

GLOBAL AND INTERIOR POINTWISE BEST APPROXIMATION RESULTS FOR THE GRADIENT OF GALERKIN SOLUTIONS FOR PARABOLIC PROBLEMS

DMITRIY LEYKEKHMEN[†] AND BORIS VEXLER[‡]

Abstract. In this paper we establish best approximation property of fully discrete Galerkin solutions of the second parabolic problems on convex polygonal and polyhedral domains in the $L^\infty(I; W^{1,\infty}(\Omega))$ norm. The discretization method consists of continuous Lagrange finite elements in space and discontinuous Galerkin methods of arbitrary order in time. The method of the proof differs from the established fully discrete error estimate techniques and uses only elliptic results and discrete maximal parabolic regularity for discontinuous Galerkin methods established by the authors in [12]. In addition, the proof does not require any relationship between spacial mesh sizes and time steps. We also establish interior best approximation property that shows more local dependence of the error at a point.

Key words. parabolic problems, finite elements, fully discrete error estimates, pointwise error estimates

AMS subject classifications.

1. Introduction. Let Ω be a convex polygonal/polyhedral domain in \mathbb{R}^N , $N = 2, 3$ and $I = (0, T)$ with some $T > 0$. We consider a second order parabolic problem

$$\begin{aligned} u_t(t, x) - \Delta u(t, x) &= f(t, x), & (t, x) \in I \times \Omega, \\ u(t, x) &= 0, & (t, x) \in I \times \partial\Omega, \\ u(0, x) &= u_0(x), & x \in \Omega. \end{aligned} \tag{1.1}$$

To discretize the problem we use continuous Lagrange finite elements in space and discontinuous Galerkin methods in time. The precise description of the method is given in section 2. Our main goal in this paper is to establish global and interior (local) space-time pointwise best approximation type results for the fully discrete error. The global estimate has the following structure:

$$\|\nabla(u - u_{kh})\|_{L^\infty(I \times \Omega)} \leq C \ell_k \ell_h \|\nabla(u - \chi)\|_{L^\infty(I \times \Omega)}, \tag{1.2}$$

where u_{kh} denotes the fully discrete solution and χ is an arbitrary element of the finite dimensional space, h stands for spacial mesh size and k for the maximal time step, and ℓ_k, ℓ_h stand for some logarithmic terms. Such results are sometimes called symmetric estimates, cf. [4, 7]. The interior (local) result provides an estimate of the error $|\nabla(u - u_{u_{kh}})(\tilde{t}, x_0)|$ for given $\tilde{t} \in (0, T]$ and $x_0 \in \Omega$ in terms of best approximation on a ball $B_d(x_0)$ and some global terms in weaker norms. Precise results are stated in section 2, see Theorem 2.1 and Theorem 2.2. For the global estimate (1.2) we assume that f and u_0 are such that $\nabla u \in C(\bar{I} \times \bar{\Omega})$. For the interior result we essentially need only $\nabla u \in C(\bar{I} \times \bar{B}_d(x_0)) \cap L^2(I \times \Omega)$. Such best approximation type results have only natural assumptions on the problem data and are desirable in many applications, for example optimal control problems governed by parabolic equations with gradient constraints. We refer to a recent paper [30] for a further discussion on the importance of best approximation results and difficulties associated with obtaining such estimates for parabolic problems.

For elliptic problems the best approximation property (1.2), which is equivalent to the stability of the Ritz projection in $W^{1,\infty}(\Omega)$ norm, is well known. First log-free result was established in [23] on convex polygonal domains. Later the result was extended to convex polyhedral domains with some restriction on angles in [2]. This restriction was removed in [11] and even extended to certain graded meshes in [6]. For parabolic problems similar results are rather scarce. Main body of the work on pointwise error estimates for parabolic problems are devoted to $L^\infty(I \times \Omega)$ error estimates, see [14] for review of the corresponding results. We are aware of only three publications dealing with pointwise error estimates for the gradient of the error.

In two space dimensions, semidiscrete error estimates were studied in [3] and the fully discrete Crank-Nicolson method was studied in [33]. Since the main motivation of both investigations was the question of superconvergence of the gradient of the error, it was assumed that the solution is sufficiently smooth. More general fully discrete error estimates using Padé time schemes were obtained in [16] for smooth domains in \mathbb{R}^N .

In both publications dealing with fully discrete error estimates, [16] and [33], the proofs are based on the splitting $u - u_{kh} = (u - R_h u) + (R_h u - u_{kh})$, where R_h is the Ritz projection. This idea was first introduced

[†]Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA (leykekhman@math.uconn.edu).

[‡]Chair of Optimal Control, Faculty of Mathematics, Technical University of Munich, Boltzmannstraße 3, 85748 Garching b. Munich, Germany (vexler@ma.tum.de).

by M. Wheeler [34] in order to obtain optimal order error estimates in L^2 norm in space. The main idea of this approach is the following: the first part of the error is treated by elliptic results and the second part satisfies a certain parabolic equation with the right-hand side involving $(u - R_h u)$, which can be treated by results from rational approximation of analytic semigroups in Banach spaces (see also [31, Thm .8.6]). However, this approach requires additional smoothness of the solution, well beyond natural regularity $\nabla u \in C(\bar{I} \times \bar{\Omega})$ of the exact solution. Our approach is completely different. It uses newly established discrete maximal parabolic regularity results [12] for discontinuous Galerkin time schemes, see section 5 below, and discrete resolvent estimates of the form:

$$\| (z + \Delta_h)^{-1} \chi \| \leq \frac{M_h}{|z|} \| \chi \|, \quad \text{for } z \in \mathbb{C} \setminus \Sigma_\gamma, \quad \text{for all } \chi \in \mathbb{V}_h = V_h + iV_h, \quad (1.3)$$

where M_h may depend on $|\ln h|$ but independent of h otherwise, V_h is the space of continuous Lagrange finite elements of degree r , Δ_h is the discrete Laplace operator, see (3.9) below, and

$$\Sigma_\gamma = \{ z \in \mathbb{C} \mid |\arg(z)| \leq \gamma \},$$

for some $\gamma \in (0, \frac{\pi}{2})$. In [14] we showed this estimate for the triple norm $\| \| v_h \| \| = \| \sigma^{\frac{N}{2}} v_h \|_{L^2(\Omega)}$ with the weight function $\sigma(x) = \sqrt{|x - x_0|^2 + K^2 h^2}$. This norm behaves similar to the $L^1(\Omega)$ norm, and we used the corresponding discrete maximal parabolic result to prove (global and interior) pointwise best approximation for function values of the solution u . Here, we will in addition require the estimate (1.3) with respect to the norm

$$\| \| v_h \| \| = \| \sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} v_h \|_{L^2(\Omega)},$$

which behaves similar to the $W^{-1,1}(\Omega)$ norm, see Theorem 4.1 below. This allows us to prove our main results of (global and interior) pointwise best approximation for the gradient of the solution.

The rest of the paper is organized as follows. In the next section we describe the discretization method and state our main results. In section 3, we review some essential elliptic results in weighted norms. Section 4 is devoted to establishing the resolvent estimate in weighted norms. In section 5, we review our discrete maximal parabolic regularity result. Finally, in sections 6 and 7, we provide proofs of the global and interior best approximation properties of the fully discrete solution.

2. Discretization and statement of main results. To introduce the time discontinuous Galerkin discretization for the problem, we partition $(0, T]$ into subintervals $I_m = (t_{m-1}, t_m]$ of length $k_m = t_m - t_{m-1}$, where $0 = t_0 < t_1 < \dots < t_{M-1} < t_M = T$. The maximal and minimal time steps are denoted by $k = \max_m k_m$ and $k_{\min} = \min_m k_m$, respectively. We impose the following conditions on the time mesh (as in [12] or [19]):

(i) There are constants $c, \beta > 0$ independent of k such that

$$k_{\min} \geq ck^\beta.$$

(ii) There is a constant $\kappa > 0$ independent of k such that for all $m = 1, 2, \dots, M - 1$

$$\kappa^{-1} \leq \frac{k_m}{k_{m+1}} \leq \kappa.$$

(iii) It holds $k \leq \frac{1}{4}T$.

The semidiscrete space X_k^q of piecewise polynomial functions in time is defined by

$$X_k^q = \{ v_k \in L^2(I; H_0^1(\Omega)) \mid v_k|_{I_m} \in \mathcal{P}_q(I_m; H_0^1(\Omega)), m = 1, 2, \dots, M \},$$

where $\mathcal{P}_q(I_m; V)$ is the space of polynomial functions of degree q in time with values in a Banach space V . We will employ the following notation for functions in X_k^q

$$v_m^+ = \lim_{\varepsilon \rightarrow 0^+} u(t_m + \varepsilon), \quad v_m^- = \lim_{\varepsilon \rightarrow 0^+} u(t_m - \varepsilon), \quad [v]_m = v_m^+ - v_m^-. \quad (2.1)$$

Next we define the following bilinear form

$$B(v, \varphi) = \sum_{m=1}^M \langle v_t, \varphi \rangle_{I_m \times \Omega} + \langle \nabla v, \nabla \varphi \rangle_{I \times \Omega} + \sum_{m=2}^M ([v]_{m-1}, \varphi_{m-1}^+)_{\Omega} + (v_0^+, \varphi_0^+)_{\Omega}, \quad (2.2)$$

where $(\cdot, \cdot)_\Omega$ and $(\cdot, \cdot)_{I_m \times \Omega}$ are the usual L^2 space and space-time inner-products, $\langle \cdot, \cdot \rangle_{I_m \times \Omega}$ is the duality product between $L^2(I_m; H^{-1}(\Omega))$ and $L^2(I_m; H_0^1(\Omega))$. We note, that the first sum vanishes for $v \in X_k^0$. The dG(q) semidiscrete (in time) approximation $u_k \in X_k^q$ of (1.1) is defined as

$$B(u_k, \varphi_k) = (f, \varphi_k)_{I \times \Omega} + (u_0, \varphi_{k,0}^+)_{\Omega} \quad \text{for all } \varphi_k \in X_k^q. \quad (2.3)$$

Rearranging the terms in (2.2), we obtain an equivalent (dual) expression of B :

$$B(v, \varphi) = - \sum_{m=1}^M \langle v, \varphi_t \rangle_{I_m \times \Omega} + (\nabla v, \nabla \varphi)_{I \times \Omega} - \sum_{m=1}^{M-1} (v_m^-, [\varphi]_m)_{\Omega} + (v_M^-, \varphi_M^-)_{\Omega}. \quad (2.4)$$

Next we define the fully discrete approximation. For $h \in (0, h_0]$, $h_0 > 0$, let \mathcal{T} denote a quasi-uniform triangulation of Ω with mesh size h , i.e., $\mathcal{T} = \{\tau\}$ is a partition of Ω into cells (triangles or tetrahedrons) τ of diameter h_τ such that for $h = \max_\tau h_\tau$,

$$\text{diam}(\tau) \leq h \leq C|\tau|^{\frac{1}{N}}, \quad \text{for all } \tau \in \mathcal{T},$$

hold. Let V_h be the set of all functions in $H_0^1(\Omega)$ that are polynomials of degree r on each τ , i.e. V_h is the usual space of conforming finite elements. To obtain the fully discrete approximation we consider the space-time finite element space

$$X_{k,h}^{q,r} = \{ v_{kh} \in L^2(I; H_0^1(\Omega)) \mid v_{kh}|_{I_m} \in \mathcal{P}_q(I_m; V_h), m = 1, 2, \dots, M \}, \quad q \geq 0, \quad r \geq 1. \quad (2.5)$$

We define a fully discrete dG(q)cG(r) solution $u_{kh} \in X_{k,h}^{q,r}$ by

$$B(u_{kh}, \varphi_{kh}) = (f, \varphi_{kh})_{I \times \Omega} + (u_0, \varphi_{kh}^+)_{\Omega} \quad \text{for all } \varphi_{kh} \in X_{k,h}^{q,r}. \quad (2.6)$$

2.1. Main results. Now we state our main results. The first result establishes the global best approximation property of the fully discrete Galerkin solution in $L^\infty(I; W^{1,\infty}(\Omega))$ norm.

THEOREM 2.1 (Global best approximation). *Let u and u_{kh} satisfy (1.1) and (2.6) respectively. Then, there exists a constant C independent of k and h such that*

$$\|\nabla(u - u_{kh})\|_{L^\infty(I \times \Omega)} \leq C \ell_k \ell_h \inf_{\chi \in X_{k,h}^{q,r}} \|\nabla(u - \chi)\|_{L^\infty(I \times \Omega)},$$

where $\ell_k = \ln \frac{T}{k}$ and $\ell_h = |\ln h|^{\frac{2N-1}{N}}$.

The proof of this theorem is given in Section 6.

For the error at a given point $x_0 \in \Omega$ we obtain a sharper results. For elliptic problems similar results were obtained in [25, 27]. We denote by $B_d = B_d(x_0)$ the ball of radius d centered at x_0 .

