-2} + h^{s-1})\|u\|_{H^s(\Omega)}, \tag{4.22a}$$

$$\|u - u_h\|_{\varepsilon,h} \le Ch^{1/2}\|f\|_{L^2(\Omega)}. \tag{4.22b}$$

Moreover, if $\bar{u} \in H^m(\Omega)$ with $2 \le m \le k+1$, then

$$\|u - u_h\|_{\varepsilon,h} \le C\big(\varepsilon^{1/2}\|f\|_{L^2(\Omega)} + h^{m-1}\|\bar{u}\|_{H^m(\Omega)}\big). \tag{4.22c}$$

Remark 2 We would like to point out that for $\varepsilon = o(h)$ the estimate (4.22c) gives a better order approximation than the uniform order $h^{1/2}$.

Proof Let $I_h u$ be the interpolant of u defined in Lemma 2 and set $w = I_h u - u_h \in X_h$. Then by (4.12) and (4.8), we have

$$\frac{1}{2}\|w\|_{\varepsilon,h} \le A_{\varepsilon,h}(I_h u - u_h, w) = A_{\varepsilon,h}(I_h u - u, w) + E_h(u, w), \tag{4.23}$$

where the consistency error is defined as

$$E_h(u, w) = A_{\varepsilon,h}(u, w) - (f, w) = \varepsilon^2 a_h(u, w) + b(u, w) - (f, w). \tag{4.24}$$

Since $u \in H^3(\Omega)$, equation (1.1) can be considered in the $H^{-1}(\Omega)$ sense, and therefore we have

$$-\varepsilon^2(\nabla\Delta u, \nabla w) + (\nabla u, \nabla w) - (f, w) = 0 \qquad \forall w \in X_h \subset H_0^1(\Omega).$$

Hence, by (4.24), (1.1b), (4.7) and integration by parts,

$$E_h(u, w) = \varepsilon^2 \sum_{F\in\mathcal{F}_h^i} \Big\langle \frac{\partial^2 u}{\partial n_F^2}, \Big[\!\!\Big[\frac{\partial w}{\partial n}\Big]\!\!\Big]\Big\rangle_F. \tag{4.25}$$

By the definition of X_h (3.9), there holds for any $\omega \in \mathbb{P}^{k-2}(F)$,

$$\Big\langle \frac{\partial^2 u}{\partial n_F^2}, \Big[\!\!\Big[\frac{\partial w}{\partial n}\Big]\!\!\Big]\Big\rangle_F = \Big\langle \frac{\partial^2 u}{\partial n_F^2} - \omega, \Big[\!\!\Big[\frac{\partial w}{\partial n}\Big]\!\!\Big]\Big\rangle_F.$$

In particular, we have

$$\Big\langle \frac{\partial^2 u}{\partial n_F^2}, \Big[\!\Big[\frac{\partial w}{\partial n}\Big]\!\Big] \Big\rangle_F = \Big\langle \Big\{\!\!\Big\{\frac{\partial^2 (u - \Pi_C u)}{\partial n^2}\Big\}\!\!\Big\}, \Big[\!\Big[\frac{\partial w}{\partial n}\Big]\!\Big] \Big\rangle_F$$
$$\le \big\| \{\!\{\partial^2 (u - \Pi_C u)/\partial n^2\}\!\} \big\|_{L^2(F)} \big\| [\![\partial w/\partial n]\!] \big\|_{L^2(F)}, \tag{4.26}$$

where we recall that Π_C is the Scott-Zhang interpolant of order k which has the following approximation properties:

$$\|u - \Pi_C u\|_{H^m(\mathcal{T}_h)} \le C\, h^{s-m} \|u\|_{H^s(\mathcal{T}_h)} \quad \text{for } 1 \le m \le s \le k+1. \tag{4.27}$$

We bound each of the factors on the right of (4.26) separately. Using that the average of $[\![\frac{\partial w}{\partial n}]\!]$ vanishes on F we have by Poincare's inequality that

$$\big\| [\![\partial w/\partial n]\!] \big\|_{L^2(F)} \le C\, h_F \big\| \nabla_F ([\![\partial w/\partial n]\!]) \big\|_{L^2(F)},$$

where ∇_F is the surface gradient on F. We then have by the inverse estimate (2.3) that

$$\big\| [\![\partial w/\partial n]\!] \big\|_{L^2(F)} \le C\, h_F^{1/2} |w|_{H^2(\mathcal{T}_F)}. \tag{4.28}$$

Of course, using the inverse estimate (2.3) twice we also have

$$\big\| [\![\partial w/\partial n]\!] \big\|_{L^2(F)} = \big\| [\![\partial w/\partial n]\!] \big\|_{L^2(F)}^{1/2} \big\| [\![\partial w/\partial n]\!] \big\|_{L^2(F)}^{1/2} \le C\, |w|_{H^1(\mathcal{T}_F)}^{1/2} |w|_{H^2(\mathcal{T}_F)}^{1/2}. \tag{4.29}$$

Next, using the trace inequality (2.2a) we obtain

$$\big\| \{\!\{\partial^2 (u - \Pi_C u)/\partial n^2\}\!\} \big\|_{L^2(F)} \tag{4.30}$$
$$\le C\,\big(h_F^{-1/2} |u - \Pi_C u|_{H^2(\mathcal{T}_F)} + |u - \Pi_C u|_{H^2(\mathcal{T}_F)}^{1/2} |u - \Pi_C u|_{H^3(\mathcal{T}_F)}^{1/2} \big).$$

Using (4.26), (4.28), (4.30) and (4.27) we obtain

$$E_h(u, w) \le C\, \varepsilon h^{s-2} \|u\|_{H^s(\Omega)} \|w\|_{\varepsilon,h}. \tag{4.31}$$

The error estimate (4.22a) then follows from (4.23), (4.18a), (4.31), the triangle inequality and (4.14a).

