POINTWISE LOCALIZED ERROR ESTIMATES FOR PARABOLIC FINITE ELEMENT EQUATIONS

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Abstract. We derive pointwise weighted error estimates for semidiscrete parabolic finite element equations. The results extend those obtained by A.H. Schatz for stationary elliptic problems. In particular, they show that the error is more localized for higher order elements.

1. Introduction

Let Ω be a bounded domain in $R^N, N \geq 2$, with a sufficiently smooth boundary. For simplicity of notation, consider the heat equation with homogeneous Neumann boundary conditions (cf. Remark 4.3 for space and time dependent coefficients),

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= f(x,t), \text{ for } x \in \Omega,\ t > 0, \\ u(x,0) &= v(x), \text{ for } x \in \Omega, \\ \partial_n u(x,t) &= 0, \text{ for } x \in \partial\Omega,\ t > 0. \end{aligned} \tag{1.1}$$

Let S_h^r, $0 < h < 1/2$ and $r \geq 2$, denote finite element spaces. For now, we may think of them as sets of continuous piecewise polynomials of total degree $r-1$ on globally quasi-uniform partitions of Ω. We seek a semidiscrete (continuous in time) approximation to the solution of (1.1), i.e., $u_h(t) : C^1(0,\infty) \to S_h^r$, which satisfies

$$\begin{aligned} (u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) &= (f, \chi), \text{ for all } \chi \in S_h^r,\ t > 0, \\ u_h(0) &= v_h \in S_h^r, \end{aligned} \tag{1.2}$$

where (v, w) denotes the $L_2(\Omega)$ inner product $\int_\Omega v(x)w(x)dx$. We will now take a look at pointwise error estimates.

In 1998, Schatz, Thomée, and Wahlbin [3] showed almost best approximation in the maximum norm. Namely,

$$\|u - u_h\|_{L_\infty(Q_T)} \leq C\|v - v_h\|_{L_\infty(\Omega)} + C\ell_{h,r} \min_{\chi \in C([0,T];S_h^r)} \|u - \chi\|_{L_\infty(Q_T)}, \tag{1.3}$$

where $Q_T = \Omega \times [0,T]$ and the logarithmic factor $\ell_{h,r} = |\ln h|$ appears and is necessary in the piecewise linear case only, i.e., when $r = 2$ (cf. [3] for a discussion of the history of such estimates).

Similar almost best approximation results in the maximum norm were proved much earlier in the stationary elliptic case. Also in 1998, Schatz [2] gave sharper results for the stationary case, which show a more local dependence of the error for higher order elements (cf. [2] for a discussion of earlier results). The aim of this paper is to extend these more localized results from the elliptic to the parabolic case.

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In order to describe our main result, we need to introduce some notation. Consider the "parabolic" weight function

$$\sigma(x,t)=\frac{h}{h+|x|+\sqrt{t}}, \text{ for } x\in R^N \text{ and } t\geq 0. \tag{1.4}$$

Notice that $\sigma(x,t)=O(1)$ if $|x|+\sqrt{t}=O(h)$ and $\sigma(x,t)=O(h)$ if $|x|+\sqrt{t}=O(1)$.

For $1\leq p\leq\infty$, s real, and fixed (x_o,T), we define the weighted norms over space-time domains Q by

$$\|u\|_{L_p(Q),\sigma,s}=\|\sigma^s(x_o-y,T-t)u(y,t)\|_{L_p(Q)}. \tag{1.5}$$

For convenience we choose the discrete initial data $v_h=P_hv$, the L_2-projection onto S_h^r, which is defined by $(P_hv,\chi)=(v,\chi)$, for all $\chi\in S_h^r$.

Our main result concerns the error at an arbitrary but fixed point (x_o,T), $x_o\in\Omega$, $T\geq 0$. We may roughly state it as follows: Let $0\leq s\leq r-2$ and $Q_T=\Omega\times[0,T]$. Then, if $v_h=P_hv$, we have

$$|(u-u_h)(x_o,T)|\leq C(T)\ell_{h,s}\min_{\chi\in C([0,T];S_h^r)}\|u-\chi\|_{L_\infty(Q_T),\sigma,s}, \tag{1.6}$$

where $C(T)$ is independent of u,u_h, and x_o, and the logarithmic term $\ell_{h,s}=|\ln h|$ appears only when $s=r-2$.

Clearly, (1.6) is sharper than (1.3) when $r\geq 3$. Indeed, choosing $(\hat{x},\widehat{T})\in Q_T$ to be a point where $|(u-u_h)(\hat{x},\widehat{T})|=\|u-u_h\|_{L_\infty(Q_T)}$, we see that (1.6) implies (1.3), but not vice versa.

One consequence of (1.6) is asymptotic error expansion inequalities (cf. [2]). We will merely give an example. For $r\geq 3$ and any $\epsilon>0$, we have

$$\begin{aligned}|(u-u_h)(x_o,T)|\leq C(T)h^r\Bigg(&\sum_{|\alpha|=r}|D_x^\alpha u(x_o,T)|\\ &+h^{1-\epsilon}\left(\|u\|_{W_\infty^{r+1}(Q_T)}+|u|_{W_\infty^{r,1/2}(Q_T)}\right)\Bigg),\end{aligned} \tag{1.7}$$

where $|u|_{W_\infty^{r,1/2}(Q_T)}=\sup_{t,t'\leq T}\frac{\|u(t)-u(t')\|_{W_\infty^r(\Omega)}}{\sqrt{|t-t'|}}$ is a Hölder semi-norm in time.

The proof of our main result (1.6) basically follows by making certain crucial changes at appropriate places in the development in [3]. Unfortunately, to make our presentation readable, we need to repeat major portions of that development.

2. Preliminaries and Statement of the Main Result

With $0<h<1/2$ a parameter, let $\tau_j^h, j=1,...,J_h$, be disjoint open sets, elements, which form a partition of Ω and fit the boundary exactly, i.e. $\overline{\Omega}=\bigcup_{j=1}^{J_h}\overline{\tau}_j^h$. For each such partition, let $S_h^r=S_h^r(\Omega)\subset W_\infty^1(\Omega)$ be a finite-dimensional space. We will use $W_p^l(D)$, with $1\leq p\leq\infty$, $l=0,1,...$, and a spatial set D to denote the standard Sobolev spaces with $\|\cdot\|_{W_p^l(D)}$ and $|\cdot|_{W_p^l(D)}$ their norms and semi-norms respectively. For a space-time set Q, we will use $W_p^l(Q)$, where l is the order of space derivatives involved. When needed, we will also use piecewise norms

$$\|u\|_{W_p^l(D)}^{(h)}=\left(\sum_{\tau_j^h\cap D\neq\emptyset}\|u\|_{W_p^l(\tau_j^h\cap D)}^p\right)^{1/p}. \tag{2.1}$$

Similarly, we have the weighted piecewise norms

$$\|u(t)\|^{(h)}_{W^l_p(D),\sigma(\cdot,T-t),s} = \sum_{0\le|\alpha|\le l} \|\sigma^s(\cdot,T-t)D^\alpha_x u(t)\|^{(h)}_{L_p(D)},$$

(2.2)

$$\|u\|^{(h)}_{W^l_p(D\times[0,T]),\sigma,s} = \left(\int_0^T \left(\|u(t)\|^{(h)}_{W^l_p(D),\sigma(\cdot,T-t),s}\right)^p dt\right)^{1/p}.$$

Next, we will state some standard assumptions about finite element spaces. Assume there exist positive constants K and $\underline{\mathrm{k}}$, and an integer $r \ge 2$, all independent of h, such that the Assumptions 2.1 through 2.4 below hold.

