

# Discrete maximal parabolic regularity for Galerkin finite element methods

Dmitriy Leykekhman · Boris Vexler

Received: date / Accepted: date

**Abstract** The main goal of the paper is to establish time semidiscrete and space-time fully discrete maximal parabolic regularity for the time discontinuous Galerkin solution of linear parabolic equations. Such estimates have many applications. They are essential, for example, in establishing optimal a priori error estimates in non-Hilbertian norms without unnatural coupling of spatial mesh sizes with time steps.

**Keywords** maximal parabolic regularity · finite elements · maximum norm · fully discrete · resolvent estimates · resolvent estimates · optimal error estimates · parabolic smoothing

## 1 Introduction

Let  $\Omega$  be a Lipschitz domain in  $\mathbb{R}^d$ ,  $d = 2, 3$  and  $I = (0, T)$ . We consider the heat equation as a model of a parabolic second order partial differential equation,

$$\begin{aligned}\partial_t u(t, x) - \Delta u(t, x) &= f(t, x), & (t, x) \in I \times \Omega, \\ u(t, x) &= 0, & (t, x) \in I \times \partial\Omega, \\ u(0, x) &= u_0(x), & x \in \Omega\end{aligned}\tag{1}$$

with a right-hand side  $f \in L^s(I; L^p(\Omega))$  for some  $1 \leq p, s \leq \infty$  and  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ .

The maximal parabolic regularity for  $u_0 \equiv 0$  says that there exists a constant  $C$  such that,

$$\|\partial_t u\|_{L^s(I; L^p(\Omega))} + \|\Delta u\|_{L^s(I; L^p(\Omega))} \leq C \|f\|_{L^s(I; L^p(\Omega))}, \quad 1 < p, s < \infty, \quad \text{for all } f \in L^s(I; L^p(\Omega)),$$

(see, e.g., [8, 19, 20]). The maximal parabolic regularity is an important analytical tool and has a number of applications, especially to nonlinear problems and/or optimal control problems when sharp regularity results are required (cf. [21, 22, 23, 25]). Our aim in this paper is to establish similar maximal parabolic regularity results for time discrete discontinuous Galerkin solutions as well as for the fully discrete Galerkin approximations. Such results are very useful, for example, in fully discrete a priori error estimates and are essential in order to keep the spatial mesh size  $h$  and the time steps  $k$  independent of each other (cf. [28]). In [27] we apply the results of this paper to establish pointwise best approximation estimates for fully discrete Galerkin solutions.

Maximal parabolic regularity with applications to semidiscrete finite element Galerkin solutions in space were analyzed for smooth domains in [14, 15] and for convex polyhedra in [29]. Time discrete results are much less

---

D. Leykekhman  
Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA  
E-mail: leykekhman@math.uconn.edu

B. Vexler  
Lehrstuhl für Optimalsteuerung, Technische Universität München, Fakultät für Mathematik, Boltzmannstraße 3, 85748 Garching b. München, Germany  
E-mail: vexler@ma.tum.de

known in the finite element community. Explicit methods are treated in [5,6,36]. Implicit Euler methods with pointwise norms in time are considered in [16,17]. A more systematic investigation of discrete maximal parabolic regularity for various time schemes was carried out by Sobolevskii and Ashyralyev and summarized in the book [1].

In this paper, we investigate maximal parabolic regularity for a family of time discontinuous Galerkin (dG) methods, which were first deeply analyzed for linear second order parabolic problems in [13]. There is a number of important properties that make the dG schemes attractive for temporal discretization of parabolic problems. For example, such schemes allow for a priori error estimates of optimal order with respect to discretization parameters, such as the size of time steps and the mesh width, as well as with respect to the regularity requirements for the solution (see, e.g., [10,11]). Different systematic approaches for a posteriori error estimation and adaptivity developed for finite element discretizations can be adapted for dG temporal discretization of parabolic equations, (see, e.g., [38,39]). Since the trial space allows for discontinuities at the time nodes, the use of different spatial discretizations for each time step can be directly incorporated into the discrete formulation, (see, e.g., [38]). Compared to the continuous Galerkin methods, dG schemes are not only A-stable but also strongly A-stable, (see, e.g., [24]). An efficient and easy to implement approach that avoids complex coefficients, which arise in the equations obtained by a direct decoupling for high order dG schemes, was developed in [37]. For the treatment of optimal control problems, Galerkin methods are particularly suitable since they expose an important property that the two approaches optimize-then-discretize, i.e., the discretization of the optimality system built up on the continuous level, and discretize-then-optimize, i.e., discretization of the state equation and subsequent construction of the optimality system on the discrete level, lead to the same discretization scheme, (see, e.g., [4]). Compared to continuous Petrov-Galerkin time-stepping schemes (see [35] for details), dG schemes also have the advantage that the adjoint state can use the same discretization as the state variable. This allows for unified numerical treatment and simplifies a priori and a posteriori error analysis, (see, e.g., [7,32,33,34]).

The main results of this paper for the time semidiscrete discontinuous Galerkin  $u_k$  solution consist roughly of two parts. First, for the homogeneous problem (i.e.  $f = 0$ ) with  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$  we show

$$\|\partial_t u_k\|_{L^\infty(I_m; L^p(\Omega))} + \|\Delta u_k\|_{L^\infty(I_m; L^p(\Omega))} + \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)} \leq \frac{C}{t_m} \|u_0\|_{L^p(\Omega)}, \quad (2)$$

for  $m = 1, 2, \dots, M$ . Then, using this smoothing result, we also establish discrete maximal parabolic regularity for the inhomogeneous problem when  $u_0 = 0$ . We show,

$$\left( \sum_{m=1}^M \|\partial_t u_k\|_{L^s(I_m; L^p(\Omega))}^s \right)^{\frac{1}{s}} + \|\Delta u_k\|_{L^s(I; L^p(\Omega))} + \left( \sum_{m=1}^M k_m \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)}^s \right)^{\frac{1}{s}} \leq C \ln \frac{T}{k} \|f\|_{L^s(I; L^p(\Omega))}, \quad (3)$$

for  $1 \leq s \leq \infty$  and  $1 \leq p \leq \infty$ , with obvious notation changes in the case of  $s = \infty$ . In the case of the lowest order piecewise constant method, i.e.,  $q = 0$ , the first terms on the left-hand side of the above estimates vanish. In contrast to the continuous case, the limiting cases  $s, p \in \{1, \infty\}$  are allowed, which explains the logarithmic factor in (3). We also provide the fully discrete analog of (2) and (3).

The rest of the paper is organized as follows. In the next section we introduce the discretization method and the resolvent estimates, which build the main analytical tool of the paper. For better communication of the ideas we first analyze the dG(0) method, which is technically much simpler, and in the following section we analyze the general dG( $q$ ) case. That is done in Sections 3 and 4, respectively. At the end of Section 4 we provide an example of how such maximal parabolic regularity results can rather easily lead to optimal order error estimates. Finally, Section 5 is devoted to fully discrete Galerkin solutions. In Section 6 we provide an extension of our results to the case of a general norm fulfilling a resolvent estimate. This generalization, being of an independent interest, is used, for example, in [27] for derivation of pointwise interior (local) error estimates of fully discrete Galerkin solutions.

## 2 Preliminaries

To introduce the time discontinuous Galerkin discretization for the problem, we partition  $I = (0, T)$  into subintervals  $I_m = (t_{m-1}, t_m]$  of length  $k_m = t_m - t_{m-1}$ , where  $0 = t_0 < t_1 < \dots < t_{M-1} < t_M = T$ . The

maximal and minimal time steps are denoted by  $k = \max_m k_m$  and  $k_{\min} = \min_m k_m$ , respectively. We impose the following conditions on the time mesh (as in [31]):

(i) There are constants  $c, \beta > 0$  independent on  $k$  such that

$$k_{\min} \geq ck^\beta.$$

(ii) There is a constant  $\kappa > 0$  independent on  $k$  such that for all  $m = 1, 2, \dots, M-1$

$$\kappa^{-1} \leq \frac{k_m}{k_{m+1}} \leq \kappa.$$

(iii) It holds  $k \leq \frac{1}{4}T$ .

The semidiscrete space  $X_k^q$  of piecewise polynomial functions in time is defined by

$$X_k^q = \{u_k \in L^2(I; H_0^1(\Omega)) : u_k|_{I_m} \in \mathcal{P}_q(H_0^1(\Omega)), m = 1, 2, \dots, M\},$$

where  $\mathcal{P}_q(V)$  is the space of polynomial functions of degree  $q$  in time with values in a Banach space  $V$ . We will employ the following notation for functions in  $X_k^q$

$$u_m^+ = \lim_{\varepsilon \rightarrow 0^+} u(t_m + \varepsilon), \quad u_m^- = \lim_{\varepsilon \rightarrow 0^+} u(t_m - \varepsilon), \quad [u]_m = u_m^+ - u_m^-. \quad (4)$$

Next we define the following bilinear form

$$B(u, \varphi) = \sum_{m=1}^M \langle \partial_t u, \varphi \rangle_{I_m \times \Omega} + (\nabla u, \nabla \varphi)_{I \times \Omega} + \sum_{m=2}^M ([u]_{m-1}, \varphi_{m-1}^+)_{\Omega} + (u_0^+, \varphi_0^+)_{\Omega}, \quad (5)$$

where  $(\cdot, \cdot)_{\Omega}$  and  $(\cdot, \cdot)_{I_m \times \Omega}$  are the usual  $L^2$  space and space-time inner-products,  $\langle \cdot, \cdot \rangle_{I_m \times \Omega}$  is the duality product between  $L^2(I_m; H^{-1}(\Omega))$  and  $L^2(I_m; H_0^1(\Omega))$ . We note, that the first sum vanishes for  $u \in X_k^0$ . The dG( $q$ ) semidiscrete (in time) approximation  $u_k \in X_k^q$  of (1) is defined as

$$B(u_k, \varphi_k) = (f, \varphi_k)_{I \times \Omega} + (u_0, \varphi_{k,0}^+)_{\Omega} \quad \text{for all } \varphi_k \in X_k^q. \quad (6)$$

Rearranging the terms in (5), we obtain an equivalent (dual) expression of  $B$ :

$$B(u, \varphi) = - \sum_{m=1}^M \langle u, \partial_t \varphi \rangle_{I_m \times \Omega} + (\nabla u, \nabla \varphi)_{I \times \Omega} - \sum_{m=1}^{M-1} (u_m^-, [\varphi]_m)_{\Omega} + (u_M^-, \varphi_M^-)_{\Omega}. \quad (7)$$

The analysis of such schemes in non-Hilbertian setting is usually done by using a semigroup approach that represents time stepping formulas as a Dunford-Taylor integral in the complex plane [41, Ch. 9]. This approach requires certain resolvent estimates. For Lipschitz domains and a given  $\gamma \in (0, \pi/2)$ , the resolvent estimate (see [40]) guarantees the existence of a constant  $C$  such that for all  $u \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ , and any  $z \in \mathbb{C} \setminus \Sigma_\gamma$  the following estimate holds:

$$\|(z + \Delta)^{-1} u\|_{L^p(\Omega)} \leq \frac{C}{1 + |z|} \|u\|_{L^p(\Omega)}, \quad (8)$$

where the Laplace operator  $-\Delta$  is supplemented with homogeneous Dirichlet boundary conditions, and

$$\Sigma_\gamma = \{z \in \mathbb{C} : |\arg(z)| \leq \gamma\}. \quad (9)$$

Using the identity  $\Delta(z + \Delta)^{-1} = \text{Id} - z(z + \Delta)^{-1}$ , one immediately obtains,

$$\|\Delta(z + \Delta)^{-1} u\|_{L^p(\Omega)} \leq C \|u\|_{L^p(\Omega)}, \quad z \in \mathbb{C} \setminus \Sigma_\gamma, \quad 1 \leq p \leq \infty, \quad u \in L^p(\Omega). \quad (10)$$

We note, that all our results for semidiscrete solutions hold if we replace the Laplace operator  $-\Delta$  with a more general self-adjoint second order elliptic operator  $A$  provided it satisfies (8).