Alternatively, by (4.26), (4.28), (4.30), (4.27) and (4.5a) we get

$$E_h(u, w) \le C\, \varepsilon h^{1/2} \|u\|_{H^3(\Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2} \|w\|_{\varepsilon,h} \le C h^{1/2} \|f\|_{L^2(\Omega)} \|w\|_{\varepsilon,h}. \tag{4.32}$$

The error estimate (4.22b) follows from (4.23), (4.18b), (4.32) and (4.14b).

Finally, by (4.26), (4.29), (4.30), (4.27) and (4.5a) we see that

$$E_h(u, w) \le C\varepsilon^2 \|u\|_{H^3(\Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2} |w|_{H^2(\mathcal{T}_h)}^{1/2} |w|_{H^1(\Omega)}^{1/2} \tag{4.33}$$
$$\le C\varepsilon^{3/2} \|u\|_{H^3(\Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2} \|w\|_{\varepsilon,h} \le C\varepsilon^{1/2} \|f\|_{L^2(\Omega)} \|w\|_{\varepsilon,h}.$$

The third estimate (4.22c) is then deduced from (4.23), (4.18c), (4.33) and (4.14c).

4.1 Remarks on imposing boundary conditions strongly

Similar to other non-conforming finite element methods (e.g. [15,17,24]), we can define a finite element method for (1.1) that imposes the normal derivative boundary conditions strongly in the finite element space. Although this approach is perhaps more natural than the one proposed above, we argue that the method does not inherit as good convergence properties.

In this case, we define the method as finding $u_h^0 \in X_h^0$ such that

$$A_{\varepsilon,h}^0(u_h^0, v) = (f, v) \qquad \forall v \in X_h^0, \tag{4.34}$$

where

$$A_{\varepsilon,h}^0(w, v) = \varepsilon^2 \sum_{T\in\mathcal{T}_h} (D^2 w, D^2 v)_T + (\nabla w, \nabla v),$$

and we recall that the finite element space X_h^0 is defined by (3.10).

It is easily to see that (4.34) is well-posed as

$$\|v\|_A^2 := A_{\varepsilon,h}^0(v, v) \tag{4.35}$$

is a norm on X_h^0. Furthermore, we have the following convergence result.

Theorem 2 *Let $u \in H^s(\Omega)$ with $(3 \le s \le k+1)$ be the solution to (1.1), and let $u_h^0 \in X_h^0$ satisfy (4.34). Then there holds*

$$\|u - u_h^0\|_A \le C\big(h^{s-1} + \varepsilon h^{s-2}\big)\|u\|_{H^s(\Omega)}, \tag{4.36a}$$

$$\|u - u_h^0\|_A \le C h^{1/2}\|u\|_{H^s(\Omega)}. \tag{4.36b}$$

Proof The proof of Theorem 2 is similar to that of Theorem 1 so we only provide a sketch.

First, by the proof of Lemma 6 along with the interpolation estimates (3.12)–(3.13), we have

$$\|u - I_h^0 u\|_A \le C\big(h^{s-1} + \varepsilon h^{s-2}\big)\|u\|_{H^s(\Omega)}, \tag{4.37a}$$

$$\|u - I_h^0 u\|_A \le C h^{1/2}\|f\|_{L^2(\Omega)}. \tag{4.37b}$$

Next, setting $w = I_h^0 u - u_h^0$, we have by (4.35) and (4.34),

$$\|w\|_A^2 = A_{\varepsilon,h}^0(I_h^0 u - u, w) + E_h^0(u, w), \tag{4.38}$$

where $E_h^0(u, w) = A_{\varepsilon,h}^0(u, w) - (f, w)$.

By the proof of Lemma 7 and by (3.12)–(3.13), we have

$$A_{\varepsilon,h}^0(u - I_h^0 u, w) \le C(1+\sigma)\big(h^{s-1} + \varepsilon h^{s-2}\big)\|u\|_{H^s(\Omega)}\|w\|_A, \tag{4.39a}$$

$$A_{\varepsilon,h}^0(u - I_h^0 u, w) \le C(1+\sigma)h^{1/2}\|f\|_{L^2(\Omega)}\|w\|_A. \tag{4.39b}$$

Next, after integrating by parts, we conclude

$$E_h^0(u, w) = \varepsilon^2 \sum_{F\in\mathcal{F}_h} \Big\langle \frac{\partial^2 u}{\partial n_F^2}, \Big[\!\!\Big[\frac{\partial w}{\partial n}\Big]\!\!\Big] \Big\rangle_F.$$

Thus, by using the same arguments in the proof of Theorem 1, we have

$$E_h^0(u, w) \le C\big(h^{s-1} + \varepsilon h^{s-2}\big)\|u\|_{H^s(\Omega)}\|w\|_A, \tag{4.40a}$$

$$E_h^0(u, w) \le C h^{1/2}\|f\|_{L^2(\Omega)}\|w\|_A. \tag{4.40b}$$

The estimate (4.36a) then follows from (4.38), (4.39a), (4.40a) and (4.37a), where as the estimate (4.36b) follows from (4.38), (4.39b), (4.40b) and (4.37b).

Remark 3 Note that the error $u - u_h^0$ does *not* satisfy an estimate of the form (4.22c). This is due to the fact that we cannot derive an estimate of the interpolation error of the form (4.14c). By carefully studying the proof of Lemma 6, we find a problem occurs when we try to estimate the term $|u - I_h^0 u|_{H^1(\Omega)}$ due to the interpolation result (3.13). Indeed, in comparison to (4.17c), we have by (3.13) and (4.5a),

$$\begin{aligned}|u - I_h^0 u|_{H^1(\Omega)} &\le C\big(|u - \bar u - I_h^0(u - \bar u)|_{H^1(\Omega)} + |\bar u - I_h^0 \bar u|_{H^1(\Omega)}\big)\\ &\le C\big(\|u - \bar u\|_{H^1(\Omega)} + |\bar u - I_h^0 \bar u|_{H^1(\Omega)}\big)\\ &\le C\big(\varepsilon^{1/2}\|f\|_{L^2(\Omega)} + |\bar u - I_h^0 \bar u|_{H^1(\Omega)}\big).\end{aligned}$$

However, by (3.13) we can only conclude

$$|\bar u - I_h^0 \bar u|_{H^1(\Omega)} \le C\|\bar u\|_{H^1(\Omega)},$$

and therefore the estimate does not yield anything.