THEOREM 2.2 (Interior best approximation). *Let u and u_{kh} satisfy (1.1) and (2.6), respectively and let $d > 4h$. Assume $x_0 \in \Omega$ and $\tilde{t} \in I_m$ for some $m = 1, 2, \dots, M$ and $B_d \subset \subset \Omega$. Then there exists a constant C independent of h , k , and d such that*

$$\begin{aligned} |\nabla(u - u_{kh})(\tilde{t}, x_0)| &\leq C \ell_k \ell_h \inf_{\chi \in X_{k,h}^{q,r}} \left\{ \|\nabla(u - \chi)\|_{L^\infty((0, \tilde{t}_m) \times B_d(x_0))} + d^{-1} \|u - \chi\|_{L^\infty((0, \tilde{t}_m) \times B_d(x_0))} \right. \\ &\quad \left. + d^{-\frac{N}{2}} \left(\|\nabla(u - \chi)\|_{L^\infty((0, \tilde{t}_m); L^2(\Omega))} + d^{-1} \|u - \chi\|_{L^\infty((0, \tilde{t}_m); L^2(\Omega))} \right) \right\}, \end{aligned}$$

with ℓ_k and ℓ_h defined as in Theorem 2.1.

The proof of this theorem is given in Section 7.

3. Elliptic estimates in weighted norms. In this section we collect some estimates for the finite element discretization of elliptic problems in weighted norms on convex polygonal/polyhedral domains mainly taken from [13]. These results will be used in the following sections within the proofs of Theorem 4.1, Theorem 2.1, and Theorem 2.2.

In this section we consider a fixed (but arbitrary) point $x_0 \in \Omega$. Associated to this point we introduce a smoothed delta function [27, Appendix], which we will denote by $\tilde{\delta}$. This function is supported in one cell, which is denoted by τ_0 with $x_0 \in \bar{\tau}_0$, and satisfies

$$(\chi, \tilde{\delta})_{\tau_0} = \chi(x_0), \quad \text{for all } \chi \in \mathcal{P}_r(\tau_0). \quad (3.1)$$

In addition we also have, see, e.g., [32, Lemma 2.2],

$$\|\tilde{\delta}\|_{W^{s,p}(\Omega)} \leq Ch^{-s-N(1-\frac{1}{p})}, \quad 1 \leq p \leq \infty, \quad s = 0, 1, 2. \quad (3.2)$$

Thus in particular $\|\tilde{\delta}\|_{L^1(\Omega)} \leq C$, $\|\tilde{\delta}\|_{L^2(\Omega)} \leq Ch^{-\frac{N}{2}}$, and $\|\tilde{\delta}\|_{L^\infty(\Omega)} \leq Ch^{-N}$. Next we introduce a weight function

$$\sigma(x) = \sqrt{|x - x_0|^2 + K^2 h^2}, \quad (3.3)$$

where $K > 0$ is a sufficiently large constant. This weight function was first introduced in [20, 21] to analyze pointwise finite element error estimates. One can easily check that σ satisfies the following properties:

$$\|\sigma^{-\frac{N}{2}}\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}}, \quad (3.4a)$$

$$|\nabla \sigma| \leq C, \quad (3.4b)$$

$$|\nabla^2 \sigma| \leq C \sigma^{-1}, \quad (3.4c)$$

$$\max_{\tau} \sigma \leq C \min_{\tau} \sigma \quad \text{for all } \tau \in \mathcal{T}. \quad (3.4d)$$

For the finite element space V_h we will utilize the L^2 projection $P_h: L^2(\Omega) \rightarrow V_h$ defined by

$$(P_h v, \chi)_{\Omega} = (v, \chi)_{\Omega} \quad \text{for all } \chi \in V_h, \quad (3.5)$$

the Ritz projection $R_h: H_0^1(\Omega) \rightarrow V_h$ defined by

$$(\nabla R_h v, \nabla \chi)_{\Omega} = (\nabla v, \nabla \chi)_{\Omega} \quad \text{for all } \chi \in V_h, \quad (3.6)$$

and the usual nodal interpolation operator $i_h: C_0(\Omega) \rightarrow V_h$ with usual approximation properties (cf., e.g., [5, Theorem 3.1.5])

$$\|u - i_h u\|_{L^q(\Omega)} \leq Ch^{2+N(\frac{1}{q}-\frac{1}{p})} \|u\|_{W^{2,p}(\Omega)}, \quad \text{for } q \geq p > \frac{N}{2}, \quad (3.7)$$

as well as the Scott-Zhang interpolation operator $i_h^{SZ}: W_0^{1,1}(\Omega) \rightarrow V_h$ with the approximation properties (cf., e.g., [28]) for $N = 3$:

$$h \|\nabla(u - i_h^{SZ} u)\|_{L^2(\Omega)} + \|u - i_h^{SZ} u\|_{L^2(\Omega)} \leq Ch^{\frac{3}{2}} \|u\|_{W^{2,\frac{3}{2}}(\Omega)} \quad \text{for all } u \in W^{2,\frac{3}{2}}(\Omega) \cap W_0^{1,1}(\Omega). \quad (3.8)$$

Moreover we introduce the discrete Laplace operator $\Delta_h: V_h \rightarrow V_h$ defined by

$$(-\Delta_h v_h, \chi)_{\Omega} = (\nabla v_h, \nabla \chi)_{\Omega}, \quad \text{for all } \chi \in V_h. \quad (3.9)$$

The next lemma states an approximation result for the Ritz projection in the $L^\infty(\Omega)$ norm.

LEMMA 3.1. *There exists a constant $C > 0$ independent of h , such that*

$$\|v - R_h v\|_{L^\infty(\Omega)} \leq Ch |\ln h| \|\nabla v\|_{L^\infty(\Omega)}.$$

For smooth domains such a result was established in [22, 25, 26] (logfree for higher order elements), for polygonal domains in [24] and [8, Theorem 3.2] (for mildly graded meshes), and for convex polyhedral domains it follows from stability of the Ritz projection in the $L^\infty(\Omega)$ norm in [13, Theorem 12].

The following lemma is a superapproximation result in weighted norms.

LEMMA 3.2 (Lemma 3 in [13]). *Let $v_h \in V_h$. Then the following estimates hold for any $\alpha, \beta \in \mathbb{R}$ and K (in the definition (3.3) of the weight σ) large enough:*

$$\|\sigma^\alpha (\text{Id} - i_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} + h \|\sigma^\alpha \nabla (\text{Id} - i_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} \leq ch \|\sigma^{\alpha+\beta-1} v_h\|_{L^2(\Omega)}, \quad (3.10a)$$

$$\|\sigma^\alpha(\text{Id} - P_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} + h\|\sigma^\alpha \nabla(\text{Id} - P_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} \leq ch\|\sigma^{\alpha+\beta-1} v_h\|_{L^2(\Omega)}. \quad (3.10b)$$

The next lemma describes a connection between the regularized delta function $\tilde{\delta}$ and the weight σ .

LEMMA 3.3. *There hold*

$$\|\sigma^{\frac{N}{2}} \tilde{\delta}\|_{L^2(\Omega)} + \|\sigma^{\frac{N+2}{2}} \nabla \tilde{\delta}\|_{L^2(\Omega)} + h\|\sigma^{\frac{N}{2}} \nabla \tilde{\delta}\|_{L^2(\Omega)} \leq C \quad (3.11)$$

and

$$\|\sigma^{\frac{N}{2}} P_h \tilde{\delta}\|_{L^2(\Omega)} + \|\sigma^{\frac{N+2}{2}} P_h \nabla \tilde{\delta}\|_{L^2(\Omega)} + h\|\sigma^{\frac{N}{2}} P_h \nabla \tilde{\delta}\|_{L^2(\Omega)} \leq C. \quad (3.12)$$

The proof of the first two terms in (3.11) and (3.12) respectively can be found in [9] for $N = 2$ and in [13, Lemma 4] for $N = 3$. Using similar arguments it is straightforward to show the result for the other terms.

The following two lemmas provide the flexibility in manipulating weighted norms.

LEMMA 3.4. *For each $\alpha \in \mathbb{R}$, there is a constant $C > 0$ such that for any $v \in H_0^1(\Omega) \cap H^2(\Omega)$ there holds*

$$\|\sigma^\alpha \nabla v\|_{L^2(\Omega)} \leq C (\|\sigma^{\alpha+1} \Delta v\|_{L^2(\Omega)} + \|\sigma^{\alpha-1} v\|_{L^2(\Omega)}).$$

Proof. There holds

$$\begin{aligned} \|\sigma^\alpha \nabla v\|_{L^2(\Omega)}^2 &= (\sigma^{2\alpha} \nabla v, \nabla v) = (\nabla(\sigma^{2\alpha} v), \nabla v) - 2\alpha(v\sigma^{2\alpha-1} \nabla \sigma, \nabla v) \\ &= -(\sigma^{\alpha-1} v, \sigma^{\alpha+1} \Delta v) - 2\alpha(v\sigma^{\alpha-1} \nabla \sigma, \sigma^\alpha \nabla v). \end{aligned}$$

Using $|\nabla \sigma| \leq C$ we obtain

$$\|\sigma^\alpha \nabla v\|_{L^2(\Omega)}^2 \leq \|\sigma^{\alpha-1} v\|_{L^2(\Omega)} \|\sigma^{\alpha+1} \Delta v\|_{L^2(\Omega)} + C \|\sigma^{\alpha-1} v\|_{L^2(\Omega)} \|\sigma^\alpha \nabla v\|_{L^2(\Omega)}.$$

Absorbing $\|\sigma^\alpha \nabla v\|_{L^2(\Omega)}$ we obtain the desired estimate. \square

LEMMA 3.5. *For each $\alpha \in \mathbb{R}$, there is a constant $C > 0$ such that for any $v_h \in V_h$ there holds*

$$\|\sigma^\alpha \nabla v_h\|_{L^2(\Omega)} \leq C (\|\sigma^{\alpha+1} \Delta_h v_h\|_{L^2(\Omega)} + \|\sigma^{\alpha-1} v_h\|_{L^2(\Omega)}).$$

Proof. Similar to the proof of the previous lemma we have

$$\begin{aligned} \|\sigma^\alpha \nabla v_h\|_{L^2(\Omega)}^2 &= (\sigma^{2\alpha} \nabla v_h, \nabla v_h) = (\nabla(\sigma^{2\alpha} v_h), \nabla v_h) - 2\alpha(v_h \sigma^{2\alpha-1} \nabla \sigma, \nabla v_h) \\ &= (\nabla P_h(\sigma^{2\alpha} v_h), \nabla v_h) + (\nabla(\text{Id} - P_h)(\sigma^{2\alpha} v_h), \nabla v_h) - 2\alpha(v_h \sigma^{2\alpha-1} \nabla \sigma, \nabla v_h) \\ &= -(\sigma^{\alpha-1} v_h, \sigma^{\alpha+1} \Delta_h v_h) + (\sigma^{-\alpha} \nabla(\text{Id} - P_h)(\sigma^{2\alpha} v_h), \sigma^\alpha \nabla v_h) - 2\alpha(v_h \sigma^{\alpha-1} \nabla \sigma, \sigma^\alpha \nabla v_h). \end{aligned}$$

Applying Lemma 3.2 for the second term and using $|\nabla \sigma| \leq C$ we obtain

$$\|\sigma^\alpha \nabla v_h\|_{L^2(\Omega)}^2 \leq \|\sigma^{\alpha-1} v_h\|_{L^2(\Omega)} \|\sigma^{\alpha+1} \Delta_h v_h\|_{L^2(\Omega)} + C \|\sigma^{\alpha-1} v_h\|_{L^2(\Omega)} \|\sigma^\alpha \nabla v_h\|_{L^2(\Omega)}.$$

Absorbing $\|\sigma^\alpha \nabla v_h\|_{L^2(\Omega)}$ we obtain the desired estimate. \square

In the following proofs we will make a heavy use of pointwise estimates for the Green's function.

LEMMA 3.6. *Let $G(x, y)$ denotes the elliptic Green's function of the Laplace operator on the domain Ω . Then for $N = 2, 3$ the following estimates hold,*

$$|\nabla_x G(x, y)| \leq C|x - y|^{1-N}, \quad \text{for all } x, y \in \Omega, \quad x \neq y, \quad (3.13a)$$

$$|\nabla_y G(x, y)| \leq C|x - y|^{1-N}, \quad \text{for all } x, y \in \Omega, \quad x \neq y. \quad (3.13b)$$

$$|\nabla_y \nabla_x G(x, y)| \leq C|x - y|^{-N}, \quad \text{for all } x, y \in \Omega, \quad x \neq y. \quad (3.13c)$$

The proof of the first estimate can be found in [10, Prop 1] and the second one follows from the symmetry of the Green's function and the first estimate, i.e. $|\nabla_y G(x, y)| = |\nabla_x G(y, x)| \leq C|x - y|^{1-N}$. The third estimate is also proven in [10, Prop 1].

The next lemma can be thought of as weighted Gagliardo-Nirenberg interpolation inequality.

LEMMA 3.7 (Lemma 5 in [13]). *Let $N = 3$. There exists a constant C independent of K and h such that for any $f \in H_0^1(\Omega)$, any $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq -\frac{1}{2}$ and any $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ there holds:*

$$\|\sigma^\alpha f\|_{L^2(\Omega)}^2 \leq C \|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)} \|\sigma^{\alpha+1+\beta} \nabla f\|_{L^{p'}(\Omega)},$$

provided $\|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)}$ and $\|\sigma^{\alpha+1+\beta} \nabla f\|_{L^{p'}(\Omega)}$ are bounded.

