LOCAL ERROR ESTIMATES FOR SUPG SOLUTIONS OF ADVECTION-DOMINATED ELLIPTIC LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS

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Abstract. We derive local error estimates for the discretization of optimal control problems governed by linear advection-diffusion partial differential equations (PDEs) using the streamline upwind/Petrov Galerkin (SUPG) stabilized finite element method. We show that if the SUPG method is used to solve optimization problems governed by an advection-dominated PDE the convergence properties of the SUPG method is substantially different from the convergence properties of the SUPG method applied for the solution of an advection-dominated PDE. The reason is that the solution of the optimal control problem involves another advection dominated PDE, the so-called adjoint equation, whose advection field is just the negative of the advection of the governing PDEs. For the solution of the optimal control problem, a coupled system involving both the original governing PDE as well as the adjoint PDE must be solved.

We show that in the presence of a boundary layer, the local error between the solution of the SUPG discretized optimal control problem and the solution of the infinite dimensional problem is only of first order even if the error is computed locally in a region away from the boundary layer. In the presence of interior layers, we prove optimal convergence rates for the local error in a region away from the layer between the solution of the SUPG discretized optimal control problems and the solution of the infinite dimensional problem. Numerical examples are presented to illustrate some of the theoretical results.

Key words. Optimal control, advection-diffusion equations, discretization, local error estimates, stabilized finite elements, SUPG

AMS subject classifications. 49M25, 49K20, 65N15, 65J10

1. Introduction.

The streamline upwind/Petrov Galerkin (SUPG) stabilized finite element method [6, 14] and other stabilized finite element methods are widely used for the solution of advection-dominated partial differential equations (PDEs). Unlike the standard Galerkin finite element method which produces strongly oscillatory solutions for mesh sizes that are larger than the ratio of diffusion and advection, stabilized finite elements generate 'good' solutions for moderately sized meshes [17, 22, 23, 24, 25, 29, 31].

If the SUPG method is used to solve optimization problems governed by an advection-dominated PDE, however, then the convergence properties of the SUPG method can be substantially different from the convergence properties of the SUPG method applied for the solution of an advection-dominated PDE. The reason is that the solution of the optimal control problem involves another PDE, the so-called adjoint equation, in addition to the original governing PDE. Both PDEs are coupled through the optimality conditions. The adjoint equation is also a linear advection-diffusion equation, in which the advection is equal to the negative advection of the governing PDE. The right hand side or the boundary data for the adjoint equation are determined by the objective function of the optimal control problem. For the solution of the optimal control problem, both PDEs, the original governing PDE as well as the adjoint PDE must be solved. The behavior of the SUPG method applied to these

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coupled PDEs is different than that of the SUPG method applied to the governing PDE only. The goal of this paper is to analyze the behavior of the SUPG method applied to a simple elliptic linear quadratic optimal control problem. In particular, we investigate the error between the solution of the optimal control problem and the the solution of an SUPG discretization of the optimal control problem in subdomains that do not contain interior or boundary layers.

Our model problem is given as follows.

$$\text{Minimize } \frac{1}{2}\int_\Omega (y(x)-\widehat{y}(x))^2 dx + \frac{\alpha}{2}\int_\Omega u^2(x)dx \tag{1.1a}$$

subject to

$$-\epsilon\Delta y(x) + \mathbf{c}(x)\cdot\nabla y(x) + r(x)y(x) = f(x) + u(x), \qquad x\in\Omega, \tag{1.1b}$$
$$y(x) = 0, \qquad x\in\partial\Omega, \tag{1.1c}$$

where $\mathbf{c}, f, r, \widehat{y}$ are given functions and $\epsilon, \alpha > 0$ are given scalars. We refer to u as the control, to y as the state, and to (1.1b,c) as the state equation. Under suitable assumptions it can be shown (see Section 2 for details) that u, y solve (1.1) if and only if there exists a function λ such that the state equation (1.1b,c), the adjoint equation

$$-\epsilon\Delta\lambda(x) - \mathbf{c}(x)\cdot\nabla\lambda(x) + (r(x) - \nabla\cdot\mathbf{c}(x))\lambda(x) = -(y(x)-\hat{y}(x)), \quad x\in\Omega, \tag{1.2a}$$
$$\lambda(x) = 0, \qquad x\in\partial\Omega, \tag{1.2b}$$

and the equation

$$\lambda(x) = \alpha u(x), \qquad x\in\Omega \tag{1.3}$$

are satisfied. Like the original state equation (1.1b,c), the adjoint equation (1.2) is also an advection-diffusion equation, but with advection $-\mathbf{c}$ instead of $\mathbf{c}$. The goal of this paper is to analyze the SUPG method applied to the coupled system (1.1b,c)–(1.3). In particular we derive weighted error estimates for the error between the solution y, u, λ of (1.1) and the solution y_h, u_h, λ_h of an SUPG discretization of (1.1). These weighted estimates are used to derive error estimates on subdomains $\Omega_0 \subset \Omega$ that do not contain interior or boundary layers of the solution.

Stabilized finite element methods for optimal control problems have been studied in a number of papers [1, 4, 5, 8, 9, 13, 21, 30] and the research on this topic is still very active. The paper [8] studies the SUPG method for the discretization of (1.1). It shows that there is a difference between discretizing the optimal control problem (1.1) using the SUPG method for the discretization of state equation (1.1b,c) (this is the discretize-then-optimize approach) and the discretization of the optimality system (1.1b,c), (1.2), (1.3) where the state equation (1.1b,c) and the adjoint equation (1.2) are both discretized by applying the SUPG method to both (this the optimize-then-discretize approach). Estimates for the global error between the solution of the original and the discretized problems are derived for both approaches, discretize-then-optimize and optimize-then-discretize. The paper [1] provides an analogous study for optimal boundary control problems governed by the Oseen equations using finite element methods with Galerkin/Least-Squares (GLS) stabilization. The paper [9] proposes stabilizing the optimality system (1.1b,c), (1.2), (1.3), i.e., it proposes an optimize-then-discretize approach. The stabilization used in [9], however, is different from the optimize-then-discretize SUPG approach discussed in [8]. In particular, the discrete optimality system (1.1b,c), (1.2), (1.3) in [9] is symmetric, while the optimize-then-discretize approach in [8] leads to nonsymmetric systems.

The stabilization methods applied in [1, 8, 9] are residual based. For example, the SUPG stabilization for (1.1b,c) involves the residual $-\epsilon\Delta y + \mathbf{c}\cdot\nabla y + ry - f - u$ of the state equation. This dependence on the residual is one reason why the discretize-then-optimize approach and the optimize-then-discretize approach in [1, 8] lead to different discrete problems. The papers [4, 5] study stabilizations based on local projection. One advantage of these approaches is that the tasks of discretization and optimization commute, i.e., the discretize-then-optimize approach and the optimize-then-discretize approach lead to the same discrete problem. Like [8], the papers [4, 5, 13, 21, 30] provide estimates for the error between the solution of the original and the discretized optimal control problem over the entire domain Ω. As a consequence, the constants in these estimates of the global error depend on derivatives of the solution y, u, λ of (1.1) on all of Ω. In the presence of interior or boundary layers these derivatives are large and can render the theoretical bounds for the global error meaningless. More informative for this kind of problems, the residual based *a posteriori* error estimates are derived in [13, 30] with some promising numerical examples.

The goal of this paper is to derive *a priori* bounds for the error localized in subdomains $\Omega_0 \subset \Omega$ away from regions where layers occur. The right hand sides of our error bounds involve derivatives of the solution y, u, λ of (1.1) restricted to $\Omega_0 \subset \Omega$. Since interior or boundary layers of the solution are located outside Ω_0, the right hand sides of our bounds are independent of ϵ and as a result these error bounds are much more descriptive.

We show that the presence of boundary layers may pollute the numerical solution everywhere. Specifically, we prove for 1D problems with constant coefficients that the error between the solution of the SUPG discretized optimal control problem and the solution of the infinite dimensional problem is only of first order even if the error is computed locally in a region away from the boundary layer. Numerical tests indicate that this is also true for 2D problems. This is in sharp contrast to the case of a single equation where it has been shown analytically that the numerical layers do not pollute the SUPG solution into domains of smoothness [17, 29]. We also show that the interior layers do not pollute the solution away from the layers. For 2D problems with constant coefficients we prove optimal error estimates over the regions away from the layers. To summarize, the main message of this paper is that as a rule, the SUPG solution to optimal control problems governed by linear advection-diffusion partial differential behaves very differently than in the case of a single equation. Any boundary layers in either forward or adjoint problem pollute the numerical solution everywhere in the entire domain, even in subregions where the exact solution is smooth, but the interior layers do not. Numerical examples in Section 5 illustrate our theoretical results.

The reason why the error between the SUPG solution and the solution of the infinite dimensional optimal control problem is only of first order is that the boundary layers are not sufficiently resolved. The discretization errors in the boundary layers are transported via the adjoint and the state equation into the domain. Thus the source of this order reduction is not due to the fact that we use a residual based stabilization. We expect this to happen also when other stabilizations, such as the local projection based methods studied, e.g., in [4, 5, 13, 21, 30], are used. This was already observed in [2] who studied finite difference methods for coupled systems of singularly perturbed ordinary differential equations.

The rest of the paper is organized as follows. In the next section we state the problem and the standard existence and regularity results. Section 3 is devoted to showing optimal estimates in the case of interior layers. The main result of this section is Theorem 3.2, with a proof in two dimensions. Section 4 studies one-dimensional problems with boundary layers. Two cental results are Theorem 4.7, which establishes first order convergence for piecewise linear elements, and Theorem 4.11, which shows the optimal global L^2 norm convergence. Theorem 4.11 is of interest by itself. Due to rather technical proofs, we only treat the problems with constant coefficients. In Section 5 we provide numerical illustrations of our theoretical findings. Finally, in the last section we conclude our paper with the brief summary of our main results.

