

WEAK DISCRETE MAXIMUM PRINCIPLE OF FINITE ELEMENT METHODS IN CONVEX POLYHEDRA

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ABSTRACT. We prove that the Galerkin finite element solution u_h of the Laplace equation in a convex polyhedron Ω , with a quasi-uniform tetrahedral partition of the domain and with finite elements of polynomial degree $r \geq 1$, satisfies the following weak maximum principle:

$$\|u_h\|_{L^\infty(\Omega)} \leq C \|u_h\|_{L^\infty(\partial\Omega)},$$

with a constant C independent of the mesh size h . By using this result, we show that Ritz projection operator R_h is stable in L^∞ norm uniformly in h for $r \geq 2$, i.e.

$$\|R_h u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\Omega)}.$$

Thus we remove a logarithmic factor appearing in the previous results for convex polyhedral domains.

1. Introduction

Let S_h be a finite element space of piecewise polynomials of degree $r \geq 1$ subject to a quasi-uniform tetrahedral partition of a convex polyhedron $\Omega \subset \mathbb{R}^3$, where h denotes the mesh size of the tetrahedral partition. Let \mathring{S}_h be the subspace of S_h consisting of functions with zero boundary values.

A function $u_h \in S_h$ is called discrete harmonic if it satisfies the following equation:

$$(1.1) \quad (\nabla u_h, \nabla \chi_h) = 0 \quad \forall \chi_h \in \mathring{S}_h.$$

In this article, we establish the following result, called weak maximum principle of finite element methods.

Theorem 1.1. *A discrete harmonic function u_h satisfies the following estimate:*

$$(1.2) \quad \|u_h\|_{L^\infty(\Omega)} \leq C \|u_h\|_{L^\infty(\partial\Omega)},$$

where the constant C is independent of the mesh size h .

As an application of the weak maximum principle, we show that the Ritz projection $R_h : H_0^1(\Omega) \rightarrow \mathring{S}_h$ defined by

$$(\nabla(u - R_h u), \nabla v_h) = 0 \quad \forall v_h \in \mathring{S}_h$$

is stable in L^∞ for finite elements of degree $r \geq 2$, i.e.

$$\|R_h u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\Omega)} \quad \forall u \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

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Although this result is well-known for smooth domains [26, 28], for convex polyhedral domains the result was available only with an additional logarithmic factor [19, Theorem 12].

In the finite element literature, the maximum principle has attracted a lot of attention; see [7, 8, 24, 29, 30], to mention a few. However, the sufficient conditions for discrete maximum principle put serious restrictions on the geometry of the mesh. For piecewise linear elements in two-dimensions, the angles of the triangles must be less than $\pi/2$, or the sum of opposite angles of the triangles that share an edge must be less than π (for example, see [30, §5]). For quadratic elements in two dimensions, discrete maximum principle holds only for equilateral triangles [14]. The situation in three dimensions is more complicated [4, 17, 18, 31], essentially it is hard to guarantee the discrete maximum principle even for piecewise linear elements. In this respect, there stands out the work of Schatz [25], who proved that a weak maximum principle in the sense of (1.2) holds for a wide class of finite elements on general quasi-uniform triangulation of any two dimensional polygonal domain Ω . By utilizing the weak maximum principle, Schatz also established the stability of the Ritz projection in L^∞ and $W^{1,\infty}$ norms. Such stability results have a wide range of applications, for example to pointwise error estimates of finite element methods for parabolic problems [16, 20, 21], Stokes systems [3], nonlinear problems [10, 11, 22], obstacle problems [6], optimal control problems [1, 2], to name a few. As far as we know, [25] is the only paper that establishes weak maximum principle and L^∞ stability estimate (without the logarithmic factor) for the Ritz projection on nonsmooth domains.

In three dimensions the situation is less satisfactory. The stability of the Ritz projection in L^∞ and $W^{1,\infty}$ norms are available on smooth domains [26, 28] and convex polyhedral domains [13, 19]. However, on convex polyhedral domains in [19], the L^∞ -stability constant depends logarithmically on the mesh size h , and it is not obvious how the logarithmic factor can be removed there. There are no results on the weak maximum principles in three dimensions even on smooth domains or convex polyhedra. The objective of this paper is to close this gap for convex polyhedral domains. In order to obtain the result, we have to modify the argument in [25] by considering error analysis in L^p norm for some $1 < p < 2$. In the case of convex polyhedral domains, the L^2 -norm based argument used in [25] would yield an additional logarithmic factor. Unfortunately, the current analysis does not allow us to extend the results to nonconvex polyhedral domains or graded meshes. These would be the subject of future research.

The paper is organized as follows. In section 2 we state some preliminary results that we use later in our arguments. In section 3, we reduce the proof of the weak discrete maximum principle to a specific error estimate. Section 4 is devoted to the proof of this estimate, which constitutes the main technical part of the paper. Finally, section 5, gives an application of the weak discrete maximum principle to showing the stability of the Ritz projection in L^∞ norm uniformly in h for higher order elements.

In the rest of this article, we denote by C a generic positive constant, which may be different at different occurrences but will be independent of the mesh size h .

2. Preliminary results

In this section, we present several well-known results that are used in our analysis. First result concerns global regularity of the weak solution $v \in H_0^1(\Omega)$ to the problem

$$(2.3) \quad (\nabla v, \nabla \chi) = (f, \chi) \quad \forall \chi \in H_0^1(\Omega).$$

On the general convex domains we naturally have the H^2 regularity (cf. [12]). However, on convex polyhedral domains, we have the following sharper $W^{2,p}(\Omega)$ regularity result (cf. [9, Corollary 3.12]).

Lemma 2.1. *Let Ω be a convex polyhedron. Then there exists a constant $p_0 > 2$ depending on Ω such that for any $1 < p < p_0$ and $f \in L^p(\Omega)$, the solution v of (2.3) is in $W^{2,p}(\Omega)$ and*

$$\|v\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

The next result addresses the problem (2.3) when the source function f is supported in some part of Ω . The following lemma traces the dependence of the stability constant on the diameter of the support.