### 3 Estimates for dG(0)

For the ease of the presentation, we first establish the results for the lowest order piecewise constant discretization dG(0). In this case, we use the following notation,

$$u_{k,m} = u_k|_{I_m}, \quad u_{k,m}^+ = u_{k,m+1}, \quad u_{k,m}^- = u_{k,m}, \quad m = 1, 2, \dots, M-1. \quad (11)$$

First, we establish results for the homogeneous problem. In this case the dG(0) method is equivalent to the Backward Euler method.

#### 3.1 Results for the homogeneous problem

Let  $f = 0$ ,  $u_0 \in L^p(\Omega)$  and let  $u_k \in X_k^0$  be the semidiscrete approximation of (1) defined by

$$B(u_k, \chi_k) = (u_0, \chi_{k,1}), \quad \forall \chi_k \in X_k^0, \quad (12)$$

i.e., the dG(0) solution  $u_k$  satisfies

$$\begin{aligned} u_{k,1} - k_1 \Delta u_{k,1} &= u_0, \\ u_{k,m} - k_m \Delta u_{k,m} &= u_{k,m-1}, \quad m = 2, 3, \dots, M. \end{aligned} \quad (13)$$

The first result shows that the solution can not grow from one time step to the next one.

**Lemma 1** *Let  $u_k$  be the solution of (12). Then, for  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$  there holds*

$$\|u_{k,m}\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)} \quad \forall m = 1, 2, \dots, M.$$

*Proof* First, we assume  $u_0 \in L^\infty(\Omega)$  and establish

$$\|u_{k,m}\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \quad m = 1, 2, \dots, M. \quad (14)$$

It is sufficient to consider only a single time step,

$$u_{k,1} - k_1 \Delta u_{k,1} = u_0. \quad (15)$$

We want to show that  $\|u_{k,1}\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$ . Assume it is false. Let  $x_0 \in \Omega$  be a point where  $u_{k,1}$  attains a maximum. By [18, Theorem 3.3], we know that  $u_{k,1} \in C_0(\Omega)$ , hence, there exists an open ball  $B_\delta(x_0)$  of radius  $\delta > 0$  centered at  $x_0$  with  $\overline{B_\delta(x_0)} \subset \Omega$  such that

$$u_{k,1}(x) > \|u_0\|_{L^\infty(\Omega)} \quad \text{for all } x \in B_\delta(x_0).$$

Hence,

$$u_{k,1}(x) - u_0(x) > 0 \quad \text{on } B_\delta(x_0).$$

By the maximum principle, from

$$-\Delta u_{k,1} = \frac{1}{k_1} (u_0 - u_{k,1}) < 0 \quad \text{on } B_\delta(x_0),$$

we obtain a contradiction to the assumption that  $u_{k,1}$  has a maximum at the interior point  $x_0$ . This contradiction establishes (14). Next, using a duality argument, we will show

$$\|u_{k,1}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}. \quad (16)$$

Consider the problem, to find  $z_{k,1} \in H_0^1(\Omega)$  that satisfies,

$$z_{k,1} - k_1 \Delta z_{k,1} = z_0, \quad \text{with } z_0 = \text{sgn } u_{k,1}.$$

The solution  $z_{k,1}$  can be thought of as a single step of the dG(0) method to a parabolic problem with initial condition  $\text{sgn } u_{k,1}$ . Thus,

$$\|u_{k,1}\|_{L^1(\Omega)} = (u_{k,1}, z_0) = (z_{k,1}, u_{k,1}) + k_1 (\nabla z_{k,1}, \nabla u_{k,1}) = (u_0, z_{k,1}) \leq \|u_0\|_{L^1(\Omega)} \|z_0\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^1(\Omega)},$$

where we have used (14) for  $z_k$  and the fact that  $\|z_0\|_{L^\infty(\Omega)} = \|\text{sgn } u_{k,1}\|_{L^\infty(\Omega)} = 1$ . This establishes (16). Interpolating, we obtain the lemma for  $1 \leq p \leq \infty$ .

Next we will establish a smoothing result.

**Theorem 1 (Homogeneous smoothing estimate)** *Let  $u_k \in X_k^0$  be the solution of (12) with  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then there exists a constant  $C$  independent of  $k$  such that*

$$\|\Delta u_{k,m}\|_{L^p(\Omega)} \leq \frac{C}{t_m} \|u_0\|_{L^p(\Omega)}, \quad m = 1, 2, \dots, M.$$

*Proof* The proof is given on page 1321 in [12] for the  $L^2(\Omega)$  norm, but the proof is valid for the  $L^p(\Omega)$  norm as well by using the resolvent estimate (8) with respect to the  $L^p(\Omega)$  norm.

*Remark 1* Let  $u_k \in X_k^0$  be the solution of (12) with  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then there exists a constant  $C$  independent of  $k$  such that

$$\|u_{k,m}\|_{L^p(\Omega)} + (t_m - t_l) \|\Delta u_{k,m}\|_{L^p(\Omega)} \leq C \|u_{k,l}\|_{L^p(\Omega)}, \quad \forall m > l \geq 1.$$

From (13), we immediately obtain the following result.

**Corollary 1** *Let  $u_k \in X_k^0$  be the solution of (12) with  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then there exists a constant  $C$  independent of  $k$  such that*

$$\left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)} \leq \frac{C}{t_m} \|u_0\|_{L^p(\Omega)}, \quad m = 1, 2, \dots, M.$$

### 3.2 Results for the inhomogeneous problem

Now we consider  $u_k \in X_k^0$  to be the dG(0) solution to the parabolic equation with  $u_0 = 0$ , i.e.,  $u_k$  satisfies,

$$B(u_k, \varphi_k) = (f, \varphi_k)_{I \times \Omega}, \quad \forall \varphi_k \in X_k^0. \quad (17)$$

Thus, the dG(0) solution satisfies

$$\begin{aligned} u_{k,1} - k_1 \Delta u_{k,1} &= k_1 f_1, \\ u_{k,m} - k_m \Delta u_{k,m} &= u_{k,m-1} + k_m f_m, \quad m = 2, 3, \dots, M, \end{aligned} \quad (18)$$

where

$$f_m(\cdot) = \frac{1}{k_m} \int_{I_m} f(t, \cdot) dt.$$

Since  $f_m$  is the  $L^2$  projection of  $f$  onto the piecewise constant functions on each subinterval  $I_m$ , we have

$$\max_{1 \leq m \leq M} \|f_m\|_{L^p(\Omega)} \leq C \|f\|_{L^\infty(I; L^p(\Omega))}, \quad 1 \leq p \leq \infty, \quad (19a)$$

$$\sum_{m=1}^M k_m \|f_m\|_{L^p(\Omega)}^r \leq C \|f\|_{L^r(I; L^p(\Omega))}^r, \quad 1 \leq p \leq \infty, \quad 1 \leq r < \infty. \quad (19b)$$

We now state our main result for the dG(0) approximations.

**Theorem 2 (Maximal parabolic regularity)** *Let  $1 \leq s, p \leq \infty$  and  $u_0 = 0$ . Then, there exists a constant  $C$  independent of  $k$  such that for every  $f \in L^s(I; L^p(\Omega))$  and  $u_k$  satisfying (17), the following estimate holds:*

$$\|\Delta u_k\|_{L^s(I; L^p(\Omega))} \leq C \ln \frac{T}{k} \|f\|_{L^s(I; L^p(\Omega))}, \quad 1 \leq s \leq \infty, \quad 1 \leq p \leq \infty.$$

*Proof* Using (18), we can write the dG(0) solution as

$$u_{k,m} = \sum_{l=1}^m k_l \left( \prod_{j=1}^{m-l+1} r(-k_{m-j+1}\Delta) \right) f_l, \quad m = 1, 2, \dots, M,$$

where  $r(z) = (1+z)^{-1}$ . Then,

$$\Delta u_{k,m} = \sum_{l=1}^m k_l \left( \Delta \prod_{j=1}^{m-l+1} r(-k_{m-j+1}\Delta) \right) f_l, \quad m = 1, 2, \dots, M.$$

Hence

$$\|\Delta u_{k,m}\|_{L^p(\Omega)} \leq \sum_{l=1}^m k_l \left\| \left( \Delta \prod_{j=1}^{m-l+1} r(-k_{m-j+1}\Delta) \right) f_l \right\|_{L^p(\Omega)}, \quad m = 1, 2, \dots, M.$$

From Remark 1, since each term in the sum on the right-hand side can be thought of as a homogeneous solution with initial condition  $f_l$  at  $t = t_{l-1}$ , we have

$$\left\| \left( \Delta \prod_{j=1}^{m-l+1} r(-k_{m-j+1}\Delta) \right) f_l \right\|_{L^p(\Omega)} \leq \frac{C}{t_m - t_{l-1}} \|f_l\|_{L^p(\Omega)}.$$

Thus, we obtain

$$\|\Delta u_{k,m}\|_{L^p(\Omega)} \leq C \sum_{l=1}^m \frac{k_l}{t_m - t_{l-1}} \|f_l\|_{L^p(\Omega)}, \quad m = 1, 2, \dots, M. \quad (20)$$

For  $s = \infty$ , we obtain from the above estimate and using (19),

$$\begin{aligned} \|\Delta u_k\|_{L^\infty(I; L^p(\Omega))} &= \max_{1 \leq m \leq M} \|\Delta u_{k,m}\|_{L^p(\Omega)} \leq C \max_{1 \leq m \leq M} \sum_{l=1}^m \frac{k_l}{t_m - t_{l-1}} \|f_l\|_{L^p(\Omega)} \\ &\leq C \max_{1 \leq l \leq M} \|f_l\|_{L^p(\Omega)} \max_{1 \leq m \leq M} \sum_{l=1}^m \frac{k_l}{t_m - t_{l-1}} \leq C \ln \frac{T}{k} \|f\|_{L^\infty(I; L^p(\Omega))}, \end{aligned}$$

where in the last step we used that

$$\sum_{l=1}^m \frac{k_l}{t_m - t_{l-1}} \leq 1 + \int_0^{t_{m-1}} \frac{dt}{t_m - t} = 1 + \ln \frac{t_m}{k_m} \leq C \ln \frac{T}{k}, \quad (21)$$

by using the assumption  $k_{\min} \geq Ck^\beta$  and  $k \leq \frac{T}{4}$ .

For  $s = 1$ , we have

$$\|\Delta u_k\|_{L^1(I; L^p(\Omega))} = \sum_{m=1}^M k_m \|\Delta u_{k,m}\|_{L^p(\Omega)} \leq C \sum_{m=1}^M k_m \sum_{l=1}^m \frac{k_l}{t_m - t_{l-1}} \|f_l\|_{L^p(\Omega)}.$$

Changing the order of summation and using (19), we obtain

$$\begin{aligned} \|\Delta u_k\|_{L^1(I; L^p(\Omega))} &\leq C \sum_{l=1}^M k_l \|f_l\|_{L^p(\Omega)} \sum_{m=l}^M \frac{k_m}{t_m - t_{l-1}} \\ &\leq C \ln \frac{T}{k} \sum_{l=1}^M k_l \|f_l\|_{L^p(\Omega)} \leq C \ln \frac{T}{k} \|f\|_{L^1(I; L^p(\Omega))}, \end{aligned}$$

where we used again that

$$\sum_{m=l}^M \frac{k_m}{t_m - t_{l-1}} \leq C \ln \frac{T}{k}.$$

Interpolating between  $s = 1$  and  $s = \infty$ , we obtain the result for any  $1 \leq s \leq \infty$ .