2. Problem Statement. For $G \subset \Omega$ we define $\langle f, g\rangle_G = \int_G f(x)g(x)dx$, $\|v\|_{0,\infty,G} = \text{ ess sup}_{x\in G}|v(x)|$ or $\|\mathbf{v}\|_{0,\infty,G} = \text{ ess sup}_{x\in G}\sqrt{\sum_i v_i(x)^2}$ for vector valued $\mathbf{v}$, and

$$\|v\|_{k,G} = \left(\sum_{|\alpha|\le k}\int_G (\partial^\alpha v(x))^2 dx\right)^{1/2}, \qquad |v|_{k,G} = \left(\sum_{|\alpha|= k}\int_G (\partial^\alpha v(x))^2 dx\right)^{1/2}.$$

If $G = \Omega$, we omit Ω and write $\|v\|_{k,\infty}$ instead of $\|v\|_{k,\infty,\Omega}$, etc. If $k = 0$, we omit k and write $\|v\|_G$ instead of $\|v\|_{0,G}$, etc. If $k = 0$ and $G = \Omega$ we omit all subscripts and write $\|v\|$ instead of $\|v\|_{0,\Omega}$. Furthermore, we define the bilinear form

$$a(y, v) = \int_\Omega \epsilon\nabla y(x)\cdot\nabla v(x) + \mathbf{c}(x)\cdot\nabla y(x)v(x) + r(x)y(x)v(x)dx \tag{2.1}$$

for $y, v \in H_0^1(\Omega)$. The optimal control problem (1.1) is given by

$$\text{minimize } \frac{1}{2}\|y - \widehat{y}\|^2 + \frac{\alpha}{2}\|u\|^2, \tag{2.2a}$$

$$\begin{aligned} \text{subject to } & a(y, v) - \langle u, v\rangle = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega), \\ & y \in H_0^1(\Omega), u \in L^2(\Omega). \end{aligned} \tag{2.2b}$$

We assume that

$$\begin{gathered} \Omega \text{ is a bounded domain,} \\ f, \widehat{y} \in L^2(\Omega), \mathbf{c} \in \left(W^{1,\infty}(\Omega)\right)^n, r \in L^\infty(\Omega), \alpha > 0, \epsilon > 0, \end{gathered} \tag{2.3a}$$

and

$$r(x) - \tfrac{1}{2}\nabla\cdot\mathbf{c}(x) \ge r_0 \ge 0 \text{ a.e. in } \Omega. \tag{2.3b}$$

For the well-posedness of the optimal control problem it is sufficient to impose fewer regularity requirements on the coefficient functions than those stated in (2.3a) and (2.3b). The assumptions (2.3a) imply stronger regularity properties of the solution, which will be stated in Theorem 2.2 below and using the exponential weighted technique (cf. [3, 16]), the assumption (2.3b) can be weakened as well.

Under the assumptions (2.3), the bilinear form $a(\cdot,\cdot)$ is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$ and $H_0^1(\Omega)$-elliptic. In fact, $a(y, y) \ge \epsilon\|\nabla y\|^2 + r_0\|y\|^2$ for all $y \in H_0^1(\Omega)$ (e.g., [24, p. 165] or [22, Sec. 2.5]). Hence the theory in [20, Sec. II.1] guarantees the existence of a unique solution $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$ of (2.2).

THEOREM 2.1. *If* (2.3) *are satisfied, the optimal control problem* (2.2) *has a unique solution* $(y,u) \in H_0^1(\Omega) \times L^2(\Omega)$. *Furthermore,* $(y,u) \in H_0^1(\Omega) \times L^2(\Omega)$ *solves* (2.2) *if and only if there exists* $\lambda \in H_0^1(\Omega)$ *such that* (y,u,λ) *solve*

$$a(\psi,\lambda) = -\langle y-\hat{y},\psi\rangle \qquad \forall \psi \in H_0^1(\Omega), \tag{2.4a}$$

$$-\langle w,\lambda\rangle + \alpha\langle u,w\rangle = 0 \qquad \forall w \in L^2(\Omega), \tag{2.4b}$$

$$a(y,v) - \langle u,v\rangle = \langle f,v\rangle \qquad \forall v \in H_0^1(\Omega). \tag{2.4c}$$

The optimality system (2.4) is the weak form of (1.1b,c)–(1.3).

For some of our error estimates we need that the state y and adjoint λ is not only in $H_0^1(\Omega)\times H_0^1(\Omega)$, but even in $H^2(\Omega)\times H^2(\Omega)$. The following result gives a sufficient condition for such a regularity result. It is derived from regularity results for single advection diffusion equations. We present the result only for a constant advection, which is sufficient for our purposes. Using, for example, the ideas similar to the ones in [3], [25, L. 1.18, p 248] such restriction can be relaxed.

THEOREM 2.2. *Let* Ω *be a bounded open convex subset of* $\mathbb{R}^n$ *and let the assumptions* (2.3) *be satisfied with* $\mathbf{c}$ *being a constant vector. There exists a positive constant* C *independent of* ϵ *such that the unique solution of the optimal control problem* (2.2) *and the associated adjoint satisfy* $(y,\lambda) \in H^2(\Omega)\times H^2(\Omega)$ *and*

$$\epsilon^{3/2}\|y\|_2 + \epsilon^{1/2}\|y\|_1 + \|y\| \le C, \quad \epsilon^{3/2}\|\lambda\|_2 + \epsilon^{1/2}\|\lambda\|_1 + \|\lambda\| \le C.$$

Proof. We provide a proof that includes the case $r_0 = 0$ and any $\epsilon > 0$, not necessarily very small. In the case of $r_0 > 0$ or $\epsilon = 0(1)$ a simpler proof can be given (see also [25, L. 1.18, p 248] and [28, L. 7.2]). Throughout the proof C denotes a positive constant independent of ϵ. Without loss of generality we assume $|\mathbf{c}| = 1$, otherwise we can just divide every term of the state equation by $|\mathbf{c}|$.

Let y_0 be the solution of the state equation (2.2b) with $u = 0$ and let u, y, λ be the optimal control, state, and adjoint. Since u is optimal,

$$\frac{1}{2}\|y-\widehat{y}\|^2 + \frac{\alpha}{2}\|u\|^2 \le \frac{1}{2}\|y_0-\widehat{y}\|^2.$$

Hence

$$\|y\| \le C, \quad \|u\| \le C, \quad \|\lambda\| \le C. \tag{2.5}$$

The latter bound comes from the identity $\alpha u = \lambda$ (cf. (2.4b)). The coercivity properties of the bilinear a and (2.2b) imply

$$\epsilon\|\nabla y\|^2 + r_0\|y\|^2 \le (\|f\| + \|u\|)\|y\|.$$

Since we allow $r_0 = 0$, we use ideas from [3] and introduce an exponential function $\psi = e^{-2(x-x_0)\cdot\mathbf{c}}$, where x_0 is a point on $\partial\Omega$ such that $\|\psi\|_{L^\infty} = 1$. Notice that

$$\mathbf{c}\cdot\nabla\psi = -2\psi.$$

Let K be a positive number that will be specified later. Since $\psi y \in H_0^1$, the equations (2.1) and (2.4c) yield

$$\epsilon(\nabla y, \nabla((\psi+K)y)) + (\mathbf{c}\cdot\nabla y, (\psi+K)y) + (ry,(\psi+K)y) = (f+u,(\psi+K)y).$$

Using that

$$(\mathbf{c} \cdot \nabla y, (\psi + K)y) = -\frac{1}{2}(y, \mathbf{c} \cdot \nabla \psi y) = (y, \psi y),$$

we have

$$\epsilon(\nabla y, (\psi + K)\nabla y) + (y, (\psi + r\psi + rK)y) = (f + u, (\psi + K)y) - \epsilon(\nabla y, y\nabla\psi). \tag{2.6}$$

Since

$$\psi \geq e^{-2L}, \quad |\nabla \psi| \leq 2, \quad r \geq r_0 \geq 0, \quad \text{and} \quad \|\psi\|_{L^\infty} = 1,$$

where $L = diam(\Omega)$, equation (2.6) implies

$$\epsilon(e^{-2L} + K)\|\nabla y\|^2 + e^{-2L}\|y\|^2 \leq (1 + K)\|f + u\|\|y\| + 2\epsilon\|\nabla y\|\|y\|. \tag{2.7}$$

The arithmetic-geometric mean inequality $ab \leq \delta a^2 + \frac{1}{4\delta}b^2$ yields

$$(1 + K)\|f + u\|\|y\| \leq e^{2L}(1 + K)^2\|f + u\|^2 + \frac{e^{-2L}}{4}\|y\|^2 \tag{2.8}$$

and

$$2\epsilon\|\nabla y\|\|y\| \leq \frac{\epsilon(e^{-2L} + K)}{2}\|\nabla y\|^2 + \frac{2\epsilon}{e^{-2L} + K}\|y\|^2. \tag{2.9}$$

If we insert (2.8) and (2.9) into (2.7) and choose $K \geq 4\epsilon e^{2L} - e^{-2L}$, i.e., choose K such that

$$\frac{2\epsilon}{e^{-2L} + K} \leq \frac{e^{-2L}}{2},$$

then we obtain

$$\epsilon^{1/2}\|\nabla y\| + \|y\| \leq C(\|f\| + \|u\|), \tag{2.10}$$

where the constant C depends on diameter of Ω, more precisely on e^{2L}, but not on y or ϵ.

Since y solves $-\epsilon\Delta y = f + u - \mathbf{c} \cdot \nabla y - ry$ and $f + u - \mathbf{c} \cdot \nabla y - ry \in L^2(\Omega)$, the solution y is in $H^2(\Omega)$ and obeys (cf. [10, 11])

$$\epsilon\|y\|_2 \leq C\|f + u - \mathbf{c} \cdot \nabla y - ry\| \leq C(\|f\| + \|u\| + \|\nabla y\| + \|r\|_{L^\infty}\|y\|) \leq C\epsilon^{-1/2},$$

where we have used (2.10) and (2.5) to derive the last inequality. This implies the desired result. The estimate for the adjoint λ can be obtained analogously. ∎

Since $\alpha > 0$, we can use (2.4b) to eliminate u from the optimality system. The continuous optimality system can be reduced to the following coupled system of two equations,

$$a(\psi, \lambda) + \langle y, \psi \rangle = \langle \hat{y}, \psi \rangle \qquad \forall \psi \in H_0^1(\Omega), \tag{2.11a}$$

$$\alpha a(y, v) - \langle \lambda, v \rangle = \alpha \langle f, v \rangle \qquad \forall v \in H_0^1(\Omega). \tag{2.11b}$$

We apply the SUPG method to (2.11). Let $\{\mathcal{T}_h\}_{h>0}$ be a family of quasi-uniform triangulations of Ω [7]. To approximate the state equation we use the spaces

$$V_h = \left\{ v_h \in H_0^1(\Omega) : v_h|_T \in P_k(T) \text{ for all } T \in \mathcal{T}_h \right\}, \quad k \geq 1, \tag{2.12}$$

where $P_k(T)$ denotes the space of polynomials of degree k defined on element T.