LEMMA 3.8. *Let $D = \partial_{x_i}$, $i = 1, \dots, N$ denote any partial derivative. Then for $N = 2, 3$ there holds*

$$\|\sigma^{\frac{N-2}{2}} \Delta^{-1} D \tilde{\delta}\|_{L^2(\Omega)} + \|\sigma^{\frac{N}{2}} \nabla \Delta^{-1} D \tilde{\delta}\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}} \quad (3.14)$$

and for $N = 3$ there holds

$$\|\Delta^{-1} D \tilde{\delta}\|_{L^3(\Omega)} + \|\nabla \Delta^{-1} D \tilde{\delta}\|_{L^{\frac{3}{2}}(\Omega)} \leq C h^{-1}. \quad (3.15)$$

Proof. Consider the following elliptic problem

$$\begin{aligned} -\Delta g(x) &= D \tilde{\delta}(x), & x \in \Omega, \\ g(x) &= 0, & x \in \partial\Omega. \end{aligned} \quad (3.16)$$

Thus, in order to obtain the estimate (3.14) we need to establish

$$\|\sigma^{\frac{N-2}{2}} g\|_{L^2(\Omega)} + \|\sigma^{\frac{N}{2}} \nabla g\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}}.$$

To estimate the first term, we will be using the following Green's function representation

$$g(x) = \int_{\tau_0} G(x, y) \partial_{y_i} \tilde{\delta}(y) dy = - \int_{\tau_0} \partial_{y_i} G(x, y) \tilde{\delta}(y) dy. \quad (3.17)$$

Define $B_h = B_{3h}(x_0) \cap \Omega$ and $B_h^c = \Omega \setminus B_h$ and consider two cases: $x \in B_h$ and $x \in B_h^c$. In the case $x \in B_h$, we obtain using polar coordinates centered at x and using (3.2), (3.17), and Lemma 3.6,

$$|g(x)| \leq \|\tilde{\delta}\|_{L^\infty(\tau_0)} \int_{\tau_0} |\nabla_y G(x, y)| dy \leq C h^{-N} \int_{\tau_0} |x - y|^{1-N} dy \leq C h^{-N} \int_0^{ch} d\rho \leq C h^{1-N}.$$

Hence by the Hölder inequality and using that $\sigma \leq Ch$ on B_h , we have

$$\|\sigma^{\frac{N-2}{2}} g\|_{L^2(B_h)} \leq C h^{\frac{N}{2}} h^{\frac{N-2}{2}} \|g\|_{L^\infty(B_h)} \leq C.$$

In the case $x \in B_h^c$, we have for any $y \in \tau_0$ by the triangle inequality

$$|x - y| \geq |x - x_0| - |y - x_0| \geq |x - x_0| - h$$

and therefore again by (3.17) and Lemma 3.6

$$|g(x)| \leq \|\tilde{\delta}\|_{L^1(\tau_0)} \frac{C}{(|x - x_0| - h)^{N-1}} \leq \frac{C}{(|x - x_0| - h)^{N-1}}.$$

Hence, using polar coordinates with $\rho = |x - x_0|$, we obtain

$$\|\sigma^{\frac{N-2}{2}} g\|_{L^2(B_h^c)}^2 \leq C \int_{B_h^c} \frac{(|x - x_0| + Kh)^{N-2}}{(|x - x_0| - h)^{2N-2}} dx \leq C \int_{3h}^{\text{diam}(\Omega)} \frac{(\rho + Kh)^{N-2}}{(\rho - h)^{2N-2}} \rho^{N-1} d\rho \leq C |\ln h|.$$

Thus, we established

$$\|\sigma^{\frac{N-2}{2}} g\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}}. \quad (3.18)$$

To estimate the second term in (3.14) we apply Lemma 3.4 and obtain

$$\|\sigma^{\frac{N}{2}} \nabla g\|_{L^2(\Omega)} \leq C \left(\|\sigma^{\frac{N+2}{2}} D\tilde{\delta}\| + \|\sigma^{\frac{N-2}{2}} g\| \right) \leq C + C \|\sigma^{\frac{N-2}{2}} g\| \leq C |\ln h|^{\frac{1}{2}},$$

where we have used Lemma 3.3 and (3.18).

The first term in (3.15) is estimated as follows. There holds

$$\|g\|_{L^3(\Omega)}^3 = \|g\|_{L^3(B_h)}^3 + \|g\|_{L^3(B_h^c)}^3.$$

For the term on B_h we obtain as above

$$\|g\|_{L^3(B_h)}^3 \leq Ch^3 \|g\|_{L^\infty(B_h)}^3 \leq Ch^{-3}.$$

For the second term we have

$$\|g\|_{L^3(B_h^c)}^3 \leq C \int_{B_h^c} \frac{1}{(|x-x_0|-h)^6} dx \leq C \int_{3h}^{\text{diam}(\Omega)} \frac{1}{(\rho-h)^6} \rho^2 d\rho \leq Ch^{-3}.$$

In order to estimate $\|\nabla g\|_{L^{\frac{3}{2}}(\Omega)}$ we use the pointwise representation

$$\nabla g(x) = \int_{\tau_0} \nabla_x G(x, y) \partial_{y_i} \tilde{\delta}(y) dy, \quad (3.19)$$

apply Lemma 3.6, and obtain for $x \in B_h$

$$|\nabla g(x)| \leq \|\tilde{\delta}\|_{W^{1,\infty}(\tau_0)} \int_{\tau_0} |\nabla_x G(x, y)| dy \leq Ch^{-N-1} \int_{\tau_0} |x-y|^{1-N} dy \leq Ch^{-N-1} \int_0^{ch} d\rho \leq Ch^{-N}.$$

Hence, for $N = 3$, we have

$$\|\nabla g\|_{L^{\frac{3}{2}}(B_h)} \leq Ch^{-3} (h^3)^{\frac{2}{3}} = Ch^{-1}.$$

For $x \in B_h^c$ we integrate by parts in (3.19),

$$\nabla g(x) = - \int_{\tau_0} \partial_{y_i} \nabla_x G(x, y) \tilde{\delta}(y) dy,$$

and obtain using estimate (3.13c) from Lemma 3.6

$$|\nabla g(x)| \leq \|\tilde{\delta}\|_{L^1(\tau_0)} \frac{C}{(|x-x_0|-h)^3} \leq \frac{C}{(|x-x_0|-h)^3}.$$

Thus,

$$\|\nabla g\|_{L^{\frac{3}{2}}(B_h^c)}^{\frac{3}{2}} \leq C \int_{B_h^c} \frac{1}{(|x-x_0|-h)^{\frac{9}{2}}} dx \leq C \int_{3h}^{\text{diam}(\Omega)} \frac{\rho^2}{(\rho-h)^{\frac{9}{2}}} d\rho \leq Ch^{-\frac{3}{2}}.$$

This completes the proof. \square

We will also require a discrete version of the Lemma 3.8.

LEMMA 3.9. For $N = 2, 3$, we have

$$\|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} P_h D\tilde{\delta}\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}}.$$

Proof. Let g be solution of (3.16) and let $g_h \in V_h$ satisfy

$$-\Delta_h g_h = P_h D\tilde{\delta}. \quad (3.20)$$

Notice that $g_h = R_h g$. Thus in order to establish the lemma, we need to show

$$\|\sigma^{\frac{N}{2}} \nabla g_h\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}}.$$

For $N = 2$ we apply Lemma 3.5 and obtain

$$\|\sigma \nabla g_h\|_{L^2(\Omega)} \leq C \left(\|\sigma^2 P_h D \tilde{\delta}\|_{L^2(\Omega)} + \|g_h\|_{L^2(\Omega)} \right) \leq C + C \|g_h\|_{L^2(\Omega)},$$

where we have used Lemma 3.3. Thus, for $N = 2$ it remains to prove

$$\|g_h\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}}.$$

To prove this estimate, we use Lemma 3.8, global error estimates in the $L^2(\Omega)$, the H^2 regularity, and the property (3.2) of $\tilde{\delta}$. Thus, we obtain

$$\begin{aligned} \|g_h\|_{L^2(\Omega)} &\leq \|g\|_{L^2(\Omega)} + \|g - g_h\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}} + Ch^2 \|g\|_{H^2(\Omega)} \\ &\leq C |\ln h|^{\frac{1}{2}} + Ch^2 \|D \tilde{\delta}\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}}. \end{aligned}$$

The case $N = 3$ is more challenging. By the triangle inequality we get

$$\|\sigma^{\frac{3}{2}} \nabla g_h\|_{L^2(\Omega)} \leq \|\sigma^{\frac{3}{2}} \nabla g\|_{L^2(\Omega)} + \|\sigma^{\frac{3}{2}} \nabla (g - g_h)\|_{L^2(\Omega)}. \quad (3.21)$$

For the first term we have by Lemma 3.8

$$\|\sigma^{\frac{3}{2}} \nabla g\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}}.$$

For the second term we apply [13, Lemma 10], which gives

$$\|\sigma^{\frac{3}{2}} \nabla (g - g_h)\|_{L^2(\Omega)} \leq Ch \left(\|\sigma^{\frac{3}{2}} \Delta_h g_h\|_{L^2(\Omega)} + \|\sigma^{\frac{1}{2}} \nabla g_h\|_{L^2(\Omega)} \right).$$

For the term $\|\sigma^{\frac{3}{2}} \Delta_h g_h\|_{L^2(\Omega)}$ we get by Lemma 3.3

$$\|\sigma^{\frac{3}{2}} \Delta_h g_h\|_{L^2(\Omega)} = \|\sigma^{\frac{3}{2}} P_h D \tilde{\delta}\|_{L^2(\Omega)} \leq Ch^{-1}.$$

Inserting this estimate into (3.21) we obtain

$$\|\sigma^{\frac{3}{2}} \nabla g_h\|_{L^2(\Omega)} \leq C |\ln h|^{\frac{1}{2}} + Ch \|\sigma^{\frac{1}{2}} \nabla g_h\|_{L^2(\Omega)}. \quad (3.22)$$

Thus, it remains to estimate $\|\sigma^{\frac{1}{2}} \nabla g_h\|_{L^2(\Omega)}$. To this end we apply Lemma 3.5 and obtain

$$\|\sigma^{\frac{1}{2}} \nabla g_h\|_{L^2(\Omega)} \leq C \left(\|\sigma^{\frac{3}{2}} P_h D \tilde{\delta}\|_{L^2(\Omega)} + \|\sigma^{-\frac{1}{2}} g_h\|_{L^2(\Omega)} \right).$$

Using Lemma 3.3 we obtain

$$\|\sigma^{\frac{1}{2}} \nabla g_h\|_{L^2(\Omega)} \leq Ch^{-1} + C \|\sigma^{-\frac{1}{2}} g_h\|_{L^2(\Omega)}. \quad (3.23)$$

To estimate $\|\sigma^{-\frac{1}{2}} g_h\|_{L^2(\Omega)}$ we use Lemma 3.7, with $\alpha = \beta = -\frac{1}{2}$ and $p = 3$, to obtain

$$\|\sigma^{-\frac{1}{2}} g_h\|_{L^2(\Omega)} \leq C \|g_h\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}} \|\nabla g_h\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}} \leq C \|g_h\|_{L^3(\Omega)}^{\frac{1}{2}} \|\nabla g\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}}, \quad (3.24)$$

where in the last step we used stability of the Ritz projection in the $W^{1, \frac{3}{2}}(\Omega)$ seminorm, see [11]. Using the inverse and the triangle inequalities,

$$\begin{aligned} \|g_h\|_{L^3(\Omega)} &\leq \|g\|_{L^3(\Omega)} + \|g - g_h\|_{L^3(\Omega)} \leq \|g\|_{L^3(\Omega)} + \|i_h g - g_h\|_{L^3(\Omega)} + \|g - i_h g\|_{L^3(\Omega)} \\ &\leq \|g\|_{L^3(\Omega)} + Ch^{-\frac{1}{2}} \|i_h g - g_h\|_{L^2(\Omega)} + \|g - i_h g\|_{L^3(\Omega)} \\ &\leq \|g\|_{L^3(\Omega)} + Ch^{-\frac{1}{2}} \|g - g_h\|_{L^2(\Omega)} + Ch^{-\frac{1}{2}} \|g - i_h g\|_{L^2(\Omega)} + \|g - i_h g\|_{L^3(\Omega)}. \end{aligned}$$

Using the approximation theory (3.7), the standard L^2 estimate, and the properties of $\tilde{\delta}$ function, we have

$$h^{-\frac{1}{2}} \|g - g_h\|_{L^2(\Omega)} + h^{-\frac{1}{2}} \|g - i_h g\|_{L^2(\Omega)} + \|g - i_h g\|_{L^3(\Omega)} \leq Ch^{\frac{3}{2}} \|g\|_{H^2(\Omega)} \leq Ch^{\frac{3}{2}} \|D\tilde{\delta}\|_{L^2(\Omega)} \leq Ch^{-1}. \quad (3.25)$$

By Lemma 3.8 we have

$$\|g\|_{L^3(\Omega)} + \|\nabla g\|_{L^{\frac{3}{2}}(\Omega)} \leq Ch^{-1}.$$

Inserting this in (3.24) and (3.23) we obtain

$$\|\sigma^{\frac{1}{2}} \nabla g_h\| \leq Ch^{-1}.$$

Using (3.22) we establish the lemma for $N = 3$. \square

4. Weighted resolvent estimates. In this section we will prove the weighted resolvent estimates in two and three dimensions. Since in this section (only) we will be dealing with complex valued function spaces, we need to modify the definition of the L^2 -inner product as

$$(u, v)_{\Omega} = \int_{\Omega} u(x) \bar{v}(x) dx,$$

where \bar{v} is the complex conjugate of v . Moreover we introduce the spaces $\mathbb{V} = H_0^1(\Omega) + iH_0^1(\Omega)$ and $\mathbb{V}_h = V_h + iV_h$.