*Remark 2* The appearance of the logarithmic term is natural for the critical values  $s = 1$ ,  $p = 1$ ,  $s = \infty$ , or  $p = \infty$ , since the corresponding maximal parabolic regularity results for the continuous problem hold only for  $1 < s, p < \infty$ . For  $s = 2$  or  $p = 2$ , the power of the logarithm can be lowered. Thus, for  $p = 2$ , from [33] we know,

$$\|\Delta u_k\|_{L^2(I;L^2(\Omega))} \leq C \|f\|_{L^2(I;L^2(\Omega))}$$

and from (20), we have

$$\|\Delta u_k\|_{L^s(I;L^2(\Omega))} \leq C \ln \frac{T}{k} \|f\|_{L^s(I;L^2(\Omega))}, \quad 1 \leq s \leq \infty.$$

Interpolating between  $s = 2$  and  $s = \infty$  and between  $s = 2$  and  $s = 1$ , we obtain

$$\|\Delta u_k\|_{L^s(I;L^2(\Omega))} \leq C \left( \ln \frac{T}{k} \right)^{\frac{|s-2|}{s}} \|f\|_{L^s(I;L^2(\Omega))}, \quad \text{for any } 1 \leq s \leq \infty.$$

Similarly, we can obtain,

$$\|\Delta u_k\|_{L^2(I;L^p(\Omega))} \leq C \left( \ln \frac{T}{k} \right)^{\frac{|p-2|}{p}} \|f\|_{L^2(I;L^p(\Omega))}, \quad \text{for any } 1 \leq p \leq \infty.$$

**Corollary 2 (Maximal parabolic regularity for jumps)** *Let  $1 \leq s, p \leq \infty$  and  $u_0 = 0$ . Then, there exists a constant  $C$  independent of  $k$  such that for every  $f \in L^s(I;L^p(\Omega))$  and  $u_k$  satisfying (17), the following estimate holds,*

$$\begin{aligned} \max_{1 \leq m \leq M} \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)} &\leq C \ln \frac{T}{k} \|f\|_{L^\infty(I;L^p(\Omega))}, \quad 1 \leq p \leq \infty, \\ \left( \sum_{m=1}^M k_m \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)}^s \right)^{\frac{1}{s}} &\leq C \ln \frac{T}{k} \|f\|_{L^s(I;L^p(\Omega))}, \quad 1 \leq s < \infty, \quad 1 \leq p \leq \infty, \end{aligned}$$

where the jump term  $[u_k]_0$  at  $t = 0$  is defined as  $u_{k,1}$ .

*Proof* Since by (18) on each time subinterval  $I_m$  we have

$$k_m^{-1} [u_k]_{m-1} = \Delta u_{k,m} + f_m, \quad m = 1, 2, \dots, M,$$

by using Theorem 2, we have

$$\max_{1 \leq m \leq M} k_m^{-1} \|[u_k]_{m-1}\|_{L^p(\Omega)} \leq \max_{1 \leq m \leq M} (\|\Delta u_{k,m}\|_{L^p(\Omega)} + \|f_m\|_{L^p(\Omega)}) \leq C \ln \frac{T}{k} \|f\|_{L^\infty(I;L^p(\Omega))}.$$

Similarly, using Theorem 2, for  $1 \leq s < \infty$  we have

$$\sum_{m=1}^M k_m \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)}^s \leq C_s \sum_{m=1}^M k_m (\|\Delta u_{k,m}\|_{L^p(\Omega)}^s + \|f_m\|_{L^p(\Omega)}^s) \leq C_s \left( \ln \frac{T}{k} \right)^s \|f\|_{L^s(I;L^p(\Omega))}^s,$$

where the constant  $C_s$  depends on  $s$ . By taking the  $s$ -root we obtain the corollary.

#### 4 Estimates for dG( $q$ )

In this section we will establish the dG( $q$ ) version of the results from the previous section. It is convenient to introduce some additional notation. Let  $q \geq 1$  and  $\psi_l(t) \in P_q([0, 1])$ ,  $l = 0, 1, \dots, q$  be the standard Lagrange basis functions on the interval  $[0, 1]$ , i.e.,  $\psi_l \left( \frac{j}{q} \right) = \delta_{lj}$ , where  $\delta_{lj}$  is the Kronecker symbol. Then for any  $u_k \in X_k^q$  on the time interval  $I_m = (t_{m-1}, t_m]$  we have

$$u_k|_{I_m} = \sum_{l=0}^q U_l^m(x) \psi_l \left( \frac{t - t_{m-1}}{k_m} \right), \quad (22)$$

with  $U_l^m \in H_0^1(\Omega)$  independent of  $t$ . In this notation, we have

$$u_{k,m}^+ = U_0^{m+1} \quad \text{and} \quad u_{k,m}^- = U_q^m.$$

#### 4.1 Results for the homogeneous problem

Let  $u_k \in X_k^q$  be the semidiscrete in time solution to the parabolic equation with  $f \equiv 0$ , i.e.,

$$B(u_k, \varphi_k) = (u_0, \varphi_{k,0}^+), \quad \forall \varphi_k \in X_k^q. \quad (23)$$

Alternatively, on a single interval  $I_m$ , we have

$$\begin{aligned} U_l^1 &= r_{l,0}(-k_1 \Delta) u_0, \quad l = 0, 1, \dots, q, \\ U_l^m &= r_{l,0}(-k_m \Delta) U_q^{m-1}, \quad l = 0, 1, \dots, q, \quad m = 2, 3, \dots, M, \end{aligned} \quad (24)$$

where the rational functions  $r_{l,0}$  are of the form,

$$r_{l,0}(\lambda) = \frac{p_{l,0}(\lambda)}{\hat{p}(\lambda)}, \quad l = 0, 1, \dots, q, \quad (25)$$

with  $\hat{p}$  being a polynomial of degree  $q + 1$  with no roots on the right-half complex plane and  $p_{l,0}$ ,  $l = 0, 1, \dots, q$  being polynomials of degree  $q$  (cf. [12], page 1322). Since  $r_{q,0}(\lambda)$  is a subdiagonal Padé approximation of  $e^{-\lambda}$ , we also have (cf. [9])

$$r_{q,0}(0) = p_{q,0}(0) = \hat{p}(0) = 1 \quad \text{and} \quad |r_{q,0}(\lambda) - e^{-\lambda}| = O(|\lambda|^{2q+2}), \quad (26)$$

as  $\lambda \rightarrow 0$ . The rational functions  $r_{l,0}$  satisfy the following properties, which we will often use

$$r_{l,0}(0) = 1, \quad \text{and} \quad r_{l,0}(\lambda) - 1 = \frac{\lambda \tilde{p}_l(\lambda)}{\hat{p}(\lambda)}, \quad l = 0, 1, \dots, q, \quad (27)$$

where  $\tilde{p}_l(\lambda)$  are some polynomials of degree  $q$ . The first property follows, for example, by considering the homogeneous Neumann problem with initial condition  $u_0 = 1$ . Then the exact solution  $u$  and the dG( $q$ ) solution  $u_k$  are the same and equal to 1, i.e.,  $u = u_k = 1$ . Hence, all nodal values  $U_l^m = 1$  for all  $m = 1, 2, \dots, M$  and  $l = 0, 1, \dots, q$ . For example for  $m = 1$ , we have

$$1 = U_l^1 = r_{l,0}(-k_1 \Delta) u_0 = r_{l,0}(-k_1 \Delta) 1 = r_{l,0}(0),$$

and as a result  $r_{l,0}(0) = 1$ . The second property in (27) is just a consequence of the first one.

*Remark 3* The dG(1) solution  $u_k$  on each subinterval  $I_m$  is of the form

$$U_0^m \left( \frac{t_m - t}{k_m} \right) + U_1^m \left( \frac{t - t_{m-1}}{k_m} \right)$$

and the rational functions are  $\hat{p}(\lambda) = 1 + \frac{2}{3}\lambda + \frac{\lambda^2}{6}$ ,  $r_{0,0}(\lambda) = 1 + \frac{2}{3}\lambda$ , and  $r_{1,0}(\lambda) = 1 - \frac{\lambda}{3}$ .

For later proof we require two supplementary results.

**Lemma 2** *Let the rational function  $r(z)$  be of the form  $r(z) = \frac{p(z)}{\hat{p}(z)}$ , where  $\hat{p}(z)$  is a polynomial of degree  $q + 1$  with no roots on the right half complex plane and  $p(z)$  is a polynomial of degree  $q$ , for some  $q \geq 0$ . Then, there exists a constant  $C$  independent of  $k > 0$ , such that for any  $g \in L^p(\Omega)$*

$$\|r(-k\Delta)g\|_{L^p(\Omega)} \leq C \|g\|_{L^p(\Omega)}. \quad (28)$$

*Proof* For simplicity we assume that the roots  $z_1, z_2, \dots, z_q$  of  $\hat{p}$  are pairwise distinct. If it is not the case, the argument can be slightly modified. For  $q = 0$  we have  $r(z) = \frac{c_0}{z - z_0}$  and the desired estimate follows directly by the resolvent estimate (8), since

$$r(-k\Delta)g = -\frac{c_0}{k} \left( \frac{z_0}{k} + \Delta \right)^{-1} g$$

and therefore by (8)

$$\|r(-k\Delta)g\|_{L^p(\Omega)} \leq \frac{|c_0|}{k} \frac{C}{1 + \frac{|z_0|}{k}} \|g\|_{L^p(\Omega)} \leq \frac{C|c_0|}{|z_0|} \|g\|_{L^p(\Omega)}.$$



For  $q > 0$  we use the partial fraction decomposition

$$r(z) = \sum_{i=0}^q \frac{c_i}{z - z_i}$$

with some  $c_i \in \mathbb{C}$ . Applying the estimate for  $q_0$  to each summand we obtain

$$\|r(-k\Delta)g\|_{L^p(\Omega)} \leq C \left( \sum_{i=0}^q \frac{|c_i|}{|z_i|} \right) \|g\|_{L^p(\Omega)},$$

which completes the proof.

**Lemma 3** *Let the rational function  $r(z)$  be of the form  $r(z) = \frac{z\hat{p}(z)}{\hat{p}(z)}$ , where  $\hat{p}(z)$  is a polynomial of degree  $q + 1$  with no roots on the right-half complex plane and  $p(z)$  is a polynomial of degree  $q$ , for some  $q \geq 0$ . Then for any  $g \in L^p(\Omega)$  with  $\Delta g \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ , there exists a constant  $C$  independent of  $k$  such that*

$$\|r(-k\Delta)g\|_{L^p(\Omega)} \leq Ck\|\Delta g\|_{L^p(\Omega)}.$$

*Proof* This lemma is just a consequence of the previous one. We set  $\tilde{r}(z) = \frac{p(z)}{\hat{p}(z)}$  and obtain:

$$r(-k\Delta)g = -k\Delta\tilde{r}(-k\Delta)g = -k\tilde{r}(-k\Delta)\Delta g.$$

The result follows by Lemma 2.