The SUPG discretization of (2.11) is given by

$$a_h^a(\psi,\lambda_h) + \langle y_h,\psi\rangle^a = \langle \widehat{y},\psi\rangle^a \qquad \forall \psi\in V_h, \tag{2.13a}$$
$$\alpha a_h^s(y_h,v) - \langle \lambda_h, v\rangle^s = \alpha\langle f,v\rangle_h^s \qquad \forall v\in V_h, \tag{2.13b}$$

where

$$\begin{aligned} a_h^a(\psi,\lambda_h) =& a(\psi,\lambda_h) \\ &+ \sum_{T_e\in\mathcal{T}_h} \tau_e\langle -\epsilon\Delta\lambda_h - \mathbf{c}\cdot\nabla\lambda_h + (r-\nabla\cdot\mathbf{c})\lambda_h, -\mathbf{c}\cdot\nabla\psi\rangle_{T_e}, \end{aligned} \tag{2.14a}$$

$$\langle y_h,\psi\rangle^a = \langle y_h,\psi\rangle + \sum_{T_e\in\mathcal{T}_h}\tau_e\langle y_h, -\mathbf{c}\cdot\nabla\psi\rangle_{T_e}, \tag{2.14b}$$

$$a_h^s(y_h,v) = a(y_h,v) + \sum_{T_e\in\mathcal{T}_h}\tau_e\langle -\epsilon\Delta y_h + \mathbf{c}\cdot\nabla y_h + r y_h, \mathbf{c}\cdot\nabla v\rangle_{T_e}, \tag{2.14c}$$

$$\langle \lambda_h, v\rangle^s = \langle\lambda_h, v\rangle + \sum_{T_e\in\mathcal{T}_h}\tau_e\langle\lambda_h,\mathbf{c}\cdot\nabla v\rangle_{T_e} \tag{2.14d}$$

for some positive parameters τ_e. We note that the SUPG discretization of the reduced system (2.11) leads to a finite dimensional system that is equivalent to that obtained from applying the SUPG discretization to the optimality system (2.4) (this is the optimize-then-discretize approach in [8]).

We define a bilinear form $\mathcal{A}^{SUPG}(\cdot,\cdot)$ on $(V_h\times V_h)\times(V_h\times V_h)$ for the reduced system (2.14) by

$$\begin{aligned} &\mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{\phi_h,\varphi_h\}) \\ &= \alpha a_h^s(y_h,\phi_h) - \langle\lambda_h,\phi_h\rangle^s + a_h^a(\varphi_h,\lambda_h) + \langle y_h,\varphi_h\rangle^a. \end{aligned} \tag{2.15}$$

Since the discrete system is obtained by applying the SUPG method (which is strongly consistent) to the optimality system, we have the usual Galerkin orthogonality,

$$\mathcal{A}^{SUPG}\big(\{y-y_h,\lambda-\lambda_h\},\{\phi_h,\varphi_h\}\big) = 0, \quad \forall\{\phi_h,\varphi_h\}\in V_h\times V_h. \tag{2.16}$$

3. The Case of Interior Layers. We restrict the discussion of this section to a bounded domain $\Omega\subset\mathbb{R}^2$. According to the continuous theory, see [28, p. 473] or [29, L. 23.1], the exact solution to an advection-diffusion problem at any fixed point $x_0\in\Omega$ is influenced by the force term only from within an $\epsilon\log(1/\epsilon)$-neighborhood in the streamline (downwind) direction and within a $\sqrt{\epsilon}\log(1/\epsilon)$-neighborhood in the crosswind direction. The same behavior can be observed from the properties of the corresponding Green's function. In the presence of interior layers only, the exact solution may vary strongly in the crosswind direction, but not in the streamline direction. Since the adjoint equation has similar properties, the same behavior of the solution can be expected from the coupled system. Our main goal of this section is to show that similarly to the single equation (cf. [17, 29]), the interior layers do not pollute the numerical solution to the coupled system. We will accomplish that by weighted error estimates, where the purpose of the weighting function is essentially to isolate the domains of smoothness from the layers. The analysis is rather technical and in order to avoid unnecessary technicalities we will make several simplifications:

- $\epsilon\le h$, i.e. we consider only the advection-dominating case.

- The advection $\mathbf{c}$ is a constant vector.
- The reaction term $r \equiv 1$. This simplification is not essential, the same analysis can be applied to $r(x) \geq r_0 > 0$.
- $\tau_e := \tau = C_1 h$, for some fixed constant C_1. This assumption is logical since we are in advection-dominating regime and the mesh is quasi-uniform.

To state the main result of this section we introduce a weight function ω, which is $O(1)$ on

$$\Omega_0 = \{\mathbf{x} \in \Omega : \ A_1 \leq (\mathbf{x} \times \mathbf{c}) \leq A_2\} \tag{3.1}$$

for some constants A_1 and A_2 and decays exponentially outside of a slightly larger subdomain. Here we used the convection that for two dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ the cross product is defined by $\mathbf{a} \times \mathbf{b} := a_1 b_2 - a_2 b_1$, which is just a z-component of the cross-product if we think of vectors $\mathbf{a}$ and $\mathbf{b}$ as three dimensional vectors with z component to be zero. Thus, the set Ω_0 defines a strip along $\mathbf{c}$ of width $|A_2 - A_1|$.

More precisely, the weight ω is a positive function with the following properties:

$$\begin{aligned}
\omega(\mathbf{x}) &= O(1), && \text{for } \mathbf{x} \in \Omega_0, \\
|\omega(\mathbf{x})| &\leq C e^{-((\mathbf{x}\times\mathbf{c}) - A_2)/K\sqrt{h}}, && \text{for } (\mathbf{x} \times \mathbf{c}) \geq A_2, \\
|\omega(\mathbf{x})| &\leq C e^{-(A_1 - (\mathbf{x}\times\mathbf{c}))/K\sqrt{h}}, && \text{for } (\mathbf{x} \times \mathbf{c}) \leq A_1.
\end{aligned}$$

Here K is a sufficiently large number and $\sqrt{h}$ is potentially the size of the numerical crosswind layer.

REMARK 3.1. *Estimating the actual size of the numerical crosswind layer is a delicate process. Generally, for the standard SUPG method on unstructured quasi-uniform meshes, like in our case, it is only known that the size of the numerical crosswind layer is at most of order $h^{1/2}$. In [17], the authors modified the SUPG method by adding extra crosswind diffusion to the method and showed analytically that it is at most of order $h^{3/4}$ for piecewise linear elements. In [31], in the case of aligned meshes and bilinear elements, the authors showed that it is of size h. The numerical investigations by Semper, [26, 27], suggest that for linear elements on the meshes not aligned with the advection it is about of order $h^{0.7}$ and of order h on the aligned meshes. To the best of our knowledge there is no theoretical or numerical results for higher order elements.*

Now we state our weighted result. Notice that we have traced the dependence of the error on the stabilization parameter α.

THEOREM 3.2. *Let $\{y, \lambda\}$ and $\{y_h, \lambda_h\}$ satisfy* (2.16). *In addition to the assumptions at the beginning of Section 3 assume $\epsilon\tau \leq h^2/(4C_{inv}^2 C_\omega^2)$ and $h \leq C_2\alpha$, for constants C_{inv} and C_ω defined below in* (3.4) *and* (3.8), *and some fixed constant C_2. Then there exists a constant C independent of h, ϵ, y, λ, α such that*

$$\alpha Q_\omega^2(y - y_h) + Q_\omega^2(\lambda - \lambda_h) \leq C(\alpha \min_{\chi_1 \in V_h} L_\omega^2(y - \chi_1) + \min_{\chi_2 \in V_h} L_\omega^2(\lambda - \chi_2)),$$

where

$$Q_\omega^2(v) := \epsilon\|\omega\nabla v\|^2 + \|\omega v\|^2 + \sum_{T_e \in \mathcal{T}_h} \tau \|\omega \mathbf{c} \cdot \nabla v\|_{T_e}^2 \tag{3.2}$$

and

$$L_\omega^2(v) := h^{-1}\|\omega v\|^2 + h\|\omega\nabla v\|^2 + h^3\|\omega\Delta v\|^2. \tag{3.3}$$

The proof of Theorem 3.2 goes along the lines of [15] and [17], where the corresponding result was established for a single equation. Since in the above mentioned papers only piecewise linear elements were treated, high order stabilizations terms were meaningless. In our situation we have to deal with those high order stabilization terms in addition to the coupled terms. Consequently, the proof is more technical. Furthermore, in the course of the proof we also need to trace the dependence of the error on the parameter α. This adds additional technical difficulties. Theorem 3.2 will be proven in Section 3.2.

To give an application of Theorem 3.2 , let Ω_0 be as in (3.1) and define

$$\Omega_s = \{\ A_1 - sK\sqrt{h}|\log h| \leq (\mathbf{x}\times\mathbf{c}) \leq A_2 + sK\sqrt{h}|\log h|\ \} \cap \Omega, \quad s > 0.$$

COROLLARY 3.3. *Under the assumptions of Theorem 3.2 and assuming $y, \lambda \in H^2(\Omega)$, there exists a constant C independent of y, λ, ϵ, and h such that for any $s > 0$,*

$$\begin{aligned}
&\alpha\left(\epsilon\|\nabla(y-y_h)\|^2_{\Omega_0} + \|y-y_h\|^2_{\Omega_0} + \sum_{T_e\in\mathcal{T}_h\cap\Omega_0}\tau\|\mathbf{c}\cdot\nabla(y-y_h)\|^2_{T_e}\right)\\
&\quad + \epsilon\|\nabla(\lambda-\lambda_h)\|^2_{\Omega_0} + \|\lambda-\lambda_h\|^2_{\Omega_0} + \sum_{T_e\in\mathcal{T}_h\cap\Omega_0}\tau\|\mathbf{c}\cdot\nabla(\lambda-\lambda_h)\|^2_{T_e}\\
&\leq C\alpha\left(h^3\|y\|^2_{2,\Omega_s} + h^{2s+3}\|y\|^2_{2,\Omega}\right) + C\left(h^3\|\lambda\|^2_{2,\Omega_s} + h^{2s+3}\|\lambda\|^2_{2,\Omega}\right).
\end{aligned}$$

The right hand side in the error estimate of Corollary 3.3 depends on local and global norms of the state and the adjoint. The local norms associated with h^3 are independent of ϵ if Ω_s does not contain interior layers. The global norms may depend on negative powers of ϵ. However, they are associated with the higher order terms h^{2s+3}. Thus negative powers of ϵ can be compensated by h^{2s} for sufficiently large s, provided that for these values of s the subdomain Ω_s does not contain interior layers. In this case the local convergence rates are optimal for H^2 regular solutions.

Corollary 3.3 will be proven in Section 3.2.