Lemma 2.2. *For any bounded Lipschitz domain Ω , there exist positive constants $\alpha \in (0, \frac{1}{2})$ and C (depending on Ω) such that for $\frac{3}{2} - \alpha \leq p \leq 3 + \alpha$ and $f \in L^p(\Omega)$ such that $\text{supp}(f) \subset S_{d_*}(x_0)$ with $\text{dist}(x_0, \partial\Omega) \leq d_*$, the solution of (2.3) satisfies*

$$\|v\|_{W^{1,p}(\Omega)} \leq C d_* \|f\|_{L^p(\Omega)}.$$

Proof. For any $\chi \in W^{1,p'}(\Omega)$ there holds

$$\begin{aligned} |(\nabla v, \nabla \chi)| &= |(f, \chi)| \leq \|f\|_{L^p(S_{d_*}(x_0))} \|\chi\|_{L^{p'}(S_{d_*}(x_0))} \\ &\leq C d_* \|f\|_{L^p(S_{d_*}(x_0))} \|\nabla \chi\|_{L^{p'}(S_{d_*}(x_0))} \\ &\leq C d_* \|f\|_{L^p(\Omega)} \|\nabla \chi\|_{L^{p'}(\Omega)}. \end{aligned}$$

If $\vec{w} \in L^{p'}(\Omega)^3$ then we let $\chi \in H_0^1(\Omega)$ be the weak solution of

$$\begin{cases} \Delta \chi = \nabla \cdot \vec{w} & \text{in } \Omega \\ \chi = 0 & \text{on } \partial\Omega \end{cases}$$

The solution χ defined above satisfies

$$\nabla \cdot (\vec{w} - \nabla \chi) = 0,$$

and, according to [15, Theorem B], there exists a positive constant $\alpha \in (0, \frac{1}{2})$ such that

$$\|\nabla \chi\|_{L^{p'}(\Omega)} \leq C \|\vec{w}\|_{L^{p'}(\Omega)} \quad \text{for } \frac{3}{2} - \alpha \leq p \leq 3 + \alpha.$$

By using these properties, we have

$$|(\nabla v, \vec{w})| = |(\nabla v, \nabla \chi)| \leq C d_* \|f\|_{L^p(\Omega)} \|\nabla \chi\|_{L^{p'}(\Omega)} \leq C d_* \|f\|_{L^p(\Omega)} \|\vec{w}\|_{L^{p'}(\Omega)}.$$

The duality pairing estimate above implies the desired result. \square

The next lemma concerns basic properties of harmonic functions on convex domains. The result is essentially the same as in [27, Lemma 8.3].

Lemma 2.3. *Let D and D_d be two subdomains satisfying $D \subset D_d \subset \Omega$, with*

$$D_d = \{x \in \Omega : \text{dist}(x, D) \leq d\},$$

where d is a positive constant. If $v \in H_0^1(\Omega)$ and v is harmonic on D_d , i.e.

$$(\nabla v, \nabla w) = 0, \quad \forall w \in H_0^1(D_d),$$

then the following estimates hold:

$$(2.4a) \quad |v|_{H^2(D)} \leq Cd^{-1} \|v\|_{H^1(D_d)},$$

$$(2.4b) \quad \|v\|_{H^1(D)} \leq Cd^{-1} \|v\|_{L_2(D_d)}.$$

Finally, we need the best approximation property of the Ritz projection in $W^{1,p}$ norm. In [13], the best approximation property of the Ritz projection in $W^{1,\infty}$ norm was established on convex polyhedral domains. Together with the standard best approximation property in H^1 norm we obtain

$$(2.5) \quad \|v - R_h v\|_{W^{1,p}(\Omega)} \leq C \min_{\chi \in \mathcal{S}_h} \|v - \chi\|_{W^{1,p}(\Omega)} \quad \forall v \in H_0^1(\Omega) \cap W^{1,p}(\Omega),$$

for any $2 \leq p \leq \infty$. Extension of the above result to $1 < p \leq \infty$ follows by duality (cf. [5, §8.5]). These can be summarized as below.

Lemma 2.4. *On a convex polyhedron Ω , the following estimate holds for any fixed $p \in (1, \infty]$:*

$$\|v - R_h v\|_{W^{1,p}(\Omega)} \leq Ch \|v\|_{W^{2,p}(\Omega)} \quad \forall v \in H_0^1(\Omega) \cap W^{2,p}(\Omega).$$

3. Basic estimates

In [26, Corollary 5.1], the following interior error estimate was established

$$\|u - u_h\|_{L^\infty(\Omega_1)} \leq Ch^l |\ln h|^{\bar{r}} |u|_{W^{l,\infty}(\Omega_2)} + Cd^{-3/q-p} \|u - u_h\|_{W^{-p,q}(\Omega_2)},$$

for $0 \leq l \leq r$, where $\bar{r} = 1$ for $r = 1$, $\bar{r} = 0$ for $r \geq 2$ and $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, with $\text{dist}(\Omega_1, \partial\Omega_2) \geq d \geq kh$ and $\text{dist}(\Omega_2, \partial\Omega) \geq d \geq kh$. Choosing $u = 0$, $p = 0$ and $q = 2$ in the above estimate, we obtain that there exists a constant C independent of h such that

$$(3.1) \quad \|u_h\|_{L^\infty(\Omega_1)} \leq Cd^{-\frac{3}{2}} \|u_h\|_{L^2(\Omega_2)}.$$

Let $x_0 \in \bar{\Omega}$ be a point satisfying

$$|u_h(x_0)| = \|u_h\|_{L^\infty(\Omega)} \quad \text{with} \quad d = \text{dist}(x_0, \partial\Omega).$$