From this result, we see the advantages of imposing Neumann boundary conditions weakly. In particular, if $\varepsilon \ll h$, then by (4.22c) we will observe convergence rates of $O(h^{m-1})$ in the energy norm instead of convergence rates of $O(h^{1/2})$.

5 L^2 error estimates

In this section we prove error estimates in the L^2 norm for ε small relative to h. Again, we see the advantage of using Nitsche's method. We also state an ε-independent estimates.

Theorem 3 *Let u and u_h satisfy* (1.1) *and* (4.8) *respectively. Then, for $\varepsilon \le h$ and for $\bar u \in H^m(\Omega)$ with $2 \le m \le k+1$, there exists a constant C such that*

$$\|u - u_h\|_{L^2(\Omega)} \le C(1+\sigma)^2\Big(\big(h\varepsilon^{1/2} + \varepsilon\big)\|f\|_{L^2(\Omega)} + h^m\|\bar u\|_{H^m(\Omega)}\Big).$$

In particular, we have

$$\|u - u_h\|_{L^2(\Omega)} \le C(1+\sigma)^2\big(\varepsilon + h^2\big)\|f\|_{L^2(\Omega)}.$$

Proof We can decompose the error as

$$u - u_h = (u - \bar u) + (\bar u - u_h). \tag{5.1}$$

By (4.5b), we have

$$\|u - \bar u\|_{L^2(\Omega)} \le C\varepsilon\|f\|_{L^2(\Omega)}. \tag{5.2}$$

Thus we only need to estimate $\|\bar u - u_h\|_{L^2(\Omega)}$.

Let $\bar z \in H_0^1(\Omega)$ be the solution to the following problem:

$$(\nabla \bar z, \nabla w) = (\bar u - u_h, w), \quad \forall w \in H_0^1(\Omega). \tag{5.3}$$

We then have

$$\|\bar u - u_h\|_{L^2(\Omega)}^2 = \big(\nabla(\bar z - I_h \bar z), \nabla(\bar u - u_h)\big) + \big(\nabla I_h \bar z, \nabla(\bar u - u_h)\big) =: J_1 + J_2. \tag{5.4}$$

By (3.11), the triangle inequality, (4.5b) and (4.11),

$$J_1 \le Ch\|\bar{u}-u_h\|_{H^1(\Omega)}\|\bar{z}\|_{H^2(\Omega)} \le Ch\big(\varepsilon^{1/2}\|f\|_{L^2(\Omega)} + \|u-u_h\|_{\varepsilon,h}\big)\|\bar{z}\|_{H^2(\Omega)}. \tag{5.5}$$

To estimate J_2 we have by (4.8), (1.3) and (4.7),

$$\begin{aligned} J_2 &= \varepsilon^2 a_h(u_h, I_h\bar{z}) \\ &= \varepsilon^2 \sum_{T\in\mathcal{T}_h} (D^2 u_h, D^2 I_h\bar{z})_T - \varepsilon^2 \sum_{F\in\mathcal{F}_h^b} \Big\langle \frac{\partial u_h}{\partial n_F}, \frac{\partial^2 I_h\bar{z}}{\partial n_F^2}\Big\rangle_F \\ &\quad + \varepsilon^2\sigma \sum_{F\in\mathcal{F}_h^b} h_F^{-1}\Big\langle \frac{\partial u_h}{\partial n_F}, \frac{\partial(I_h\bar{z}-\bar{z})}{\partial n_F}\Big\rangle_F - \varepsilon^2 \sum_{F\in\mathcal{F}_h^b} \Big\langle \frac{\partial^2 u_h}{\partial n_F^2}, \frac{\partial(I_h\bar{z}-\bar{z})}{\partial n_F}\Big\rangle_F \\ &\quad + \varepsilon^2\sigma \sum_{F\in\mathcal{F}_h^b} h_F^{-1}\Big\langle \frac{\partial u_h}{\partial n_F}, \frac{\partial \bar{z}}{\partial n_F}\Big\rangle_F - \varepsilon^2 \sum_{F\in\mathcal{F}_h^b} \Big\langle \frac{\partial^2 u_h}{\partial n_F^2}, \frac{\partial \bar{z}}{\partial n_F}\Big\rangle_F \\ &=: K_1 + K_2 + K_3 + K_4 + K_5 + K_6. \end{aligned} \tag{5.6}$$

Using the Cauchy-Schwarz, inverse inequalities, (4.5a), (4.22b) and (3.11), we have

$$K_1 \le C\varepsilon^2 h^{-1}\|u_h\|_{H^1(\Omega)}\|I_h\bar{z}\|_{H^2(\mathcal{T}_h)} \le C\varepsilon\|f\|_{L^2(\Omega)}\|\bar{z}\|_{H^2(\Omega)}, \tag{5.7}$$

where we used that $\varepsilon \le h$.