3.1. Preliminary Results. First we recall the standard inverse inequality, which holds for quasi-uniform meshes, and the approximation property of the finite dimensional subspaces. The proofs can be found in many textbooks on finite elements, such as [7].

3.1.1. Inverse inequalities. For $T \in \mathcal{T}_h$ and $v \in V_h$, there exists a constant C_{inv} independent of T and v, such that

$$\|\nabla v\|_{L^p(T)} \leq C_{inv}h^{-1}\|v\|_{L^p(T)}, \quad 1 \leq p \leq \infty. \tag{3.4}$$

3.1.2. Local approximation. There exists a local interpolant operator $I : H^{k+1}(\Omega) \to V_h$ such that for any $T \in \mathcal{T}_h$,

$$h\|\nabla(v - I(v))\|_T + \|v - I(v)\|_T \leq Ch^{k+1}|v|_{k+1,T}. \tag{3.5}$$

3.1.3. The weight function. In addition to the properties of ω described above, we assume that ω satisfies,

$$D_{\mathbf{c}}\omega(\mathbf{x}) = 0, \quad \text{for all } \mathbf{x}\in\Omega, \text{ i.e. } \omega \text{ is constant in the direction } \mathbf{c}, \tag{3.6}$$

$$|D_{\mathbf{c}^\perp}\omega| \leq CK^{-1}h^{-1/2}\omega, \tag{3.7}$$

$$RO(S,\omega) \leq C_\omega, \quad \text{for any ball } S \text{ of radius } Kh, \tag{3.8}$$

where

$$D_{\mathbf{c}} = \mathbf{c}\cdot\nabla, \quad D_{\mathbf{c}^\perp} = \mathbf{c}^\perp\cdot\nabla, \quad \text{and} \quad RO(S,v) = \max_{x\in S}|v(x)|/\min_{x\in S}|v(x)|.$$

The explicit construction of such a function is given in [17].

3.1.4. Superapproximation. Next we will need the superapproximation result. The proof of this result is essentially contained in [17], Lemma 2.2, or [29], Lemma 25.1, where piecewise linear elements were treated. The extension to higher order elements is straightforward.

LEMMA 3.4. *Let $v \in V_h$. Set $E_\omega(v) = \omega^2 v - I(\omega^2 v)$, where I is the interpolant from above. There exists a constant C independent of h and v, such that for any triangle $T \in \mathcal{T}_h$,*

$$\|\omega^{-1}E_\omega(v)\|_T + h\|\omega^{-1}\nabla E_\omega(v)\|_T \le Ch^{1/2}K^{-1}\|\omega v\|_T.$$

As we have mentioned above the proof of Theorem 3.2 follows from the following three lemmas. The proofs of these lemmas are rather standard, although technical. Due to space limitations we state the results without proofs. The details of the proofs can be examined in the full version of the paper which is available as a technical report [12]. In the following lemmas we use Q_ω and L_ω as defined in the statement of Theorem 3.2, (3.2) and (3.3) respectively.

LEMMA 3.5. *Let $\{y_h,\lambda_h\} \in V_h \times V_h$ and assume $\epsilon\tau \le \frac{h^2}{4C_{inv}^2 C_\omega^2}$. Then*

$$\alpha Q_\omega^2(y_h) + Q_\omega^2(\lambda_h) \le 2\mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{\omega^2 y_h, \omega^2\lambda_h\}).$$

LEMMA 3.6. *In addition to the assumptions of Lemma 3.5, assume $h \le C_2\alpha$ for some fixed constant C_2. Then, there exists a constant C independent of h and K such that*

$$\mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{\omega^2 y_h - I(\omega^2 y_h), \omega^2\lambda_h - I(\omega^2\lambda_h)\}) \le CK^{-1}\left(\alpha Q_\omega^2(y_h) + Q_\omega^2(\lambda_h)\right).$$

LEMMA 3.7. *In addition to the assumptions of Lemma 3.5 and Lemma 3.6, let $\{y,\lambda\}$ and $\{y_h,\lambda_h\}$ satisfy* (2.16). *Then there exist a constant C and an arbitrarily small constant δ both independent of h and K, such that*

$$\begin{aligned}&\mathcal{A}^{SUPG}(\{y,\lambda\},\{I(\omega^2 y_h), I(\omega^2\lambda_h)\})\\ &\le C\left((\delta + K^{-1})(\alpha Q_\omega^2(y_h) + Q_\omega^2(\lambda_h)) + \alpha L_\omega^2(y) + L_\omega^2(\lambda)\right).\end{aligned}$$

REMARK 3.8. *Global analysis of the SUPG method for a single equation with high order stabilization terms is presented in [18], Chapter 9.2. Similar to our analysis, their argument also requires the corresponding restrictions on ϵ and τ.*

3.2. Proof of Theorem 3.2 and Corollary 3.3. Assuming Lemmas 3.5-3.7, we are ready to present a proof of Theorem 3.2.

Proof. By the triangle inequality, we have

$$\alpha Q_\omega^2(y-y_h) + Q_\omega^2(\lambda-\lambda_h) \le \alpha Q_\omega^2(y_h-\chi_1) + Q_\omega^2(\lambda_h-\chi_2) + C(\alpha L_\omega^2(y-\chi_1) + L_\omega^2(\lambda-\chi_2)), \tag{3.9}$$

where we have used that

$$Q_\omega^2(v-\chi) \le C L_\omega^2(v-\chi),$$

which follows from the properties of ω and the assumption $\epsilon \le h$. Hence, it is enough to show that for any $\chi_1, \chi_2 \in V_h$,

$$\alpha Q_\omega^2(y_h-\chi_1) + Q_\omega^2(\lambda_h-\chi_2) \le C(\alpha L_\omega^2(y-\chi_1) + L_\omega^2(\lambda-\chi_2)).$$

Notice, the SUPG method is invariant on V_h, i.e. if the exact solution $\{y,\lambda\} \in V_h \times V_h$, then $\{y_h,\lambda_h\} = \{y,\lambda\}$. Consequently, for $\chi_1,\chi_2 \in V_h$, we can replace $\{y_h,\lambda_h\}$ with $\{y_h-\chi_1, \lambda_h-\chi_2\}$ and $\{y,\lambda\}$ with $\{y-\chi_1, \lambda-\chi_2\}$. Therefore, it is sufficient to establish

$$\alpha Q_\omega^2(y_h) + Q_\omega^2(\lambda_h) \le C(\alpha L_\omega^2(y) + L_\omega^2(\lambda)),$$

whenever (2.16) holds. By Lemma 3.5,

$$\alpha Q_\omega^2(y_h) + Q_\omega^2(\lambda_h) \le 2\mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{\omega^2 y_h, \omega^2\lambda_h\}).$$

Since $\omega^2 v$ is not in V_h even if $v \in V_h$, to treat $\mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{\omega^2 y_h,\omega^2\lambda_h\})$ we add and subtract $\mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{I(\omega^2 y_h), I(\omega^2\lambda_h)\})$ and use the orthogonality property (2.16). Thus,

$$\begin{aligned}
&\mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{\omega^2 y_h,\omega^2\lambda_h\}) \\
&= \mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{\omega^2 y_h - I(\omega^2 y_h), \omega^2\lambda_h - I(\omega^2\lambda_h)\}) \\
&\quad + \mathcal{A}^{SUPG}(\{y,\lambda\},\{I(\omega^2 y_h), I(\omega^2\lambda_h)\}).
\end{aligned} \tag{3.10}$$

Applying Lemma 3.6 and Lemma 3.7 to the right hand side of (3.10) and choosing K large enough and δ small enough, we complete the proof of Theorem 3.2. ◻

Next we prove Corollary 3.3.

Proof. By Theorem 3.2, taking $\chi_1 = I(y)$ and $\chi_2 = I(\lambda)$, where I is the interpolant defined in Section 3.1.2, we obtain

$$\begin{aligned}
&\alpha\big(\epsilon\|\nabla(y-y_h)\|_{\Omega_0}^2 + \|y-y_h\|_{\Omega_0}^2 + \sum_{T_e\in\mathcal{T}_h\cap\Omega_0} \tau\|\mathbf{c}\cdot\nabla(y-y_h)\|_{T_e}^2\big) \\
&+ \epsilon\|\nabla(\lambda-\lambda_h)\|_{\Omega_0}^2 + \|\lambda-\lambda_h\|_{\Omega_0}^2 + \sum_{T_e\in\mathcal{T}_h\cap\Omega_0} \tau\|\mathbf{c}\cdot\nabla(\lambda-\lambda_h)\|_{T_e}^2 \\
&\le \alpha Q_\omega^2(y-y_h) + Q_\omega^2(\lambda-\lambda_h) \le C\left(\alpha L_\omega^2(y-I(y)) + L_\omega^2(\lambda-I(\lambda))\right).
\end{aligned}$$

Let v denote either y or λ. By the triangle inequality

$$\begin{aligned}
&L_\omega^2(v-I(v)) \\
&\le h^{-1}\|v-I(v)\|_{\Omega_s}^2 + h\|\nabla(v-I(v))\|_{\Omega_s}^2 + h^3\|\Delta(v-I(v))\|_{\Omega_s}^2 \\
&\quad + h^{-1}\|\omega(v-I(v))\|_{\Omega\setminus\Omega_s}^2 + h\|\omega\nabla(v-I(v))\|_{\Omega\setminus\Omega_s}^2 + h^3\|\omega\Delta(v-I(v))\|_{\Omega\setminus\Omega_s}^2.
\end{aligned}$$

Using the approximation theory, we have

$$h^{-1}\|v-I(v)\|_{\Omega_s}^2 + h\|\nabla(v-I(v))\|_{\Omega_s}^2 + h^3\|\Delta(v-I(v))\|_{\Omega_s}^2 \le Ch^3\|v\|_{2,\Omega_s}^2.$$

For the remaining terms we use the fact that $\omega = O(h^s)$ on $\Omega\setminus\Omega_s$ and as a result,

$$\begin{aligned}
&h^{-1}\|\omega(v-I(v))\|_{\Omega\setminus\Omega_s}^2 + h\|\omega\nabla(v-I(v))\|_{\Omega\setminus\Omega_s}^2 + h^3\|\omega\Delta(v-I(v))\|_{\Omega\setminus\Omega_s}^2 \\
&\le Ch^{2s+3}\|v\|_{2,\Omega\setminus\Omega_s}^2.
\end{aligned}$$

The last two estimates establish the corollary. ◻

4. The Case of Boundary Layers. The presence of boundary layers makes the estimates from the previous section rather meaningless. For example, $\|y\|_{H^2}$ is typically of order $\epsilon^{-3/2}$ and dominates $h^{3/2}$ for $\epsilon < h$. The numerical results of Section 5 show that the presence of boundary layers pollutes the numerical solution everywhere, even into the subdomains where the exact solution is very smooth. The numerical examples suggest that the first order convergence rates are the best possible in L^2, L^∞, or H^1 norms even over subdomains that are far away from the layers regardless the degree of polynomials used. This is in sharp contrast to the case of a single equation, where it has been shown analytically that the SUPG method does not pollute the solution into the domain of smoothness and the error is nearly optimal there (cf. [17, 23]). A rough explanation of this phenomena is the following. Assume we can solve one equation directly. Then we know that the error is nearly optimal in the interior of the domain. On the other hand without any special mesh adaptation the SUPG method can not resolve the boundary layer and the error is of order one over such layer (cf. Figures 5.1). The size of the numerical boundary layer is typically of order $h|\log h|$. Thus even if the error is consistent for one equation it will not be for the second one. In other words, the error equation will have a perturbation term which will be the source of the pollution. Of course, for the coupled system, the argument is more complicated. The main result of this section is to prove the first order error estimates in local L^2 norm (cf. Theorem 4.7) for the coupled system, and as a result for the advection-dominated optimal control problems. The numerical illustrations of Section 5 show the first order convergence rate is sharp. We will also show that the global L^2 error for the coupled system converges at the rate $h^{1/2}$ (cf. Theorem 4.11). This rate is sharp even for a single equation.