In the continuous case for Lipschitz domains the following result was shown in [29]: For any $\gamma \in (0, \frac{\pi}{2})$ there exists a constant $C = C_{\gamma}$ such that

$$\|(z + \Delta)^{-1} v\|_{L^p(\Omega)} \leq \frac{C}{|z|} \|v\|_{L^p(\Omega)}, \quad z \in \mathbb{C} \setminus \Sigma_{\gamma}, \quad 1 \leq p \leq \infty, \quad v \in L^p(\Omega), \quad (4.1)$$

where Σ_{γ} is defined by

$$\Sigma_{\gamma} = \{z \in \mathbb{C} \mid |\arg z| \leq \gamma\}. \quad (4.2)$$

In the finite element setting, it is also known that

$$\|(z + \Delta_h)^{-1} \chi\|_{L^p(\Omega)} \leq \frac{C}{|z|} \|\chi\|_{L^p(\Omega)}, \quad \text{for all } z \in \mathbb{C} \setminus \Sigma_{\gamma}, \quad \chi \in \mathbb{V}_h \quad (4.3)$$

for $1 \leq p \leq \infty$. For smooth domains such result is established in [1] and for convex polyhedral domains in [15, 17]. In [14, Theorem 7] we also established the following weighted resolvent estimate:

$$\|\sigma^{\frac{N}{2}} (z + \Delta_h)^{-1} \chi\|_{L^2(\Omega)} \leq \frac{C |\ln h|}{|z|} \|\sigma^{\frac{N}{2}} \chi\|_{L^2(\Omega)}, \quad \text{for all } z \in \mathbb{C} \setminus \Sigma_{\gamma}, \quad \chi \in \mathbb{V}_h. \quad (4.4)$$

Our goal in this section is to establish a further resolvent estimate in the weighted norm, which will be required later.

THEOREM 4.1. *Let $N = 2, 3$. For any $\gamma \in (0, \frac{\pi}{2})$, there exists a constant C independent of h and z such that*

$$\|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} (z + \Delta_h)^{-1} \chi\|_{L^2(\Omega)} \leq \frac{C |\ln h|^{\frac{N-1}{N}}}{|z|} \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} \chi\|_{L^2(\Omega)}, \quad \text{for all } z \in \mathbb{C} \setminus \Sigma_{\gamma}, \quad \chi \in \mathbb{V}_h,$$

where Σ_{γ} is defined in (4.2). Before we provide a proof of the above theorem we collect some preliminary results.

4.1. Preliminary resolvent results. The following lemma will be often used if dealing resolvent estimates.

LEMMA 4.2. *Let for each $z \in \mathbb{C} \setminus \Sigma_{\gamma}$ the numbers $\alpha_z, \beta_z \in \mathbb{R}_+$ be given and let $F_z = -z\alpha_z^2 + \beta_z^2$. Then there exists a constant C_{γ} such that*

$$|z|\alpha_z^2 + \beta_z^2 \leq C_{\gamma} |F_z| \quad \text{for all } z \in \mathbb{C} \setminus \Sigma_{\gamma}.$$

Proof. We consider the polar representation $-z\alpha_z^2 = |z|\alpha_z^2 e^{i\phi_z}$ with $|\phi_z| \leq \pi - \gamma$, since $\gamma \leq |\arg z| \leq \pi$. This results in

$$|z|\alpha_z^2 e^{i\phi_z} + \beta_z^2 = F_z.$$

Multiplying it by $e^{-i\phi_z/2}$ and taking real parts, we have

$$|z|\alpha_z^2 + \beta_z^2 \leq (\cos(\phi_z/2))^{-1}|F_z| \leq (\sin(\gamma/2))^{-1}|F_z| = C_\gamma|F_z|.$$

□

The following result is a best approximation type estimate in H^1 norm.

LEMMA 4.3. *Let $w \in \mathbb{V}$ and let $w_h \in \mathbb{V}_h$ with $e = w - w_h$ be defined by the orthogonality relation*

$$z(e, \chi) - (\nabla e, \nabla \chi) = 0, \quad \text{for all } \chi \in \mathbb{V}_h. \quad (4.5)$$

Then there exists a constant $C > 0$ such that for any $\chi \in \mathbb{V}_h$

$$\|\nabla(w - w_h)\|_{L^2(\Omega)} \leq C \inf_{\chi \in \mathbb{V}_h} (h^{-1}\|w - \chi\|_{L^2(\Omega)} + \|\nabla(w - \chi)\|_{L^2(\Omega)}).$$

Proof. Although the proof is straightforward, we will provide it for a completeness. Using (4.5), for any $\chi \in \mathbb{V}_h$ we have

$$-z\|e\|^2 + \|\nabla e\|^2 = -z(e, e) + (\nabla e, \nabla e) = -z(e, w - \chi) + (\nabla e, \nabla(w - \chi)) := F.$$

Using the Cauchy-Schwarz inequality, we obtain

$$|F| \leq |z|\|e\|\|w - \chi\| + \|\nabla e\|\|\nabla(w - \chi)\|$$

Hence, by Lemma 4.2 and the the Young's inequality, we have

$$\begin{aligned} |z|\|e\|^2 + \|\nabla e\|^2 &\leq C_\gamma (|z|\|e\|\|w - \chi\| + \|\nabla e\|\|\nabla(w - \chi)\|) \\ &\leq \frac{|z|}{2}\|e\|^2 + \frac{C_\gamma^2}{2}|z|\|w - \chi\|^2 + \frac{1}{2}\|\nabla e\|^2 + \frac{C_\gamma^2}{2}\|\nabla(w - \chi)\|^2. \end{aligned}$$

Canceling, we obtain for all $z \in \mathbb{C} \setminus \Sigma_\gamma$

$$|z|\|e\|^2 + \|\nabla e\|^2 \leq C_\gamma^2 (|z|\|w - \chi\|^2 + \|\nabla(w - \chi)\|^2). \quad (4.6)$$

Now we consider two cases: $|z| \leq h^{-2}$ and $|z| > h^{-2}$.

Case 1: $|z| \leq h^{-2}$.

Using that $|z| \leq h^{-2}$ from (4.6) we immediately obtain

$$\|\nabla e\| \leq C_\gamma (h^{-1}\|w - \chi\| + \|\nabla(w - \chi)\|).$$

Case 2: $|z| > h^{-2}$.

In this case from (4.6), we conclude

$$\|e\|^2 \leq C_\gamma^2 \left(\|w - \chi\|^2 + \frac{1}{|z|} \|\nabla(w - \chi)\|^2 \right) \leq C_\gamma^2 (\|w - \chi\|^2 + h^2 \|\nabla(w - \chi)\|^2).$$

To estimate $\|\nabla e\|$ we use the triangle and the inverse estimate to obtain

$$\begin{aligned} \|\nabla e\| &\leq \|\nabla(w - \chi)\| + \|\nabla(\chi - w_h)\| \\ &\leq \|\nabla(w - \chi)\| + C_{\text{inv}} h^{-1} \|\chi - w_h\| \\ &\leq \|\nabla(w - \chi)\| + C_{\text{inv}} h^{-1} (\|\chi - w\| + \|e\|) \\ &\leq C_{\text{inv}} (1 + C_\gamma) h^{-1} \|w - \chi\| + (C_{\text{inv}} C_\gamma + 1) \|\nabla(w - \chi)\|. \end{aligned}$$

Combining both cases, completes the proof. □

We will also need the following lemma.

LEMMA 4.4. *Let $w_h \in \mathbb{V}_h$ be the solution of*

$$z(w_h, \varphi)_\Omega - (\nabla w_h, \nabla \varphi)_\Omega = (f, \varphi)_\Omega, \quad \text{for all } \varphi \in \mathbb{V}_h$$

for some $f \in L^{\frac{3}{2}}(\Omega) + iL^{\frac{3}{2}}(\Omega)$. There exists a constant $c > 0$ such that

$$\|\nabla w_h\|_{L^3(\Omega)} \leq C \|f\|_{L^{\frac{3}{2}}(\Omega)}.$$

Proof. Let $w = (z + \Delta)^{-1}f$. From the resolvent estimates [29] we have

$$\|(z + \Delta)^{-1}f\|_{L^{\frac{3}{2}}(\Omega)} \leq \frac{C}{|z|} \|f\|_{L^{\frac{3}{2}}(\Omega)} \quad \text{and} \quad \|\Delta(z + \Delta)^{-1}f\|_{L^{\frac{3}{2}}(\Omega)} \leq C \|f\|_{L^{\frac{3}{2}}(\Omega)}.$$

Therefore $\Delta w \in L^{\frac{3}{2}}(\Omega)$ and using the elliptic regularity, see [10, Corollary 1], we can conclude that $w \in W^{2, \frac{3}{2}}(\Omega)$ with

$$\|w\|_{W^{2, \frac{3}{2}}(\Omega)} \leq C \|f\|_{L^{\frac{3}{2}}(\Omega)}. \quad (4.7)$$

Since $W^{2, \frac{3}{2}}(\Omega)$ is not embedded into $C(\Omega)$, we use the Scott-Zhang interpolant i_h^{SZ} . Thus, by the triangle inequality we have

$$\|\nabla w_h\|_{L^3(\Omega)} \leq \|\nabla w\|_{L^3(\Omega)} + \|\nabla(w - i_h^{SZ}w)\|_{L^3(\Omega)} + \|\nabla(w_h - i_h^{SZ}w)\|_{L^3(\Omega)} := J_1 + J_2 + J_3.$$

Using the Sobolev embedding $W^{2, \frac{3}{2}}(\Omega) \hookrightarrow W^{1, 3}(\Omega)$ and (4.7) we have

$$J_1 \leq \|w\|_{W^{2, \frac{3}{2}}(\Omega)} \leq C \|f\|_{L^{\frac{3}{2}}(\Omega)}.$$

Similarly, using stability of i_h^{SZ} we have

$$J_2 \leq \|w\|_{W^{2, \frac{3}{2}}(\Omega)} \leq C \|f\|_{L^{\frac{3}{2}}(\Omega)}.$$

To estimate J_3 , we first use the inverse inequality, Lemma 4.3, and (3.8), we have

$$\begin{aligned} J_3 &\leq Ch^{-\frac{1}{2}} \|\nabla(w_h - i_h^{SZ}w)\|_{L^2(\Omega)} \leq Ch^{-\frac{1}{2}} (\|\nabla(w_h - w)\|_{L^2(\Omega)} + \|\nabla(w - i_h^{SZ}w)\|_{L^2(\Omega)}) \\ &\leq Ch^{-\frac{1}{2}} (h^{-1} \|w - i_h^{SZ}w\|_{L^2(\Omega)} + \|\nabla(w - i_h^{SZ}w)\|_{L^2(\Omega)}) \leq C \|w\|_{W^{2, \frac{3}{2}}(\Omega)} \leq C \|f\|_{L^{\frac{3}{2}}(\Omega)}. \end{aligned}$$

Combining estimates for J_1 , J_2 , and J_3 we obtain the lemma. \square

The following lemma is needed for the proof of our main resolvent estimate Theorem 4.1.

LEMMA 4.5. *Let $N = 2, 3$. For a given $\chi \in \mathbb{V}_h$, let $u_h = (z + \Delta_h)^{-1}\chi$, or equivalently*

$$z(u_h, \varphi)_\Omega + (\Delta_h u_h, \varphi)_\Omega = (\chi, \varphi)_\Omega, \quad \text{for all } \varphi \in \mathbb{V}_h. \quad (4.8)$$

Then for any $\gamma \in (0, \frac{\pi}{2})$, there exists a constant C independent of h and z such that

$$\|\sigma^{\frac{N-2}{2}} \Delta_h^{-1} u_h\|_{L^2(\Omega)} \leq \frac{C |\ln h|^{\frac{N-1}{N}}}{|z|} \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} \chi\|_{L^2(\Omega)} \quad \text{for all } z \in \mathbb{C} \setminus \Sigma_\gamma. \quad (4.9)$$

Proof. Most arguments will be using L^2 inner-products and L^2 norms over the whole domain Ω . To simplify the notation in this proof we will denote $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$ and $(\cdot, \cdot)_\Omega$ by (\cdot, \cdot) .