The first assumption expresses the global quasi-uniformity of the partition of Ω and a so called trace inequality at the boundary on each element.

2.1. **Quasi-Uniformity and Trace.** (i) Each τ^h_j contains a ball of radius $\underline{\mathrm{k}}h$ and is contained in a ball of radius Kh.
(ii) For $0 < h < \frac{1}{2}$ and $j = 1, 2, ..., J_h$,

$$\int_{\partial\tau^h_j} |\nabla u| dS_j \le K\left(h^{-1}|u|_{W^1_1(\tau^h_j)} + |u|_{W^2_1(\tau^h_j)}\right), \ \forall v \in W^2_1(\tau^h_j).$$

For $D \subset \Omega, S^r_h(D)$ will denote the restriction of S^r_h to D. The second assumption is a standard inverse property.

2.2. **Inverse Property.** Let $\chi \in S^r_h(D)$, where D is any union of closures of elements. Then for $0 \le k \le l \le r-1$, $1 \le q \le p \le \infty$,

$$\|\chi\|^{(h)}_{W^l_p(D)} \le Kh^{-(l-k)-N\left(\frac{1}{q}-\frac{1}{p}\right)}\|\chi\|^{(h)}_{W^k_q(D)}.$$

Our third assumption is about local approximation properties of the finite element spaces. For D a subset of Ω we let $D_d = \{x \in \Omega : dist(x, D) \le d\}$.

2.3. **Local Approximation.** There exists a linear operator $I_h : W^1_1(\Omega) \to S^r_h(\Omega)$ such that for any D the following holds:

$$\|I_h v - v\|^{(h)}_{W^k_p(D)} \le Kh^{l-k}\|v\|_{W^l_p(D_{Kh})}, \text{ for } \ 0 \le k \le l \le r, \ 1 \le p \le \infty.$$

Finally, we assume the existence of a convenient "delta function".

2.4. **Regularized Delta function.** Let $x_o \in \overline{\Omega} \cap \overline{\tau}^h_j$. There exists a function $\tilde{\delta}_{x_o} \in C^1$ with support in τ^h_j such that

$$\chi(x_o) = \int_{\tau^h_j} \chi\tilde{\delta}_{x_o} dx, \ \forall\chi \in S^r_h,$$

and

$$\|\tilde{\delta}_{x_o}\|_{W^l_p} \le Kh^{-l-N(1-1/p)}, \ \text{ for } 1 \le p \le \infty, \ l = 0, 1.$$

We can state now our main result, which expresses how the approximate solution at a point depends on the continuous solution.

Theorem 1. *Suppose that Assumptions 2.1 through 2.4 hold and u and $u_h \in S^r_h$ satisfy (1.1) and (1.2) respectevily, with $v_h = P_h v$. Let $x_o \in \Omega$, $T \ge 0$, and let s satisfy $0 \le s \le r-2$, $r \ge 2$. Then there exists a constant $C(T)$ independent of x_o, u, u_h, and h such that*

$$|u_h(x_o, T)| \le C(T)\ell_{h,s}\|u\|_{L_\infty(Q_T),\sigma,s},$$

where $Q_T = \Omega \times [0,T]$*, the weighted norm* $\|.\|_{L_\infty(Q_T),\sigma,s}$ *is given by (1.5),* $\ell_{h,s} = 1$ *if* $s < r-2$*, and* $\ell_{h,s} = |\ln h|$ *if* $s = r-2$*.*

From this we can deduce the localized error estimate (1.6).

Corollary 1. *Under the assumptions of Theorem 1, we have*

$$|(u-u_h)(x_o,T)| \le C(T)\ell_{h,s} \min_{\chi \in C([0,T];S_h^r)} \|u-\chi\|_{L_\infty(Q_T),\sigma,s}.$$

Proof. Let $\chi \in C^1([0,T];S_h^r)$. Then $u_h - \chi$ is the approximate solution of (1.2) with f such that the weak exact solution of (1.1) is $u-\chi$. From Theorem 1 it follows that

$$|(u_h-\chi)(x_o,T)| \le C(T)\ell_{h,s}\|u-\chi\|_{L_\infty(Q_T),\sigma,s}.$$

Using the triangle inequality we have

$$\begin{aligned}|(u_h-u)(x_o,T)| &\le |(u_h-\chi)(x_o,T)| + |(u-\chi)(x_o,T)| \\ &\le C(T)\ell_{h,s}\|u-\chi\|_{L_\infty(Q_T),\sigma,s} + |(u-\chi)(x_o,T)| \le C(T)\ell_{h,s}\|u-\chi\|_{L_\infty(Q_T),\sigma,s},\end{aligned}$$

which proves Corollary 1 for χ smooth in time. The general proof follows from the density of $C^1[0,T]$ in $C[0,T]$. □

To prove the asymptotic error expansion inequality (1.7) we assume, in addition to Assumption 2.3, that there exists a function $\chi \in S_h^r$ such that

$$\|u-\chi\|_{L_\infty(D)} \le Ch^r|u|_{W_\infty^r(D_{Kh})}. \tag{2.3}$$

Corollary 2. *Suppose that the conditions of Theorem 1 are satisfied with* $r \ge 3$ *and in addition that (2.3) holds. Then for any* $(x_o,T) \in Q$ *and any* $\epsilon > 0$

$$\begin{aligned}|(u-u_h)(x_o,T)| \le C(T)h^r\Bigg(&\sum_{|\alpha|=r}|D_x^\alpha u(x_o,T)| \\ &+ h^{1-\epsilon}\left(\|u\|_{W_\infty^{r+1}(Q_T)} + |u|_{W_\infty^{r,1/2}(Q_T)}\right)\Bigg),\end{aligned}$$

where $|u|_{W_\infty^{r,1/2}(Q_T)} = \sup_{t',t\le T}\frac{\|u(t')-u(t)\|_{W_\infty^r(\Omega)}}{\sqrt{|t'-t|}}$ *is a Hölder semi-norm in time.*