If $d \geq 2kh$ then we can choose $\Omega_1 = S_{d/2}(x_0)$ and $\Omega_2 = S_d(x_0)$. In this case, the following interior L^∞ estimate holds (cf. [26, Corollary 5.1] and [25, Lemma 2.1 (ii)]):

$$|u_h(x_0)| \leq Cd^{-\frac{3}{2}} \|u_h\|_{L^2(S_d(x_0))}.$$

Otherwise, we have $d \leq 2kh$. In this case, the inverse inequality of finite element functions implies

$$|u_h(x_0)| \leq Ch^{-\frac{3}{2}} \|u_h\|_{L^2(S_h(x_0))}.$$

Hence, either $d \geq 2kh$ or $d \leq 2kh$, the following estimate holds:

$$(3.2) \quad |u_h(x_0)| \leq C\rho^{-\frac{3}{2}} \|u_h\|_{L^2(S_\rho(x_0))}, \quad \text{with} \quad \rho = d + 2kh.$$

To estimate the term $\|u_h\|_{L^2(S_\rho(x_0))}$ on the right hand side of the inequality above, we use the following duality property:

$$\|u_h\|_{L^2(S_\rho(x_0))} = \sup_{\substack{\text{supp}(\varphi) \subset S_\rho(x_0) \\ \|\varphi\|_{L^2(S_\rho(x_0))} \leq 1}} |(u_h, \varphi)|,$$

which implies the existence of a function $\varphi \in C_0^\infty(\Omega)$ with the following properties:

$$(3.3) \quad \text{supp}(\varphi) \subset S_\rho(x_0), \quad \|\varphi\|_{L^2(S_\rho(x_0))} \leq 1$$

and

$$(3.4) \quad \|u_h\|_{L^2(S_\rho(x_0))} \leq 2|(u_h, \varphi)|.$$

For this function φ , we define $v \in H_0^1(\Omega)$ to be the solution of the PDE problem (in weak form):

$$(3.5) \quad (\nabla v, \nabla \chi) = (\varphi, \chi) \quad \forall \chi \in H_0^1(\Omega),$$

and let $v_h \in \dot{S}_h$ be the finite element solution of

$$(\nabla v_h, \nabla \chi_h) = (\varphi, \chi_h) \quad \forall \chi_h \in \dot{S}_h.$$

Then v_h is the Ritz projection of v , satisfying

$$(3.6) \quad (\nabla(v - v_h), \nabla \chi_h) = 0 \quad \forall \chi_h \in \dot{S}_h.$$

Let u be the solution of the PDE problem (in weak form)

$$(3.7) \quad \begin{cases} (\nabla u, \nabla \chi) = 0 & \forall \chi \in H_0^1(\Omega), \\ u = u_h & \text{on } \partial\Omega. \end{cases}$$

Then the continuous maximum principle of the PDE problem implies

$$(3.8) \quad \|u\|_{L^\infty(\Omega)} \leq \|u_h\|_{L^\infty(\partial\Omega)}.$$

Notice, that u_h is the Ritz projection of u , i.e.

$$\begin{cases} (\nabla(u - u_h), \nabla \chi_h) = 0 & \forall \chi_h \in \dot{S}_h, \\ u - u_h = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, we have

$$\begin{aligned} & \|u_h\|_{L^2(S_\rho(x_0))} \\ & \leq 2|(u_h, \varphi)| && \text{(here we used (3.4))} \\ & = 2|(u_h - u, \varphi) + (u, \varphi)| \\ & = 2|(\nabla(u_h - u), \nabla v) + (u, \varphi)| && \text{(here we used (3.5))} \\ & = 2|(\nabla u_h, \nabla v) + (u, \varphi)| && \text{(here we used (3.7))} \\ & \leq 2|(\nabla u_h, \nabla v)| + 2\|u\|_{L^\infty(\Omega)} \|\varphi\|_{L^1(\Omega)} \\ (3.9) \quad & \leq 2|(\nabla u_h, \nabla v)| + C\rho^{\frac{3}{2}} \|u_h\|_{L^\infty(\partial\Omega)} \|\varphi\|_{L^2(S_\rho(x_0))}, \end{aligned}$$

where we have used (3.8) and the Hölder inequality in deriving the last inequality.

To estimate $|(\nabla u_h, \nabla v)|$, we note that

$$\begin{aligned} (\nabla u_h, \nabla v) &= (\nabla u_h, \nabla(v - v_h)) && \text{(here we use (1.1) and } v_h \in \dot{S}_h) \\ &= (\nabla(u_h - \chi_h), \nabla(v - v_h)) && \forall \chi_h \in \dot{S}_h. \quad \text{(here we use (3.6)).} \end{aligned}$$

We simply choose χ_h to be equal to u_h at interior nodes and $\chi_h = 0$ on $\partial\Omega$; thus $u_h(x) - \chi_h(x)$ is zero when $\text{dist}(x, \partial\Omega) \geq h$, and

$$\|u_h - \chi_h\|_{L^\infty(\Omega)} \leq \|u_h\|_{L^\infty(\partial\Omega)}.$$

If we define

$$A_h = \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq h\},$$

then

$$\begin{aligned} |(\nabla u_h, \nabla v)| &\leq \|\nabla(u_h - \chi_h)\|_{L^\infty(A_h)} \|\nabla(v - v_h)\|_{L^1(A_h)} \\ &\leq Ch^{-1} \|u_h - \chi_h\|_{L^\infty(A_h)} \|\nabla(v - v_h)\|_{L^1(A_h)} \\ (3.10) \quad &\leq Ch^{-1} \|u_h\|_{L^\infty(\partial\Omega)} \|\nabla(v - v_h)\|_{L^1(A_h)}. \end{aligned}$$

Then, substituting (3.9) and (3.10) into (3.2), we obtain

$$(3.11) \quad \|u_h\|_{L^\infty(\Omega)} \leq C(\rho^{-\frac{3}{2}} h^{-1} \|\nabla(v - v_h)\|_{L^1(A_h)} + 1) \|u_h\|_{L^\infty(\partial\Omega)}.$$