We will consider the cases $N = 2$ and $N = 3$ separately. Thus, for $N = 2$, we need to show

$$\|\Delta_h^{-1} u_h\| \leq \frac{C |\ln h|^{\frac{1}{2}}}{|z|} \|\sigma \nabla \Delta_h^{-1} \chi\|. \quad (4.10)$$

To accomplish that, we test (4.8) with $\varphi = -\Delta_h^{-2}u_h$. We obtain

$$-z(u_h, \Delta_h^{-2}u_h) - (\Delta_h u_h, \Delta_h^{-2}u_h) = -(\chi, \Delta_h^{-2}u_h).$$

Using that $(u_h, \Delta_h^{-2}u_h) = \|\Delta_h^{-1}u_h\|^2$ and $(\Delta_h u_h, \Delta_h^{-2}u_h) = -\|\nabla \Delta_h^{-1}u_h\|^2$ we obtain

$$-z\|\Delta_h^{-1}u_h\|^2 + \|\nabla \Delta_h^{-1}u_h\|^2 = -(\chi, \Delta_h^{-2}u_h) = -(\Delta_h^{-1}\chi, \Delta_h^{-1}u_h). \quad (4.11)$$

Using Lemma 4.2 we obtain

$$|z|\|\Delta_h^{-1}u_h\|^2 + \|\nabla \Delta_h^{-1}u_h\|^2 \leq C_\gamma |(\Delta_h^{-1}\chi, \Delta_h^{-1}u_h)|, \quad \text{for } z \in \mathbb{C} \setminus \Sigma_\gamma.$$

For the right-hand side we have by the Cauchy-Schwarz and the Young's inequalities,

$$|(\Delta_h^{-1}\chi, \Delta_h^{-1}u_h)| \leq \|\Delta_h^{-1}\chi\| \|\Delta_h^{-1}u_h\| \leq \frac{|z|}{2C_\gamma} \|\Delta_h^{-1}u_h\|^2 + \frac{C}{|z|} \|\Delta_h^{-1}\chi\|^2.$$

With the Sobolev $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ in two space dimensions, the Poincare inequality, and using the property of σ (3.4a), we obtain

$$\|\Delta_h^{-1}\chi\| \leq C \|\Delta_h^{-1}\chi\|_{W^{1,1}(\Omega)} \leq C \|\nabla \Delta_h^{-1}\chi\|_{L^1(\Omega)} \leq C |\ln h|^{\frac{1}{2}} \|\sigma \nabla \Delta_h^{-1}\chi\|.$$

Thus, we have

$$|z|\|\Delta_h^{-1}u_h\|^2 + \|\nabla \Delta_h^{-1}u_h\|^2 \leq \frac{C |\ln h|}{|z|} \|\sigma \nabla \Delta_h^{-1}\chi\|^2 + \frac{|z|}{2} \|\Delta_h^{-1}u_h\|^2.$$

Kicking back $\frac{|z|}{2} \|\Delta_h^{-1}u_h\|^2$, we establish (4.10) and hence the lemma for $N = 2$.

For $N = 3$, we need to show

$$\|\sigma^{\frac{1}{2}} \Delta_h^{-1}u_h\| \leq \frac{C |\ln h|^{\frac{2}{3}}}{|z|} \|\sigma^{\frac{3}{2}} \nabla \Delta_h^{-1}\chi\|. \quad (4.12)$$

To accomplish that, we test (4.8) with $\varphi = -\Delta_h^{-1}P_h(\sigma \Delta_h^{-1}u_h)$. We obtain

$$-z(u_h, \Delta_h^{-1}P_h(\sigma \Delta_h^{-1}u_h)) - (\Delta_h u_h, \Delta_h^{-1}P_h(\sigma \Delta_h^{-1}u_h)) = -(\chi, \Delta_h^{-1}P_h(\sigma \Delta_h^{-1}u_h)).$$

Using that

$$(u_h, \Delta_h^{-1}P_h(\sigma \Delta_h^{-1}u_h)) = (\Delta_h^{-1}u_h, P_h(\sigma \Delta_h^{-1}u_h)) = \|\sigma^{\frac{1}{2}} \Delta_h^{-1}u_h\|^2$$

and

$$\begin{aligned} (\Delta_h u_h, \Delta_h^{-1}P_h(\sigma \Delta_h^{-1}u_h)) &= (\Delta_h \Delta_h^{-1}u_h, P_h(\sigma \Delta_h^{-1}u_h)) \\ &= -(\nabla \Delta_h^{-1}u_h, \nabla P_h(\sigma \Delta_h^{-1}u_h)) \\ &= -(\nabla \Delta_h^{-1}u_h, \nabla(\sigma \Delta_h^{-1}u_h)) - (\nabla \Delta_h^{-1}u_h, \nabla(P_h - \text{Id})(\sigma \Delta_h^{-1}u_h)) \\ &= -\|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1}u_h\|^2 - (\nabla \Delta_h^{-1}u_h, \nabla \sigma \Delta_h^{-1}u_h) \\ &\quad - (\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1}u_h, \sigma^{-\frac{1}{2}} \nabla(P_h - \text{Id})(\sigma \Delta_h^{-1}u_h)), \end{aligned}$$

we obtain

$$-z\|\sigma^{\frac{1}{2}} \Delta_h^{-1}u_h\|^2 + \|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1}u_h\|^2 = F, \quad (4.13)$$

where

$$\begin{aligned} F = F_1 + F_2 + F_3 &:= -(\chi, \Delta_h^{-1}P_h(\sigma \Delta_h^{-1}u_h)) - (\nabla \Delta_h^{-1}u_h, \nabla \sigma \Delta_h^{-1}u_h) \\ &\quad - (\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1}u_h, \sigma^{-\frac{1}{2}} \nabla(P_h - \text{Id})(\sigma \Delta_h^{-1}u_h)). \end{aligned}$$

Using Lemma 4.2 we conclude

$$|z| \|\sigma^{\frac{1}{2}} \Delta_h^{-1} u_h\|^2 + \|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1} u_h\|^2 \leq C_\gamma |F|, \quad \text{for } z \in \mathbb{C} \setminus \Sigma_\gamma. \quad (4.14)$$

By the Cauchy-Schwarz and the Young's inequalities,

$$\begin{aligned} |F_1| &\leq \|\sigma^{\frac{1}{2}} \Delta_h^{-1} \chi\| \|\sigma^{\frac{1}{2}} \Delta_h^{-1} u_h\| \leq \frac{CC_\gamma}{|z|} \|\sigma^{\frac{1}{2}} \Delta_h^{-1} \chi\|^2 + \frac{|z|}{4C_\gamma} \|\sigma^{\frac{1}{2}} \Delta_h^{-1} u_h\|^2 \\ &\leq \frac{CC_\gamma}{|z|} \|\sigma^{\frac{3}{2}} \nabla \Delta_h^{-1} \chi\|^2 + \frac{|z|}{4C_\gamma} \|\sigma^{\frac{1}{2}} \Delta_h^{-1} u_h\|^2, \end{aligned}$$

where in the last step we again use Lemma 3.7 with $\alpha = \frac{1}{2}$, $\beta = 0$, and $p = 2$. To estimate F_2 we use the Cauchy-Schwarz and the Young's inequalities, as well as the fact that $|\nabla \sigma| \leq C$.

$$|F_2| \leq C \|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1} u_h\| \|\sigma^{-\frac{1}{2}} \Delta_h^{-1} u_h\| \leq \frac{1}{4C_\gamma} \|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1} u_h\|^2 + CC_\gamma \|\sigma^{-\frac{1}{2}} \Delta_h^{-1} u_h\|^2.$$

Using Lemma 3.7 with $\alpha = \beta = -\frac{1}{2}$, $p = \frac{3}{2}$ and $p' = 3$, we obtain

$$\|\sigma^{-\frac{1}{2}} \Delta_h^{-1} u_h\|^2 \leq C \|\Delta_h^{-1} u_h\|_{L^{\frac{3}{2}}(\Omega)} \|\nabla \Delta_h^{-1} u_h\|_{L^3(\Omega)}.$$

Using the properties of σ and the Hölder inequality, we have

$$\|\Delta_h^{-1} u_h\|_{L^{\frac{3}{2}}(\Omega)} \leq C |\ln h|^{\frac{1}{6}} \|\sigma^{\frac{1}{2}} \Delta_h^{-1} u_h\|,$$

and as a result

$$|F_2| \leq \frac{1}{4C_\gamma} \|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1} u_h\|^2 + \frac{|z|}{4C_\gamma} \|\sigma^{\frac{1}{2}} \Delta_h^{-1} u_h\|^2 + \frac{C}{|z|} |\ln h|^{\frac{1}{3}} \|\nabla \Delta_h^{-1} u_h\|_{L^3(\Omega)}^2. \quad (4.15)$$

Finally, using the Cauchy-Schwarz inequality, Lemma 3.2, and the Young's inequality, we obtain

$$|F_3| \leq C \|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1} u_h\| \|\sigma^{-\frac{1}{2}} \Delta_h^{-1} u_h\| \leq \frac{1}{4C_\gamma} \|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1} u_h\|^2 + C_\gamma \|\sigma^{-\frac{1}{2}} \Delta_h^{-1} u_h\|^2.$$

Similarly to the estimate of F_2 above we obtain,

$$|F_3| \leq \frac{1}{4C_\gamma} \|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1} u_h\|^2 + \frac{|z|}{4C_\gamma} \|\sigma^{\frac{1}{2}} \Delta_h^{-1} u_h\|^2 + \frac{C}{|z|} |\ln h|^{\frac{1}{3}} \|\nabla \Delta_h^{-1} u_h\|_{L^3(\Omega)}^2. \quad (4.16)$$

Combining estimates for F_1 , F_2 , and F_3 , inserting them into (4.14) and kicking back, we obtain

$$|z| \|\sigma^{\frac{1}{2}} \Delta_h^{-1} u_h\|^2 + \|\sigma^{\frac{1}{2}} \nabla \Delta_h^{-1} u_h\|^2 \leq \frac{C}{|z|} \|\sigma^{\frac{3}{2}} \nabla \Delta_h^{-1} \chi\|^2 + \frac{C}{|z|} |\ln h|^{\frac{1}{3}} \|\nabla \Delta_h^{-1} u_h\|_{L^3(\Omega)}^2. \quad (4.17)$$

Thus, in order to establish the lemma for $N = 3$, we need to show

$$\|\nabla \Delta_h^{-1} u_h\|_{L^3(\Omega)} \leq C |\ln h|^{\frac{1}{2}} \|\sigma^{\frac{3}{2}} \nabla \Delta_h^{-1} \chi\|. \quad (4.18)$$

This estimates follows by Lemma 4.4, Sobolev embedding theorem $W^{1,1}(\Omega) \hookrightarrow L^{\frac{3}{2}}(\Omega)$ combined with the Poincare inequality, and the properties of σ . Indeed,

$$\|\nabla \Delta_h^{-1} u_h\|_{L^3(\Omega)} \leq C \|\Delta_h^{-1} \chi\|_{L^{\frac{3}{2}}(\Omega)} \leq C \|\nabla \Delta_h^{-1} \chi\|_{L^1(\Omega)} \leq C |\ln h|^{\frac{1}{2}} \|\sigma^{\frac{3}{2}} \nabla \Delta_h^{-1} \chi\|.$$

This concludes the proof of the lemma. \square

4.2. Proof of Theorem 4.1. For an arbitrary $\chi \in \mathbb{V}_h$, the solution to resolvent equation u_h satisfies

$$z(u_h, \varphi) + (\Delta_h u_h, \varphi) = (\chi, \varphi), \quad \text{for all } \varphi \in \mathbb{V}_h. \quad (4.19)$$

First we test (4.19) with $\varphi = \Delta_h^{-1} P_h(\sigma^N u_h)$ to obtain

$$z(u_h, \Delta_h^{-1} P_h(\sigma^N u_h)) + (\Delta_h u_h, \Delta_h^{-1} P_h(\sigma^N u_h)) = (\chi, \Delta_h^{-1} P_h(\sigma^N u_h)).$$

Using that

$$(\Delta_h u_h, \Delta_h^{-1} P_h(\sigma^N u_h)) = (u_h, P_h(\sigma^N u_h)) = (u_h, \sigma^N u_h) = \|\sigma^{\frac{N}{2}} u_h\|^2$$

and

$$\begin{aligned} (u_h, \Delta_h^{-1} P_h(\sigma^N u_h)) &= (\Delta_h^{-1} u_h, P_h(\sigma^N u_h)) = (\Delta_h^{-1} u_h, \sigma^N u_h) = (\sigma^N \Delta_h^{-1} u_h, \Delta_h \Delta_h^{-1} u_h) \\ &= -(\nabla(P_h \sigma^N \Delta_h^{-1} u_h), \nabla \Delta_h^{-1} u_h) \\ &= -(\nabla(\sigma^N \Delta_h^{-1} u_h), \nabla \Delta_h^{-1} u_h) - (\nabla(P_h - \text{Id})(\sigma^N \Delta_h^{-1} u_h), \nabla \Delta_h^{-1} u_h) \\ &= -\|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} u_h\|^2 - N(\sigma^{N-1} \nabla \sigma \Delta_h^{-1} u_h, \nabla \Delta_h^{-1} u_h) - (\nabla(P_h - \text{Id})(\sigma^N \Delta_h^{-1} u_h), \nabla \Delta_h^{-1} u_h), \end{aligned}$$

we obtain

$$-z\|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} u_h\|^2 + \|\sigma^{\frac{N}{2}} u_h\|^2 = F, \quad (4.20)$$

where

$$F = F_1 + F_2 + F_3$$

$$:= (\chi, \Delta_h^{-1} P_h(\sigma^N u_h)) + Nz(\sigma^{N-1} \nabla \sigma \Delta_h^{-1} u_h, \nabla \Delta_h^{-1} u_h) + z(\sigma^{-\frac{N}{2}} \nabla(P_h - \text{Id})(\sigma^N \Delta_h^{-1} u_h), \sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} u_h).$$