We will present a proof only in the one dimensional setting for piecewise linear elements. Throughout this section we assume that

$$\alpha = 1, \quad c = 1, \quad r = 1 \quad \text{ and } \Omega = (0, 1).$$

Even in this simple setting, the errors measured in a local L^2 norm away from the boundary layer only exhibit first order convergence. Therefore, for more general problems in higher dimensions better convergence rates can not be expected. This is confirmed by our numerical experiments. Our numerical experiments also show that in general not much can be gained from using higher order finite elements.

Since we assume $\alpha = 1$, $c = 1$, $r = 1$ and $\Omega = (0, 1)$, our problem is

$$a_h^a(\psi, \lambda_h) + \langle y_h, \psi\rangle^a = \langle \widehat{y}, \psi\rangle^a \qquad \forall \psi \in V_h, \tag{4.1a}$$

$$a_h^s(y_h, v) - \langle \lambda_h, v\rangle^s = \langle f, v\rangle_h^s \qquad \forall v \in V_h, \tag{4.1b}$$

where

$$a_h^a(\psi, \lambda_h) = \epsilon\langle \psi', \lambda_h'\rangle + \langle \psi', \lambda_h\rangle + \langle \psi, \lambda_h\rangle + \sum_{T_e \in \mathcal{T}_h} \tau\langle -\lambda_h' + \lambda_h, -\psi'\rangle_{T_e}, \tag{4.2a}$$

$$\langle y_h, \psi\rangle^a = \langle y_h, \psi\rangle + \sum_{T_e \in \mathcal{T}_h} \tau\langle y_h, -\psi'\rangle_{T_e}, \tag{4.2b}$$

$$a_h^s(y_h, v) = \epsilon\langle y_h', v'\rangle + \langle y_h', v\rangle + \langle y_h, v\rangle + \sum_{T_e \in \mathcal{T}_h} \tau\langle y_h' + y_h, v'\rangle_{T_e}, \tag{4.2c}$$

$$\langle \lambda_h, v\rangle^s = \langle \lambda_h, v\rangle + \sum_{T_e \in \mathcal{T}_h} \tau\langle \lambda_h, v'\rangle_{T_e}. \tag{4.2d}$$

4.1. Estimates of Discrete Green's Functions. Our proof makes heavy use of the discrete Green's function $G = G_{x_0} \in V_h$, which for any point $x_0 \in \Omega = (0,1)$ is the unique solution of

$$a_h^s(\chi, G) = \chi(x_0), \quad \forall \chi \in V_h. \tag{4.3}$$

Define

$$\Omega_s^- = \{\mathbf{x} \in \Omega : x - x_0 \le sKh\ell_h, \}, \quad s > 0, \tag{4.4}$$

for some constant K sufficiently large. Here and in the rest of the paper we put

$$\ell_h \equiv |\log h|.$$

The next result shows that the discrete Green's function is small outside of Ω_s^-.

LEMMA 4.1. *For ν sufficiently large, there exists $s > 0$ such that $\|G\|_{W_\infty^1(\Omega\backslash\Omega_s^-)} \le Ch^\nu$.*

The proof of these result in two dimensions is given in [23, L. 2.1]. The same argument can easily be adapted to one dimensional setting.

REMARK 4.2. *Since $G \in V_h$ and ν is arbitrary, by the inverse estimates we also have $\|G\| \le Ch^\nu$ in almost any other norm over $\Omega\backslash\Omega_s^-$.*

Following the ideas of [23], we can show the following result.

LEMMA 4.3. *Let $G = G_{x_0}$ be the one dimensional discrete Green's function defined by* (4.3)*. Then there is a constant C independent of h such that the following estimates hold:*

$$\|G'\| \le Ch^{-1/2}\ell_h^{1/2}, \quad \|G\| \le C\ell_h^{1/2}, \quad |G(x_0)| \le C\ell_h.$$

Proof. Let x_m be the first meshpoint in $\Omega\backslash\Omega_s^-$ to the right of x_0. Then by the Fundamental Theorem of Calculus

$$-G(x_0) = \sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} G'(s)ds - G(x_m), \tag{4.5}$$

where x_i are the meshpoints. Now since $|x_m - x_0| \le Ch\ell_h$

$$G(x_0) \le Ch^{1/2}\ell_h^{1/2}\|G'\| + Ch^\nu.$$

On the other hand

$$G(x_0) = a_h^s(G,G) = \epsilon\langle G', G'\rangle + \langle G', G\rangle + \langle G, G\rangle + \sum_{T_e \in \mathcal{T}_h} \tau\langle G' + G, G'\rangle_{T_e}.$$

Using that $\langle G', G\rangle = 0$ and $\sum_{T_e \in \mathcal{T}_h} \tau\langle G, G'\rangle_{T_e} = 0$, we have

$$a_h^s(G,G) = \epsilon\|G'\|^2 + \|G\|^2 + \sum_{T_e \in \mathcal{T}_h} \tau\|G'\|_{T_e}^2 = (\epsilon + \tau)\|G'\|^2 + \|G\|^2.$$

Thus using that $\epsilon \le h$ and $\tau = C_1 h$, we have for ν sufficiently large,

$$h\|G'\|^2 + \|G\|^2 \le Ca_h^s(G,G) = CG(x_0) \le Ch^{1/2}\ell_h^{1/2}\|G'\| + Ch^\nu \le \frac{h}{2}\|G'\|^2 + C\ell_h$$

and we can conclude that

$$\|G'\| \le Ch^{-1/2}\ell_h^{1/2} \quad \text{and} \quad \|G\| \le C\ell_h^{1/2}.$$

Using the estimate for $\|G'\|$ in (4.5) we see that $G(x_0) \le C\ell_h$. ∎

Later in the analysis we will be interested in the case when x_0 is near the boundary. In the case when x_0 is near the boundary that has no layer, we can show that the L^2 norm of the corresponding discrete Green's function is small. More precisely, we have the following result.

COROLLARY 4.4. *Assume $\Omega = (0,1)$ and let $G = G_{x_0}$ be the one dimensional discrete Green's function with $x_0 \in D^-$, where $D^- = \{x \in \Omega : 0 \le x \le Mh\ell_h\}$, for some $M > 0$. Then there exists a constant C independent of h such that*

$$\|G\| \le Ch^{1/2}\ell_h^{3/2}.$$

Proof. Define

$$D_s^- = \{x \in \Omega : 0 \le x \le sMh\ell_h\}, \quad s > 0.$$

Then, $\|G\| \le \|G\|_{\Omega\setminus D_s^-} + \|G\|_{D_s^-}$. By Remark 4.2 for some s we have that $\|G\|_{\Omega\setminus D_s^-}$ is small. Thus, we only need to estimate $\|G\|_{D_s^-}$.

For any $x \in D_s^-$ by the Fundamental Theorem of Calculus

$$G(x) = \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} G'(s)ds + \int_{x_m}^{x} G'(s)ds,$$

where $0 = x_0 < x_1 < \cdots < x_m \le x$. Since the mesh is quasi-uniform $x_{i+1} - x_i = O(h)$ and $m = O(M\ell_h)$. By the Cauchy-Schwartz inequality

$$|G(x)| \le \|G'\|_{D_s^-} \left(\sum_{i=0}^{m-1} (x_{i+1} - x_i)^{1/2} + (x - x_m)^{1/2} \right).$$

Squaring both sides we obtain,

$$|G(x)|^2 \le C\|G'\|_{D_s^-}^2 \left(\sum_{i=0}^{m-1} (x_{i+1} - x_i) + (x - x_m) \right) = C\|G'\|^2 x.$$

Integrating over D_s^- and using the estimates from Lemma 4.3, we obtain

$$\|G(x)\|_{D_s^-}^2 \le C\|G'\|^2 h^2 \ell_h^2 \le Ch\ell_h^3.$$

∎

In the following section we will also require the adjoint discrete Green's function $G^* = G^*_{x_0} \in V_h$, which is the unique solution of

$$a_h^a(G^*, \chi) = \chi(x_0), \quad \forall \chi \in V_h. \tag{4.6}$$

Analogously to Lemma 4.1 one can show the following estimate.

LEMMA 4.5. *If*

$$\Omega_s^+ = \{\mathbf{x} \in \Omega : x - x_0 \ge sKh\ell_h, \}, \quad s > 0, \tag{4.7}$$

then for ν sufficiently large there exists $s > 0$ such that

$$\|G^*\|_{W^1_\infty(\Omega\setminus\Omega^+_s)} \leq Ch^\nu.$$

We can also show that G^* satisfies the same global estimates as G in Lemma 4.3 and we can establish the following result which corresponds to Corollary 4.4.