Bounding K_2 we use that $\partial u/\partial n = 0$ on $\partial\Omega$. Thus by the Cauchy-Schwarz, the inverse inequality (2.3), and (3.11),

$$\begin{aligned} K_2 &\le \varepsilon^2 \Bigg(\sum_{F\in\mathcal{F}_h^b} h_F^{-1}\big\|\partial(u-u_h)/\partial n_F\big\|_{L^2(F)}^2\Bigg)^{1/2} \Bigg(\sum_{F\in\mathcal{F}_h^b} h_F\big\|\partial^2 I_h\bar{z}/\partial n_F^2\big\|_{L^2(F)}^2\Bigg)^{1/2} \\ &\le C\varepsilon\|u-u_h\|_{\varepsilon,h}\|I_h\bar{z}\|_{H^2(\mathcal{T}_h)} \le C\varepsilon\|u-u_h\|_{\varepsilon,h}\|\bar{z}\|_{H^2(\Omega)}. \end{aligned} \tag{5.8}$$

Below we will need the following estimate which follows from the trace inequality (2.2b) and the interpolation estimate (4.14a)

$$\big\|\partial(I_h\bar{z}-\bar{z})/\partial n\big\|_{L^2(\partial F)} \le h^{1/2}\|\bar{z}\|_{H^2(\mathcal{T}_F)}. \tag{5.9}$$

We easily see using (5.9) that we have the bound

$$K_3 \le C\sigma\varepsilon\|u-u_h\|_{\varepsilon,h}\|\bar{z}\|_{H^2(\Omega)}. \tag{5.10}$$

Bounding K_4, we have use (5.9) and the inverse estimate (2.3)

$$K_4 \le C\varepsilon^2\|u_h\|_{H^2(\mathcal{T}_h)}\|\bar{z}\|_{H^2(\Omega)}.$$

Note that by the triangle and inverse inequalities, (3.11), (4.5a) and (4.22b),

$$\begin{aligned} \|u-u_h\|_{H^2(\mathcal{T}_h)} &\le \|u-I_h u\|_{H^2(\mathcal{T}_h)} + h^{-1}\|I_h u - u_h\|_{H^1(\Omega)} \\ &\le C\|u\|_{H^2(\Omega)} + h^{-1}\|u-u_h\|_{H^1(\Omega)} \\ &\le C\big(\varepsilon^{-1/2} + h^{-1/2}\big)\|f\|_{L^2(\Omega)} \le C\varepsilon^{-1/2}\|f\|_{L^2(\Omega)} \end{aligned} \tag{5.11}$$

since $\varepsilon \le h$. Thus, $\|u_h\|_{H^2(\mathcal{T}_h)} \le C\varepsilon^{-1/2}\|f\|_{L^2(\Omega)}$, and therefore

$$K_4 \le \varepsilon^{3/2}\|f\|_{L^2(\Omega)}\|\bar{z}\|_{H^2(\Omega)}. \tag{5.12}$$

Bounding K_5, we again use that $\partial u/\partial n = 0$ on $\partial\Omega$. Thus by the Cauchy-Schwarz inequality, the trace inequality (2.2b), the trace inequality (2.1), (5.11), (4.22b), and the fact $\varepsilon \le h$,

$$\begin{aligned} K_5 &\le \sigma\varepsilon^2 \left(\sum_{F\in\mathcal{F}_h^b} h_F^{-2}\big\|\partial(u-u_h)/\partial n_F\big\|^2_{L^2(F)}\right)^{1/2} \|\partial\bar{z}/\partial n\|_{L^2(\partial\Omega)} \\ &\le C\sigma\varepsilon^2 \left(\sum_{F\in\mathcal{F}_h^b} \big(h_F^{-3}\|u-u_h\|^2_{H^1(\mathcal{T}_F)} + h_F^{-1}\|u-u_h\|^2_{H^2(\mathcal{T}_F)}\big)\right)^{1/2} \|\bar{z}\|_{H^2(\Omega)} \\ &\le C\sigma\varepsilon^2\big(h^{-3/2}\|u-u_h\|_{H^1(\Omega)} + h^{-1/2}\|u-u_h\|_{H^2(\mathcal{T}_h)}\big)\|\bar{z}\|_{H^2(\Omega)} \\ &\le C\sigma\varepsilon^2\big(h^{-1} + h^{-1/2}\varepsilon^{-1/2}\big)\|f\|_{L^2(\Omega)}\|\bar{z}\|_{H^2(\Omega)} \\ &\le C\sigma\varepsilon\|f\|_{L^2(\Omega)}\|\bar{z}\|_{H^2(\Omega)}. \end{aligned} \tag{5.13}$$

Similarly we can bound K_6 as follows

$$\begin{aligned} K_6 &\le \varepsilon^2 \left(\sum_{F\in\mathcal{F}_h^b} \|\partial^2 u_h/\partial n_F^2\|^2_{L^2(F)}\right)^{1/2} \|\partial\bar{z}/\partial n\|_{L^2(\partial\Omega)} \\ &\le \varepsilon^2 h^{-1/2}\|u_h\|_{H^2(\mathcal{T}_h)}\|\bar{z}\|_{H^2(\Omega)} \le \varepsilon\|f\|_{L^2(\Omega)}\|\bar{z}\|_{H^2(\Omega)}, \end{aligned} \tag{5.14}$$

where again, we used the hypothesis $\varepsilon \le h$.

Applying the estimates (5.7)–(5.10) and (5.12)–(5.14) to (5.6) we have

$$J_2 \le C(1+\sigma)\big(\varepsilon\|f\|_{L^2(\Omega)} + \varepsilon\|u-u_h\|_{\varepsilon,h}\big)\|\bar{z}\|_{H^2(\Omega)}.$$

Combining this last inequality with (5.5) and (5.4), and using the H^2 regularity of $\bar{z}$, we conclude

$$\|\bar{u}-u_h\|_{L^2(\Omega)} \le C(1+\sigma)\Big((h\varepsilon^{1/2}+\varepsilon)\|f\|_{L^2(\Omega)} + (h+\varepsilon)\|u-u_h\|_{\varepsilon,h}\Big).$$

It then follows from (5.1), (5.2) and (4.22c) that

$$\begin{aligned} \|u-u_h\|_{L^2(\Omega)} &\le C(1+\sigma)\Big((h\varepsilon^{1/2}+\varepsilon)\|f\|_{L^2(\Omega)} + (h+\varepsilon)\|u-u_h\|_{\varepsilon,h}\Big) \\ &\le C(1+\sigma)\Big((h\varepsilon^{1/2}+\varepsilon)\|f\|_{L^2(\Omega)} + h\|u-u_h\|_{\varepsilon,h}\Big) \\ &\le C(1+\sigma)^2\Big((h\varepsilon^{1/2}+\varepsilon)\|f\|_{L^2(\Omega)} + h^m\|\bar{u}\|_{H^m(\Omega)}\Big). \end{aligned}$$

Next we state an ε-independent estimate.