By Lemma 4.2 we conclude

$$|z|\|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} u_h\|^2 + \|\sigma^{\frac{N}{2}} u_h\|^2 \leq C_\gamma |F|, \quad \text{for } z \in \mathbb{C} \setminus \Sigma_\gamma.$$

To estimate F_1 we notice that

$$\begin{aligned} (\chi, \Delta_h^{-1} P_h(\sigma^N u_h)) &= (\Delta_h^{-1} \chi, P_h(\sigma^N u_h)) = (\sigma^N \Delta_h^{-1} \chi, u_h) \\ &= (P_h(\sigma^N \Delta_h^{-1} \chi), \Delta_h \Delta_h^{-1} u_h) \\ &= -(\nabla P_h(\sigma^N \Delta_h^{-1} \chi), \nabla \Delta_h^{-1} u_h) \\ &= -(\nabla(\sigma^N \Delta_h^{-1} \chi), \nabla \Delta_h^{-1} u_h) - (\nabla(P_h - \text{Id})(\sigma^N \Delta_h^{-1} \chi), \nabla \Delta_h^{-1} u_h) \\ &= -(\sigma^N \nabla \Delta_h^{-1} \chi, \nabla \Delta_h^{-1} u_h) - N(\sigma^{N-1} \nabla \sigma \Delta_h^{-1} \chi, \nabla \Delta_h^{-1} u_h) \\ &\quad - (\sigma^{-\frac{N}{2}} \nabla(P_h - \text{Id})(\sigma^N \Delta_h^{-1} \chi), \sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} u_h). \end{aligned}$$

Using $|\nabla \sigma| \leq C$, the Cauchy-Schwarz inequality, and the Young's inequality, we obtain,

$$\begin{aligned} |F_1| &\leq \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} u_h\| \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} \chi\| + C \|\sigma^{\frac{N-2}{2}} \Delta_h^{-1} \chi\| \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} u_h\| \\ &\quad + \|\sigma^{-\frac{N}{2}} \nabla(P_h - \text{Id})(\sigma^N \Delta_h^{-1} \chi)\| \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} u_h\| \\ &\leq \frac{CC_\gamma}{|z|} \left(\|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} \chi\|^2 + \|\sigma^{\frac{N-2}{2}} \Delta_h^{-1} \chi\|^2 \right) + \frac{|z|}{4C_\gamma} \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} u_h\|^2, \end{aligned}$$

where in the last step we used Lemma 3.2 to obtain

$$\|\sigma^{-\frac{N}{2}} \nabla(P_h - \text{Id})(\sigma^N \Delta_h^{-1} \chi)\| \leq C \|\sigma^{\frac{N-2}{2}} \Delta_h^{-1} \chi\|.$$

For $N = 2$, using the Sobolev embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ and the Poincare inequality, we obtain

$$\|v_h\| \leq C \|v_h\|_{W^{1,1}(\Omega)} \leq C \|\nabla v_h\|_{L^1(\Omega)}, \quad \text{for all } v_h \in V_h.$$

Using in addition the property of σ (3.4a), we obtain

$$\|\Delta_h^{-1}\chi\| \leq C\|\nabla\Delta_h^{-1}\chi\|_{L^1(\Omega)} \leq C|\ln h|^{\frac{1}{2}}\|\sigma\nabla\Delta_h^{-1}\chi\|.$$

For $N = 3$, we use Lemma 3.7 with $\alpha = \frac{1}{2}$, $\beta = 0$, and $p = 2$, to obtain

$$\|\sigma^{\frac{1}{2}}\Delta_h^{-1}\chi\| \leq C\|\sigma^{\frac{3}{2}}\nabla\Delta_h^{-1}\chi\|.$$

Thus,

$$|F_1| \leq \frac{CC_\gamma|\ln h|^{3-N}}{|z|}\|\sigma^{\frac{N}{2}}\nabla\Delta_h^{-1}\chi\|^2 + \frac{|z|}{4C_\gamma}\|\sigma^{\frac{N}{2}}\nabla\Delta_h^{-1}u_h\|^2.$$

To estimate F_2 we use the Cauchy-Schwarz and the Young's inequalities,

$$|F_2| \leq C|z|\|\sigma^{\frac{N-2}{2}}\Delta_h^{-1}u_h\|\|\sigma^{\frac{N}{2}}\nabla\Delta_h^{-1}u_h\| \leq \frac{|z|}{4C_\gamma}\|\sigma^{\frac{N}{2}}\nabla\Delta_h^{-1}u_h\|^2 + CC_\gamma|z|\|\sigma^{\frac{N-2}{2}}\Delta_h^{-1}u_h\|^2.$$

To estimate F_3 we use Lemma 3.2, the Cauchy-Schwarz and the Young's inequalities,

$$|F_3| \leq C|z|\|\sigma^{-\frac{N}{2}}\nabla(P_h - \text{Id})(\sigma^N\Delta_h^{-1}u_h)\|\|\sigma^{\frac{N}{2}}\nabla\Delta_h^{-1}u_h\| \leq C_\gamma|z|\|\sigma^{\frac{N-2}{2}}\Delta_h^{-1}u_h\|^2 + \frac{|z|}{4C_\gamma}\|\sigma^{\frac{N}{2}}\nabla\Delta_h^{-1}u_h\|^2.$$

Combining estimates for F_1 , F_2 , F_3 and kicking back, we obtain

$$|z|\|\sigma^{\frac{N}{2}}\nabla\Delta_h^{-1}u_h\|^2 + \|\sigma^{\frac{N}{2}}u_h\|^2 \leq \frac{C|\ln h|^{3-N}}{|z|}\|\sigma^{\frac{N}{2}}\nabla\Delta_h^{-1}\chi\|^2 + C|z|\|\sigma^{\frac{N-2}{2}}\Delta_h^{-1}u_h\|^2. \quad (4.21)$$

Now applying Lemma 4.5 to the last term concludes the proof of the theorem.

5. Discrete maximal parabolic estimates. In this section we state stability results for inhomogeneous problems that are central in establishing our main results. Since we apply the following results for different norms on V_h , namely, for $L^p(\Omega)$, weighted $L^2(\Omega)$, and weighted $H^{-1}(\Omega)$ norms, we state them for a general Banach norm $\|\cdot\|$.

Let $\|\cdot\|$ be a norm on V_h (naturally extended to a norm on \mathbb{V}_h) such that for some $\gamma \in (0, \frac{\pi}{2})$ the following resolvent estimate holds,

$$\|(z + \Delta_h)^{-1}\chi\| \leq \frac{M_h}{|z|}\|\chi\|, \quad \text{for all } z \in \mathbb{C} \setminus \Sigma_\gamma, \quad \chi \in \mathbb{V}_h, \quad (5.1)$$

where Σ_γ is defined in (4.2) and the constant M_h is independent of z .

This assumption is fulfilled for $\|\cdot\| = \|\cdot\|_{L^p(\Omega)}$, $1 \leq p \leq \infty$, with a constant $M_h \leq C$ independent of h , see [18], for $\|\cdot\| = \|\sigma^{\frac{N}{2}}(\cdot)\|_{L^2(\Omega)}$ with $M_h \leq C|\ln h|$, see [14, Theorem 7], and for $\|\cdot\| = \|\sigma^{\frac{N}{2}}\nabla\Delta_h^{-1}(\cdot)\|_{L^2(\Omega)}$ with $M_h \leq C|\ln h|^{\frac{N-1}{N}}$, see Theorem 4.1.

We consider the inhomogeneous heat equation (1.1), with $u_0 = 0$ and its discrete approximation $u_{kh} \in X_{k,h}^{q,r}$ defined by

$$B(u_{kh}, \varphi_{kh}) = (f, \varphi_{kh}), \quad \text{for all } \varphi_{kh} \in X_{k,h}^{q,r}. \quad (5.2)$$

The next result is a discrete maximal parabolic regularity result [12, Theorem 14].

LEMMA 5.1 (Discrete maximal parabolic regularity). *Let $\|\cdot\|$ be a norm on V_h fulfilling the resolvent estimate (5.1) and let $1 \leq s \leq \infty$. Let u_{kh} be a solution of (5.2). Then, there exists a constant C independent of k and h such that*

$$\begin{aligned} \left(\sum_{m=1}^M \int_{I_m} \|\partial_t u_{kh}(t)\|^s dt \right)^{\frac{1}{s}} + \left(\sum_{m=1}^M \int_{I_m} \|\Delta_h u_{kh}(t)\|^s dt \right)^{\frac{1}{s}} + \left(\sum_{m=1}^M k_m \|k_m^{-1}[u_{kh}]_{m-1}\|^s \right)^{\frac{1}{s}} \\ \leq CM_h \ln \frac{T}{k} \left(\int_I \|P_h f(t)\|^s dt \right)^{\frac{1}{s}}, \end{aligned}$$

with obvious change of notation in the case $s = \infty$. For $m = 1$ the jump is understood as $[u_{kh}]_0 = u_{kh,0}^+$.

6. Proofs of pointwise global best approximation results. We are now ready to establish our main results.

6.1. Proof of Theorem 2.1. *Proof.* Let $\tilde{t} \in (0, T]$ and let $x_0 \in \Omega$ be an arbitrary but fixed point. Without loss of generality we assume $\tilde{t} \in I_M = (t_{M-1}, T]$. Note, that the case $\tilde{t} = 0$ is trivial, since $u_{kh}(0) = P_h u_0$ and the statement of the Theorem is fulfilled by the stability of the L^2 -Projection in the $W^{1,\infty}(\Omega)$ norm.

We consider the following regularized Green's function

$$\begin{aligned} -\tilde{g}_t(t, x) - \Delta \tilde{g}(t, x) &= D\tilde{\delta}_{x_0}(x)\tilde{\theta}(t) & (t, x) \in I \times \Omega, \\ \tilde{g}(t, x) &= 0, & (t, x) \in I \times \partial\Omega, \\ \tilde{g}(T, x) &= 0, & x \in \Omega, \end{aligned} \quad (6.1)$$

where $\tilde{\delta}_{x_0}$ is the smoothed Dirac introduced in (3.1), D denotes an arbitrary partial derivative in space, and $\tilde{\theta} \in C^\infty(0, T)$ is the regularized Delta function in time with properties $\text{supp}(\tilde{\theta}) \subset I_M$, $\|\tilde{\theta}\|_{L^1(I_M)} \leq C$ and

$$(\tilde{\theta}, \varphi_k)_{I_M} = \varphi_k(\tilde{t}), \quad \text{for all } \varphi_k \in X_k^q.$$

Let \tilde{g}_{kh} be dG(q)cG(r) approximation of \tilde{g} , i.e. $B(\tilde{g} - \tilde{g}_{kh}, \varphi_{kh}) = 0$. Then we have

$$\begin{aligned} -Du_{kh}(\tilde{t}, x_0) &= (u_{kh}, D\tilde{\delta}_{x_0}\tilde{\theta}) = B(u_{kh}, \tilde{g}) = B(u_{kh}, \tilde{g}_{kh}) = B(u, \tilde{g}_{kh}) \\ &= -\sum_{m=1}^M (u, \partial_t \tilde{g}_{kh})_{I_m \times \Omega} + (\nabla u, \nabla \tilde{g}_{kh})_{I \times \Omega} - \sum_{m=1}^M (u_m, [\tilde{g}_{kh}]_m)_\Omega = J_1 + J_2 + J_3, \end{aligned}$$

where in the sum with jumps we included the last term by setting $\tilde{g}_{kh, M+1} = 0$ and defining consequently $[\tilde{g}_{kh}]_M = -\tilde{g}_{kh, M}$. Using the Hölder inequality and the $W^{1,\infty}(\Omega)$ estimate for the Ritz-projection, see Lemma 3.1 we have

$$\begin{aligned} J_1 &= -\sum_{m=1}^M (R_h u, \Delta_h \Delta_h^{-1} \partial_t \tilde{g}_{kh})_{I_m \times \Omega} + ((I - R_h)u, \partial_t \tilde{g}_{kh})_{I_m \times \Omega} \\ &= \sum_{m=1}^M (\nabla R_h u, \nabla \Delta_h^{-1} \partial_t \tilde{g}_{kh})_{I_m \times \Omega} - ((I - R_h)u, \partial_t \tilde{g}_{kh})_{I_m \times \Omega} \\ &\leq \sum_{m=1}^M \|\nabla u\|_{L^\infty(I_m \times \Omega)} \|\nabla \Delta_h^{-1} \partial_t \tilde{g}_{kh}\|_{L^1(I_m; L^1(\Omega))} + \|(I - R_h)u\|_{L^\infty(I_m \times \Omega)} \|\partial_t \tilde{g}_{kh}\|_{L^1(I_m; L^1(\Omega))} \\ &\leq C |\ln h|^{\frac{1}{2}} \|\nabla u\|_{L^\infty(I \times \Omega)} \sum_{m=1}^M \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} \partial_t \tilde{g}_{kh}\|_{L^1(I_m; L^2(\Omega))} \\ &\quad + Ch |\ln h| \|\nabla u\|_{L^\infty(I \times \Omega)} \sum_{m=1}^M \|\partial_t \tilde{g}_{kh}\|_{L^1(I_m; L^1(\Omega))} \end{aligned}$$