Proof. By Corollary 1 and (2.3) we have

$$|(u-u_h)(x_o,T)| \le C(T)h^r\ell_{h,s}|u|_{W_\infty^r(Q_T),\sigma,s}. \tag{2.4}$$

For any $(x,t) \in Q_T$ and multi-index α with $|\alpha| = r$,

$$|D_x^\alpha u(x,t)| \le |D_x^\alpha u(x_o,T)| + \|u\|_{W_\infty^{r+1}(Q_T)}|x-x_o| + \frac{\|u(T)-u(t)\|_{W_\infty^r(\Omega)}}{\sqrt{T-t}}\sqrt{T-t}.$$

Choosing $s = 1-\epsilon$, we have

$$\begin{aligned}\left(\frac{h}{|x-x_o|+\sqrt{T-t}+h}\right)^{1-\epsilon}|D_x^\alpha u(x,t)| \le C\Bigg(&|D_x^\alpha u(x_o,T)| \\ &+ h^{1-\epsilon}\left(\|u\|_{W_\infty^{r+1}(Q_T)} + |u|_{W_\infty^{r,1/2}(Q_T)}\right)\Bigg).\end{aligned}$$

From this Corollary 2 follows. □

Finally, we will introduce two results that will be used repeatedly later. The first result concerns the detailed behavior of the L_2-projection $P_h : L_1(\Omega) \to S_h^r(\Omega)$.

Lemma 2.1. *(i) With $\tilde{\delta}_{x_o}$ given in Assumption 2.4 we have*

$$(P_h v)(x_o) = (v, P_h \tilde{\delta}_{x_o}), \text{ for } x_o \in \Omega, v \in L_1(\Omega).$$

(ii) There exist constants C and $c > 0$ such that

$$|P_h \tilde{\delta}_{x_o}(y)| \leq C h^{-N} e^{-c\frac{|x_o - y|}{h}}, \quad \forall x_o, y \in \Omega.$$

(iii) There exists a constant C such that $\|P_h\|_{L_p} \leq C$, for $1 \leq p \leq \infty$.

Such results can be found in [5] (cf. Lemma 7.2 in particular).

The second result is the following estimates for the Green's function in the continuous problem (1.1).

Lemma 2.2. *If $f \equiv 0$, the solution of (1.1) may be represented in terms of a Green's function $G(x, y; t, 0), t > 0, x, y \in \Omega$, as*

$$u(x,t) = \int_\Omega G(x, y; t, 0)\, v(y)\, dy.$$

Assume that the boundary $\partial\Omega$ is sufficiently smooth. Then for any integer l_0 and multi-integer l, there exist constants C and $c > 0$ such that for the Green's function $G(x, y; t, s), 0 \leq s < t$, and $x, y \in \Omega$, we have

$$|D_t^{l_0} D_x^l G(x, y; t, s)| \leq C(|x - y| + (t - s)^{1/2})^{-(N+2l_0+|l|)} e^{-c\frac{|x-y|^2}{t-s}}.$$

A proof is given in [1].

3. Proof of Theorem 1: Part 1

In this section, the proof will be reduced to certain technical estimates for approximate Green's functions which will be proved in the next section.

Since for $r = 2$ the result is already in [3], we will only consider the case $r \geq 3$.

Let x_o be any fixed point in Ω. Assume $x_o \in \bar{\tau}_0 = \bar{\tau}_{j_0}^h$ and let $\tilde{\delta} = \tilde{\delta}_{x_o}$ be the regularized delta function of Assumption 2.4 and $\tilde{\delta}_h = \tilde{\delta}_{x_o,h} = P_h \tilde{\delta}$.

We define the adjoint discrete Green's function $\Gamma_h = \Gamma_{x_o,h}(x, t) \in S_h^r$ by

$$(\Gamma_{h,t}, \chi) - (\nabla \Gamma_h, \nabla \chi) = 0, \text{ for } T > t \geq 0, \ \forall \chi \in S_h^r, \quad \text{with } \Gamma_h(T) = \tilde{\delta}_h. \tag{3.1}$$

Further, let $\Gamma = \Gamma_{x_o}(x, t)$ be the solution of the continuous dual problem

$$\Gamma_t + \Delta \Gamma = 0, \text{ for } T > t \geq 0, \quad \text{with } \Gamma(T) = \tilde{\delta}. \tag{3.2}$$

This may be thought of as a regularized continuous Green's function.

Consider now the equality

$$\begin{aligned}
\frac{d}{dt}(u_h(t), \Gamma_h(t)) &= (u_{h,t}(t), \Gamma_h(t)) + (u_h(t), \Gamma_{h,t}(t)) \\
&= (u_{h,t}(t), \Gamma_h(t)) + (\nabla u_h(t), \nabla \Gamma_h(t)) = (f(t), \Gamma_h(t)) \\
&= (u_t(t), \Gamma_h(t)) + (\nabla u(t), \nabla \Gamma_h(t)).
\end{aligned}$$

Integrating with respect to time from 0 to T, then integrating by parts and using that $u_h(x_o, T) = (u_h(T), \tilde{\delta}_h)$, we obtain

$$\begin{aligned}
u_h(x_o, T) &= (v_h, \Gamma_h(0)) + \int_0^T (u_t(t), \Gamma_h(t)) dt + \int_0^T (\nabla u(t), \nabla \Gamma_h(t)) dt \\
&= (v_h - v, \Gamma_h(0)) + (u(T), \tilde{\delta}_h) - \int_0^T (u(t), \Gamma_{h,t}(t)) dt + \int_0^T (\nabla u(t), \nabla \Gamma_h(t)) dt.
\end{aligned}$$

Using that $v_h = P_h v$, $(u, \Gamma_t) - (\nabla u, \nabla \Gamma) = 0$, and letting $F = \Gamma_h - \Gamma$, we have

$$u_h(x_o, T) = (u(T), \tilde{\delta}_h) + \int_0^T (u(t), F_t(t))dt + \int_0^T (\nabla u(t), \nabla F(t))dt. \tag{3.3}$$

Using Lemma 2.1(ii), the first term is bounded as desired for any $s > 0$,

$$|(u(T), \tilde{\delta}_h)| \le \|u(T)\|_{L_\infty(\Omega),\sigma(\cdot,0),s} \|\tilde{\delta}_h\|_{L_1(\Omega),\sigma(\cdot,0),-s} \le C\|u(T)\|_{L_\infty(\Omega),\sigma(\cdot,0),s}.$$

The second term is bounded by $\|u\|_{L_\infty(Q),\sigma,s}\|F_t\|_{L_1(Q),\sigma,-s}$.