COROLLARY 4.6. *Assume $\Omega = (0, 1)$ and let $G^* = G^*_{x_0}$ be the one dimensional discrete Green's function with $x_0 \in D^+$, where $D^+ = \{x \in \Omega : 1 - Mh\ell_h \leq x \leq 1\}$, for some $M > 0$. Then there exists a constant C independent of h such that*

$$\|G^*\| \leq Ch^{1/2}\ell_h^{3/2}.$$

4.2. Local error estimates. To describe the main theorem of this section, the weighted error estimates, we need the following domain decomposition $\Omega = \Omega^- \cup \Omega^0 \cup \Omega^+$, where

$$\begin{aligned} \Omega^- &= \{x \in \Omega : 0 \leq x \leq Kh\ell_h\} \\ \Omega^0 &= \{x \in \Omega : Kh\ell_h \leq x \leq 1 - Kh\ell_h\} \\ \Omega^+ &= \{x \in \Omega : 1 - Kh\ell_h \leq x \leq 1\}, \end{aligned} \tag{4.8}$$

with constant K sufficiently large. In addition we define two weight functions $\omega_+ > 0$ and $\omega_- > 0$ with the following properties:

$$\omega_+(x) \equiv 1, \qquad \text{for } x \in \Omega^- \cup \Omega^0, \tag{4.9}$$

$$\omega_+(x) \leq Ce^{(x-1)/K\,h}, \qquad \text{for } x \in \Omega^+, \tag{4.10}$$

$$\omega_+(x)' < 0. \tag{4.11}$$

and $\omega_-(x) = \omega_+(1 - x)$. Note that by construction $\omega_- = \omega_+$ on Ω^0.

THEOREM 4.7. *Let $\{y_h, \lambda_h\}$ be the SUPG solution of* (4.1) *and let y, λ, f, and $\widehat{y}$ satisfy*

$$\|y\|_{L^\infty(\Omega^+)} + \|y\|_{H^2(\Omega^-\cup\Omega^0)} + \|y\|_{W^1_1(\Omega^+)} \leq C \tag{4.12a}$$

$$\|\lambda\|_{L^\infty(\Omega^-)} + \|\lambda\|_{H^2(\Omega^0\cup\Omega^+)} + \|\lambda\|_{W^1_1(\Omega^-)} \leq C \tag{4.12b}$$

$$\|f\|_{L^\infty(\Omega^-)} + \|f\|_{L^2(\Omega^0\cup\Omega^+)} + \|\widehat{y}\|_{L^2(\Omega^-\cup\Omega^0)} + \|\widehat{y}\|_{L^\infty(\Omega^+)} \leq C. \tag{4.12c}$$

Then there exists a constant C independent of y, λ, h, and ϵ, such that

$$Q^2_{\omega_+}(y - y_h) + Q^2_{\omega_-}(\lambda - \lambda_h) \leq C\left(\min_{\chi_1\in V_h} L^2_{\omega_+}(y - \chi_1) + \min_{\chi_2\in V_h} L^2_{\omega_-}(\lambda - \chi_2) + h^2\ell_h^6\right),$$

where this time

$$Q^2_\omega(v) = (\epsilon + \tau)\|\omega v'\|^2 + \frac{1}{2}\|(\omega|\omega'|)^{1/2}v\|^2 + \|\omega v\|^2$$

and

$$L^2_\omega(v) := h^{-1}\|\omega v\|^2 + h\|\omega v'\|^2.$$

The proof of Theorem 4.7 will be given later.

REMARK 4.8. *The assumptions on the data and the exact solution are natural, since according to Theorem 2, in [19], the exact solution* $\{y, \lambda\}$ *satisfies the following estimates,*

$$\left|y^{(n)}(x)\right| \leq C\left(1+\epsilon^{-n} e^{-(1-x)/\epsilon}\right), \quad n=0,1$$
$$\left|\lambda^{(n)}(x)\right| \leq C\left(1+\epsilon^{-n} e^{-x/\epsilon}\right), \quad n=0,1.$$

REMARK 4.9. *In the proof we did not try to obtain sharp estimates in terms of the power of the logarithmic terms. In the case of* $\epsilon \ll h$ *this power can be reduced. Furthermore, the numerical results suggest that the estimates in this case should be log free.*

Theorem 4.7 implies the following local error estimate.

COROLLARY 4.10. *Under the assumptions of Theorem 4.7 there exists a constant* C *independent of* y, λ, ϵ, *and* h *such that,*

$$\|y-y_h\|_{\Omega^-\cup\Omega^0} + \|\lambda-\lambda_h\|_{\Omega^0\cup\Omega^+} \leq Ch\ell_h^3.$$

The proof is similar to that of Corollary 3.3. The numerical results in Section 5 show that the above estimates, modulo logarithmic terms, are sharp.

4.3. Outline of the proof. The proof of Theorem 4.7 is more technical and complicated than in the case of the interior layers (cf. Section 3) or in the case of a single equation (cf. [17, 23]). The main difficulty is due to the presence of the coupled terms of the form

$$\langle \lambda_h-\lambda, \omega_+^2 y_h\rangle - \langle y_h-y, \omega_-^2\lambda_h\rangle. \tag{4.13}$$

All other terms are similar to the single equation case and of almost optimal order over the domains of smoothness. In the interior layer case $\omega_+=\omega_-=\omega$ and since we use the same finite dimensional subspaces for both λ_h and y_h

$$\langle \lambda_h, \omega_+^2 y_h\rangle - \langle y_h, \omega_-^2\lambda_h\rangle = 0.$$

In the presence of boundary layers there is no such cancellation. Numerical results presented in Section 5 show the first order convergence rates in L^2 or H^1 norms even over the subdomains of smoothness. Thus, the observed order reduction is due to the presence of the coupling terms similar to the ones in (4.13). Hence the main challenge is to show that these terms are if fact of order h.

We examine the coupling terms more closely. Consider for example,

$$\langle \lambda_h-\lambda, \omega_+^2 y_h\rangle = \langle \lambda_h-\lambda, \omega_+^2 y_h\rangle_{\Omega^-} + \langle \lambda_h-\lambda, \omega_+^2 y_h\rangle_{\Omega^0\cup\Omega^+}.$$

Since λ is smooth on $\Omega^0\cup\Omega^+$, we expect $\|\lambda-\lambda_h\|_{\Omega^0\cup\Omega^+}$ to be of optimal order and as a result the term $\langle \lambda_h-\lambda, \omega_+^2 y_h\rangle_{\Omega^0\cup\Omega^+}$ should not pose much trouble. On the other hand, in general, λ has a boundary layer over Ω^- and since the SUPG method does not resolve the layer, $\|\lambda-\lambda_h\|_{L^\infty(\Omega^-)} = O(1)$. If for example we use the Cauchy-Schwartz inequality and use that $\omega_+=1$ on Ω^- we obtain

$$\langle \lambda_h-\lambda, \omega_+^2 y_h\rangle_{\Omega^-} \leq \|\lambda-\lambda_h\|_{\Omega^-}\|y_h\|_{\Omega^-}.$$

The size of Ω^- is of order $h\ell_h$ and we expect $\|\lambda - \lambda_h\|_{\Omega^-} = O(h^{1/2}\ell_h^{1/2})$. This result we in fact establish in Theorem 4.11. Hence the main problem is to show $\|y_h\|_{\Omega^-} = O(h^{3/2})$. Looking at the exact solution y we can see that such estimate is feasible. For example, by the Mean Value Theorem, $\|y\|_{L^\infty(\Omega^-)} = O(h)$ and using that the size of of Ω^- is of order $h\ell_h$ we do obtain $\|y\|_{\Omega^-} = O(h^{3/2})$. For y_h the analysis is more complicated due to the fact that it is not clear to us how to establish directly that $\|y_h\|_{L^\infty} \le C$ or even $\|y_h\|_{L^\infty} \le C\ell_h$. In Proposition 4.12 we prove the somewhat weaker result

$$\|y_h\|_{L^\infty(\Omega^-)} \le Ch\ell_h^3 + \frac{1}{16}\|\lambda_h\|^2.$$

Still, this result is sufficient to derive the desired estimates. In fact, it implies that $\|y_h\|_{L^\infty} \le C\ell_h^{3/2}$. The rest of this section is devoted to provide the technical details following the above outline.

4.4. Global L^2 error estimates.. The following result, which has an independent interest, shows that even if the exact solution has boundary layers, the error in the global L^2 norm converges almost at the rate $h^{1/2}$. This result is sharp even in the case of a single equation.

THEOREM 4.11. *Let $\{y_h, \lambda_h\}$ be the SUPG solution of (4.1) and let y, λ, f, and $\widehat{y}$ satisfy the assumption of Theorem 4.7. Then there exists a constant C independent of y, λ, h, and ϵ, such that*

$$\|y - y_h\| + \|\lambda - \lambda_h\| \le Ch^{1/2}\ell_h^{3/2}.$$

Proof. From (2.16) for any $\phi_h, \varphi_h \in V_h$,

$$\mathcal{A}^{SUPG}(\{y_h, \lambda_h\}, \{\phi_h, \varphi_h\}) = \mathcal{A}^{SUPG}(\{y, \lambda\}, \{\phi_h, \varphi_h\}).$$

Choosing $\phi_h = -\lambda_h$ and $\varphi_h = y_h$ we obtain,

$$\begin{aligned}
&\mathcal{A}^{SUPG}(\{y_h, \lambda_h\}, \{-\lambda_h, y_h\}) \\
&= \langle \lambda_h, \lambda_h\rangle^s + \langle y_h, y_h\rangle^a + a_h^s(y_h, -\lambda_h) + a_h^a(y_h, \lambda_h) \\
&= \|\lambda_h\|^2 + \|y_h\|^2 \\
&= \mathcal{A}^{SUPG}(\{y, \lambda\}, \{-\lambda_h, y_h\}) \\
&= \langle \lambda, \lambda_h\rangle^s + \langle y, y_h\rangle^a + a_h^s(y, -\lambda_h) + a_h^a(y_h, \lambda) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.14}$$

We start with

$$I_1 = \langle \lambda, \lambda_h\rangle^s = \langle \lambda, \lambda_h\rangle + \sum_{T_e \in \mathcal{T}_h} \tau\langle \lambda, \lambda_h'\rangle_{T_e}.$$

By the Cauchy-Schwarz and the arithmetic-geometric mean inequalities,

$$\langle \lambda, \lambda_h\rangle \le \|\lambda\|\|\lambda_h\| \le 4\|\lambda\|^2 + \frac{1}{16}\|\lambda_h\|^2.$$

By the Cauchy-Schwarz, the inverse inequality, and the arithmetic-geometric mean inequalities and the assumption $\tau = C_1 h$,

$$\tau\langle \lambda, \lambda_h'\rangle_{T_e} \le C_1 C_{inv}\|\lambda\|_{T_e}\|\lambda_h\|_{T_e} \le C\|\lambda\|_{T_e}^2 + \frac{1}{16}\|\lambda_h\|_{T_e}^2.$$

Thus summing over all elements we obtain,

$$I_1 \leq C\|\lambda\|^2 + \frac{1}{8}\|\lambda_h\|^2. \tag{4.15}$$

Very similarly we can obtain

$$I_2 \leq C\|y\|^2 + \frac{1}{8}\|y_h\|^2. \tag{4.16}$$

The analysis of I_3 and I_4 is more involved and we require the following result.