Theorem 4 *Let u and u_h satisfy* (1.1) *and* (4.8) *respectively. There exists a constant C such that*

$$\|u-u_h\|_{L^2(\Omega)} \le Ch\,\|f\|_{L^2(\Omega)}.$$

The proof of Theorem 4 is based on the duality argument

$$\begin{aligned} \varepsilon^2\Delta z - \Delta z &= u - u_h \qquad &&\text{in } \Omega, \\ z = \frac{\partial z}{\partial n} &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

and by using similar techniques as those found in Theorems 3 and 1. Since the proof is long and similar to ones above, we omit it.

6 Other examples of methods imposing boundary conditions weakly

The above error estimates are not limited to just the above finite element method. The weak treatment of the boundary conditions can be applied to any method. In this section, we briefly discuss how some finite element method may be altered to enjoy the same properties above.

C^1 finite element methods

Let $X_h = X_C \subset H^2(\Omega) \cap H_0^1(\Omega)$ denote a C^1 conforming finite element space with vanishing trace on the boundary. Some well-known examples of X_C are the family of Argyris elements, macro Hsieh-Clough-Tocher elements, and singular Zienkiewicz elements [5,9]. In this case, we keep the bilinear form $b(\cdot,\cdot)$ and define the bilinear form $a_h(\cdot,\cdot)$ as

$$a_h(v,w) = \int_\Omega D^2 v : D^2 w \, dx - \sum_{F\in\mathcal{F}_h^b} \int_F \Big(\frac{\partial^2 v}{\partial n_F^2}\frac{\partial w}{\partial n_F} + \frac{\partial v}{\partial n_F}\frac{\partial^2 w}{\partial n_F^2} - \frac{\sigma}{h_F}\frac{\partial v}{\partial n_F}\frac{\partial w}{\partial n_F}\Big) ds.$$

C^0 finite element methods

Let $X_h = X_L \subset H_0^1(\Omega)$ denote the Lagrange finite element space consisting of polynomials of degree $k \geq 2$; that is,

$$X_L = \big\{ v_h \in H_0^1(\Omega);\ v_h\big|_T \in \mathbb{P}^k(T)\ \forall T \in \mathcal{T}_h \big\}.$$

We then take the bilinear form $a_h(\cdot,\cdot)$ as

$$\begin{aligned} a_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T D^2 v : D^2 w\, dx - \sum_{F\in\mathcal{F}_h} \int_F \Big(& \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] + \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n^2} \right\}\!\!\right\} \\ & - \frac{\sigma}{h_F} \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] \Big) ds. \end{aligned}$$

We note that this finite element method with $\varepsilon = 1$ was considered in [6,10]. Unlike the C^1 finite element methods above, the weak enforcement of the Neumann boundary condition arises naturally in the method's formulation.

Discontinuous Galerkin methods

In the case of DG methods (see [3,16] for example), we take $X_h = X_{DG}$ with

$$X_{DG} = \big\{ v_h \in L^2(\Omega);\ v_h\big|_T \in \mathbb{P}^k(T)\ \forall T \in \mathcal{T}_h \big\}$$

for some integer $k \geq 2$. We then define the bilinear form as

$$\begin{aligned} a_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T D^2 v : D^2 w\, dx + \sum_{F\in\mathcal{F}_h} \int_F \Big(& \left\{\!\!\left\{ \frac{\partial \Delta v}{\partial n} \right\}\!\!\right\} [\![w]\!] + [\![v]\!] \left\{\!\!\left\{ \frac{\partial \Delta w}{\partial n} \right\}\!\!\right\} \\ & - \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] - \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n^2} \right\}\!\!\right\} + \frac{\sigma_1}{h_F} \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] + \frac{\sigma_2}{h_F^3} [\![v]\!] [\![w]\!] \Big) ds. \end{aligned}$$

We also replace the bilinear form $b(\cdot,\cdot)$ with

$$b_h(v,w) = \sum_{T\in\mathcal{T}_h}\int_T \nabla v\cdot\nabla w\,dx - \sum_{F\in\mathcal{F}_h}\Big(\left\{\!\!\left\{\frac{\partial v}{\partial n}\right\}\!\!\right\}[\![w]\!] + [\![v]\!]\left\{\!\!\left\{\frac{\partial w}{\partial n}\right\}\!\!\right\} - \frac{\sigma_3}{h_F}[\![v]\!]\,[\![w]\!]\Big)ds.$$

In comparison to other methods, both boundary conditions are imposed weakly. Our analysis shows that there is no obvious advantages to imposing function values on the boundary weakly for the problem in hand.

7 Discussion on local error estimates and pollution effects

In this section we discuss the local error estimates and a pollution effect from the boundary layers. Our analysis above shows that for ε small the presence of boundary layers has a substantial effect on global *a priori* error estimates when the Neumann boundary conditions are imposed strongly (cf. Section 4.1). As a result, the use of high order elements do not necessary improve the convergence rates. On the other hand, since the boundary layers are contained in a small neighborhood of the boundary, one would expect better convergence rates in the interior of the domain. However as was shown in [22, 23] for conformal C^1 elements, the improvement is only minor. In those papers, Semper showed that for $\varepsilon < h$ the presence of boundary layers pollutes the finite element method everywhere even far away from the boundary, and that the error in the interior is controlled by the global error in the L^2 norm; this error happens to be of order h. Numerical results using C^1 Hermite polynomials in one dimension presented in those papers confirm this statement (cf. Table 2 and 3 in [23]). We stress that this strong pollution effect is not due to the fact that C^1 basis are used, but due to the treatment of Neumann boundary conditions strongly.