To treat the third term in (3.3), which we will call I, we first integrate by parts over each element, so that

$$\begin{aligned} I &= \int_0^T \sum_{j=1}^{J_h} \left[\int_{\tau_j^h} \nabla u(t) \nabla F(t) dx \right] dt \\ &= \int_0^T \sum_{j=1}^{J_h} \left[-\int_{\tau_j^h} u(t) \Delta F(t) dx + \int_{\partial \tau_j^h} u(t) \partial_{n_A} F(t) dS_j \right] dt. \end{aligned}$$

Then, using the trace inequality in Assumption 2.1, which also holds in the weighted norms, we can bound the third term as

$$|I| \le C\|u\|_{L_\infty(Q),\sigma,s} \int_0^T \left(\|F(t)\|^{(h)}_{W_1^2(\Omega),\sigma(\cdot,T-t),-s} + h^{-1}\|F(t)\|_{W_1^1(\Omega),\sigma(\cdot,T-t),-s} \right) dt.$$

Writing $F = (\Gamma_h - I_h\Gamma) + (I_h\Gamma - \Gamma)$, using the inverse property 2.2 and the triangle inequality, we obtain

$$\begin{aligned} \|F\|^{(h)}_{W_1^2(\Omega),\sigma(\cdot,T-t),-s} &\le Ch^{-1}\|F\|_{W_1^1(\Omega),\sigma(\cdot,T-t),-s} \\ &\quad + Ch^{-1}\|I_h\Gamma - \Gamma\|_{W_1^1(\Omega),\sigma(\cdot,T-t),-s} + \|I_h\Gamma - \Gamma\|^{(h)}_{W_1^2(\Omega),\sigma(\cdot,T-t),-s}. \end{aligned}$$

Putting it all together, we have from (3.3)

$$\begin{aligned} &|u(x_o, T)| \le C\|u(T)\|_{L_\infty(\Omega),\sigma(\cdot,0),s} \\ &+ C\|u\|_{L_\infty(Q),\sigma,s} \int_0^T \Big(\|F_t(t)\|_{L_1(\Omega),\sigma(\cdot,T-t),-s} + h^{-1}\|F(t)\|_{W_1^1(\Omega),\sigma(\cdot,T-t),-s} \\ &+ \|(I_h\Gamma - \Gamma)(t)\|^{(h)}_{W_1^2(\Omega),\sigma(\cdot,T-t),-s} + h^{-1}\|(I_h\Gamma - \Gamma)(t)\|_{W_1^1(\Omega),\sigma(\cdot,T-t),-s} \Big) dt. \end{aligned} \tag{3.4}$$

Thus we need to prove

$$I_1 = \|I_h\Gamma - \Gamma\|^{(h)}_{W_1^2(Q),\sigma,-s} + h^{-1}\|I_h\Gamma - \Gamma\|_{W_1^1(Q),\sigma,-s} \le C\ell_{h,s} \tag{3.5}$$

and

$$I_2 = \|F_t\|_{L_1(Q),\sigma,-s} + h^{-1}\ell_{h,s}^{-1}\|F\|_{W_1^1(Q),\sigma,-s} \le C, \tag{3.6}$$

which we shall do in the next section.

Observe that once we have shown (3.5) and (3.6), the first term on the right in (3.4) can be absorbed into the second term.

4. Proof of Theorem 1: Part 2

In this section we will prove (3.5) and (3.6). For simplicity, we shall make a change of variables $t = T - t$ and we let $Q = Q_T$ below. Hence, the definitions of Γ_h and Γ given in (3.1) and (3.2) are replaced respectively by

$$(\Gamma_{h,t}, \chi) + (\nabla\Gamma_h, \nabla\chi) = 0, \text{ for } t > 0, \ \forall\chi \in S_h^r, \quad \text{with } \Gamma_h(0) = \tilde{\delta}_h \tag{4.1}$$

and

$$\Gamma_t - \Delta\Gamma = 0, \text{ for } t > 0, \quad \text{with } \Gamma(0) = \tilde{\delta}. \tag{4.2}$$

As a consequence of this time-reversal, we set

$$\|u\|_{L_p(Q),\sigma,-s} = \left(\int_Q (\sigma^{-s}(x_o - y, t)|u(y,t)|)^p dydt\right)^{1/p},$$

(which deviates from the previous notation in (1.5) and (2.2)), for the rest of the paper. We note that (3.5) and (3.6) do not change when expressed in this new notation.

Now, to prove (3.5) and (3.6), we shall decompose Q into "parabolic annuli". For j an integer, let $d_j = 2^{-j}$ and $Q_j = \{(x,t) \in Q : d_j \le \rho(x,t) \le 2d_j\}$, where $\rho(x,t) = |x - x_o| + \sqrt{t}$ denotes the "parabolic" distance to $(x_o, 0)$, and similarly $\Omega_j = \{x \in \Omega : d_j \le |x - x_o| \le 2d_j\}$. Then, with J_0 fixed small enough so that $\rho(x,t) \le 2d_{J_0} = 2^{1-J_0}$ in Q, and any $J_* > J_0$,

$$Q = \left(\bigcup_{j=J_0}^{J_*} Q_j\right) \cup Q_*, \quad \text{where } Q_* = \{(x,t) \in Q : \rho(x,t) \le d_{J_*}\}.$$

We shall refer to Q_* as the "innermost" set. Ultimately, we shall choose $J_* = J_*(h)$ such that $d_{J_*} \approx C_* h$ for small h, where C_* is a sufficiently large number to be chosen later. Note that then $J_* \approx C \ln|h|$. Constants C will, as usual, change freely but will be independent of C_*. We shall write $\sum_{*,j}$ when the innermost set is included and $\sum_j$ when it is not. We also define $Q'_j = Q_{j-1} \cup Q_j \cup Q_{j+1}$ and $\Omega'_j = Q'_j \cap \{t = 0\}$.