PROPOSITION 4.12. *Under the assumptions of Theorem 4.11, there exists a constant C independent of h, y, and λ such that*

$$\|\lambda_h\|_{L^\infty(\Omega^+)} \leq Ch\ell_h^3 + \frac{1}{16}\|y_h\|^2 \quad \textit{and} \quad \|y_h\|_{L^\infty(\Omega^-)} \leq Ch\ell_h^3 + \frac{1}{16}\|\lambda_h\|^2.$$

Assuming the proposition for now, we give an estimate for I_3. Recall,

$$I_3 = a_h^s(y, -\lambda_h) = \epsilon\langle y', -\lambda_h'\rangle + \langle y', -\lambda_h\rangle + \langle y, -\lambda_h\rangle + \sum_{T_e\in\mathcal{T}_h} \tau\langle y' + y, -\lambda_h'\rangle_{T_e}.$$

By the assumptions of the theorem, y is smooth on $\Omega^- \cup \Omega^0$, but is rather rough on Ω^+, only in W_1^1. Thus initially we split the first term on the right hand side as

$$\epsilon\langle y', -\lambda_h'\rangle = \epsilon\langle y', -\lambda_h'\rangle_{\Omega^-\cup\Omega^0} + \epsilon\langle y', -\lambda_h'\rangle_{\Omega^+} = J_1 + J_2.$$

By the Cauchy-Schwarz, the inverse inequality, the arithmetic-geometric mean inequality, and the assumption $\epsilon \leq h$

$$J_1 \leq CC_{inv}\|y'\|_{\Omega^-\cup\Omega^0}\|\lambda_h\| \leq C\|y'\|^2_{\Omega^-\cup\Omega^0} + \frac{1}{16}\|\lambda_h\|^2.$$

By the Hölder and the inverse inequalities, (4.12a), and Proposition 4.12,

$$J_2 \leq CC_{inv}\|y'\|_{L^1(\Omega^+)}\|\lambda_h\|_{L^\infty(\Omega^+)} \leq C\|\lambda_h\|_{L^\infty(\Omega^+)} \leq Ch\ell_h^3 + \frac{1}{16}\|y_h\|^2.$$

Thus, we can we conclude,

$$\epsilon\langle y', -\lambda_h'\rangle \leq Ch\ell_h^3 + \frac{1}{16}\|\lambda_h\|^2 + \frac{1}{16}\|y_h\|^2. \tag{4.17}$$

Very similarly we can treat the second and the third terms $\langle y', -\lambda_h\rangle + \langle y, -\lambda_h\rangle$. To treat the stabilization terms we split the sum,

$$\begin{aligned}
&\sum_{T_e\in\mathcal{T}_h} \tau\langle y' + y, -\lambda_h'\rangle_{T_e} \\
&= \sum_{T_e\in\mathcal{T}_h\cap(\Omega^-\cup\Omega^0)} \tau\langle y' + y, -\lambda_h'\rangle_{T_e} + \sum_{T_e\in\mathcal{T}_h\cap\Omega^+} \tau\langle y' + y, -\lambda_h'\rangle_{T_e} \\
&= S_1 + S_2.
\end{aligned}$$

To treat S_1, we use the Cauchy-Schwarz inequality, the assumption $\tau = C_1 h$, the inverse inequality, and the arithmetic-geometric mean inequality to obtain

$$\begin{aligned}
S_1 &\leq \sum_{T_e\in\mathcal{T}_h\cap(\Omega^-\cup\Omega^0)} \|y' + y\|_{T_e} C_1 C_{inv}\|\lambda_h\|_{T_e} \\
&\leq \sum_{T_e\in\mathcal{T}_h\cap(\Omega^-\cup\Omega^0)} C\|y' + y\|^2_{T_e} + \frac{1}{16}\|\lambda_h\|^2_{T_e} \\
&\leq C\|y' + y\|^2_{\Omega^-\cup\Omega^0} + \frac{1}{16}\|\lambda_h\|^2.
\end{aligned}$$

To treat S_2, we use the Hölder inequality, the assumption $\tau = C_1 h$, the inverse inequality, and (4.12a),

$$
\begin{aligned}
S_2 &\le \sum_{T_e \in \mathcal{T}_h \cap \Omega^+} \|y' + y\|_{0,1,T_e} C_1 C_{inv} \|\lambda_h\|_{0,\infty,T_e} \\
&\le C \|\lambda_h\|_{0,\infty,\Omega^+} \sum_{T_e \in \mathcal{T}_h \cap \Omega^+} \|y' + y\|_{0,1,T_e} \\
&= C \|\lambda_h\|_{0,\infty,\Omega^+} \|y' + y\|_{0,1,\Omega^+} \le C \|\lambda_h\|_{0,\infty,\Omega^+} \\
&\le C h \ell_h^3 + \frac{1}{16} \|y_h\|^2,
\end{aligned}
$$

where in the last step we used the result of Proposition 4.12. Thus, we have shown,

$$
I_3 \le C\big(h\ell_h^3 + \|y'\|^2_{\Omega^- \cup \Omega^0} + \|y\|^2_{\Omega^- \cup \Omega^0}\big) + \frac{1}{4}\|\lambda_h\|^2 + \frac{1}{4}\|y_h\|^2. \tag{4.18}
$$

Similarly we can obtain

$$
I_4 \le C\big(h\ell_h^3 + \|\lambda'\|^2_{\Omega^0 \cup \Omega^+} + \|\lambda\|^2_{\Omega^0 \cup \Omega^+}\big) + \frac{1}{4}\|\lambda_h\|^2 + \frac{1}{4}\|y_h\|^2. \tag{4.19}
$$

Using (4.14), (4.15), (4.16), (4.18), and (4.19) we obtain

$$
\|\lambda_h\|^2 + \|y_h\|^2 \le C\left(\|\lambda'\|^2_{\Omega^0 \cup \Omega^+} + \|\lambda\|^2 + \|y'\|^2_{\Omega^- \cup \Omega^0} + \|y\|^2 + h\ell_h^3\right).
$$

Using the triangle inequality, assumptions of the theorem and that $diam(\Omega^-)$ and $diam(\Omega^+)$ are of order $h\ell_h$,

$$
\begin{aligned}
\|\lambda\|^2 + \|y\|^2 &\le \|\lambda\|^2_{\Omega^0 \cup \Omega^+} + \|y\|^2_{\Omega^- \cup \Omega^0} + \|\lambda\|^2_{\Omega^-} + \|y\|^2_{\Omega^+} \\
&\le \|\lambda\|^2_{\Omega^0 \cup \Omega^+} + \|y\|^2_{\Omega^- \cup \Omega^0} + C h\ell_h (\|\lambda\|^2_{0,\infty,\Omega^-} + \|y\|^2_{0,\infty,\Omega^+}) \\
&\le \|\lambda\|^2_{\Omega^0 \cup \Omega^+} + \|y\|^2_{\Omega^- \cup \Omega^0} + C h\ell_h.
\end{aligned}
$$

Thus,

$$
\|\lambda_h\|^2 + \|y_h\|^2 \le C\left(h\ell_h^3 + \|\lambda\|^2_{1,\Omega^0 \cup \Omega^+} + \|y\|^2_{1,\Omega^- \cup \Omega^0}\right). \tag{4.20}
$$

Since the SUPG method is invariant on V_h, (4.20) imply

$$
\|\lambda_h - I(\lambda)\|^2 + \|y_h - I(y)\|^2 \le C\big(h\ell_h^3 + \|\lambda - I(\lambda)\|^2_{1,\Omega^0 \cup \Omega^+} + \|y - I(y)\|^2_{1,\Omega^- \cup \Omega^0}\big),
$$

where $I(v)$ denotes the interpolant of v. Hence by the triangle inequality and the estimate above we have,

$$
\begin{aligned}
\|\lambda_h - \lambda\|^2 + \|y_h - y\|^2 &\le \|\lambda_h - I(\lambda)\|^2 + \|y_h - I(y)\|^2 + \|\lambda - I(\lambda)\|^2 + \|y - I(y)\|^2 \\
&\le \|\lambda - I(\lambda)\|^2 + \|y - I(y)\|^2 \\
&\quad + C\big(h\ell_h^3 + \|\lambda - I(\lambda)\|^2_{1,\Omega^0 \cup \Omega^+} + \|y - I(y)\|^2_{1,\Omega^- \cup \Omega^0}\big) \\
&\le \|\lambda - I(\lambda)\|^2_{\Omega^-} + \|y - I(y)\|^2_{\Omega^+} \\
&\quad + C\big(h\ell_h^3 + \|\lambda - I(\lambda)\|^2_{1,\Omega^0 \cup \Omega^+} + \|y - I(y)\|^2_{1,\Omega^- \cup \Omega^0}\big).
\end{aligned}
$$

Using the local approximation properties of the interpolant, Section 3.1.2 and the assumptions of the theorem on λ and y and the assumption $\epsilon \le h$, we conclude

$$
\begin{aligned}
\|\lambda_h - \lambda\|^2 + \|y_h - y\|^2 &\le C h\ell_h^3 + C h\ell_h (\|\lambda\|^2_{0,\infty,\Omega^-} + \|y\|^2_{0,\infty,\Omega^+}) \\
&\quad + C h^2 \left(|\lambda|^2_{2,\Omega^0 \cup \Omega^+} + |y|^2_{2,\Omega^- \cup \Omega^0}\right) \le C h\ell_h^3.
\end{aligned}
$$

Taking the square root we conclude the proof of the theorem. ∎

REMARK 4.13. *The case $\alpha \neq 1$ can be handled similarly. One just needs to take $\phi_h = -\alpha\lambda_h$ and $\varphi_h = y_h$ at the beginning of the proof.*