The proof of Theorem 1.1 will be completed if we can establish

$$(3.12) \quad \rho^{-\frac{3}{2}} h^{-1} \|\nabla(v - v_h)\|_{L^1(A_h)} \leq C.$$

4. Estimate of $\rho^{-\frac{3}{2}} h^{-1} \|\nabla(v - v_h)\|_{L^1(A_h)}$

Let $R_0 = \text{diam}(\Omega)$ and $d_j = R_0 2^{-j}$ for $j = 0, 1, 2, \dots$. We define a sequence of subdomains

$$A_j = \{x \in \Omega : d_{j+1} \leq |x - x_0| \leq d_j\}, \quad j = 0, 1, 2, \dots$$

For each j we denote A_j^l to be a subdomain slightly bigger than A_j , defined by

$$A_j^l = A_{j-l} \cup \dots \cup A_j \cup A_{j+1} \cup \dots \cup A_{j+l} \quad l = 1, 2, \dots$$

Let $J = \lceil \ln_2(R_0/8\rho) \rceil + 1$, with $\lceil \ln_2(R_0/8\rho) \rceil$ denoting the greatest integer not exceeding $\ln_2(R_0/8\rho)$. Then

$$2\rho \leq d_{J+1} \leq 4\rho$$

and

$$(4.13) \quad \text{measure}(A_j \cap A_h) \leq Chd_j^2.$$

By using these subdomains defined above, we have

$$\begin{aligned} &\rho^{-\frac{3}{2}} h^{-1} \|\nabla(v - v_h)\|_{L^1(A_h)} \\ &\leq \rho^{-\frac{3}{2}} h^{-1} \left(\sum_{j=0}^J \|\nabla(v - v_h)\|_{L^1(A_h \cap A_j)} + \|\nabla(v - v_h)\|_{L^1(A_h \cap S_{4\rho}(x_0))} \right) \\ &\leq C\rho^{-\frac{3}{2}} h^{-1} \sum_{j=0}^J h^{\frac{1}{2}} d_j \|\nabla(v - v_h)\|_{L^2(A_h \cap A_j)} \\ (4.14) \quad &+ C\rho^{-\frac{1}{2}} h^{-\frac{1}{2}} \|\nabla(v - v_h)\|_{L^2(A_h \cap S_{4\rho}(x_0))}, \end{aligned}$$

where the Hölder inequality and (4.13) were used in deriving the last inequality.

Using global error estimate in H^1 norm, Lemma 2.1 with $p = 2$ and (3.3), we obtain

$$\begin{aligned} \rho^{-\frac{1}{2}} h^{-\frac{1}{2}} \|\nabla(v - v_h)\|_{L^2(A_h \cap S_{4\rho}(x_0))} &\leq C\rho^{-\frac{1}{2}} h^{-\frac{1}{2}} h \|v\|_{H^2(\Omega)} \\ &\leq C\rho^{-\frac{1}{2}} h^{-\frac{1}{2}} h \|\varphi\|_{L^2(\Omega)} \end{aligned}$$

$$\leq C,$$

where we have used $\rho \geq h$ and $\|\varphi\|_{L^2(\Omega)} \leq 1$ in deriving the last inequality. Substituting the last inequality into (4.14) yields

$$(4.15) \quad \rho^{-\frac{3}{2}} h^{-1} \|\nabla(v - v_h)\|_{L^1(A_h)} \leq C \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^J d_j \|\nabla(v - v_h)\|_{L^2(A_j)} + C.$$

Now, we use the following interior energy error estimate (proved in [23, Theorem 5.1], also see [25, Lemma 2.1 (i)]):

$$(4.16) \quad \begin{aligned} & \|\nabla(v - v_h)\|_{L^2(A_j)} \\ & \leq C \|\nabla(v - I_h v)\|_{L^2(A_j^1)} + C d_j^{-1} \|v - I_h v\|_{L^2(A_j^1)} + C d_j^{-1} \|v - v_h\|_{L^2(A_j^1)} \\ & \leq (Ch + Ch^2 d_j^{-1}) \|v\|_{H^2(A_j^2)} + C d_j^{-1} \|v - v_h\|_{L^2(A_j^1)} \\ & \leq Ch d_j^{\frac{1}{2} - \frac{3}{p}} \|v\|_{W^{1,p}(A_j^3)} + C d_j^{-1} \|v - v_h\|_{L^2(A_j^1)} \quad \text{for } \frac{6}{5} < p < 2, \end{aligned}$$

where we have used $d_j \geq h$ and the following inequality in deriving the last inequality:

$$(4.17) \quad \|v\|_{H^2(A_j^2)} \leq C d_j^{\frac{1}{2} - \frac{3}{p}} \|v\|_{W^{1,p}(A_j^3)} \quad \text{for } \frac{6}{5} < p < 2.$$

The inequality above follows from Lemma 2.3, the Hölder inequality and Sobolev embedding, i.e.

$$\begin{aligned} \|v\|_{H^2(A_j^2)} & \leq C d_j^{-2} \|v\|_{L^2(A_j^3)} \\ & \leq C d_j^{-2 + \frac{3}{2} - \frac{3}{q}} \|v\|_{L^q(A_j^3)} \quad \text{if } q > 2 \\ & \leq C d_j^{\frac{1}{2} - \frac{3}{p}} \|v\|_{W^{1,p}(A_j^3)} \quad \text{for } \frac{3}{q} = \frac{3}{p} - 1 \text{ and } \frac{6}{5} < p < 2 \text{ (so that } q > 2). \end{aligned}$$

This proves that (4.16) holds for $\frac{6}{5} < p < 2$.