Proposition 4.1. *There exists a constant C such that $I_1 \le CC_*^{1+N/2+s}\ell_{h,s}$, where I_1 is defined in (3.5), and $\ell_{h,s}$ is as in Theorem 1.*

Proof. First we shall bound the first term in I_1 on Q_*. Since on Q_* the weight $\sigma^{-s} \le (C_* + 1)^s$, it is sufficient to estimate $\|I_h\Gamma - \Gamma\|_{W_1^2(Q_*)}$. Using Cauchy-Schwarz's inequality, the local approximation Assumption 2.3, a standard parabolic energy estimate, and Assumption 2.4, we have

$$\|I_h\Gamma - \Gamma\|^{(h)}_{W_1^2(Q'_*)} \le C(C_*h)^{1+N/2}\|\Gamma\|_{W_2^2(Q)} \le C(C_*h)^{1+N/2}\|\tilde{\delta}\|_{W_2^1(\Omega)} \le CC_*^{1+N/2}.$$

To estimate I_1 on $Q \backslash Q'_*$ we note that for any $|\alpha| \le r$, $|D_x^\alpha \Gamma(x,t)| \le \rho(x,t)^{-N-r}$. On each Q_j, $\sigma^{-1} \le 3(d_j/h)$ and $meas(Q_j) \le Cd_j^{N+2}$, and thus

$$\begin{aligned}
\|I_h\Gamma - \Gamma\|^{(h)}_{W_1^2(Q\backslash Q'_*),\sigma,-s} &\le \sum_{j=J_0}^{J_*/2} \|I_h\Gamma - \Gamma\|^{(h)}_{W_1^2(Q_j),\sigma,-s} \\
\le C\sum_{j=J_0}^{J_*/2} (d_j/h)^s \|I_h\Gamma - \Gamma\|^{(h)}_{W_1^2(Q_j)} &\le C\sum_{j=J_0}^{J_*/2} h^{r-2-s} d_j^s \|\rho^{-N-r}\|_{L_1(Q'_j)}
\end{aligned}$$

$$\leq C\sum_{j=J_0}^{J_*} h^{r-2-s}d_j^{2+s-r} \leq \begin{cases} C, & \text{if } r-2-s>0 \\ C|\ln h|, & \text{if } r-2-s=0. \end{cases}$$

The proof is very similar for the other term in I_1. □

To conclude this paper, it now remains to prove the following.

Proposition 4.2. *There exist constants C and C_*, with the latter large enough, such that $I_2 \leq CC_*^{1+N/2+s}$, where I_2 is defined in (3.6).*

Proof. In this proof, almost all norms occurring will be L_2 based. We shall write $\|v\|_\Omega$ and $|||v|||_Q$ for L_2-norms over space and space-time sets Ω and Q respectively, and $\|v\|_{k,\Omega}$ and $|||v|||_{k,Q}$ when up to k spatial derivatives are included. Time derivatives will be displayed explicitly.

Using that $\sigma^{-s} \leq (3d_j/h)^s$ and Cauchy-Schwarz's inequality

$$I_2 \leq \sum_{*,j}(2d_j)^{1+N/2}(3d_j/h)^s\left(|||F_t|||_{Q_j} + h^{-1}\ell_{h,s}^{-1}|||F|||_{1,Q_j}\right).$$

The part of I_2 over Q_*, which we will call K_*, is bounded by

$$\begin{aligned} K_* &\leq CC_*^s(C_*h)^{1+N/2}\left[|||F_t|||_Q + h^{-1}\ell_{h,s}^{-1}|||F|||_{1,Q}\right] \\ &\leq CC_*^s(C_*h)^{1+N/2}\left[|||\Gamma_t|||_Q + |||\Gamma_{h,t}|||_Q + h^{-1}\ell_{h,s}^{-1}(|||\Gamma|||_{1,Q} + |||\Gamma_h|||_{1,Q})\right]. \end{aligned} \tag{4.3}$$

The remaining terms are bounded by $Cd_j^{1+N/2}(d_j/h)^sK_j$, where

$$K_j = |||F_t|||_{Q_j} + \mu_j|||F|||_{1,Q_j} \text{ with } \mu_j = h^{-1}\ell_{h,s}^{-1} + d_j^{-1}; \tag{4.4}$$

here the term $d_j^{-1}|||F|||_{1,Q_j}$ is added for the purpose of a later "kickback argument". Thus so far we have

$$I_2 \leq K_* + CK, \text{ where } K = \sum_j d_j^{1+N/2}(d_j/h)^sK_j. \tag{4.5}$$

First, we will estimate K_* to show that

$$K_* \leq CC_*^{1+N/2+s}. \tag{4.6}$$

This follows by standard energy arguments and using Assumptions 2.2 and 2.4. For example, taking $\chi = \Gamma_{h,t}$ in (4.1), we have $\|\Gamma_{h,t}\|_\Omega^2 + \frac{1}{2}\frac{d}{dt}(\nabla\Gamma, \nabla\Gamma) = 0$. Integrating, neglecting $(\nabla\Gamma(T), \nabla\Gamma(T)) \geq 0$, and using the inverse property 2.4,

$$\int_0^T \|\Gamma_{h,t}\|_\Omega^2 \leq (\nabla\Gamma(0), \nabla\Gamma(0)) = \|\nabla\tilde{\delta}_h\|_\Omega^2 \leq Ch^{-2-N}.$$

Hence, $|||\Gamma_{h,t}|||_Q \leq Ch^{-1-N/2}$. Similarly, we can estimate the other terms in (4.3) to show (4.6).

To treat the terms involved in K_j in (4.4), we shall use the local energy-based estimates from [3] for functions $e = z_h - z$ satisfying

$$(e_t, \chi) + (\nabla e, \nabla\chi) = 0, \ \forall\chi \in S_h^r, \ t > 0. \tag{4.7}$$

Proposition 4.3. *For any $q > 0$, there exists C such that the following holds: Let $e = z_h - z$ satisfy (4.7). Then*

$$|||e_t|||_{Q_j} + d_j^{-1}|||e|||_{1,Q_j} \leq C(I_j(e(0)) + X_j(\zeta) + H_j(e) + d_j^{-2}|||e|||_{Q_j'}), \tag{4.8}$$

where $\zeta = I_h z - z$ *and*

$$I_j(e(0)) = \|e(0)\|_{1,\Omega_j'} + d_j^{-1}\|e(0)\|_{\Omega_j'},$$
$$X_j(\zeta) = d_j|||\zeta_t|||_{1,Q_j'} + |||\zeta_t|||_{Q_j'} + d_j^{-1}|||\zeta|||_{1,Q_j'} + d_j^{-2}|||\zeta|||_{Q_j'},$$
$$H_j(e) = (h/d_j)^q(|||e_t|||_{Q_j'} + d_j^{-1}|||e|||_{1,Q_j'}).$$

Remark 4.1. In [3], this result was showed under the additional assumption $z(0) = 0$ on Ω_j'. Using the techniques of [4], this extra assumption can easily be removed. Also, using the more direct techniques from [4], the Assumption A.5 in [3] concerning "scaling" is not needed.