The above result may look surprising if compared to analogous result of singularly perturbed reaction-diffusion problem,

$$-\varepsilon^2\Delta u + u = f \quad \text{in } \Omega, \tag{7.1a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{7.1b}$$

Indeed, Schatz and Wahlbin [19], proved that the standard conformal finite element approximation to the above problem converges optimally on any interior subdomain; i.e., there is no pollution from the boundary layers into the interior.

At first glance this may seem puzzling since both problems are singularly perturbed elliptic problems, but their conformal finite elements behave so differently. The fundamental issue lies in the difference between the low order terms of each respective equation and its Galerkin approximation (i.e. difference between the Ritz and L^2 projections). Some additional insight to that phenomena can be obtained from looking at one dimensional problem. In one dimension one may decompose the differential operator as

$$\varepsilon^2\frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial x^2} = (-\varepsilon\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})(-\varepsilon\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x})$$

and rewrite (1.1) as a system

$$\left(-\varepsilon\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial x}\right)u = v \tag{7.2a}$$

$$\left(-\varepsilon\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)v = f, \tag{7.2b}$$

with the following Dirichlet boundary conditions for u and v: $u(0) = u(1) = 0$ and $v(0) = \varepsilon^2 u''(0)$, $v(1) = \varepsilon^2 u''(1)$. Thus we obtain a singularly perturbed system of advection-diffusion equations. Thus for example, if we solve (7.2b) by finite element method on a quasi-uniform mesh, we can not resolve the boundary layer at $x = 0$. This will result in a perturbation term on the right hand side for (7.2a) near $x = 0$, which is now on inflow part of the boundary for (7.2a) and will pollute the whole solution on the entire domain. The pollution is of size h (the width of the numerical layer) and that is precisely the convergence rate observed numerically in [22]. By treating the Neumann boundary conditions weakly we essentially reduce the size of the numerical layer from h to ε. In the case of $\varepsilon < h$, this improvement allows us to obtain better global error estimates in energy and L^2 norms. In particular, assuming the reduced solution is smooth, from (4.22c) for $\varepsilon^{1/2} \le h^k$, we obtain optimal error estimates in the energy norm. We also expect to obtain better interior error estimates. In fact, we expect to observe optimal error estimates in the energy norm in interior sub-domains, and this is exactly what is seen in our numerical experiments below. However, we do not pursue this here as this is outside of the scope of the present paper.

8 Numerical Results

In this section we provide several numerical examples that show the approximation properties of the method, the robustness of the method with respect to the penalty parameter σ, and illustrate the advantages of weakly imposed Neumann boundary conditions in certain situations. All computations are performed on uniform meshes.

8.1 Example 1

In the first example we show how our method performs for a smooth solution. We take the exact solution to be

$$u(x,y) = \sin^2(\pi x)\sin^2(\pi y),$$

with the following data

$$f(x,y) = \varepsilon^2\Delta^2 u - \Delta u, \quad \sigma = 20, \quad \Omega = (0,1)^2.$$

We note that f depends on ϵ, however, this should not affect convergence rates in this case. In Tables 1 and 2 we report the converges rates in various norms for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-5}$, respectively. Since the exact solution has no layers, we observe the expected rates of convergence in L^2, H^1, and H^2 norms.

Table 1. Rates of convergence for $\varepsilon = 10^{-2}$

h	L^2 error	rates	H^1 error	rates	H^2 error	rates
1	$4.02E-02$		$6.32E-01$		$1.67E+01$	
1/2	$1.74E-02$	1.21	$3.82E-01$	0.73	$9.70E+00$	0.78
1/4	$4.31E-03$	2.01	$1.60E-01$	1.26	$6.26E+00$	0.63
1/8	$9.20E-04$	2.23	$5.46E-02$	1.55	$3.19E+00$	0.97
1/16	$1.73E-04$	2.38	$1.58E-02$	1.77	$1.59E+00$	1.00
1/32	$3.36E-05$	2.36	$4.21E-03$	1.91	$8.07E-01$	0.98
1/64	$7.30E-06$	2.31	$1.09E-03$	2.05	$4.08E-01$	1.03
1/128	$1.73E-06$	2.07	$2.76E-04$	1.98	$2.05E-01$	0.99
1/256	$4.28E-07$	2.02	$6.94E-05$	1.99	$1.03E-01$	1.00

Table 2. Rates of convergence for $\varepsilon = 10^{-5}$

h	L^2 error	rates	H^1 error	rates	H^2 error	rates
1	$3.96E-02$		$6.33E-01$		$1.71E+01$	
1/2	$1.66E-02$	1.25	$3.82E-01$	0.73	$1.01E+01$	0.77
1/4	$3.81E-03$	2.12	$1.59E-01$	1.26	$6.73E+00$	0.58
1/8	$7.00E-04$	2.44	$5.37E-02$	1.57	$3.60E+00$	0.90
1/16	$1.06E-04$	2.70	$1.55E-02$	1.78	$1.75E+00$	1.03
1/32	$1.44E-05$	2.87	$4.13E-03$	1.91	$8.70E-01$	1.01
1/64	$1.89E-06$	3.08	$1.07E-03$	2.05	$4.37E-01$	1.04
1/128	$2.41E-07$	2.97	$2.70E-04$	1.98	$2.19E-01$	0.99
1/256	$3.05E-08$	2.98	$6.82E-05$	1.99	$1.10E-01$	1.00

8.2 Example 2

In the second example we show that our method is robust with respect to the penalty parameter σ. For this example we take the same exact solution and data as in Example 1. In Table 3 we report the errors for various values of σ. From the results we can see that the method is robust for a large range of σ, and surprisingly, the method even works for $\sigma = 0$.