**Lemma 4** *Let the rational function  $r(z)$  be of the form  $r(z) = \frac{z\hat{p}(z)}{\hat{p}(z)}$ , where  $\hat{p}(z)$  is a polynomial of degree  $q + 1$  with no roots on the right half complex plane and  $p(z)$  is a polynomial of degree  $q$ , for some  $q \geq 1$ . Then, there exists a constant  $C$  independent of  $k$ , such that for any  $g \in L^p(\Omega)$*

$$\|r(-k\Delta)g\|_{L^p(\Omega)} \leq C\|g\|_{L^p(\Omega)}. \quad (29)$$

*Proof* We set  $\tilde{r}(z) = \frac{p(z)}{\hat{p}(z)}$  and obtain:

$$\|r(-k\Delta)g\|_{L^p(\Omega)} \leq k\|\Delta\tilde{r}(-k\Delta)g\|_{L^p(\Omega)}.$$

The estimate

$$\|\Delta\tilde{r}(-k\Delta)g\|_{L^p(\Omega)} \leq \frac{C}{k}\|g\|_{L^p(\Omega)}$$

is provided on the top of page 1322 in [12] using a decomposition  $r(z) = r_1(z) + r_2(z)$ , where  $r_1(z) = \frac{c}{z - z_0}$ , with  $z_0$  being a root of  $\hat{p}(z)$  and  $c$  such that the degree of the polynomial in the numerator of  $r_2(z)$  is less or equal  $q - 1$ . Then the estimate for  $\Delta\tilde{r}_1(-k\Delta)g$  follows directly by applying a dG(0) type argument and the term  $\Delta\tilde{r}_2(-k\Delta)g$  is estimated using the Dunford-Taylor formula.

Next we provide some properties of the dG( $q$ ) solutions of the homogeneous problem.

**Lemma 5** *Let  $u_k$  be the solution of (23) with  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then,*

$$\|u_k\|_{L^\infty(I_m; L^p(\Omega))} \leq C\|u_0\|_{L^p(\Omega)}, \quad \forall m = 1, 2, \dots, M.$$

*Proof* The proof is given in [12, Thm. 5.1] for the  $L^2(\Omega)$  norm, but the proof is valid for the  $L^p(\Omega)$  norm as well by using the resolvent estimate (8) with respect to the  $L^p(\Omega)$  norm.

**Theorem 3 (Homogeneous smoothing estimate)** *Let  $u_k$  be the solution of (23) with  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then there exists a constant  $C$  independent of  $k$  such that*

$$\|\Delta u_k\|_{L^\infty(I_m; L^p(\Omega))} \leq \frac{C}{t_m}\|u_0\|_{L^p(\Omega)}, \quad m = 1, 2, \dots, M.$$

*Proof* Again the proof is given in [12, Thm. 5.1] for the  $L^2(\Omega)$  norm, but the proof is valid for the  $L^p(\Omega)$  norm as well by using the resolvent estimate (8) with respect to the  $L^p(\Omega)$  norm.

*Remark 4* Notice that the statement of Theorem 3 is equivalent to

$$\|\Delta U_l^m\|_{L^p(\Omega)} \leq \frac{C}{t_m} \|u_0\|_{L^p(\Omega)}, \quad m = 1, 2, \dots, M, \quad l = 0, 1, \dots, q, \quad (30)$$

which we will use in the following proofs.

*Remark 5* Let  $u_k$  be the solution of (23). Then there exists a constant  $C$  independent of  $k$  such that

$$\|u_{k,m}^-\|_{L^p(\Omega)} + (t_m - t_n) \|\Delta u_{k,m}\|_{L^\infty(I_m; L^p(\Omega))} \leq C \|u_{k,n}^-\|_{L^p(\Omega)}, \quad m > n, \quad n = 1, 2, \dots, M,$$

or in terms of nodal values

$$\|U_q^m\|_{L^p(\Omega)} + (t_m - t_n) \|\Delta U_l^m\|_{L^p(\Omega)} \leq C \|U_q^n\|_{L^p(\Omega)}, \quad m > n, \quad n = 1, 2, \dots, M, \quad l = 0, 1, \dots, q. \quad (31)$$

**Theorem 4 (Homogeneous smoothing estimate for jumps)** *Let  $u_k$  be the solution of (23) with  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then there exists a constant  $C$  independent of  $k$  such that*

$$\left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)} \leq \frac{C}{t_m} \|u_0\|_{L^p(\Omega)}, \quad m = 1, 2, \dots, M,$$

where  $[u_k]_0 = U_0^1 - u_0$ .

*Proof* For  $m > 1$ , using (24), we have

$$[u_k]_{m-1} = U_0^m - U_q^{m-1} = r_{0,0}(-k_m \Delta) U_q^{m-1} - U_q^{m-1} = (r_{0,0}(-k_m \Delta) - \text{Id}) U_q^{m-1}.$$

Using (27) and Lemma 3, we obtain

$$\left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)} \leq C \|\Delta U_q^{m-1}\|_{L^p(\Omega)}.$$

Now by Remark 4 and the assumption on the time mesh (ii), we obtain

$$\|\Delta U_q^{m-1}\|_{L^p(\Omega)} \leq \frac{C}{t_{m-1}} \|u_0\|_{L^p(\Omega)} \leq \frac{C}{t_m} \|u_0\|_{L^p(\Omega)}.$$

That finishes the proof for this case.

For  $m = 1$ , by Lemma 5 we have,

$$\left\| \frac{[u_k]_0}{k_1} \right\|_{L^p(\Omega)} = \frac{1}{k_1} \|U_0^1 - u_0\|_{L^p(\Omega)} \leq \frac{C}{k_1} \|u_0\|_{L^p(\Omega)} = \frac{C}{t_1} \|u_0\|_{L^p(\Omega)}.$$

Similarly, we can obtain the corresponding result for the time derivative.

**Theorem 5 (Homogeneous smoothing estimate for time derivatives)** *Let  $u_k$  be the solution of (23) with  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then there exists a constant  $C$  independent of  $k$  such that*

$$\|\partial_t u_k\|_{L^\infty(I_m; L^p(\Omega))} \leq \frac{C}{t_m} \|u_0\|_{L^p(\Omega)}.$$

*Proof* For  $m > 1$ , using (22) and (24), we have

$$\partial_t u_k|_{I_m} = k_m^{-1} \sum_{l=0}^q U_l^m(x) \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right) = k_m^{-1} \sum_{l=0}^q r_{l,0}(-k_m \Delta) \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right) U_q^{m-1}(x).$$

By the fact that  $\sum_{l=0}^q \psi_l \left( \frac{t - t_{m-1}}{k_m} \right) = 1$  we have  $\sum_{l=0}^q \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right) = 0$ . Using (27), i.e.,  $r_{l,0}(0) = 1$  we obtain

$$\sum_{l=0}^q r_{l,0}(z) \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right) = \frac{z \tilde{p}_t(z)}{\hat{p}(z)},$$

where  $\hat{p}(z)$  is the same polynomial as in (25) and  $\tilde{p}_t(z)$  is some polynomial of degree  $q - 1$  whose coefficients are time dependent, but uniformly bounded on  $I_m$ . Thus again by Lemma 3, we obtain

$$\|\partial_t u_k\|_{L^\infty(I_m; L^p(\Omega))} \leq C \|\Delta U_q^{m-1}\|_{L^p(\Omega)}.$$

Remark 4 and the assumption on the time mesh (ii), finishes the proof for  $m > 1$ .

For  $m = 1$ , by Lemma 5 we have,

$$\|\partial_t u_k\|_{L^\infty(I_1; L^p(\Omega))} \leq C k_1^{-1} \sum_{l=0}^q \|U_l^1\|_{L^p(\Omega)} \|\psi_l'\|_{L^\infty(I_1)} \leq \frac{C}{t_1} \|u_0\|_{L^p(\Omega)}.$$

## 4.2 Results for the inhomogeneous problem

In this section we establish properties of the dG( $q$ ) solution  $u_k \in X_k^q$  to the inhomogeneous parabolic equation with  $u_0 = 0$ , that satisfies,

$$B(u_k, \varphi_k) = (f, \varphi_k), \quad \forall \varphi_k \in X_k^q. \quad (32)$$

Alternatively, on a single time interval  $I_m$ , we have

$$\begin{aligned} U_l^1 &= k_1 \sum_{j=0}^q r_{l,j} (-k_1 \Delta) f_j^1, \quad l = 0, 1, \dots, q, \\ U_l^m &= r_{l,0} (-k_m \Delta) U_q^{m-1} + k_m \sum_{j=0}^q r_{l,j} (-k_m \Delta) f_j^m, \quad l = 0, 1, \dots, q, \quad m = 2, 3, \dots, M, \end{aligned} \quad (33)$$

where

$$f_j^m(\cdot) = \frac{1}{k_m} \int_{I_m} f(t, \cdot) \psi_j \left( \frac{t - t_{m-1}}{k_m} \right) dt,$$

and the rational functions

$$r_{l,j} = \frac{p_{l,j}(\lambda)}{\hat{p}(\lambda)}, \quad l, j = 0, 1, \dots, q, \quad (34)$$

are as in the homogenous case with  $\hat{p}$  being a polynomial of degree  $q + 1$  with no roots on the right half complex plane and  $p_{l,j}$ ,  $l, j = 0, 1, \dots, q$  being polynomials of degree  $q$  (cf. [12], page 1322).

Notice that for  $m = 1, 2, \dots, M$ ,

$$\|f_j^m\|_{L^p(\Omega)} \leq C \|f\|_{L^\infty(I_m; L^p(\Omega))} \quad \text{and} \quad \|f_j^m\|_{L^p(\Omega)} \leq C k_m^{-1} \|f\|_{L^1(I_m; L^p(\Omega))}. \quad (35)$$

**Theorem 6 (Maximal parabolic regularity)** *Let  $u_k$  satisfy (32) with  $f \in L^s(I; L^p(\Omega))$  for  $1 \leq s, p \leq \infty$ . There exists a constant  $C$  independent of  $k$  and  $f$  such that*

$$\|\Delta u_k\|_{L^s(I; L^p(\Omega))} \leq C \ln \frac{T}{k} \|f\|_{L^s(I; L^p(\Omega))}.$$

*Proof* Using (33), we have the following representation

$$U_l^m = k_m G_l^m + r_{l,0} (-k_m \Delta) \sum_{n=1}^{m-1} k_n \left( \prod_{j=1}^{m-n-1} r_{q,0} (-k_{m-j-1} \Delta) \right) G_q^n, \quad (36)$$

where

$$G_l^m = \sum_{j=0}^q r_{l,j} (-k_m \Delta) f_j^m, \quad m = 1, 2, \dots, M.$$

with the usual convention that  $\prod_{j=1}^0$  is an empty product. The proof now follows along the lines of Theorem 2. Taking the Laplacian of both sides we obtain

$$\Delta U_l^m = k_m \Delta G_l^m + \Delta r_{l,0}(-k_m \Delta) \sum_{n=1}^{m-1} k_n \left( \prod_{j=1}^{m-n-1} r_{q,0}(-k_{m-j-1} \Delta) \right) G_q^n,$$

and as a result

$$\|\Delta U_l^m\|_{L^p(\Omega)} \leq \|k_m \Delta G_l^m\|_{L^p(\Omega)} + \left\| \Delta r_{l,0}(-k_m \Delta) \sum_{n=1}^{m-1} k_n \left( \prod_{j=1}^{m-n-1} r_{q,0}(-k_{m-j-1} \Delta) \right) G_q^n \right\|_{L^p(\Omega)}.$$

By Lemma 4, we have

$$\|k_m \Delta G_l^m\|_{L^p(\Omega)} \leq C \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)}, \quad l = 0, 1, \dots, q, \quad (37a)$$

and by Lemma 2 we also have

$$\|G_l^m\|_{L^p(\Omega)} \leq C \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)}, \quad l = 0, 1, \dots, q. \quad (37b)$$