Multiplying (4.8) by d_j and dividing it by $h\ell_{h,s}$, we get,

$$(h\ell_{h,s})^{-1}|||e|||_{1,Q_j} \le Cd_j(h\ell_{h,s})^{-1}(I_j(e(0)) + X_j(\zeta) + H_j(e) + d_j^{-2}|||e|||_{Q_j'}). \tag{4.9}$$

Adding (4.8) to (4.9), and noticing that $1 + d_j(h\ell_{h,s})^{-1} = \mu_j d_j$, we have

$$|||e_t|||_{Q_j} + \mu_j|||e|||_{1,Q_j} \le C\mu_j d_j(I_j(e(0)) + X_j(\zeta) + H_j(e) + d_j^{-2}|||e|||_{Q_j'}). \tag{4.10}$$

Applying equation (4.10) with $e = F$ to estimate the terms K_j defined in (4.4), we get

$$K_j = |||F_t|||_{Q_j} + \mu_j|||F|||_{1,Q_j} \le C(\hat{I}_j(F(0)) + \hat{X}_j(\zeta) + \hat{H}_j(F) + \mu_j d_j^{-1}|||F|||_{Q_j'}), \tag{4.11}$$

where $\zeta = I_h\Gamma - \Gamma$ and

(*i*) $$\hat{I}_j(F(0)) = \mu_j d_j\|F(0)\|_{1,\Omega_j'} + \mu_j\|F(0)\|_{\Omega_j'},$$

(*ii*) $$\hat{X}_j(\zeta) = \mu_j d_j^2|||\zeta_t|||_{1,Q_j'} + \mu_j d_j|||\zeta_t|||_{Q_j'} + \mu_j|||\zeta|||_{1,Q_j'} + \mu_j d_j^{-1}|||\zeta|||_{Q_j'},$$

(*iii*) $$\hat{H}_j(F) = (h/d_j)^q(\mu_j d_j|||F_t|||_{Q_j'} + \mu_j|||F|||_{1,Q_j'}).$$

Next we will show that the contribution of $(\hat{I}_j + \hat{X}_j + \hat{H}_j)$ to K is bounded.

Let us start with the initial terms $\hat{I}_j$. Using the fact that $\tilde{\delta} = 0$ on Ω_j', the inverse property 2.2 and Lemma 2.1 (ii), from (4.11) (i) we have

$$\hat{I}_j = \mu_j d_j\|\tilde{\delta}_h\|_{1,\Omega_j'} + \mu_j\|\tilde{\delta}_h\|_{\Omega_j'} \le Ch^{-1}(d_j/h)\|\tilde{\delta}_h\|_{\Omega_j''} \le Ch^{-1-N/2}(d_j/h)e^{-cd_j/h}.$$

Hence the contribution of $\hat{I}_j$ to K is bounded by

$$\sum_j d_j^{1+N/2}(d_j/h)^s\hat{I}_j \le C\sum_j(d_j/h)^{2+N/2+s}e^{-cd_j/h} \le C. \tag{4.12}$$

Now we will estimate the approximation terms in $\hat{X}_j(\zeta)$. To be brief, we will estimate just the first and the third terms in $\hat{X}_j(\zeta)$; the other terms may be estimated in a very similar fashion. Thus, by the approximation theory and the Green's function estimates of Lemma 2.2, the first term in $\hat{X}_j(\zeta)$ can be estimated as

$$\mu_j d_j^2|||(I_h\Gamma - \Gamma)_t|||_{1,Q_j'} \le C\mu_j d_j^2 h^{r-1}|||\Gamma_t|||_{r,Q_j''} \le C\mu_j d_j(h/d_j)^{r-1}d_j^{-1-N/2}$$

and the third term as

$$\mu_j|||I_h\Gamma - \Gamma|||_{1,Q_j'} \le C\mu_j h^{r-1}|||\Gamma|||_{r,Q_j''} \le C\mu_j d_j(h/d_j)^{r-1}d_j^{-1-N/2},$$

where Q_j'' is defined as $(Q_j')'$. Hence the contribution of $\hat{X}_j$ to K is bounded by

$$\sum_j d_j^{1+N/2}(d_j/h)^s\hat{X}_j \le C\sum_j(h/d_j)^{r-1-s} + C\ell_{h,s}^{-1}\sum_j(h/d_j)^{r-2-s} \le C. \tag{4.13}$$

To bound $\hat{H}_j$ we write $|F| \le |\Gamma_h| + |\Gamma|$ and replace Q'_j by the full set Q. Thus,

$$\hat{H}_j \le (h/d_j)^q[\mu_j d_j(|||\Gamma_{h,t}|||_Q + |||\Gamma_t|||_Q) + \mu_j(|||\Gamma_h|||_{1,Q} + |||\Gamma|||_{1,Q})].$$

Now, by standard energy estimates (cf. the proof of (4.6)),

$$\mu_j d_j(|||\Gamma_{h,t}|||_Q + |||\Gamma_t|||_Q) + \mu_j(|||\Gamma_h|||_{1,Q} + |||\Gamma|||_{1,Q}) \le C\mu_j d_j h^{-1-N/2}.$$

Taking $q = N/2 + r$, we have $\hat{H}_j \le C\mu_j h d_j^{-1-N/2}(h/d_j)^{r-2}$, and noticing that $\mu_j h = \ell_{h,s}^{-1} + h/d_j$, the contribution of $\hat{H}_j$ to K is bounded by

$$\sum_j d_j^{1+N/2}(d_j/h)^s\hat{H}_j \le C\sum_j(h/d_j)^{r-1-s} + C\ell_{h,s}^{-1}\sum_j(h/d_j)^{r-2-s} \le C. \tag{4.14}$$

Thus, from (4.11), (4.12),(4.13), and (4.14) we obtain

$$K = \sum_j d_j^{1+N/2}(d_j/h)^s K_j \le C + C\sum_j d_j^{N/2}(d_j/h)^s\mu_j|||F|||_{Q'_j}. \tag{4.15}$$

In the following lemma we will estimate $|||F|||_{Q'_j}$ by a duality argument. If we put $r = 2$ below, this is Lemma 4.2 in [3].