Table 3. Error for various values of σ with $h = 1/128$

σ	L^2 error	H^1 error	H^2 error
0	$1.75E-06$	$2.76E-04$	$2.05E-01$
$1E-03$	$1.75E-06$	$2.76E-04$	$2.05E-01$
$1E-02$	$1.75E-06$	$2.76E-04$	$2.05E-01$
$1E-01$	$1.75E-06$	$2.76E-04$	$2.05E-01$
1	$1.77E-06$	$2.76E-04$	$2.05E-01$
$1E+01$	$1.38E-06$	$2.76E-04$	$2.08E-01$
$1E+02$	$1.94E-06$	$2.77E-04$	$2.06E-01$
$1E+03$	$2.42E-06$	$2.83E-04$	$2.15E-01$
$1E+04$	$2.72E-06$	$2.88E-04$	$2.23E-01$
$1E+05$	$2.76E-06$	$2.89E-04$	$2.25E-01$
$1E+06$	$2.77E-06$	$2.89E-04$	$2.25E-01$

8.3 Example 3

In the third example, we show that in some situations it is very advantageous to impose Neumann boundary conditions weakly. This time we take the exact solution to the reduced problem (1.3) to be

$$\overline{u}(x,y) = \sin(\pi x)\sin(\pi y)$$

with the following data

$$f(x,y) = -\Delta\overline{u}(x,y), \quad \sigma = 20, \quad \varepsilon = 10^{-6}, \quad \Omega = (0,1)^2, \quad \Omega_0 = (0.2, 0.8)^2.$$

The exact solution u for (1.1) with the given data is not known. However, the exact solution has strong boundary layers. Since we take ε to be very small, we can estimate the error well by using the solution to the reduced problem as (4.5b) and (4.5c) show.

In Tables 4 and 5, we compare the errors and the corresponding rates of convergence in global L^2 and H^1, norms with weakly and strongly imposed boundary conditions. From the tables, we can see the optimal rates of convergence for weakly imposed boundary conditions in both norms and the greatly reduced rates of convergence with strongly imposed Neumann boundary conditions.

Table 4. Rates of convergence for $\varepsilon = 10^{-6}$ on Ω with weakly imposed BC

h	$L^2(\Omega)$ error	rates	$H^1(\Omega)$ error	rates
1	$2.35E-02$		$4.26E-01$	
1/2	$7.91E-03$	1.57	$2.10E-01$	1.02
1/4	$1.82E-03$	2.12	$8.02E-02$	1.39
1/8	$3.15E-04$	2.53	$2.53E-02$	1.67
1/16	$4.66E-05$	2.72	$7.12E-03$	1.81
1/32	$6.37E-06$	2.87	$1.89E-03$	1.91
1/64	$8.33E-07$	3.08	$4.88E-04$	2.05
1/128	$1.07E-07$	2.96	$1.24E-04$	1.98

Table 5. Rates of convergence for $\varepsilon = 10^{-6}$ on Ω with strongly imposed BC

h	$L^2(\Omega)$ error	rates	$H^1(\Omega)$ error	rates
1	$1.28E-01$		$1.05E+00$	
1/2	$7.61E-02$	0.75	$8.01E-01$	0.39
1/4	$4.23E-02$	0.85	$5.93E-01$	0.43
1/8	$2.27E-02$	0.90	$4.33E-01$	0.45
1/16	$1.19E-02$	0.93	$3.13E-01$	0.46
1/32	$6.07E-03$	0.97	$2.24E-01$	0.48
1/64	$3.07E-03$	1.03	$1.59E-01$	0.51
1/128	$1.54E-03$	0.99	$1.13E-01$	0.49

In Tables 6 and 7, we compare the errors and the corresponding rates of convergence in global L^2, H^1, and H^2 norms with weakly and strongly imposed boundary conditions on a subdomain Ω_0 that is sufficiently far away from the layers. Thus, the exact solution is very smooth on Ω_0. From the tables we can see the optimal rates of convergence for the method with weakly imposed boundary conditions in all three norms, but only first order convergence when the Neumann boundary conditions are treated strongly. In this example we can clearly observe the pollution effect from the numerical boundary layers described in Section 7 into the interior of the domain and the advantages of the weak treatment of the boundary conditions for such problems.

Table 6. Rates of convergence for $\varepsilon = 10^{-6}$ on Ω_0 with weakly imposed BC

h	$L^2(\Omega_0)$ error	rates	$H^1(\Omega_0)$ error	rates	$H^2(\Omega_0)$ error	rates
1	$1.44E-02$		$2.73E-01$		$3.91E+00$	
1/2	$5.58E-03$	1.37	$1.50E-01$	0.87	$3.20E+00$	0.29
1/4	$1.05E-03$	2.41	$4.67E-02$	1.68	$1.54E+00$	1.05
1/8	$1.96E-04$	2.42	$1.58E-02$	1.56	$8.46E-01$	0.87
1/16	$2.38E-05$	3.01	$3.63E-03$	2.09	$3.50E-01$	1.26
1/32	$3.36E-06$	2.82	$1.00E-03$	1.86	$1.84E-01$	0.93
1/64	$4.32E-07$	3.10	$2.53E-04$	2.08	$9.12E-02$	1.06
1/128	$5.60E-08$	2.95	$6.50E-05$	1.96	$4.64E-02$	0.97