By applying Lemma 2.2 to (4.16) with $p = \frac{3}{2}$, we obtain

$$(4.18) \quad \begin{aligned} & \|\nabla(v - v_h)\|_{L^2(A_j)} \\ & \leq Ch d_j^{-\frac{3}{2}} \rho \|\varphi\|_{L^{\frac{3}{2}}(S_\rho(x_0))} + C d_j^{-1} \|v - v_h\|_{L^2(A_j^1)} \\ & \leq Ch d_j^{-\frac{3}{2}} \rho^{\frac{3}{2}} + C d_j^{-1} \|v - v_h\|_{L^2(A_j^1)}, \end{aligned}$$

where the last inequality is due to the following Hölder inequality:

$$\|\varphi\|_{L^{\frac{3}{2}}(S_\rho(x_0))} \leq C \rho^{\frac{1}{2}} \|\varphi\|_{L^2(S_\rho(x_0))} \quad \text{with } \|\varphi\|_{L^2(S_\rho(x_0))} \leq 1.$$

From (4.18) we see that

$$(4.19) \quad d_j \|\nabla(v - v_h)\|_{L^2(A_j)} \leq C \rho^{\frac{3}{2}} h^{\frac{1}{2}} \left(\frac{h}{d_j}\right)^{\frac{1}{2}} + C \|v - v_h\|_{L^2(A_j^1)}.$$

Then, substituting (4.19) into (4.15), we have

$$\begin{aligned} & \rho^{-\frac{3}{2}} h^{-1} \|\nabla(v - v_h)\|_{L^1(A_h)} \\ & \leq C \sum_{j=0}^J \left(\frac{h}{d_j}\right)^{\frac{1}{2}} + C \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^J \|v - v_h\|_{L^2(A_j^1)} \end{aligned}$$

$$(4.20) \quad \leq C + C\rho^{-\frac{3}{2}}h^{-\frac{1}{2}} \sum_{j=0}^J \|v - v_h\|_{L^2(A_j^1)}.$$

It remains to estimate $\sum_{j=0}^J \|v - v_h\|_{L^2(A_j^1)}$. To this end, we let χ be a smooth cut-off function satisfying

$$\chi = 1 \text{ on } A_j^1 \quad \text{and} \quad \chi = 0 \text{ outside } A_j^2.$$

Then

$$(4.21) \quad \begin{aligned} \|v - v_h\|_{L^6(A_j^1)} &\leq \|\chi(v - v_h)\|_{L^6(\Omega)} \\ &\leq \|\chi(v - v_h)\|_{H^1(\Omega)} \quad (\text{Sobolev embedding } H^1(\Omega) \hookrightarrow L^6(\Omega)) \\ &\leq \|\nabla(v - v_h)\|_{L^2(A_j^2)} + Cd_j^{-1}\|v - v_h\|_{L^2(A_j^2)}. \end{aligned}$$

By using (4.21) and the interpolation inequality (for $1 < p < 2$)

$$(4.22) \quad \|v - v_h\|_{L^2(A_j^1)} \leq \|v - v_h\|_{L^p(A_j^1)}^{1-\theta} \|v - v_h\|_{L^6(A_j^1)}^\theta \quad \text{with} \quad \frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{6},$$

we obtain

$$\begin{aligned} &\|v - v_h\|_{L^2(A_j^1)} \\ &\leq \|v - v_h\|_{L^p(A_j^1)}^{1-\theta} (\|\nabla(v - v_h)\|_{L^2(A_j^2)} + Cd_j^{-1}\|v - v_h\|_{L^2(A_j^2)})^\theta \\ &= (\varepsilon^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)})^{1-\theta} (\varepsilon \|\nabla(v - v_h)\|_{L^2(A_j^2)} + C\varepsilon d_j^{-1}\|v - v_h\|_{L^2(A_j^2)})^\theta \\ &\leq \varepsilon^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)} + \varepsilon \|\nabla(v - v_h)\|_{L^2(A_j^2)} + C\varepsilon d_j^{-1}\|v - v_h\|_{L^2(A_j^2)}, \end{aligned}$$

where ε can be an arbitrary positive number. By choosing $\varepsilon = d_j(\rho/d_j)^\sigma$ with $\sigma \in (0, 1)$, we obtain

$$\begin{aligned} \|v - v_h\|_{L^2(A_j^1)} &\leq \left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)} \\ &\quad + \left(\frac{\rho}{d_j}\right)^\sigma (d_j \|\nabla(v - v_h)\|_{L^2(A_j^2)} + C\|v - v_h\|_{L^2(A_j^2)}). \end{aligned}$$

Hence,

$$\begin{aligned} &\rho^{-\frac{3}{2}}h^{-\frac{1}{2}} \sum_{j=0}^J \|v - v_h\|_{L^2(A_j^1)} \\ &\leq C\rho^{-\frac{3}{2}}h^{-\frac{1}{2}} \sum_{j=0}^J \left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)} \\ &\quad + C\rho^{-\frac{3}{2}}h^{-\frac{1}{2}} \sum_{j=0}^J \left(\frac{\rho}{d_j}\right)^\sigma (d_j \|\nabla(v - v_h)\|_{L^2(A_j^2)} + C\|v - v_h\|_{L^2(A_j^2)}) \\ &\leq C\rho^{-\frac{3}{2}}h^{-\frac{1}{2}} \sum_{j=0}^J \left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)} \\ &\quad + C\rho^{-\frac{3}{2}}h^{-\frac{1}{2}} \sum_{j=0}^J \left(\frac{\rho}{d_j}\right)^\sigma \|v - v_h\|_{L^2(A_j^2)}, \end{aligned}$$

where we have used (4.19) in deriving the last inequality. Note that

$$\begin{aligned} & \sum_{j=0}^J \left(\frac{\rho}{d_j} \right)^\sigma \|v - v_h\|_{L^2(A_j^2)} \\ & \leq C \left(\frac{\rho}{d_j} \right)^\sigma \|v - v_h\|_{L^2(S_{8\rho}(x_0))} + 2 \sum_{j=0}^J \left(\frac{\rho}{d_j} \right)^\sigma \|v - v_h\|_{L^2(A_j^1)}. \end{aligned}$$