Applying the discrete maximal parabolic regularity result from Lemma 5.1 with respect to $\|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1}(\cdot)\|_{L^2(\Omega)}$ and with respect to the $L^1(\Omega)$ norm we get

$$\begin{aligned} J_1 &\leq C \ln \frac{T}{k} \|\nabla u\|_{L^\infty(I \times \Omega)} \left(|\ln h|^{\frac{1}{2} + \frac{N-1}{N}} \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} P_h D\tilde{\delta}\|_{L^2(\Omega)} \|\tilde{\theta}\|_{L^1(I_M)} + h |\ln h| \|P_h D\tilde{\delta}\|_{L^1(\Omega)} \|\tilde{\theta}\|_{L^1(I_M)} \right) \\ &\leq C |\ln h|^{\frac{2N-1}{N}} \ln \frac{T}{k} \|\nabla u\|_{L^\infty(I \times \Omega)}, \end{aligned} \quad (6.2)$$

where in the last step we used Lemma 3.9, Lemma 3.3 and the fact that $\|\tilde{\theta}\|_{L^1(I_M)} \leq C$. Similarly, using the Hölder inequality, properties of σ , Lemma 5.1, and Lemma 3.9, we have

$$\begin{aligned} J_2 &= (\nabla u, \nabla g_{kh})_{I \times \Omega} \leq \|\nabla u\|_{L^\infty(I \times \Omega)} \|\nabla \tilde{g}_{kh}\|_{L^1(I; L^1(\Omega))} \\ &\leq C |\ln h|^{\frac{1}{2}} \|\nabla u\|_{L^\infty(I \times \Omega)} \|\sigma^{\frac{N}{2}} \nabla \tilde{g}_{kh}\|_{L^1(I; L^2(\Omega))} \\ &\leq C |\ln h|^{\frac{1}{2} + \frac{N-1}{N}} \ln \frac{T}{k} \|\nabla u\|_{L^\infty(I \times \Omega)} \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} P_h D\tilde{\delta}\|_{L^2(\Omega)} \|\tilde{\theta}\|_{L^1(I_M)} \\ &\leq C |\ln h|^{\frac{2N-1}{N}} \ln \frac{T}{k} \|\nabla u\|_{L^\infty(I \times \Omega)}. \end{aligned} \quad (6.3)$$

Similarly to the estimate of J_1 , using the Hölder inequality, properties of σ , and Lemma 3.1 we have

$$\begin{aligned}
J_3 &= - \sum_{m=1}^M (R_h u_m, [\tilde{g}_{kh}]_m)_\Omega + ((I - R_h)u_m, [\tilde{g}_{kh}]_m)_\Omega \\
&= \sum_{m=1}^M (\nabla u_m, [\nabla \Delta_h^{-1} \tilde{g}_{kh}]_m)_\Omega - ((I - R_h)u_m, [\tilde{g}_{kh}]_m)_\Omega \\
&\leq \sum_{m=1}^M \|\nabla u_m\|_{L^\infty(\Omega)} \|[\nabla \Delta_h^{-1} \tilde{g}_{kh}]_m\|_{L^1(\Omega)} + \sum_{m=1}^M \|(I - R_h)u_m\|_{L^\infty(\Omega)} \|[\tilde{g}_{kh}]_m\|_{L^1(\Omega)} \\
&\leq C |\ln h|^{\frac{1}{2}} \|\nabla u\|_{L^\infty(I \times \Omega)} \sum_{m=1}^M \|\sigma^{\frac{N}{2}} [\nabla \Delta_h^{-1} \tilde{g}_{kh}]_m\|_{L^2(\Omega)} + Ch |\ln h| \|\nabla u\|_{L^\infty(I \times \Omega)} \sum_{m=1}^M \|[\tilde{g}_{kh}]_m\|_{L^1(\Omega)}.
\end{aligned}$$

Applying the discrete maximal parabolic regularity result from Lemma 5.1 with respect to $\|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1}(\cdot)\|_{L^2(\Omega)}$ and with respect to the $L^1(\Omega)$ norm we get

$$\begin{aligned}
J_3 &\leq C \ln \frac{T}{k} \|\nabla u\|_{L^\infty(I \times \Omega)} \left(|\ln h|^{\frac{1}{2} + \frac{N-1}{N}} \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} P_h D \tilde{\delta}\|_{L^2(\Omega)} \|\tilde{\theta}\|_{L^1(I_M)} + h |\ln h| \|P_h D \tilde{\delta}\|_{L^1(\Omega)} \|\tilde{\theta}\|_{L^1(I_M)} \right) \\
&\leq C |\ln h|^{\frac{2N-1}{N}} \ln \frac{T}{k} \|\nabla u\|_{L^\infty(I \times \Omega)},
\end{aligned} \tag{6.4}$$

where in the last step we again used Lemma 3.9, Lemma 3.3, and the fact that $\|\tilde{\theta}\|_{L^1(I_M)} \leq C$. Combining the estimates for J_1 , J_2 , and J_3 , and taking supremum over all partial derivatives, we conclude that

$$|\nabla u_{kh}(\tilde{t}, x_0)| \leq C \ell_h \ell_k \|\nabla u\|_{L^\infty(I \times \Omega)}.$$

Using that the dG(q)cG(r) method is invariant on $X_{k,h}^{q,r}$, by replacing u and u_{kh} with $u - \chi$ and $u_{kh} - \chi$ for any $\chi \in X_{k,h}^{q,r}$, and using the triangle inequality we obtain Theorem 2.1. \square

7. Proof of pointwise interior best approximation results.

7.1. Proof of Theorem 2.2. To obtain the interior estimate we introduce a smooth cut-off function ω with the properties that

$$\omega(x) \equiv 1, \quad x \in B_d \tag{7.1a}$$

$$\omega(x) \equiv 0, \quad x \in \Omega \setminus B_{2d} \tag{7.1b}$$

$$|\nabla \omega| \leq C d^{-1}, \quad |\nabla^2 \omega| \leq C d^{-2}, \tag{7.1c}$$

where $B_d = B_d(x_0)$ is a ball of radius d centered at x_0 .

As in the proof of Theorem 2.1, we obtain

$$-Du_{kh}(\tilde{t}, x_0) = B(u_{kh}, \tilde{g}_{kh}) = B(u, \tilde{g}_{kh}) = B(\omega u, \tilde{g}_{kh}) + B((1 - \omega)u, \tilde{g}_{kh}), \tag{7.2}$$

where \tilde{g}_{kh} is the solution of (6.1). The first term can be estimated using the global result from Theorem 2.1. To this end we introduce $\tilde{u} = \omega u$ and the dG(q)cG(r) solution $\tilde{u}_{kh} \in X_{k,h}^{q,r}$ defined by

$$B(\tilde{u}_{kh} - \tilde{u}, \varphi_{kh}) = 0 \quad \text{for all } \varphi_{kh} \in X_{k,h}^{q,r}.$$

There holds

$$\begin{aligned}
B(\tilde{u}, \tilde{g}_{kh}) &= B(\tilde{u}_{kh}, \tilde{g}_{kh}) = -D\tilde{u}_{kh}(\tilde{t}, x_0) \leq C \ell_k \ell_h \|\nabla \tilde{u}\|_{L^\infty(I \times \Omega)} \\
&\leq C \ell_k \ell_h (d^{-1} \|u\|_{L^\infty(I \times B_{2d})} + \|\nabla u\|_{L^\infty(I \times B_{2d})}).
\end{aligned}$$

This results in

$$|\nabla u_{kh}(\tilde{t}, x_0)| \leq C \ell_k \ell_h (d^{-1} \|u\|_{L^\infty(I \times B_{2d})} + \|\nabla u\|_{L^\infty(I \times B_{2d})}) + B((1 - \omega)u, \tilde{g}_{kh}). \tag{7.3}$$

It remains to estimate the term $B((1-\omega)u, \tilde{g}_{kh})$. Using the dual expression (2.4) of the bilinear form B we obtain

$$\begin{aligned} B((1-\omega)u, \tilde{g}_{kh}) &= - \sum_{m=1}^M ((1-\omega)u, \partial_t \tilde{g}_{kh})_{I_m \times \Omega} + (\nabla((1-\omega)u), \nabla \tilde{g}_{kh})_{I \times \Omega} \\ &\quad - \sum_{m=1}^M ((1-\omega)u_m, [\tilde{g}_{kh}]_m)_\Omega = J_1 + J_2 + J_3, \end{aligned} \quad (7.4)$$

where again in the sum with jumps we included the last term by setting $\tilde{g}_{kh, M+1} = 0$ and defining consequently $[\tilde{g}_{kh}]_M = -\tilde{g}_{kh, M}$. For J_1 , adding and subtracting $(R_h(1-\omega)u, \partial_t \tilde{g}_{kh})_{I \times \Omega}$, we obtain

$$J_1 = - \sum_{m=1}^M (R_h(1-\omega)u, \partial_t \tilde{g}_{kh})_{I_m \times \Omega} + \sum_{m=1}^M ((I - R_h)(1-\omega)u, \partial_t \tilde{g}_{kh})_{I_m \times \Omega} = J_{11} + J_{12}.$$

Using that $\sigma^{-\frac{N}{2}} \leq Cd^{-\frac{N}{2}}$ on $\Omega \setminus B_d$ and $(1-\omega) \leq 1$, we obtain

$$\begin{aligned} J_{11} &= \sum_{m=1}^M (\nabla((1-\omega)u), \nabla \Delta_h^{-1} \partial_t \tilde{g}_{kh})_{I_m \times \Omega} \\ &= \sum_{m=1}^M (\sigma^{-\frac{N}{2}} \nabla((1-\omega)u), \sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} \partial_t \tilde{g}_{kh})_{I_m \times \Omega} \\ &\leq Cd^{-\frac{N}{2}} \sum_{m=1}^M \|\nabla((1-\omega)u)\|_{L^\infty(I_m; L^2(\Omega))} \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} \partial_t \tilde{g}_{kh}\|_{L^1(I_m; L^2(\Omega))} \\ &\leq Cd^{-\frac{N}{2}} (d^{-1} \|u\|_{L^\infty(I; L^2(\Omega))} + \|\nabla u\|_{L^\infty(I; L^2(\Omega))}) \sum_{m=1}^M \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} \partial_t \tilde{g}_{kh}\|_{L^1(I_m; L^2(\Omega))}. \end{aligned}$$

Applying the discrete maximal parabolic regularity result from Lemma 5.1 with respect to $\|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1}(\cdot)\|_{L^2(\Omega)}$ we get

$$\begin{aligned} J_{11} &\leq C \ln \frac{T}{k} |\ln h|^{\frac{N-1}{N}} d^{-\frac{N}{2}} (d^{-1} \|u\|_{L^\infty(I; L^2(\Omega))} + \|\nabla u\|_{L^\infty(I; L^2(\Omega))}) \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} P_h D \tilde{\theta}\|_{L^2(\Omega)} \|\tilde{\theta}\|_{L^1(I_M)} \\ &\leq C \ln \frac{T}{k} |\ln h|^{\frac{N-1}{N} + \frac{1}{2}} d^{-\frac{N}{2}} (d^{-1} \|u\|_{L^\infty(I; L^2(\Omega))} + \|\nabla u\|_{L^\infty(I; L^2(\Omega))}), \end{aligned} \quad (7.5)$$

where in the last step we used Lemma 3.9, Lemma 3.3 and the fact that $\|\tilde{\theta}\|_{L^1(I_M)} \leq C$.