On the other hand by Remark 5 for any  $l = 0, 1, \dots, q$ , since each term in the sum on the right-hand side can be thought of as a homogeneous solution with initial condition  $G_q^n$  at  $t = t_{n-1}$ , we have

$$\left\| \Delta r_{l,0}(-k_m \Delta) \sum_{n=1}^{m-1} k_n \left( \prod_{j=1}^{m-n-1} r_{q,0}(-k_{m-j-1} \Delta) \right) G_q^n \right\|_{L^p(\Omega)} \leq C \sum_{n=1}^{m-1} \frac{k_n}{t_m - t_{n-1}} \|G_q^n\|_{L^p(\Omega)}. \quad (38)$$

To establish the result for  $s = \infty$ , we observe

$$\begin{aligned} \|\Delta u_k\|_{L^\infty(I; L^p(\Omega))} &= \max_{1 \leq m \leq M} \max_{0 \leq l \leq q} \|\Delta U_l^m\|_{L^p(\Omega)} \\ &\leq C \max_{1 \leq m \leq M} \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)} + C \max_{1 \leq m \leq M} \sum_{n=1}^{m-1} \frac{k_n}{t_m - t_{n-1}} \|G_q^n\|_{L^p(\Omega)} \\ &\leq C \max_{1 \leq m \leq M} \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)} \left( 1 + \max_{1 \leq m \leq M} \sum_{n=1}^{m-1} \frac{k_n}{t_m - t_{n-1}} \right) \\ &\leq C \ln \frac{T}{k} \max_{1 \leq m \leq M} \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)}, \end{aligned}$$

where in the last step we used (21). Using (35) we can conclude that for  $s = \infty$

$$\|\Delta u_k\|_{L^\infty(I; L^p(\Omega))} \leq C \ln \frac{T}{k} \max_{1 \leq m \leq M} \|f\|_{L^\infty(I_m; L^p(\Omega))} \leq C \ln \frac{T}{k} \|f\|_{L^\infty(I; L^p(\Omega))}.$$

Similarly, for  $s = 1$ , we have

$$\begin{aligned} \|\Delta u_k\|_{L^1(I; L^p(\Omega))} &\leq \sum_{m=1}^M k_m \max_{0 \leq l \leq q} \|\Delta U_l^m\|_{L^p(\Omega)} \\ &\leq C \sum_{m=1}^M k_m \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)} + C \sum_{m=1}^M k_m \sum_{n=1}^{m-1} \frac{k_n}{t_m - t_{n-1}} \|G_q^n\|_{L^p(\Omega)} \\ &\leq C \sum_{m=1}^M k_m \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)} + C \sum_{m=1}^M k_m \sum_{n=1}^{m-1} \frac{k_n}{t_m - t_{n-1}} \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)} \\ &\leq C \sum_{m=1}^M k_m \sum_{n=1}^m \frac{k_n}{t_m - t_{n-1}} \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)}. \end{aligned}$$

Changing the order of summation and using (21) we obtain,

$$\begin{aligned} \sum_{m=1}^M k_m \sum_{n=1}^m \frac{k_n}{t_m - t_{n-1}} \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)} &\leq \sum_{n=1}^M k_n \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)} \sum_{m=n}^M \frac{k_m}{t_m - t_{n-1}} \\ &\leq C \ln \frac{T}{k} \sum_{n=1}^M k_n \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)}. \end{aligned}$$

Thus, by using (35), we have

$$\|\Delta u_k\|_{L^1(I; L^p(\Omega))} \leq C \ln \frac{T}{k} \sum_{m=1}^M k_m \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)} \leq C \ln \frac{T}{k} \|f\|_{L^1(I; L^p(\Omega))}.$$

Interpolating between  $s = 1$  and  $s = \infty$  we obtain the result for any  $1 \leq s \leq \infty$ .

*Remark 6* As in the case of dG(0) the appearance of a logarithmic term is natural, since in contrast to the continuous case the choices  $s, p \in \{1, \infty\}$  are allowed. The power of the logarithm can be improved for  $p = 2$  or  $s = 2$ . In fact, we can obtain the following estimates,

$$\|\Delta u_k\|_{L^s(I; L^2(\Omega))} \leq C \left( \ln \frac{T}{k} \right)^{\frac{|s-2|}{s}} \|f\|_{L^s(I; L^2(\Omega))},$$

and

$$\|\Delta u_k\|_{L^2(I; L^p(\Omega))} \leq C \left( \ln \frac{T}{k} \right)^{\frac{|p-2|}{p}} \|f\|_{L^2(I; L^p(\Omega))}.$$

**Theorem 7 (Maximal parabolic regularity for jumps)** *Let  $u_k$  satisfy (32) with  $f \in L^s(I; L^p(\Omega))$  for  $1 \leq s, p \leq \infty$ . Then there exists a constant  $C$  independent of  $k$  and  $f$  such that*

$$\begin{aligned} \max_{1 \leq m \leq M} \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)} &\leq C \ln \frac{T}{k} \|f\|_{L^\infty(I; L^p(\Omega))}, \quad \text{for } s = \infty, \\ \left( \sum_{m=1}^M k_m \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)}^s \right)^{\frac{1}{s}} &\leq C \ln \frac{T}{k} \|f\|_{L^s(I; L^p(\Omega))}, \quad \text{for } 1 \leq s < \infty. \end{aligned}$$

*Proof* Using (33) and (36), we have the following representation for the jump terms

$$\begin{aligned} \frac{[u_k]_{m-1}}{k_m} &= \frac{U_0^m - U_q^{m-1}}{k_m} \\ &= G_0^m + k_m^{-1} (r_{0,0}(-k_m \Delta) U_q^{m-1} - U_q^{m-1}) = G_0^m + k_m^{-1} (r_{0,0}(-k_m \Delta) - \text{Id}) U_q^{m-1}. \end{aligned}$$

Using that  $r_{0,0} - 1$  satisfies (27) and using Lemma 3, Lemma 2, and proceeding similarly to the proof of Theorem 6, we have

$$\begin{aligned} k_m^{-1} \|[u_k]_{m-1}\|_{L^p(\Omega)} &\leq C (\|G_0^m\|_{L^p(\Omega)} + \|\Delta U_q^{m-1}\|_{L^p(\Omega)}) \\ &\leq C \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)} + C \sum_{n=1}^{m-1} \frac{k_n}{t_m - t_{n-1}} \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)} \\ &\leq C \sum_{n=1}^m \frac{k_n}{t_m - t_{n-1}} \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)}. \end{aligned} \tag{39}$$

Now, the proof of the cases  $s = 1$  and  $s = \infty$  is identical to the one of the previous Theorem 6 and we have

$$\begin{aligned} \max_{1 \leq m \leq M} \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)} &\leq C \ln \frac{T}{k} \|f\|_{L^\infty(I; L^p(\Omega))}, \quad 1 \leq p \leq \infty, \\ \sum_{m=1}^M k_m \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)} &\leq C \ln \frac{T}{k} \|f\|_{L^1(I; L^p(\Omega))}, \quad 1 \leq p \leq \infty. \end{aligned}$$

For  $1 < s < \infty$  using the Hölder inequality with  $\frac{1}{s} + \frac{1}{s'} = 1$ , we obtain,

$$\begin{aligned} \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)} &\leq C \sum_{n=1}^m \frac{k_n}{t_m - t_{n-1}} \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)} \\ &\leq C \left( \sum_{n=1}^m \frac{k_n}{t_m - t_{n-1}} \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)}^s \right)^{1/s} \left( \sum_{n=1}^m \frac{k_n}{t_m - t_{n-1}} \right)^{1/s'} \\ &\leq C \left( \ln \frac{T}{k} \right)^{1/s'} \left( \sum_{n=1}^m \frac{k_n}{t_m - t_{n-1}} \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)}^s \right)^{1/s}. \end{aligned} \quad (40)$$

Hence

$$\sum_{m=1}^M k_m \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)}^s \leq C \left( \ln \frac{T}{k} \right)^{s/s'} \sum_{m=1}^M k_m \sum_{n=1}^m \frac{k_n}{t_m - t_{n-1}} \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)}^s.$$

Changing the order of summation, we obtain

$$\begin{aligned} \sum_{m=1}^M k_m \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^p(\Omega)}^s &\leq C \left( \ln \frac{T}{k} \right)^{s/s'} \sum_{n=1}^M k_n \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)}^s \sum_{m=n}^M \frac{k_m}{t_m - t_{n-1}} \\ &\leq C \left( \ln \frac{T}{k} \right)^{1+s/s'} \sum_{n=1}^M k_n \max_{0 \leq j \leq q} \|f_j^n\|_{L^p(\Omega)}^s = C \left( \ln \frac{T}{k} \right)^s \|f\|_{L^s(I; L^p(\Omega))}^s. \end{aligned}$$

Taking the  $s$ -root we finish the proof.

**Theorem 8** *Let  $u_k$  satisfy (32). Then there exists a constant  $C$  independent of  $k$  and  $f$  such that*

$$\left( \sum_{m=1}^M \|\partial_t u_k\|_{L^s(I_m; L^p(\Omega))}^s \right)^{\frac{1}{s}} \leq C \ln \frac{T}{k} \|f\|_{L^s(I; L^p(\Omega))}, \quad 1 \leq s < \infty, \quad 1 \leq p \leq \infty.$$

*Proof* Similarly to the proof of Theorem 4, using (22) and (33), we have

$$\begin{aligned} \partial_t u_k|_{I_m} &= k_m^{-1} \sum_{l=0}^q U_l^m(x) \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right) + \sum_{l=0}^q G_l^m(x) \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right) \\ &= k_m^{-1} \sum_{l=0}^q r_{l,0}(-k_m \Delta) \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right) U_q^{m-1}(x) + \sum_{l=0}^q G_l^m(x) \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right). \end{aligned}$$

Using (27) and  $\sum_{l=0}^q \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right) = 0$ , we can conclude that

$$\sum_{l=0}^q r_{l,0}(z) \psi_l' \left( \frac{t - t_{m-1}}{k_m} \right) = \frac{z \tilde{p}_t(z)}{\hat{p}(z)},$$

where  $\hat{p}(z)$  is the same polynomial as in (25) and  $\tilde{p}_t(z)$  is some polynomial of degree  $q$  whose coefficients are time dependent, but uniformly bounded on  $I_m$ . Thus again by Lemma 3 and Lemma 4, we obtain

$$\|\partial_t u_k\|_{L^\infty(I_m; L^p(\Omega))} \leq C \|\Delta U_q^{m-1}\|_{L^p(\Omega)} + C \max_{0 \leq j \leq q} \|f_j^m\|_{L^p(\Omega)}.$$

The rest of the proof is identical to the proof of the previous theorem.

### 4.3 Application to optimal order error estimates.

As an application of the maximal parabolic regularity, we show optimal convergence rates for the  $dG(q)$  solution. First, we establish that the error is bounded by a certain projection error. A similar result was obtained for the  $L^2(I; L^2(\Omega))$  norm in [32]. First, we define a projection  $\pi_k$  for  $u \in C(I, L^2(\Omega))$  with  $\pi_k u|_{I_m} \in P_q(L^2(\Omega))$  for  $m = 1, 2, \dots, M$  on each subinterval  $I_m$  by

$$(\pi_k u - u, \phi)_{I_m \times \Omega} = 0, \quad \forall \phi \in P_{q-1}(I_m, L^2(\Omega)), \quad q > 0, \quad (41a)$$

$$\pi_k u(t_m^-) = u(t_m^-). \quad (41b)$$

In the case  $q = 0$ ,  $\pi_k u$  is defined solely by the second condition.