Combining the last two estimates, we obtain

$$\begin{aligned} & \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^J \|v - v_h\|_{L^2(A_j^1)} \\ & \leq C \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^J \left(\frac{\rho}{d_j} \right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)} \\ & \quad + C \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \left(\frac{\rho}{d_j} \right)^\sigma \|v - v_h\|_{L^2(S_{8\rho}(x_0))} + C \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^J \left(\frac{\rho}{d_j} \right)^\sigma \|v - v_h\|_{L^2(A_j^1)}. \end{aligned}$$

If $d_j \geq \kappa\rho$ for sufficiently large constant κ , then the last term can be absorbed by the left side. Hence, we have

$$\begin{aligned} & \sum_{j=0}^J \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \|v - v_h\|_{L^2(A_j^1)} \leq \sum_{j=0}^J C \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \left(\frac{\rho}{d_j} \right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)} \\ (4.23) \quad & \quad + C \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \left(\frac{\rho}{d_j} \right)^\sigma \|v - v_h\|_{L^2(S_{8\rho}(x_0))}. \end{aligned}$$

It remains to estimate $\|v - v_h\|_{L^p(A_j^1)}$ and $\|v - v_h\|_{L^2(S_{8\rho}(x_0))}$. To this end, we let $\psi \in C_0^\infty(A_j^1)$ be a function satisfying

$$\|v - v_h\|_{L^p(A_j^1)} \leq 2(v - v_h, \psi) \quad \text{and} \quad \|\psi\|_{L^q(A_j^1)} \leq 1, \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Let $w \in H_0^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta w = \psi & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

Then using Lemma 2.4 and Lemma 2.1, we obtain

$$\begin{aligned} (v - v_h, \psi) &= (\nabla(v - v_h), \nabla w) \\ &= (\nabla(v - v_h), \nabla(w - I_h w)) \\ &\leq \|\nabla(v - v_h)\|_{L^p(\Omega)} \|\nabla(w - I_h w)\|_{L^q(\Omega)} \\ &\leq Ch^2 \|v\|_{W^{2,p}(\Omega)} \|w\|_{W^{2,q}(\Omega)} \\ &\leq Ch^2 \|\varphi\|_{L^p(\Omega)} \|\psi\|_{L^q(\Omega)} \\ &\leq Ch^2 \|\varphi\|_{L^p(S_\rho(x_0))} \\ &\leq Ch^2 \rho^{\frac{3}{p} - \frac{3}{2}} \|\varphi\|_{L^2(S_\rho(x_0))} \|\psi\|_{L^q(A_j^1)} \\ &\leq Ch^2 \rho^{\frac{3}{p} - \frac{3}{2}}, \end{aligned}$$

where we have used $\|\varphi\|_{L^2(S_\rho(x_0))} \leq 1$ and $\|\psi\|_{L^q(A_j^1)} \leq 1$ in deriving the last inequalities. This implies

$$\|v - v_h\|_{L^p(A_j^1)} \leq Ch^2 \rho^{\frac{3}{p} - \frac{3}{2}} \quad \text{and} \quad \|v - v_h\|_{L^2(S_{8\rho}(x_0))} \leq Ch^2.$$

By substituting these estimates into (4.23), we obtain

$$(4.24) \quad \sum_{j=0}^J \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \|v - v_h\|_{L^2(A_j^1)} \leq \sum_{j=0}^J C \left(\frac{h}{\rho}\right)^{\frac{3}{2}} \left(\frac{\rho}{d_j}\right)^{\frac{3}{p} - \frac{3}{2} - \frac{\theta\sigma}{1-\theta}} + C.$$

Since $p < 2$, by choosing sufficiently small σ we have $\frac{3}{p} - \frac{3}{2} - \frac{\theta\sigma}{1-\theta} > 0$ and therefore

$$(4.25) \quad \sum_{j=0}^J \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \|v - v_h\|_{L^2(A_j^1)} \leq \sum_{j=0}^J C \left(\frac{h}{\rho}\right)^{\frac{3}{2}} \left(\frac{\rho}{d_j}\right)^{\frac{3}{p} - \frac{3}{2} - \frac{\theta\sigma}{1-\theta}} + C \leq C.$$

Then, substituting this into (4.20), we obtain

$$(4.26) \quad \rho^{-\frac{3}{2}} h^{-1} \|\nabla(v - v_h)\|_{L^1(\Lambda_h)} \leq C.$$

This proves the desired result.

5. Application to the Ritz projection

In this section, we adopt Schatz's argument to prove the maximum-norm stability of the Ritz projection. This argument uses the weak maximum principle established above to remove a logarithmic factor for finite elements of degree $r \geq 2$ in convex polyhedral domains under the following assumption:

- (A) The tetrahedral partition of Ω can be extended to a larger convex domain $\tilde{\Omega}$ quasi-uniformly, with $\Omega \subset\subset \tilde{\Omega}$.

The logarithmic factor has been removed in previous articles only for $r \geq 2$ on smooth and two-dimensional polygonal domains.