Table 7. Rates of convergence for $\varepsilon = 10^{-6}$ on Ω_0 with strongly imposed BC

h	$L^2(\Omega_0)$ error	rates	$H^1(\Omega_0)$ error	rates	$H^2(\Omega_0)$ error	rates
1	$5.88E-02$		$4.87E-01$		$1.09E+01$	
1/2	$5.04E-02$	0.22	$4.99E-01$	-0.03	$1.91E+01$	-0.81
1/4	$2.64E-02$	0.93	$2.88E-01$	0.79	$1.82E+01$	0.08
1/8	$1.55E-02$	0.77	$5.21E-02$	2.47	$3.60E+00$	2.33
1/16	$7.22E-03$	1.09	$4.56E-03$	3.47	$3.63E-01$	3.27
1/32	$3.82E-03$	0.92	$1.79E-03$	1.35	$1.85E-01$	0.97
1/64	$1.92E-03$	1.04	$7.55E-04$	1.30	$9.18E-02$	1.06
1/128	$9.72E-04$	0.98	$3.71E-04$	1.03	$4.67E-02$	0.98

Acknowledgments

This work of the first author was supported by the National Science Foundation (grant DMS-0713833). The work of the second author was supported by the National Science Foundation (grant DMS-0811167). The work of the third author was supported by the National Science Foundation (grants DMS-09-02683 & DMS-11-15421).

References

1. G. Awanou, *Robustness of a spline element method with constraints*, J. Sci. Comput., 36(3):421–432, 2008.
2. Y. Bazilevs and T. J. R. Hughes, *Weak imposition of Dirichlet boundary conditions in fluid mechanics*, Comput. & Fluids, 36(1):12–26, 2007.
3. G. Baker, *Finite element methods for elliptic equations using non-conforming elements*, Math. Comp., 31(137):45–59, 1977.
4. J. G. Besjes, *Singular perturbation problems for linear elliptic differential operators of arbitrary order. I. Degeneration to elliptic operators*, J. Math. Anal. Appl., 49:24–46, 1975.
5. S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods (Third edition)*, Springer, 2008.
6. S. C. Brenner and L.-Y. Sung, *C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains*, J. Sci. Comput., 22/23:83–118, 2005.
7. S. C. Brenner, T. Gudi, and L.-Y. Sung, *An a posteriori error estimator for a quadratic C^0-interior penalty method for the biharmonic problem*, IMA J. Numer. Anal., 30(3):777–798, 2010.

8. S. C. Brenner and M. Neilan, *C^0 penalty methods for a fourth order elliptic singular perturbation problem*, SIAM J. Numer. Anal., 49:869–892, 2011.
9. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
10. G. Engel, K. Garikipati, T. J. R. Hughes, M. G. Larson, L. Mazzei, and R. L. Taylor, *Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity,* Comput. Meth. Appl. Mech. Eng., 191:3669–3750, 2002.
11. L. S. Frank, *Singular Perturbations in Elasticity Theory*, Analysis and its Applications, 1., IOS Press, Amsterdam, 1997.
12. P. Grisvard, *Elliptic Problems on Nonsmooth Domains*, Monographs and studies in mathematics, vol. 24, Pitman Publishing Inc., 1985.
13. P. Hansbo and M. Juntunen, *Weakly imposed Dirichlet boundary conditions for the Brinkman model of porous media flow*, Appl. Numer. Math., 59 (2009), pp. 1274–1289.
14. M. Juntunen, R. Stenberg, *Analysis of finite element methods for the Brinkman problem*, Calcolo, 47 (2010), pp. 129–147.
15. L. S. D. Morley, *The triangular equilibrium element in the solution of plate bending problems*, Aero. Quart., 19:149–169, 1968.
16. I. Mozolevski, E. Süli, P. R. Bösing, *hp-version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation*, J. Sci. Comput., 30 (2007), no. 3, 465-491.
17. T. K. Nilssen, X. C. Tai, and R. Winther, *A robust nonconforming H^2-element*, Math. Comp., 70(234):489–505, 2001.
18. J. A. Nitsche, *Über ein Variationspirinzip zur Lösung Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unteworfen sind*, Abh. Math. Sem. Univ. Hamburg, 36:9–15, 1971.
19. A. H. Schatz and L. B. Wahlbin, *On the finite element method for singularly perturbed reaction-diffusion problems in two and one dimensions*, Math. Comp., 40:47–89, 1983.
20. F. Schieweck, *On the role of boundary conditions for CIP stabilization of higher order finite elements*, Electron. Trans. Numer. Anal., 32:1–16, 2008.
21. L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54(190):483–493, 1990.
22. B. Semper, *Conforming finite element approximations for a fourth-order singular perturbation problem*, SIAM J. Numer. Anal., 29(4):1043–1058, 1992.
23. B. Semper, *Locking in finite-element approximations to long thin extensible beams*, IMA J. Numer. Anal., 14(1):97–109, 1994.
24. Z. Shi, *Error estimates of Morley element*, Math. Numer. Sinica, 12(2):113–118, 1990.
25. R. Stenberg, *On some techniques for approximating boundary conditions in the finite element method*, MJ. Comput. Appl. Math. 63(1-3):139–148, 1995.
26. X.-C. Tai and R. Winther, *A discrete de Rham complex with enhanced smoothness*, Calcolo, 43:287–306, 2006.
27. M. Wang, *The necessity and sufficiency of the patch test for the convergence of finite element methods*, SIAM J. Numer. Anal., 39(2):363–384, 2001.
28. M. Wang, J. Xu, and Y-C. Hu, *Modified Morley element method for a fourth order elliptic singular perturbation problem*, J. Comput. Math., 24(2):113–120, 2006.
29. M. Wang and X. Meng, *A robust finite element method for a 3-D elliptic singular perturbation problem*, J. Comput. Math., 25(6):631–644, 2007.
30. P. Xie, D. Shi, and H. Li, *A new robust C^0-type nonconforming triangular element for singular perturbation problems*, Appl. Math. Comput., 217(8):3832–3843, 2010.