**Theorem 9** *Let  $u$  be the solution to (1) with  $u \in C(\bar{I}; L^p(\Omega))$  and  $u_k$  be its  $dG(q)$  approximation (6), for  $q \geq 0$ . Then there exists a constant  $C$  independent of  $k$  such that*

$$\|u - u_k\|_{L^s(I; L^p(\Omega))} \leq C \ln \frac{T}{k} \|u - \pi_k u\|_{L^s(I; L^p(\Omega))}, \quad 1 \leq s, p < \infty,$$

where the projection  $\pi_k$  is defined above in (41).

*Proof* Put  $e := u - u_k = (u - \pi_k u) + (\pi_k u - u_k) := \eta_k + \xi_k$ . For  $1 \leq s, p < \infty$ , we have

$$\|e\|_{L^s(I; L^p(\Omega))} = \sup_{\substack{\psi \in L^{s'}(I; L^{p'}(\Omega)) \\ \|\psi\|_{L^{s'}(I; L^{p'}(\Omega))} = 1}} (e, \psi)_{I \times \Omega}, \quad \frac{1}{s} + \frac{1}{s'} = 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

For each such  $\psi$ , we consider a dual problem for  $z_k \in X_k^q$  satisfying

$$B(\varphi_k, z_k) = (\varphi_k, \psi)_{I \times \Omega} \quad \text{for all } \varphi_k \in X_k^q.$$

Thus, we have

$$(e, \psi)_{I \times \Omega} = (\eta_k, \psi)_{I \times \Omega} + (\xi_k, \psi)_{I \times \Omega} := J_1 + J_2.$$

Using the Hölder inequality, we find

$$J_1 \leq \|\eta_k\|_{L^s(I; L^p(\Omega))} \|\psi\|_{L^{s'}(I; L^{p'}(\Omega))} \leq \|\eta_k\|_{L^s(I; L^p(\Omega))}.$$

On the other hand using that  $B(u - u_k, \chi_k) = 0$  for any  $\chi_k \in X_k^q$ , we obtain

$$\begin{aligned} J_2 = B(\xi_k, z_k) &= -B(\eta_k, z_k) = -\sum_{m=1}^M (\eta_k, \partial_t z_k)_{I_m \times \Omega} - (\nabla \eta_k, \nabla z_k)_{I_m \times \Omega} + (\eta_{k,m}^-, [z_k]_m)_{I_m} \\ &= -(\nabla \eta_k, \nabla z_k)_{I \times \Omega}, \end{aligned}$$

where we used that the first sum vanishes due to (41a) and the sum involving jumps due to (41b). Integrating by parts in space, using the Hölder inequality and Theorem 6, we obtain

$$\begin{aligned} J_2 &= -(\nabla \eta_k, \nabla z_k)_{I \times \Omega} = (\eta_k, \Delta z_k)_{I \times \Omega} \leq \|\eta_k\|_{L^s(I; L^p(\Omega))} \|\Delta z_k\|_{L^{s'}(I; L^{p'}(\Omega))} \\ &\leq C \ln \frac{T}{k} \|\eta_k\|_{L^s(I; L^p(\Omega))} \|\psi\|_{L^{s'}(I; L^{p'}(\Omega))} \leq C \ln \frac{T}{k} \|\eta_k\|_{L^s(I; L^p(\Omega))}. \end{aligned}$$

Combining the estimates for  $J_1$  and  $J_2$  we obtain the result.

If the exact solution is sufficiently smooth then the above result easily leads to an optimal convergence rate, modulo a logarithmic term.

**Corollary 3** *Let  $u \in W^{q+1, s}(I; L^p(\Omega))$  be the solution to (1) and  $u_k$  be its  $dG(q)$  approximation for  $q \geq 0$ . Then there exists a constant  $C$  independent of  $k$  such that*

$$\|u - u_k\|_{L^s(I; L^p(\Omega))} \leq C k^{q+1} \ln \frac{T}{k} \|u\|_{W^{q+1, s}(I; L^p(\Omega))}, \quad 1 \leq s, p < \infty.$$

*Remark 7* The above result can be extended to the case of non-homogeneous Dirichlet boundary conditions. Let  $g \in C(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega))$  be given and consider the equation

$$\begin{aligned} \partial_t u(t, x) - \Delta u(t, x) &= f(t, x), & (t, x) \in I \times \Omega, \\ u(t, x) &= g(t, x), & (t, x) \in I \times \partial\Omega, \\ u(0, x) &= u_0(x), & x \in \Omega. \end{aligned}$$

It turns out, that it is convenient to use  $\pi_k g$  as boundary conditions for the semidiscrete solution, i.e.

$$u_k \in \pi_k g + X_k^q \quad : \quad B(u_k, \varphi_k) = (f, \varphi_k)_{I \times \Omega} + (u_0, \varphi_{k,0}^+)_{\Omega} \quad \text{for all } \varphi_k \in X_k^q.$$

Then following the lines of the proof of Theorem 9 and using that  $\xi_k = \pi_k u - u_k$  has homogeneous boundary conditions, i.e.,  $\xi_k \in X_k^q$ , we obtain

$$(\xi_k, \psi)_{I \times \Omega} = -(\nabla \eta_k, \nabla z_k) = (\eta_k, \Delta z_k)_{I \times \Omega} + \int_I \int_{\partial\Omega} (g - \pi_k g) \partial_n z_k \, ds \, dt.$$

Under an additional assumption on  $\Omega$  that for any  $v \in H_0^1(\Omega)$  with  $\Delta v \in L^{p'}(\Omega)$  the estimate

$$\|\partial_n v\|_{L^{p'}(\partial\Omega)} \leq c \|\Delta v\|_{L^{p'}(\Omega)}$$

holds, we obtain

$$\|u - u_k\|_{L^s(I; L^p(\Omega))} \leq C \ln \frac{T}{k} (\|u - \pi_k u\|_{L^s(I; L^p(\Omega))} + \|g - \pi_k g\|_{L^s(I; L^p(\partial\Omega))}), \quad 1 \leq s, p < \infty.$$

The above assumption is fulfilled, for example, if on  $\Omega$  the  $W^{2,p'}$  elliptic regularity holds.

## 5 Fully discrete solutions

In this section, we consider the fully discrete approximation of the equation (1). From now on we assume that the domain  $\Omega$  is a polygonal/polyhedral convex domain. For  $h \in (0, h_0]$ ;  $h_0 > 0$ , let  $\mathcal{T}$  denote a quasi-uniform triangulation of  $\Omega$  with mesh size  $h$ , i.e.,  $\mathcal{T} = \{\tau\}$  is a partition of  $\Omega$  into cells (triangles or tetrahedrons)  $\tau$  of diameter  $h_\tau$  such that for  $h = \max_\tau h_\tau$ ,

$$\text{diam}(\tau) \leq h \leq C |\tau|^{\frac{1}{d}}, \quad \forall \tau \in \mathcal{T}, \quad d = 2, 3,$$

hold. Let  $V_h$  be the set of all functions in  $H_0^1(\Omega)$  that are polynomials of degree  $r$  on each  $\tau$ , i.e.,  $V_h$  is the usual space of conforming finite elements. To obtain the fully discrete approximation we consider the space-time finite element space

$$X_{k,h}^{q,r} = \{v_{kh} : v_{kh}|_{I_m} \in \mathcal{P}_q(V_h), \, m = 1, 2, \dots, M, \quad q \geq 0, \quad r \geq 1\}. \quad (42)$$

We define a fully discrete analog  $u_{kh} \in X_{k,h}^{q,r}$  of  $u_k$  introduced in (6) by

$$B(u_{kh}, \varphi_{kh}) = (f, \varphi_{kh})_{I \times \Omega} + (u_0, \varphi_{kh}^+)_{\Omega} \quad \text{for all } \varphi_{kh} \in X_{k,h}^{q,r}. \quad (43)$$

Moreover, we introduce the discrete Laplace operator  $\Delta_h : V_h \rightarrow V_h$  by

$$(-\Delta_h v_h, \chi)_{\Omega} = (\nabla v_h, \nabla \chi)_{\Omega}, \quad \forall \chi \in V_h.$$

The semidiscrete results from the first part of the paper translate almost immediately to the fully discrete setting provided we have the corresponding resolvent estimate,

$$\|(z + \Delta_h)^{-1} \chi\|_{L^p(\Omega)} \leq \frac{C}{1 + |z|} \|\chi\|_{L^p(\Omega)}, \quad \forall z \in \mathbb{C} \setminus \Sigma_\gamma, \quad \forall \chi \in V_h, \quad 1 \leq p \leq \infty, \quad (44)$$

with some constant  $C$  independent of  $h$ . Such a result was established in [3] for smooth domains. Later it was extended to convex polyhedral domains in [30] (for some  $\gamma > 0$ ) via stability and smoothing properties of the semigroup  $E_h(t) = e^{-\Delta_h t}$  and directly for an arbitrary  $\gamma > 0$  but with logarithmic dependence of the constant  $C$  on  $h$  in [26].



### 5.1 Result for the homogeneous problem

Let  $u_{kh} \in X_{k,h}^{q,r}$  be the fully discrete dG(q)cG(r) solution to the parabolic equation with  $f \equiv 0$ , i.e.

$$B(u_{kh}, \varphi_{kh}) = (u_0, \varphi_{kh,0}^+), \quad \forall \varphi_{kh} \in X_{k,h}^{q,r}. \quad (45)$$

**Theorem 10 (Fully discrete homogeneous smoothing estimate)** *Let  $u_{kh}$  be a solution of (45) with  $u_0 \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . Then there exists a constant  $C$  independent of  $k$  and  $h$  such that*

$$\|\partial_t u_{kh}\|_{L^\infty(I_m; L^p(\Omega))} + \|\Delta_h u_{kh}\|_{L^\infty(I_m; L^p(\Omega))} + k_m^{-1} \|[u_{kh}]_{m-1}\|_{L^p(\Omega)} \leq \frac{C}{t_m} \|u_0\|_{L^p(\Omega)},$$

for  $m = 1, 2, \dots, M$ .

### 5.2 Results for the inhomogeneous problem

Let  $u_{kh} \in X_{k,h}^{q,r}$  be the dG(q)cG(r) solution to the inhomogeneous parabolic equation with  $u_0 = 0$ , i.e.

$$B(u_{kh}, \varphi_{kh}) = (f, \varphi_{kh}), \quad \forall \varphi_{kh} \in X_{k,h}^{q,r}. \quad (46)$$

**Theorem 11 (Fully discrete maximal parabolic regularity)** *Let  $u_{kh}$  satisfy (46) with  $f \in L^s(I; L^p(\Omega))$ ,  $1 \leq s, p \leq \infty$ . Then there exists a constant  $C$  independent of  $k$  and  $h$  such that*

$$\left( \sum_{m=1}^M \|\partial_t u_{kh}\|_{L^s(I_m; L^p(\Omega))}^s \right)^{\frac{1}{s}} + \|\Delta_h u_{kh}\|_{L^s(I; L^p(\Omega))} + \left( \sum_{m=1}^M k_m \left\| \frac{[u_{kh}]_{m-1}}{k_m} \right\|_{L^p(\Omega)}^s \right)^{\frac{1}{s}} \leq C \ln \frac{T}{k} \|f\|_{L^s(I; L^p(\Omega))},$$

with obvious notation changes in the case of  $s = \infty$ .

### 5.3 Application to optimal order error estimates.

Similarly to the semidiscrete case, as an application of the maximal parabolic regularity, we show optimal convergence rates for the dG(q)cG(r) solution.