Lemma 4.1. *With $m_{ij} = min(d_j^{2-r}, d_i^{2-r})(min(d_i/d_j, d_j/d_i))^{1+N/2}$, we have*

$$|||F|||_{Q'_j} \le Ch^r d_j^{1-r-N/2} + C\sum_{*,|i-j|>3} m_{ij}\left(h^r|||F_t|||_{Q_i} + h^{r-1}|||F|||_{1,Q_i}\right)$$
$$+ C\sum_{|i-j|\le 3}\left(h^2|||F_t|||_{Q_i} + h|||F|||_{1,Q_i}\right).$$

Proof. Using $[v, w]$ for the L_2 space-time inner product over Q, we have

$$|||F|||_{Q'_j} = sup\{[F, v] : \text{supp } v \subset Q'_j, |||v|||_{Q'_j} = 1\}. \tag{4.16}$$

For each such fixed v, let w solve the dual problem $-w_t - \Delta w = v$ in Q, with $w(T) = 0$. Integrating by parts we obtain

$$[F, v] = (F(0), w(0)) + [F_t, w] + [\nabla F, \nabla w]. \tag{4.17}$$

First we will estimate the first term on the right. For any $\chi_0 \in S_h^r$ and C_* large, we have

$$(F(0), w(0)) = (\tilde{\delta}_h - \tilde{\delta}, w(0) - \chi_0) = (\tilde{\delta}_h, w(0) - \chi_0)_{\Omega''_j} + (\tilde{\delta}_h - \tilde{\delta}, w(0) - \chi_0)_{\Omega\setminus\Omega''_j}.$$

Choosing $\chi_0 = I_h w(0)$, using $\|\tilde{\delta}_h\|_{\Omega''_j} \le Cd_j^{N/2}\|\tilde{\delta}_h\|_{L_\infty(\Omega''_j)}$, Assumption 2.3, the exponential decay property of P_h with $dist(x_0, \Omega''_j) \ge cd_j$, and the standard a priori estimate $\|\nabla w(0)\|_\Omega \le |||v|||_Q = 1$, we find

$$(\tilde{\delta}_h, w(0) - \chi_0)_{\Omega''_j} \le \|\tilde{\delta}_h\|_{\Omega''_j}\|w(0) - I_h w(0)\|_{\Omega''_j} \le Cd_j^{N/2}\|\tilde{\delta}_h\|_{L_\infty(\Omega''_j)}h\|\nabla w(0)\|_\Omega$$
$$\le Cd_j^{N/2}h^{-N}e^{-cd_j/h}h \le Cd_j^{N/2}h^{-N+1}(d_j/h)^{N-1-r} \le Ch^r d_j^{1-r-N/2}.$$

Using the stability of P_h in L_1, and the local approximation property 2.3,

$$(\tilde{\delta}_h - \tilde{\delta}, w(0) - \chi_0)_{\Omega\setminus\Omega''_j} \le Ch^r\|w(0)\|_{W_\infty^r(D_j)},$$

where D_j is a set containing $\Omega\backslash\Omega_j^{''}$ but whose parabolic distance to $Q_j^{'}$ is greater than Cd_j. By Duhamel's principle,

$$w(x,0)=\int_0^{(4d_j)^2}\left(\int_{Q_j'\cap\{s\}}G(x,y;s,0)v(y,s)dy\right)ds.$$

Using Lemma 2.2, Cauchy-Schwarz's inequality, and $|||v|||=1$ we have,

$$\begin{aligned}|D_x^r w(x,0)| &\le \int_0^{(4d_j)^2}\left(\int_{Q_j'\cap\{s\}}|D_x^r G|\ |v(y,s)|dy\right)ds\\ &\le C\int_0^{(4d_j)^2}\left(\int_{Q_j'\cap\{s\}}d_j^{-N-r}\ |v(y,s)|dy\right)ds\\ &\le C\int_0^{(4d_j)^2}d_j^{-N/2-r}\|v(s)\|ds\le Cd_j^{1-r-N/2}.\end{aligned}$$

Thus,

$$|(F(0),w(0))|\le Ch^r d_j^{1-r-N/2},$$

which bounds the first term in (4.17) as desired.

We now consider the two remaining terms in (4.17). We have

$$\begin{aligned}&[F_t,w]+[\nabla F,\nabla w]=[F_t,w-I_hw]+[\nabla F,\nabla(w-I_hw)]\\ &\le C\sum_{*,i}\left(|||F_t|||_{Q_i}|||w-I_hw|||_{Q_i}+|||\nabla F|||_{Q_i}|||\nabla(w-I_hw)|||_{Q_i}\right).\end{aligned}$$

To prove the lemma we need to treat three cases.

Case 1: $i-j>3$**,** $(d_i<d_j)$**.**

$$|||F_t|||_{Q_i}|||w-I_hw|||_{Q_i}+|||\nabla F|||_{Q_i}|||\nabla(w-I_hw)|||_{Q_i}\le(h^r|||F_t|||_{Q_i}+h^{r-1}|||\nabla F|||_{Q_i})|||w|||_{r,Q_i'}.$$

By Duhamel's principle

$$w(x,t)=\int_t^{(4d_j)^2}\left(\int_{Q_j'\cap\{s-t\}}G(x,y;s-t,0)v(y,s)dy\right)ds.$$

Using Lemma 2.2, Cauchy-Schwarz's inequality twice, and that $|||v|||=1$, we have

$$\begin{aligned}|D_x^r w(x,t)| &\le \int_t^{(4d_j)^2}\left(\int_{Q_j'\cap\{s-t\}}|D_x^r G|\ |v(y,s)|dy\right)ds\\ &\le C\int_t^{(4d_j)^2}\left(\int_{Q_j'\cap\{s-t\}}d_j^{-N-r}\ |v(y,s)|dy\right)ds\\ &\le C\int_t^{(4d_j)^2}d_j^{-N/2-r}\|v(s)\|ds\le Cd_j^{1-r-N/2}.\end{aligned}$$

Hence

$$|||D_x^r w|||_{Q_i'}\le Cd_i^{N/2+1}|||D_x^r w|||_{L_\infty(Q_i)}\le Cd_j^{2-r}\left(d_i/d_j\right)^{1+N/2}.$$

A similar argument can be applied to **Case 2: $j-i>3$, $(d_j<d_i)$:**

$$\begin{aligned}|D_x^r w(x,t)| &\le \int_t^{(4d_j)^2}\left(\int_{Q_j'\cap\{s-t\}} |D_x^r G|\ |v(y,s)|dy\right)ds\\ &\le C\int_t^{(4d_j)^2}\left(\int_{Q_j'\cap\{s-t\}} d_i^{-N-r}\ |v(y,s)|dy\right)ds\\ &\le C\int_t^{(4d_j)^2} d_i^{-N-r}d_j^{N/2}\|v(s)\|ds \le Cd_i^{-N-r}d_j^{1+N/2}.\end{aligned}$$

Hence

$$|||D_x^r w|||_{Q_i} \le Cd_i^{N/2+1}|||D_x^r w|||_{L_\infty(Q_i)} \le Cd_i^{2-r}\left(d_j/d_i\right)^{1+N/2}.$$