For any function $u \in H_0^1(\Omega)$, we denote by $R_h u \in \mathring{S}_h$ the Ritz projection of u , defined by

$$(5.27) \quad (\nabla(u - R_h u), \nabla \chi_h) = 0 \quad \forall \chi_h \in \mathring{S}_h.$$

Theorem 5.1. *Under assumption (A), for finite elements of degree $r \geq 2$ the Ritz projection satisfies*

$$(5.28) \quad \|R_h u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\Omega)} \quad \forall u \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Proof. Let \tilde{u} be the zero extension of u to the larger domain $\tilde{\Omega}$. Let $\mathring{S}_h(\tilde{\Omega})$ be the finite element space subject to the tetrahedral partition of $\tilde{\Omega}$ (with zero boundary values), and let \tilde{u}_h be the Ritz projection of \tilde{u} in the domain $\tilde{\Omega}$, i.e.

$$(5.29) \quad \int_{\tilde{\Omega}} \nabla(\tilde{u} - \tilde{u}_h) \cdot \nabla \chi_h \, dx = 0 \quad \forall \chi_h \in \mathring{S}_h(\tilde{\Omega}).$$

Since $u = \tilde{u}$ on Ω , it follows that

$$(5.30) \quad \|u - u_h\|_{L^\infty(\Omega)} = \|\tilde{u} - u_h\|_{L^\infty(\Omega)}$$

$$(5.31) \quad \leq \|\tilde{u} - \tilde{u}_h\|_{L^\infty(\Omega)} + \|\tilde{u}_h - u_h\|_{L^\infty(\Omega)}$$

$$(5.32) \quad := E_1 + E_2.$$

By using [26, Theorem 5.1] (which requires $r \geq 2$ to remove a logarithmic factor), we have

$$(5.33) \quad E_1 \leq C \|\tilde{u} - I_h \tilde{u}\|_{L^\infty(\Omega')} + C \|\tilde{u} - \tilde{u}_h\|_{L^2(\Omega')},$$

where Ω' is some intermediate domain satisfying $\Omega \subset\subset \Omega' \subset\subset \tilde{\Omega}$. Since the Lagrange interpolation operator I_h is stable in L^∞ , it follows that

$$(5.34) \quad \|\tilde{u} - I_h \tilde{u}\|_{L^\infty(\Omega')} \leq C \|\tilde{u}\|_{L^\infty(\Omega)} = C \|u\|_{L^\infty(\Omega)}.$$

To estimate $\|\tilde{u} - \tilde{u}_h\|_{L^2(\Omega')}$, we use a duality argument. Thus,

$$\|\tilde{u} - \tilde{u}_h\|_{L^2(\Omega')} \leq \|\tilde{u} - \tilde{u}_h\|_{L^2(\tilde{\Omega})} = \sup_{\substack{\tilde{\varphi} \in C_0^\infty(\tilde{\Omega}) \\ \|\tilde{\varphi}\|_{L^2(\tilde{\Omega})} \leq 1}} \int_{\tilde{\Omega}} (\tilde{u} - \tilde{u}_h) \tilde{\varphi} \, dx.$$

In particular, there exists a $\tilde{\varphi} \in C_0^\infty(\tilde{\Omega})$ satisfying

$$(5.35) \quad \|\tilde{\varphi}\|_{L^2(\tilde{\Omega})} \leq 1 \quad \text{and} \quad \|\tilde{u} - \tilde{u}_h\|_{L^2(\tilde{\Omega})} \leq 2 \int_{\tilde{\Omega}} (\tilde{u} - \tilde{u}_h) \tilde{\varphi} \, dx.$$

For this $\tilde{\varphi}$ we define $\tilde{\psi} \in H_0^1(\Omega)$ to be the weak solution of

$$(5.36) \quad \begin{cases} -\Delta \tilde{\psi} = \tilde{\varphi} & \text{in } \tilde{\Omega}, \\ \tilde{\psi} = 0 & \text{on } \partial \tilde{\Omega}, \end{cases}$$

and denote by $\tilde{\psi}_h \in \dot{S}_h(\tilde{\Omega})$ the Ritz projection of $\tilde{\psi}$ in $\tilde{\Omega}$, i.e.

$$(5.37) \quad \int_{\tilde{\Omega}} \nabla(\tilde{\psi} - \tilde{\psi}_h) \cdot \nabla \tilde{\chi}_h \, dx = 0 \quad \forall \tilde{\chi}_h \in \dot{S}_h(\tilde{\Omega}).$$

If we denote by $\tilde{\mathfrak{T}}$ the set of tetrahedra in the partition of $\tilde{\Omega}$, then testing (5.36) by $\tilde{u} - \tilde{u}_h$ yields

$$(5.38) \quad \begin{aligned} \int_{\tilde{\Omega}} (\tilde{u} - \tilde{u}_h) \tilde{\varphi} \, dx &= \int_{\tilde{\Omega}} \nabla(\tilde{u} - \tilde{u}_h) \cdot \nabla \tilde{\psi} \, dx \quad (\text{here we use integration by parts}) \\ &= \int_{\tilde{\Omega}} \nabla(\tilde{u} - \tilde{u}_h) \cdot \nabla(\tilde{\psi} - \tilde{\psi}_h) \, dx \quad (\text{here we use (5.29)}) \\ &= \int_{\tilde{\Omega}} \nabla \tilde{u} \cdot \nabla(\tilde{\psi} - \tilde{\psi}_h) \, dx \quad (\text{here we use (5.37)}) \\ &= \sum_{\tau \in \tilde{\mathfrak{T}}} \int_{\tau} \nabla \tilde{u} \cdot \nabla(\tilde{\psi} - \tilde{\psi}_h) \, dx \\ &= - \sum_{\tau \in \tilde{\mathfrak{T}}} \int_{\tau} \tilde{u} \Delta(\tilde{\psi} - \tilde{\psi}_h) \, dx + \int_{\partial \tau} \tilde{u} \partial_n(\tilde{\psi} - \tilde{\psi}_h) \, ds \\ &\leq C \|\tilde{u}\|_{L^\infty(\tilde{\Omega})} \sum_{\tau \in \tilde{\mathfrak{T}}} \left(\|\Delta(\tilde{\psi} - \tilde{\psi}_h)\|_{L^1(\tau)} + \|\partial_n(\tilde{\psi} - \tilde{\psi}_h)\|_{L^1(\partial \tau)} \right) \\ &\leq C \|u\|_{L^\infty(\Omega)} \left(h^{-1} \|\nabla(\tilde{\psi} - \tilde{\psi}_h)\|_{L^1(\tilde{\Omega})} + \sum_{\tau \in \tilde{\mathfrak{T}}} \|\tilde{\psi} - \tilde{\psi}_h\|_{W^{2,1}(\tau)} \right), \end{aligned}$$

where in the last step we have used $\|\tilde{u}\|_{L^\infty(\tilde{\Omega})} = \|u\|_{L^\infty(\Omega)}$ and the trace inequality

$$\|\partial_n(\tilde{\psi} - \tilde{\psi}_h)\|_{L^1(\partial \tau)} \leq Ch^{-1} \|\nabla(\tilde{\psi} - \tilde{\psi}_h)\|_{L^1(\tau)} + C \|\tilde{\psi} - \tilde{\psi}_h\|_{W^{2,1}(\tau)}.$$