**Theorem 12** *Let  $u$  be the solution to (1) with  $u \in C(\bar{I}; L^p(\Omega))$  and  $u_{kh}$  be the dG(q)cG(r) solution for  $q \geq 0$  and  $r \geq 1$ . Then there exists a constant  $C$  independent of  $k$  and  $h$  such that for  $1 \leq s, p < \infty$ ,*

$$\|u - u_{kh}\|_{L^s(I; L^p(\Omega))} \leq C \ln \frac{T}{k} \left( \|u - \pi_k u\|_{L^s(I; L^p(\Omega))} + \|P_h u - u\|_{L^s(I; L^p(\Omega))} + \|R_h u - u\|_{L^s(I; L^p(\Omega))} \right),$$

where the projection  $\pi_k$  is defined in (41),  $P_h: L^2(\Omega) \rightarrow V_h$  is the orthogonal  $L^2$  projection and  $R_h: H_0^1(\Omega) \rightarrow V_h$  is the Ritz projection.

*Proof* Put  $e := u - u_{kh} = (u - P_h \pi_k u) + (P_h \pi_k u - u_{kh}) := \eta_{kh} + \xi_{kh}$ . For  $1 \leq s, p < \infty$ , we have

$$\|e\|_{L^s(I; L^p(\Omega))} = \sup_{\substack{\psi \in L^{s'}(I; L^{p'}(\Omega)) \\ \|\psi\|_{L^{s'}(I; L^{p'}(\Omega))} = 1}} (e, \psi)_{I \times \Omega}, \quad \frac{1}{s} + \frac{1}{s'} = 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

For each such  $\psi$ , consider a dual problem

$$B(\varphi_{kh}, z_{kh}) = (\varphi_{kh}, \psi)_{I \times \Omega}.$$

Thus, we have

$$(e, \psi)_{I \times \Omega} = (\eta_{kh}, \psi)_{I \times \Omega} + (\xi_{kh}, \psi)_{I \times \Omega} := J_1 + J_2.$$

Using the Hölder inequality, the triangle inequality, the stability of the  $L^2$  projection  $P_h$  in  $L^p(\Omega)$  and the approximation properties of  $\pi_k$  and  $P_h$ , we find

$$\begin{aligned} J_1 &\leq C \|\eta_{kh}\|_{L^s(I;L^p(\Omega))} \|\psi\|_{L^{s'}(I;L^{p'}(\Omega))} \leq C \|\eta_{kh}\|_{L^s(I;L^p(\Omega))} = C \|u - P_h \pi_k u\|_{L^s(I;L^p(\Omega))} \\ &\leq C \|u - P_h u\|_{L^s(I;L^p(\Omega))} + C \|P_h(u - \pi_k u)\|_{L^s(I;L^p(\Omega))} \\ &\leq C \|u - P_h u\|_{L^s(I;L^p(\Omega))} + C \|u - \pi_k u\|_{L^s(I;L^p(\Omega))}. \end{aligned}$$

On the other hand, using that  $B(u - u_{kh}, \chi_{kh}) = 0$  for any  $\chi_{kh} \in X_{k,h}^{q,r}$ , and the properties of the  $L^2$  projection and the properties of  $\pi_k$ , we obtain

$$\begin{aligned} J_2 &= B(\xi_{kh}, z_{kh}) = -B(\eta_{kh}, z_{kh}) = \sum_{m=1}^M (\eta_{kh}, \partial_t z_{kh})_{I_m \times \Omega} - (\nabla \eta_{kh}, \nabla z_{kh})_{I_m \times \Omega} + (\eta_{kh,m}^-, [z_{kh}]_m)_\Omega \\ &= \sum_{m=1}^M (u - \pi_k u, \partial_t z_{kh})_{I_m \times \Omega} - (\nabla \eta_{kh}, \nabla z_{kh})_{I_m \times \Omega} + (u_m^- - (\pi_k u)_m^-, [z_{kh}]_m)_\Omega \\ &= -(\nabla(u - P_h \pi_k u), \nabla z_{kh})_{I \times \Omega}. \end{aligned}$$

where we used that the first sum vanishes due to (41a) and the sum involving jumps due to (41b). Using the properties of the Ritz projection, integrating by parts in space, and using the Hölder inequality and Theorem 6, we obtain

$$\begin{aligned} J_2 &= -(\nabla(u - P_h \pi_k u), \nabla z_{kh})_{I \times \Omega} = -(\nabla(R_h u - P_h \pi_k u), \nabla z_{kh})_{I \times \Omega} = (R_h u - P_h \pi_k u, \Delta_h z_{kh})_{I \times \Omega} \\ &\leq C \|P_h(R_h u - \pi_k u)\|_{L^s(I;L^p(\Omega))} \|\Delta_h z_{kh}\|_{L^{s'}(I;L^{p'}(\Omega))} \\ &\leq C \ln \frac{T}{k} \|R_h u - \pi_k u\|_{L^s(I;L^p(\Omega))} \|\psi\|_{L^{s'}(I;L^{p'}(\Omega))} \\ &\leq C \ln \frac{T}{k} (\|R_h u - u\|_{L^s(I;L^p(\Omega))} + \|u - \pi_k u\|_{L^s(I;L^p(\Omega))}). \end{aligned}$$

Combining the estimates for  $J_1$  and  $J_2$  we obtain the result.

**Corollary 4** *If the solution  $u$  to (1) satisfies  $u \in W^{q+1,s}(I;L^p(\Omega)) \cap L^s(I;W^{r+1,p}(\Omega))$  and  $\Omega$  such that elliptic  $W^{2,p'}$ -regularity holds, then there exists a constant  $C$  independent of  $k$  and  $h$  such that*

$$\|u - u_{kh}\|_{L^s(I;L^p(\Omega))} \leq C \ln \frac{T}{k} (k^{q+1} \|u\|_{W^{q+1,s}(I;L^p(\Omega))} + h^{r+1} \|u\|_{L^s(I;W^{r+1,p}(\Omega))}), \quad 1 \leq s, p < \infty.$$

## 6 Fully discrete results in general norms

For the future references we provide discrete maximal parabolic regularity results in general norms. For example, we use these results to establish pointwise best approximation estimates in [27] for fully discrete Galerkin solutions.

Let  $\Omega$  be a Lipschitz domain and let  $\mathcal{T} = \{\tau\}$  be an arbitrary partition of  $\Omega$  into cells  $\tau$  (triangles, tetrahedrons, quads, or hexahedrons, not necessary quasi-uniform). Let  $V_h$  be the set of all functions in  $H_0^1(\Omega)$  that belong to a certain polynomial space (i.e.,  $P_r$  or  $Q_r$ ) on each  $\tau$ . As before, we define a fully discrete solution  $u_{kh} \in X_{k,h}^{q,r}$  by

$$B(u_{kh}, \varphi_{kh}) = (f, \varphi_{kh})_{I \times \Omega} + (u_0, \varphi_{kh}^+)_{\Omega} \quad \text{for all } \varphi_{kh} \in X_{k,h}^{q,r}, \quad (47)$$

where

$$X_{k,h}^{q,r} = \{v_{kh} : v_{kh}|_{I_m} \in \mathcal{P}_q(V_h), m = 1, 2, \dots, M\}, \quad \text{for some } q \geq 0, \quad r \geq 1. \quad (48)$$

As in the previous section, we introduce the discrete Laplace operator  $\Delta_h : V_h \rightarrow V_h$  by

$$(-\Delta_h v_h, \chi)_\Omega = (\nabla v_h, \nabla \chi)_\Omega, \quad \forall \chi \in V_h,$$

and the orthogonal  $L^2$  projection  $P_h : L^2(\Omega) \rightarrow V_h$  by

$$(P_h v, \chi)_\Omega = (v, \chi)_\Omega, \quad \forall \chi \in V_h.$$

Let  $\|\cdot\|$  be a norm on  $V_h$  such that for some  $\gamma \in (0, \frac{\pi}{2})$  the following resolvent estimate holds,

$$\|(z + \Delta_h)^{-1}\chi\| \leq \frac{M_h}{|z|} \|\chi\|, \quad \text{for } z \in \mathbb{C} \setminus \Sigma_\gamma, \quad (49)$$

for all  $\chi \in V_h$ , where  $\Sigma_\gamma$  is defined in (9) and the constant  $M_h$  is independent of  $z$ .

For quasi-uniform meshes, this assumption is fulfilled for  $\|\cdot\| = \|\cdot\|_{L^p(\Omega)}$  with a constant  $M_h \leq C$  independent of  $h$ , see [30], as discussed and exploited above. For a weighted norm  $\|\cdot\| = \|\sigma^{\frac{N}{2}} \cdot\|_{L^2(\Omega)}$  with the weight  $\sigma_{x_0}(x) = \sqrt{|x - x_0|^2 + h^2}$  and  $M_h \leq C|\ln h|$  we established this estimate in [27], and used the corresponding result to obtain interior (local) pointwise estimates. Moreover, the resolvent estimate (49) is known also to hold in  $L^p(\Omega)$  norms on a class of non quasi-uniform meshes as well, see [2].

### 6.1 Smoothing estimates for the homogeneous problem in general norms

For the homogeneous heat equation (1), i.e.  $f = 0$  and its discrete approximation  $u_{kh} \in X_{k,h}^{q,r}$  defined by

$$B(u_{kh}, \varphi_{kh}) = (u_0, \varphi_{kh,0}^+) \quad \forall \varphi_{kh} \in X_{k,h}^{q,r}, \quad (50)$$

we have the following smoothing result.

**Theorem 13 (Fully discrete smoothing estimate in general norms)** *Let  $\|\cdot\|$  be a norm on  $V_h$  fulfilling the resolvent estimate (49). Let  $u_{kh}$  be the solution of (50). Then, there exists a constant  $C$  independent of  $k$  and  $h$  such that*

$$\sup_{t \in I_m} \|\partial_t u_{kh}(t)\| + \sup_{t \in I_m} \|\Delta_h u_{kh}(t)\| + k_m^{-1} \|[u_{kh}]_{m-1}\| \leq \frac{CM_h}{t_m} \|P_h u_0\|,$$

for  $m = 1, 2, \dots, M$ , where  $P_h: L^2(\Omega) \rightarrow V_h$  is the orthogonal  $L^2$  projection. For  $m = 1$  the jump term is understood as  $[u_{kh}]_0 = u_{kh,0}^+ - P_h u_0$ .

### 6.2 Discrete maximal parabolic estimates for the inhomogeneous problem in general norms

Now, we consider the inhomogeneous heat equation (1), with  $u_0 = 0$  and its discrete approximation  $u_{kh} \in X_{k,h}^{q,r}$  defined by

$$B(u_{kh}, \varphi_{kh}) = (f, \varphi_{kh}), \quad \forall \varphi_{kh} \in X_{k,h}^{q,r}. \quad (51)$$

**Theorem 14 (Discrete maximal parabolic regularity in general norms)** *Let  $\|\cdot\|$  be a norm on  $V_h$  fulfilling the resolvent estimate (49) and let  $1 \leq s \leq \infty$ . Let  $u_{kh}$  be a solution of (51). Then, there exists a constant  $C$  independent of  $k$  and  $h$  such that*

$$\begin{aligned} \left( \sum_{m=1}^M \int_{I_m} \|\partial_t u_{kh}(t)\|^s dt \right)^{\frac{1}{s}} + \left( \sum_{m=1}^M \int_{I_m} \|\Delta_h u_{kh}(t)\|^s dt \right)^{\frac{1}{s}} + \left( \sum_{m=1}^M k_m \|[u_{kh}]_{m-1}\|^s \right)^{\frac{1}{s}} \\ \leq CM_h \ln \frac{T}{k} \left( \int_I \|P_h f(t)\|^s dt \right)^{\frac{1}{s}}, \end{aligned}$$

where  $P_h: L^2(\Omega) \rightarrow V_h$  is the orthogonal  $L^2$  projection and with obvious notation change in the case of  $s = \infty$ . For  $m = 1$  the jump term is understood as  $[u_{kh}]_0 = u_{kh,0}^+$ .