In **Case 3: $|j-i|\le 3$, $(d_j\approx d_i)$,**

$$|||F_t|||_{Q_i}|||w-I_hw|||_{Q_i}+|||\nabla F|||_{Q_i}|||\nabla(w-I_hw)|||_{Q_i} \le (h^2|||F_t|||_{Q_i}+h|||\nabla F|||_{Q_i})|||w|||_{2,Q_i}.$$

We then use the global a priori bound $|||w|||_{2,Q}\le C|||v|||_Q = C$. This proves the lemma. □

Now we are ready to conclude the proof of Proposition 4.2. By the lemma above and (4.15), we have

(4.18)

$$\begin{aligned}K \le C + \sum_j d_j^{N/2}(d_j/h)^s\mu_j|||F|||_{Q_j'} &\le C + C\sum_j d_j^{N/2}(d_j/h)^s\mu_j h^r d_j^{1-r-N/2}\\ &+ C\sum_j d_j^{N/2}(d_j/h)^s\mu_j \sum_{*,|i-j|>3} m_{ij}\left(h^r|||F_t|||_{Q_i}+h^{r-1}|||F|||_{1,Q_i}\right)\\ &+ C\sum_j d_j^{N/2}(d_j/h)^s\mu_j \sum_{|i-j|\le 3}\left(h^2|||F_t|||_{Q_i}+h|||F|||_{1,Q_i}\right).\end{aligned}$$

We consider each term on the right separately.

The first term

$$\sum_j d_j^{N/2}(d_j/h)^s\mu_j h^r d_j^{1-r-N/2} \le C\sum_j (h/d_j)^{r-1-s} \le C$$

is bounded since $r-1-s\ge 1$. In the second term of (4.18), extending the second summation to every i and changing the order of summation, we obtain

$$\begin{aligned}&\sum_j d_j^{N/2}(d_j/h)^s\mu_j \sum_{*,|i-j|>3} m_{ij}\left(h^r|||F_t|||_{Q_i}+h^{r-1}|||F|||_{1,Q_i}\right)\\ &\le \sum_j d_j^{N/2}(d_j/h)^s\mu_j \sum_{*,i} m_{ij}\left(h^r|||F_t|||_{Q_i}+h^{r-1}|||F|||_{1,Q_i}\right)\\ &\le \sum_{*,i}\left(h|||F_t|||_{Q_i}+|||F|||_{1,Q_i}\right)\left(\sum_{j\le i}+\sum_{j>i}\right)h^{r-1}d_j^{N/2}(d_j/h)^s\mu_j m_{ij}\\ &\le \sum_{*,i} d_i^{1+N/2}\left(|||F_t|||_{Q_i}+\mu_i|||F|||_{1,Q_i}\right)(d_i/h)^s(h/d_i)^{r-1}\\ &\le CC_*^{N/2+1}+CC_*^{1-r}K,\end{aligned}$$

where we used elementary properties of geometric sums, namely

$$\begin{aligned}\sum_{j\le i} h^{r-1} d_j^{N/2} (d_j/h)^s \mu_j m_{ij} &= d_i^{N/2+1} h^{r-1-s} \sum_{j\le i} d_j^{1+s-r}\mu_j \\ &\le C d_i^{N/2+1} d_i^{1+s-r}\mu_i h^{r-1-s} \le C d_i^{N/2+1} (d_i/h)^s \mu_i (h/d_i)^{r-1}\end{aligned},$$

and

$$\begin{aligned}\sum_{j> i} h^{r-1} d_j^{N/2} (d_j/h)^s \mu_j m_{ij} &= d_i^{1-N/2-r} h^{r-1-s} \sum_{j> i} d_j^{1+N+s}\mu_j \\ &\le C d_i^{1-N/2-r} d_i^{1+N+s}\mu_i h^{r-1-s} \le C d_i^{N/2+1} (d_i/h)^s \mu_i (h/d_i)^{r-1}.\end{aligned}$$

For the third term in (4.18), noticing that when $|i-j|\le 3$, $d_i \approx d_j$ and changing the order of summation, we find

$$\begin{aligned}&\sum_j d_j^{N/2}(d_j/h)^s \mu_j \sum_{|i-j|\le 3} \left(h^2 |||F_t|||_{Q_i} + h|||F|||_{1,Q_i}\right)\\ &\le C\sum_i d_i^{N/2}(d_i/h)^s\mu_i \left(h^2|||F_t|||_{Q_i} + h|||F|||_{1,Q_i}\right)\\ &\le C\sum_i d_i^{N/2+1}(d_i/h)^s \left(|||F_t|||_{Q_i} + \mu_i|||F|||_{1,Q_i}\right)(h/d_i)\\ &\le CC_*^{-1}K.\end{aligned}$$

Putting it all together, we obtain

$$K \le CC_*^{N/2+1+s} + \left(CC_*^{1-r} + CC_*^{-1}\right)K.$$

Since $r\ge 3$, taking C_* large enough, we can kick back the last term on the right. This shows that K is bounded, and together with (4.5) and (4.6), it shows that I_2 is bounded. The proof of Proposition 4.2, and thus of Theorem 1, is now complete. □

Remark 4.2. Here, we will comment on the behavior of $C(T)$ for T large. Clearly, for general second order parabolic problems, this is connected with whether or not the Green's function estimate of Lemma 2.2 holds with constants independent of T. We note that, since Ω is bounded, the domains Q_j for d_j large (i.e. when we are far from the singularity $(x_o,0)$), actually have volume $O(d_j^2)$, not $O(d_j^{2+N})$. This makes the estimates easier. If there is exponential decay for the homogeneous problem ($f\equiv 0$), we can show that $C(T)$ is independent of T.

Remark 4.3. The proof can easily be adapted to a general second order parabolic nonsymmetric partial differential equation with space and time dependent coefficients. The only requirement is that Green's functions estimates, analogous to Lemma 2.2, hold also for the adjoint problem.

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References

1. S.D. Èĭdel'man and S.D. Ivasišen, *Investigation of the Green's matrix for a homogeneous parabolic boundary value problem*, Trans. Moscow Math. Soc. **23** (1970), 179–242.
2. A. H. Schatz, *Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids: Part 1*, Math. Comp. **67** (1998), 877–899.

3. A.H. Schatz, V. Thomée, and L.B. Wahlbin, *Stability, analyticity, and almost best approximation in maximum-norm for parabolic finite element equations*, Comm. Pure Appl. Math. **51** (1998), 1349–1385.
4. V. Thomée and L.B. Wahlbin, *Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions*, Numer. Math. **87** (2000), 373–389. Erratum, Numer. Math. **89** (2001), 191.
5. L.B. Wahlbin *Local behavior in finite element methods*, Handbook of Numerical Analysis, Vol II, Finite Element Methods (Part 1), P.G. Ciarlet and J.L. Lions, Eds, Elsevier, 1991, 355–522.

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