The estimate for J_{12} is slightly more involved since R_h is a global operator. Put $\psi = (1-\omega)u$, then pointwise in time we obtain

$$((I - R_h)\psi, \partial_t \tilde{g}_{kh})_\Omega = ((I - R_h)\psi, \partial_t \tilde{g}_{kh})_{B_{d/2}} + ((I - R_h)\psi, \partial_t \tilde{g}_{kh})_{\Omega \setminus B_{d/2}} = I_1 + I_2.$$

Using local pointwise error estimates [25], the fact that ψ is supported on $\Omega \setminus B_d$, and the standard error estimate for R_h we have

$$\begin{aligned} I_1 &\leq \|(I - R_h)\psi\|_{L^\infty(B_{d/2})} \|\partial_t \tilde{g}_{kh}\|_{L^1(B_{d/2})} \\ &\leq C \left(|\ln h| \|\psi\|_{L^\infty(B_d)} + d^{-\frac{N}{2}} \|(I - R_h)\psi\|_{L^2(\Omega)} \right) \|\partial_t \tilde{g}_{kh}\|_{L^1(\Omega)} \\ &\leq Chd^{-\frac{N}{2}} \|\nabla \psi\|_{L^2(\Omega)} \|\partial_t \tilde{g}_{kh}\|_{L^1(\Omega)} \leq Chd^{-\frac{N}{2}} (d^{-1} \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) \|\partial_t \tilde{g}_{kh}\|_{L^1(\Omega)}. \end{aligned}$$

Using that $\sigma \geq Cd$ on $\Omega \setminus B_{d/2}$ we have for I_2 :

$$\begin{aligned} I_2 &= (\sigma^{-\frac{N}{2}} (I - R_h)\psi, \sigma^{\frac{N}{2}} \partial_t \tilde{g}_{kh})_{\Omega \setminus B_{d/2}} \leq Cd^{-\frac{N}{2}} \|(I - R_h)\psi\|_{L^2(\Omega)} \|\sigma^{\frac{N}{2}} \partial_t \tilde{g}_{kh}\|_{L^2(\Omega)} \\ &\leq Chd^{-\frac{N}{2}} \|\nabla \psi\|_{L^2(\Omega)} \|\sigma^{\frac{N}{2}} \partial_t \tilde{g}_{kh}\|_{L^2(\Omega)} \leq Chd^{-\frac{N}{2}} (d^{-1} \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) \|\sigma^{\frac{N}{2}} \partial_t \tilde{g}_{kh}\|_{L^2(\Omega)}. \end{aligned}$$

Combining estimates for I_1 and I_2 and using discrete maximal parabolic regularity from Lemma 5.1 with respect to the $L^1(\Omega)$ norm and $\|\sigma^{\frac{N}{2}}(\cdot)\|_{L^2(\Omega)}$, we obtain

$$\begin{aligned} J_{12} &\leq Chd^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}) \sum_{m=1}^M \left(\|\partial_t \tilde{g}_{kh}\|_{L^1(I_m \times \Omega)} + \|\sigma^{\frac{N}{2}} \partial_t \tilde{g}_{kh}\|_{L^1(I_m;L^2(\Omega))} \right) \\ &\leq C \ln \frac{T}{k} h d^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}) \|\tilde{\theta}\|_{L^1(I_M)} \times \\ &\quad \left(\|P_h D \tilde{\delta}\|_{L^1(\Omega)} + |\ln h| \|\sigma^{\frac{N}{2}} P_h D \tilde{\delta}\|_{L^2(\Omega)} \right) \leq C \ln \frac{T}{k} |\ln h| d^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}), \end{aligned} \quad (7.6)$$

where in the last step we used Lemma 3.3. Thus, combining estimates for J_{11} and J_{12} we obtain

$$J_1 \leq C \ln \frac{T}{k} |\ln h|^{\frac{N-1}{N} + \frac{1}{2}} d^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}).$$

To estimate J_2 , we use the Hölder inequality, Lemma 5.1, and Lemma 3.9, to obtain

$$\begin{aligned} J_2 &= (\sigma^{-\frac{N}{2}} \nabla((1-\omega)u), \sigma^{\frac{N}{2}} \nabla \tilde{g}_{kh})_{I \times \Omega} \\ &\leq Cd^{-\frac{N}{2}} \|\nabla((1-\omega)u)\|_{L^\infty(I;L^2(\Omega))} \|\sigma^{\frac{N}{2}} \nabla \tilde{g}_{kh}\|_{L^1(I;L^2(\Omega))} \\ &\leq C \ln \frac{T}{k} |\ln h|^{\frac{N-1}{N}} d^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}) \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} P_h D \tilde{\delta}\|_{L^2(\Omega)} \|\tilde{\theta}\|_{L^1(I_M)} \\ &\leq C \ln \frac{T}{k} |\ln h|^{\frac{N-1}{N} + \frac{1}{2}} d^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}). \end{aligned} \quad (7.7)$$

Similarly to J_1 , to estimate J_3 , we, add and subtract $(R_h(1-\omega)u, [\tilde{g}_{kh}]_m)_\Omega$, to obtain

$$J_3 = - \sum_{m=1}^M (R_h((1-\omega)u_m), [\tilde{g}_{kh}]_m)_\Omega + \sum_{m=1}^M ((I - R_h)((1-\omega)u_m), [\tilde{g}_{kh}]_m)_\Omega = J_{31} + J_{32}.$$

Similarly to J_{11} , using that $\sigma^{-\frac{N}{2}} \leq Cd^{-\frac{N}{2}}$ on $\Omega \setminus B_d$ and $(1-\omega) \leq 1$, we obtain

$$\begin{aligned} J_{31} &= \sum_{m=1}^M (\nabla((1-\omega)u), \nabla \Delta_h^{-1} [\tilde{g}_{kh}]_m)_\Omega \\ &= \sum_{m=1}^M (\sigma^{-\frac{N}{2}} \nabla((1-\omega)u), \sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} [\tilde{g}_{kh}]_m)_\Omega \\ &\leq Cd^{-\frac{N}{2}} \sum_{m=1}^M \|\nabla((1-\omega)u)\|_{L^\infty(I_m;L^2(\Omega))} \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} [\tilde{g}_{kh}]_m\|_{L^2(\Omega)} \\ &\leq Cd^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}) \sum_{m=1}^M \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} [\tilde{g}_{kh}]_m\|_{L^2(\Omega)} \\ &\leq C \ln \frac{T}{k} |\ln h|^{\frac{N-1}{N}} d^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}) \|\sigma^{\frac{N}{2}} \nabla \Delta_h^{-1} P_h D \tilde{\delta}\|_{L^2(\Omega)} \|\tilde{\theta}\|_{L^1(I_M)} \\ &\leq C \ln \frac{T}{k} |\ln h|^{\frac{N-1}{N} + \frac{1}{2}} d^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}). \end{aligned} \quad (7.8)$$

Similarly to J_{12} we also obtain

$$J_{32} \leq C \ln \frac{T}{k} |\ln h| d^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}).$$

Combining the estimates for J_1 , J_2 , and J_3 , and taking supremum over all partial derivatives, we conclude that

$$\begin{aligned} |\nabla u_{kh}(\tilde{t}, x_0)| &\leq C \ell_k \ell_h \left(d^{-1}\|u\|_{L^\infty(I \times B_{2d})} + \|\nabla u\|_{L^\infty(I \times B_{2d})} \right. \\ &\quad \left. + d^{-\frac{N}{2}} (d^{-1}\|u\|_{L^\infty(I;L^2(\Omega))} + \|\nabla u\|_{L^\infty(I;L^2(\Omega))}) \right). \end{aligned}$$

Using that the $dG(q)cG(r)$ method is invariant on $X_{k,h}^{q,r}$, by replacing u and u_{kh} with $u - \chi$ and $u_{kh} - \chi$ for any $\chi \in X_{k,h}^{q,r}$, we obtain Theorem 2.2.

Acknowledgments. The authors would like to thank Dominik Meidner for the careful reading of the manuscript and providing valuable suggestions that help to improve the presentation of the paper.

REFERENCES

- [1] N. Y. BAKAEV, V. THOMÉE, AND L. B. WAHLBIN, *Maximum-norm estimates for resolvents of elliptic finite element operators*, Math. Comp., 72 (2003), pp. 1597–1610 (electronic).
- [2] S. C. BRENNER AND L. R. SCOTT, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer-Verlag, New York, 1994.
- [3] H. CHEN, *An L^2 - and L^∞ error analysis for parabolic finite element equations with applications to superconvergence and error expansions*, PhD thesis, Ruprecht-Karls-Universität Heidelberg, Germany, 1993.
- [4] K. CHRYSAFINOS AND N. J. WALKINGTON, *Error estimates for the discontinuous Galerkin methods for parabolic equations*, SIAM J. Numer. Anal., 44 (2006), pp. 349–366 (electronic).
- [5] P. G. CIARLET, *The finite element method for elliptic problems*, North-Holland, Amsterdam, New York, 1978.
- [6] A. DEMLOW, D. LEYKEKHMAN, A. H. SCHATZ, AND L. B. WAHLBIN, *Best approximation property in the W_∞^1 norm for finite element methods on graded meshes*, Math. Comp., 81 (2012), pp. 743–764.
- [7] T. F. DUPONT AND Y. LIU, *Symmetric error estimates for moving mesh Galerkin methods for advection-diffusion equations*, SIAM J. Numer. Anal., 40 (2002), pp. 914–927 (electronic).
- [8] K. ERIKSSON, *An adaptive finite element method with efficient maximum norm error control for elliptic problems*, Math. Models Methods Appl. Sci., 4 (1994), pp. 313–329.
- [9] K. ERIKSSON AND C. JOHNSON, *Adaptive finite element methods for parabolic problems. II. Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$* , SIAM J. Numer. Anal., 32 (1995), pp. 706–740.
- [10] S. J. FROMM, *Potential space estimates for Green potentials in convex domains*, Proc. Amer. Math. Soc., 119 (1993), pp. 225–233.
- [11] J. GUZMÁN, D. LEYKEKHMAN, J. ROSSMANN, AND A. H. SCHATZ, *Hölder estimates for Green’s functions on convex polyhedral domains and their applications to finite element methods*, Numer. Math., 112 (2009), pp. 221–243.
- [12] D. LEYKEKHMAN AND B. VEXLER, *Discrete maximal parabolic regularity for Galerkin finite element methods*, accepted to Numer. Math., (2016).
- [13] ———, *Finite element pointwise results on convex polyhedral domains*, SIAM J. Numer. Anal., 54 (2016), pp. 561–587.
- [14] ———, *Pointwise best approximation results for Galerkin finite element solutions of parabolic problems*, SIAM J. Numer. Anal., 54 (2016), pp. 1365–1384.
- [15] ———, *A priori error estimates for three dimensional parabolic optimal control problems with pointwise control*, accepted to SICON, (2016).
- [16] D. LEYKEKHMAN AND L. B. WAHLBIN, *A posteriori error estimates by recovered gradients in parabolic finite element equations*, BIT, 48 (2008), pp. 585–605.
- [17] B. LI, *Maximum-norm stability and maximal L^p regularity of FEMs for parabolic equations with Lipschitz continuous coefficients*, Numer. Math., 131 (2015), pp. 489–516.
- [18] B. LI AND W. SUN, *Regularity of the diffusion-dispersion tensor and error analysis of Galerkin FEMs for a porous medium flow*, SIAM J. Numer. Anal., 53 (2015), pp. 1418–1437.
- [19] D. MEIDNER, R. RANNACHER, AND B. VEXLER, *A priori error estimates for finite element discretizations of parabolic optimization problems with pointwise state constraints in time*, SIAM J. Control Optim., 49 (2011), pp. 1961–1997.
- [20] F. NATTERER, *Über die punktweise Konvergenz finiter Elemente*, Numer. Math., 25 (1975/76), pp. 67–77.
- [21] J. A. NITSCHKE, *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind*, Abh. Math. Sem. Univ. Hamburg, 36 (1971), pp. 9–15.
- [22] J. A. NITSCHKE, *L_∞ -convergence of finite element approximation*, in Journées “Éléments Finis” (Rennes, 1975), Univ. Rennes, Rennes, 1975, p. 18.
- [23] R. RANNACHER AND R. SCOTT, *Some optimal error estimates for piecewise linear finite element approximations*, Math. Comp., 38 (1982), pp. 437–445.
- [24] A. H. SCHATZ, *A weak discrete maximum principle and stability of the finite element method in L_∞ on plane polygonal domains. I*, Math. Comp., 34 (1980), pp. 77–91.
- [25] A. H. SCHATZ AND L. B. WAHLBIN, *Interior maximum norm estimates for finite element methods*, Math. Comp., 31 (1977), pp. 414–442.
- [26] ———, *On the quasi-optimality in L_∞ of the \dot{H}^1 -projection into finite element spaces*, Math. Comp., 38 (1982), pp. 1–22.
- [27] ———, *Interior maximum-norm estimates for finite element methods. II*, Math. Comp., 64 (1995), pp. 907–928.
- [28] L. R. SCOTT AND S. ZHANG, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.
- [29] Z. W. SHEN, *Resolvent estimates in L^p for elliptic systems in Lipschitz domains*, J. Funct. Anal., 133 (1995), pp. 224–251.
- [30] F. TANTARDINI AND A. VEESER, *The L^2 -projection and quasi-optimality of Galerkin methods for parabolic equations*, SIAM J. Numer. Anal., 54 (2016), pp. 317–340.
- [31] V. THOMÉE, *Galerkin finite element methods for parabolic problems*, vol. 25 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, second ed., 2006.
- [32] V. THOMÉE AND L. B. WAHLBIN, *Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions*, Numer. Math., 87 (2000), pp. 373–389.
- [33] V. THOMÉE, J. XU, AND N. Y. ZHANG, *Superconvergence of the gradient in piecewise linear finite-element approximation to a parabolic problem*, SIAM J. Numer. Anal., 26 (1989), pp. 553–573.

- [34] M. F. WHEELER, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 723–759.