The proofs of the above two results are identical to the proofs of the corresponding time discrete results from Section 4, provided the resolvent estimate (49) holds.

**Acknowledgements** The authors would like to thank Dominik Meidner and Konstantin Pieper for the careful reading of the manuscript and providing valuable suggestions that help to improve the presentation of the paper.

## References

1. Ashyralyev, A., Sobolevskii, P.E.: Well-posedness of parabolic difference equations, *Operator Theory: Advances and Applications*, vol. 69. Birkhäuser Verlag, Basel (1994). DOI 10.1007/978-3-0348-8518-8. URL <http://dx.doi.org/10.1007/978-3-0348-8518-8>. Translated from the Russian by A. Iacob
2. Bakaev, N.Y., Crouzeix, M., Thomée, V.: Maximum-norm resolvent estimates for elliptic finite element operators on nonquasiuniform triangulations. *M2AN Math. Model. Numer. Anal.* **40**(5), 923–937 (2006). DOI 10.1051/m2an:2006040. URL <http://dx.doi.org/10.1051/m2an:2006040>
3. Bakaev, N.Y., Thomée, V., Wahlbin, L.B.: Maximum-norm estimates for resolvents of elliptic finite element operators. *Math. Comp.* **72**(244), 1597–1610 (electronic) (2003). DOI 10.1090/S0025-5718-02-01488-6. URL <http://dx.doi.org/10.1090/S0025-5718-02-01488-6>
4. Becker, R., Meidner, D., Vexler, B.: Efficient numerical solution of parabolic optimization problems by finite element methods. *Optim. Methods Softw.* **22**(5), 813–833 (2007). DOI 10.1080/10556780701228532. URL <http://dx.doi.org/10.1080/10556780701228532>
5. Blunck, S.: Analyticity and discrete maximal regularity on  $L_p$ -spaces. *J. Funct. Anal.* **183**(1), 211–230 (2001). DOI 10.1006/jfan.2001.3740. URL <http://dx.doi.org/10.1006/jfan.2001.3740>
6. Blunck, S.: Maximal regularity of discrete and continuous time evolution equations. *Studia Math.* **146**(2), 157–176 (2001). DOI 10.4064/sm146-2-3. URL <http://dx.doi.org/10.4064/sm146-2-3>
7. Chrysafinos, K.: Discontinuous Galerkin approximations for distributed optimal control problems constrained by parabolic PDE's. *Int. J. Numer. Anal. Model.* **4**(3-4), 690–712 (2007)
8. Coulhon, T., Duong, X.T.: Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss. *Adv. Differential Equations* **5**(1-3), 343–368 (2000)
9. Egert, M., Rozendaal, J.: Convergence of subdiagonal Padé approximations of  $C_0$ -semigroups. *J. Evol. Equ.* **13**(4), 875–895 (2013). DOI 10.1007/s00028-013-0207-1. URL <http://dx.doi.org/10.1007/s00028-013-0207-1>
10. Eriksson, K., Johnson, C.: Adaptive finite element methods for parabolic problems. I. A linear model problem. *SIAM J. Numer. Anal.* **28**(1), 43–77 (1991). DOI 10.1137/0728003. URL <http://dx.doi.org/10.1137/0728003>
11. Eriksson, K., Johnson, C.: Adaptive finite element methods for parabolic problems. II. Optimal error estimates in  $L_\infty L_2$  and  $L_\infty L_\infty$ . *SIAM J. Numer. Anal.* **32**(3), 706–740 (1995). DOI 10.1137/0732033. URL <http://dx.doi.org/10.1137/0732033>
12. Eriksson, K., Johnson, C., Larsson, S.: Adaptive finite element methods for parabolic problems. VI. Analytic semigroups. *SIAM J. Numer. Anal.* **35**(4), 1315–1325 (electronic) (1998). DOI 10.1137/S0036142996310216. URL <http://dx.doi.org/10.1137/S0036142996310216>
13. Eriksson, K., Johnson, C., Thomée, V.: Time discretization of parabolic problems by the discontinuous Galerkin method. *RAIRO Modél. Math. Anal. Numér.* **19**(4), 611–643 (1985)
14. Geissert, M.: Discrete maximal  $L_p$  regularity for finite element operators. *SIAM J. Numer. Anal.* **44**(2), 677–698 (electronic) (2006). DOI 10.1137/040616553. URL <http://dx.doi.org/10.1137/040616553>
15. Geissert, M.: Applications of discrete maximal  $L_p$  regularity for finite element operators. *Numer. Math.* **108**(1), 121–149 (2007). DOI 10.1007/s00211-007-0110-1. URL <http://dx.doi.org/10.1007/s00211-007-0110-1>
16. Guidetti, D.: Backward Euler scheme, singular Hölder norms, and maximal regularity for parabolic difference equations. *Numer. Funct. Anal. Optim.* **28**(3-4), 307–337 (2007). DOI 10.1080/01630560701285594. URL <http://dx.doi.org/10.1080/01630560701285594>
17. Guidetti, D.: Maximal regularity for parabolic equations and implicit Euler scheme. In: *Mathematical Analysis Seminar, University of Bologna Department of Mathematics: Academic Year 2003/2004. Academic Year 2004/2005 (Italian)*, pp. 253–264. Tecnoprint, Bologna (2007)
18. Haller-Dintelmann, R., Meyer, C., Rehberg, J., Schiela, A.: Hölder continuity and optimal control for nonsmooth elliptic problems. *Appl. Math. Optim.* **60**(3), 397–428 (2009). DOI 10.1007/s00245-009-9077-x. URL <http://dx.doi.org/10.1007/s00245-009-9077-x>
19. Haller-Dintelmann, R., Rehberg, J.: Maximal parabolic regularity for divergence operators including mixed boundary conditions. *J. Differential Equations* **247**(5), 1354–1396 (2009). DOI 10.1016/j.jde.2009.06.001. URL <http://dx.doi.org/10.1016/j.jde.2009.06.001>
20. Hieber, M., Prüss, J.: Heat kernels and maximal  $L^p$ - $L^q$  estimates for parabolic evolution equations. *Comm. Partial Differential Equations* **22**(9-10), 1647–1669 (1997). DOI 10.1080/03605309708821314. URL <http://dx.doi.org/10.1080/03605309708821314>
21. Hömberg, D., Meyer, C., Rehberg, J., Ring, W.: Optimal control for the thermistor problem. *SIAM J. Control Optim.* **48**(5), 3449–3481 (2009/10). DOI 10.1137/080736259. URL <http://dx.doi.org/10.1137/080736259>
22. Krumbiegel, K., Rehberg, J.: Second order sufficient optimality conditions for parabolic optimal control problems with pointwise state constraints. *SIAM J. Control Optim.* **51**(1), 304–331 (2013). DOI 10.1137/120871687. URL <http://dx.doi.org/10.1137/120871687>
23. Kunisch, K., Pieper, K., Vexler, B.: Measure valued directional sparsity for parabolic optimal control problems. *SIAM J. Control Optim.* **52**(5), 3078–3108 (2014)
24. Lasaint, P., Raviart, P.A.: On a finite element method for solving the neutron transport equation. In: *Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin-Madison, Wis., 1974)*, pp. 89–123. Publication No. 33. Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York (1974)
25. Leykekhman, D., Vexler, B.: Optimal a priori error estimates of parabolic optimal control problems with pointwise control. *SIAM J. Numer. Anal.* **51**(5), 2797–2821 (2013). DOI 10.1137/120885772. URL <http://dx.doi.org/10.1137/120885772>
26. Leykekhman, D., Vexler, B.: Finite element pointwise results on convex polyhedral domains. *SIAM J. Numer. Anal.* (2016). Accepted
27. Leykekhman, D., Vexler, B.: Pointwise best approximation results for Galerkin finite element solutions of parabolic problems. submitted (2016)
28. Leykekhman, D., Vexler, B.: A priori error estimates for three dimensional parabolic optimal control problems with pointwise control (2016). Submitted

29. Li, B.: Maximum-norm stability and maximal  $L^p$  regularity of FEMs for parabolic equations with Lipschitz continuous coefficients. *Numer. Math.* (2015)
30. Li, B., Sun, W.: Maximal  $L^p$  analysis of finite element solutions for parabolic equations with nonsmooth coefficients in convex polyhedra. Preprint arXiv:1501.07345 (2015)
31. Meidner, D., Rannacher, R., Vexler, B.: A priori error estimates for finite element discretizations of parabolic optimization problems with pointwise state constraints in time. *SIAM J. Control Optim.* **49**(5), 1961–1997 (2011). DOI 10.1137/100793888. URL <http://dx.doi.org/10.1137/100793888>
32. Meidner, D., Vexler, B.: Adaptive space-time finite element methods for parabolic optimization problems. *SIAM J. Control Optim.* **46**(1), 116–142 (electronic) (2007). DOI 10.1137/060648994. URL <http://dx.doi.org/10.1137/060648994>
33. Meidner, D., Vexler, B.: A priori error estimates for space-time finite element approximation of parabolic optimal control problems. Part I: Problems without control constraints. *SIAM J. Control Optim.* **47**(3), 1150–1177 (2008)
34. Meidner, D., Vexler, B.: A priori error estimates for space-time finite element approximation of parabolic optimal control problems. Part II: Problems with control constraints. *SIAM J. Control Optim.* **47**(3), 1301–1329 (2008)
35. Meidner, D., Vexler, B.: A priori error analysis of the Petrov-Galerkin Crank-Nicolson scheme for parabolic optimal control problems. *SIAM J. Control Optim.* **49**(5), 2183–2211 (2011). DOI 10.1137/100809611. URL <http://dx.doi.org/10.1137/100809611>
36. Portal, P.: Maximal regularity of evolution equations on discrete time scales. *J. Math. Anal. Appl.* **304**(1), 1–12 (2005). DOI 10.1016/j.jmaa.2004.09.003. URL <http://dx.doi.org/10.1016/j.jmaa.2004.09.003>
37. Richter, T., Springer, A., Vexler, B.: Efficient numerical realization of discontinuous Galerkin methods for temporal discretization of parabolic problems. *Numer. Math.* **124**(1), 151–182 (2013). DOI 10.1007/s00211-012-0511-7. URL <http://dx.doi.org/10.1007/s00211-012-0511-7>
38. Schmich, M., Vexler, B.: Adaptivity with dynamic meshes for space-time finite element discretizations of parabolic equations. *SIAM J. Sci. Comput.* **30**(1), 369–393 (2007/08). DOI 10.1137/060670468. URL <http://dx.doi.org/10.1137/060670468>
39. Schötzau, D., Wihler, T.P.: A posteriori error estimation for  $hp$ -version time-stepping methods for parabolic partial differential equations. *Numer. Math.* **115**(3), 475–509 (2010). DOI 10.1007/s00211-009-0285-8. URL <http://dx.doi.org/10.1007/s00211-009-0285-8>
40. Shen, Z.W.: Resolvent estimates in  $L^p$  for elliptic systems in Lipschitz domains. *J. Funct. Anal.* **133**(1), 224–251 (1995). DOI 10.1006/jfan.1995.1124. URL <http://dx.doi.org/10.1006/jfan.1995.1124>
41. Thomée, V.: Galerkin finite element methods for parabolic problems, *Springer Series in Computational Mathematics*, vol. 25, second edn. Springer-Verlag, Berlin (2006)