By using a priori energy estimate and H^2 regularity, we have

$$\begin{aligned}
\|\nabla(\tilde{\psi} - \tilde{\psi}_h)\|_{L^1(\tilde{\Omega})} &\leq \|\nabla(\tilde{\psi} - \tilde{\psi}_h)\|_{L^2(\tilde{\Omega})} \\
&\leq \|\nabla(\tilde{\psi} - \tilde{I}_h\tilde{\psi})\|_{L^2(\tilde{\Omega})} \\
&\leq Ch\|\tilde{\psi}\|_{H^2(\tilde{\Omega})} \\
(5.39) \quad &\leq Ch\|\tilde{\varphi}\|_{L^2(\tilde{\Omega})} \leq Ch.
\end{aligned}$$

Let \tilde{I}_h be the Scott-Zhang interpolant. Then by the triangle and inverse inequalities, we have

$$\begin{aligned}
&\sum_{\tau \in \tilde{\mathfrak{T}}} \|\tilde{\psi} - \tilde{\psi}_h\|_{W^{2,1}(\tau)} \\
&\leq C \sum_{\tau \in \tilde{\mathfrak{T}}} \left(\|\tilde{\psi} - \tilde{I}_h\tilde{\psi}\|_{W^{2,1}(\tau)} + \|\tilde{I}_h\tilde{\psi} - \tilde{\psi}_h\|_{W^{2,1}(\tau)} \right) \\
&\leq C \left(\sum_{\tau \in \tilde{\mathfrak{T}}} \|\tilde{\psi} - \tilde{I}_h\tilde{\psi}\|_{W^{2,1}(\tau)} + h^{-1}\|\tilde{I}_h\tilde{\psi} - \tilde{\psi}_h\|_{W^{1,1}(\tilde{\Omega})} \right) \\
&\leq C \left(\sum_{\tau \in \tilde{\mathfrak{T}}} \|\tilde{\psi} - \tilde{I}_h\tilde{\psi}\|_{W^{2,1}(\tau)} + h^{-1}\|\tilde{\psi} - \tilde{I}_h\tilde{\psi}\|_{W^{1,1}(\tilde{\Omega})} + h^{-1}\|\tilde{\psi} - \tilde{\psi}_h\|_{W^{1,1}(\tilde{\Omega})} \right).
\end{aligned}$$

Similarly as (5.39), we can prove the following estimate:

$$h^{-1}\|\tilde{\psi} - \tilde{I}_h\tilde{\psi}\|_{W^{1,1}(\tilde{\Omega})} + h^{-1}\|\tilde{\psi} - \tilde{\psi}_h\|_{W^{1,1}(\tilde{\Omega})} \leq C,$$

and by using the properties of \tilde{I}_h (cf. [5, Theorem 4.8.3.8]),

$$\sum_{\tau \in \tilde{\mathfrak{T}}} \|\tilde{\psi} - \tilde{I}_h\tilde{\psi}\|_{W^{2,1}(\tau)} \leq C \sum_{\tau \in \tilde{\mathfrak{T}}} \|\tilde{\psi}\|_{W^{2,1}(\tau)} \leq C\|\tilde{\psi}\|_{H^2(\tilde{\Omega})} \leq C\|\tilde{\varphi}\|_{L^2(\tilde{\Omega})} \leq C.$$

Now we substitute these estimates into (5.38). This yields

$$(5.40) \quad \|\tilde{u} - \tilde{u}_h\|_{L^2(\tilde{\Omega})} \leq C\|u\|_{L^\infty(\Omega)}.$$

Then, by substituting (5.34) and (5.40) into (5.33), we obtain

$$(5.41) \quad E_1 \leq C\|u\|_{L^\infty(\Omega)}.$$

To estimate E_2 , we use the fact that $\tilde{u}_h - u_h$ is discrete harmonic in Ω , i.e.

$$\int_{\Omega} \nabla(\tilde{u}_h - u_h) \cdot \nabla \chi_h \, dx = \int_{\Omega} \nabla(\tilde{u} - u) \cdot \nabla \chi_h \, dx = 0 \quad \forall \chi_h \in \mathring{S}_h(\Omega).$$

Thus, by the weak discrete maximum principle proved in Theorem 1.1 and using the fact that $u_h = 0$ and $\tilde{u} = 0$ on $\partial\Omega$, we have

$$\begin{aligned}
(5.42) \quad E_2 &= \|\tilde{u}_h - u_h\|_{L^\infty(\Omega)} \\
&\leq C\|\tilde{u}_h - u_h\|_{L^\infty(\partial\Omega)} \\
&= C\|\tilde{u}_h\|_{L^\infty(\partial\Omega)} \quad (\text{use } u_h = 0 \text{ on } \partial\Omega) \\
&= C\|\tilde{u}_h - \tilde{u}\|_{L^\infty(\partial\Omega)} \quad (\text{use } \tilde{u} = 0 \text{ on } \partial\Omega) \\
&\leq C\|\tilde{u}_h - \tilde{u}\|_{L^\infty(\Omega)} \\
&= E_1,
\end{aligned}$$

which has already been estimated. Hence, substituting (5.41) and (5.42) into (5.30), we obtain

$$(5.43) \quad \|u - u_h\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\Omega)}.$$

This completes the proof of Theorem 5.1. \square

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