REMARK 4.14. *The proof of the above result could be significantly simplified if*

$$\|y_h\|_{0,\infty} + \|\lambda_h\|_{0,\infty} \leq C, \tag{4.21}$$

were known for some C, that may depend on ℓ_h. Using the Green's function estimates of Lemma 4.3, such estimate is easy to obtain for a single equation. It is not obvious to us how to obtain such estimate for the coupled system directly. However, the result of Theorem 4.11 does imply (4.21)*, whenever the assumptions of Theorem 4.11 hold. This can easily be established by the inverse inequality, the triangle inequality, and the approximation theory. Thus,*

$$\begin{aligned}
\|y_h\|_{0,\infty,\Omega^-} + \|y_h\|_{0,\infty,\Omega^+} &\leq C_{inv}h^{-1/2}(\|y_h\|_{\Omega^-} + \|y_h\|_{\Omega^+})\\
&\leq Ch^{-1/2}(\|y - y_h\| + \|y\|_{\Omega^-} + \|y\|_{\Omega^+})\\
&\leq Ch^{-1/2}(Ch^{1/2}\ell_h^{3/2} + Ch^{1/2}\ell_h^{1/2}\|y\|_{0,\infty,\Omega}) \leq C\ell_h^{3/2}
\end{aligned}$$

and similarly,

$$\begin{aligned}
\|y_h\|_{0,\infty,\Omega^0} &\leq \|y - y_h\|_{0,\infty,\Omega^0} + \|y\|_{0,\infty,\Omega^0}\\
&\leq \|y - I(y)\|_{0,\infty,\Omega^0} + \|y_h - I(y)\|_{0,\infty,\Omega^0} + \|y\|_{0,\infty,\Omega^0}\\
&\leq \|y - I(y)\|_{0,\infty,\Omega^0} + C_{inv}h^{-1/2}\|y_h - I(y)\|_{\Omega^0} + \|y\|_{0,\infty,\Omega^0}\\
&\leq C\|y\|_{0,\infty,\Omega} + C_{inv}h^{-1/2}(\|y_h - y\|_{\Omega^0} + \|y - I(y)\|_{\Omega^0})\\
&\leq C\|y\|_{0,\infty,\Omega} + C_{inv}h^{-1/2}(Ch^{1/2}\ell_h^{3/2} + Ch^2\|y\|_{2,\Omega^0}) \leq C\ell_h^{3/2}.
\end{aligned}$$

Estimates for λ_h are very similar.

In two dimensions, even for a single equation, such estimates are not known. On general quasi-uniform meshes, the sharpest result so far was obtained by Niijima, [23], which says that $\|y_h\|_{0,\infty} \leq Ch^{-1/8}\ell_h$ for $\epsilon \leq h^{3/2}$. On structured meshes some improvements are possible (cf. [31]).

REMARK 4.15. *The above argument does not require $r = 1$ or even $r > 0$. The same proof works for any $r(x) \geq 0$.*

4.5. Proof of Proposition 4.12. We will only provide a proof for $\|\lambda_h\|_{0,\infty,\Omega^+}$. The proof for $\|y_h\|_{0,\infty,\Omega^-}$ is very similar. Let $z_0 \in \Omega^+$ be such that $\|\lambda_h\|_{0,\infty,\Omega^+} = |\lambda_h(z_0)|$ and $G^* = G^*_{z_0} \in V_h$ be the discrete Green's function defined in (4.6). Then using the Green's function representation we have

$$\lambda_h(z_0) = a_h^a(G^*, \lambda_h) = \langle \widehat{y}, G^*\rangle^a - \langle y_h, G^*\rangle^a.$$

Define

$$\Omega_s^+ = \{x \in \Omega : 1 - sKh\ell_h \leq x \leq 1\}, \quad \text{for } s > 0. \tag{4.22}$$

By Lemma 4.5, G^* is small on $\Omega\backslash\Omega_s^+$, for some s. Hence by the assumption of the theorem and using that the diameter of Ω_s^+ is of order $h\ell_h$, Lemma 4.3, and Corollary 4.6,

$$\begin{aligned}
\langle \widehat{y}, G^*\rangle &= \langle \widehat{y}, G^*\rangle_{\Omega\backslash\Omega_s^+} + \langle \widehat{y}, G^*\rangle_{\Omega_s^+}\\
&\leq \|\widehat{y}\|_{\Omega\backslash\Omega_s^+}\|G^*\|_{\Omega\backslash\Omega_s^+} + \|\widehat{y}\|_{0,\infty,\Omega_s^+}\|G^*\|_{0,1,\Omega_s^+}\\
&\leq Ch^\nu + Ch^{1/2}\ell_h^{1/2}\|G^*\|_{\Omega_s^+} \leq Ch^\nu + Ch\ell_h^2 \leq Ch\ell_h^2.
\end{aligned}$$

Similarly, using that $\tau = C_1 h$,

$$\begin{aligned}
&\sum_{T_e\in\mathcal{T}_h} \tau\langle \widehat{y}, -(G^*)'\rangle_{T_e} \\
&= \sum_{T_e\in\mathcal{T}_h\cap(\Omega\setminus\Omega_s^+)} \tau\langle \widehat{y}, -(G^*)'\rangle_{T_e} + \sum_{T_e\in\mathcal{T}_h\cap\Omega_s^+} \tau\langle \widehat{y}, -(G^*)'\rangle_{T_e} \\
&\leq \sum_{T_e\in\mathcal{T}_h\cap(\Omega\setminus\Omega_s^+)} \tau\|\widehat{y}\|_{T_e}\|(G^*)'\|_{T_e} + \sum_{T_e\in\mathcal{T}_h\cap\Omega_s^+} \tau\|\widehat{y}\|_{0,\infty,T_e}\|(G^*)'\|_{0,1,T_e} \\
&\leq Ch^\nu \sum_{T_e\in\mathcal{T}_h\cap(\Omega\setminus\Omega_s^+)} \|\widehat{y}\|_{T_e}^2 + C\|\widehat{y}\|_{0,\infty,\Omega_s^+} \sum_{T_e\in\mathcal{T}_h\cap\Omega_s^+} h^{3/2}\|(G^*)'\|_{T_e} \\
&\leq Ch^\nu\|\widehat{y}\|_{\Omega\setminus\Omega_s^+}^2 + C\|\widehat{y}\|_{0,\infty,\Omega_s^+} \sum_{T_e\in\mathcal{T}_h\cap\Omega_s^+} \left(h + h^2\|(G^*)'\|_{T_e}^2\right) \\
&\leq Ch^\nu + Ch\ell_h + Ch^2\|(G^*)'\|^2 \leq Ch^\nu + Ch\ell_h + Ch^2h^{-1}\ell_h \leq Ch\ell_h.
\end{aligned}$$

By the Cauchy-Schwarz inequality and Corollary 4.6,

$$\langle y_h, G^*\rangle \leq \|y_h\|\|G^*\| \leq Ch^{1/2}\ell^{3/2}\|y_h\| \leq \frac{1}{32}\|y_h\|^2 + Ch\ell_h^3.$$

Similarly, using the inverse inequality and Corollary 4.6,

$$\begin{aligned}
\sum_{T_e\in\mathcal{T}_h} \tau\langle y_h, -(G^*)'\rangle_{T_e} &\leq \sum_{T_e\in\mathcal{T}_h} \tau\|y_h\|_{T_e}\|(G^*)'\|_{T_e} \leq C\sum_{T_e\in\mathcal{T}_h} \|y_h\|_{T_e}\|G^*\|_{T_e} \\
&\leq \frac{1}{32}\sum_{T_e\in\mathcal{T}_h} \|y_h\|_{T_e}^2 + C\sum_{T_e\in\mathcal{T}_h} \|G^*\|_{T_e}^2 = \frac{1}{32}\|y_h\|^2 + C\|G^*\|^2 \\
&\leq Ch\ell_h + \frac{1}{32}\|y_h\|^2.
\end{aligned}$$

Thus,

$$\|\lambda_h\|_{0,\infty,\Omega^+} \leq Ch\ell_h^3 + \frac{1}{16}\|y_h\|^2. \tag{4.23}$$

The analysis for $\|y_h\|_{0,\infty,\Omega^-}$ is very similar.

4.6. Proof of Theorem 4.7. Step 1: Initial estimate for the uncoupled terms.

From arguments in [17] or [23] it follows that for some constant C_0

$$Q_{\omega_+}^2(y_h) \leq C_0 a_h^s(y_h, \omega_+^2 y_h) \quad \text{and} \quad Q_{\omega_-}^2(\lambda_h) \leq C_0 a_h^a(\omega_-^2\lambda_h, \lambda_h).$$

Thus,

$$\begin{aligned}
&\frac{1}{C_0}\left(Q_{\omega_+}^2(y_h) + Q_{\omega_-}^2(\lambda_h)\right) - \langle\lambda_h, \omega_+^2 y_h\rangle^s + \langle y_h, \omega_-^2\lambda_h\rangle^a \\
&\leq a_h^s(y_h, \omega_+^2 y_h) - \langle\lambda_h, \omega_+^2 y_h\rangle^s + a_h^a(\omega_-^2\lambda_h, \lambda_h) + \langle y_h, \omega_-^2\lambda_h\rangle^a \\
&= \mathcal{A}^{SUPG}(\{y_h, \lambda_h\}, \{\omega_+^2 y_h, \omega_-^2\lambda_h\}).
\end{aligned}$$

Adding and subtracting $\mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{I(\omega_+^2 y_h), I(\omega_-^2\lambda_h)\})$ and using the orthogonality relation (2.16), we have

$$\begin{aligned}
\mathcal{A}^{SUPG}&(\{y_h,\lambda_h\},\{\omega_+^2 y_h,\omega_-^2\lambda_h\}) = \mathcal{A}^{SUPG}(\{y,\lambda\},\{I(\omega_+^2 y_h), I(\omega_-^2\lambda_h)\})\\
&+\mathcal{A}^{SUPG}(\{y_h,\lambda_h\},\{E_{\omega_+}(y_h),E_{\omega_-}(\lambda_h)\})\\
&= a_h^s(y,I(\omega_+^2 y_h)) - \langle\lambda, I(\omega_+^2 y_h)\rangle^s + a_h^a(I(\omega_-^2\lambda_h),\lambda) + \langle y, I(\omega_-^2\lambda_h)\rangle^a\\
&+ a_h^s(y_h,E_{\omega_+}(y_h)) - \langle\lambda_h, E_{\omega_+}(y_h)\rangle^s + a_h^a(E_{\omega_-}(\lambda_h),\lambda_h) + \langle y_h, E_{\omega_-}(\lambda_h)\rangle^a,
\end{aligned}$$

where $E_\omega(v) = \omega^2 v - I(\omega^2 v)$. Similarly to Lemma 3.6 and Lemma 3.7, we can show

$$a_h^s(y_h,E_{\omega_+}(y_h)) + a_h^a(E_{\omega_-}(\lambda_h)),\lambda_h) \le CK^{-1}\left(Q^2_{\omega_+}(y_h)+Q^2_{\omega_-}(\lambda_h)\right),$$

and

$$\begin{aligned}
&a_h^s(y,I(\omega_+^2 y_h)) + a_h^a(I(\omega_-^2\lambda_h),\lambda)\\
&\le C(\delta+K^{-1})\left(Q^2_{\omega_+}(y_h)+Q^2_{\omega_-}(\lambda_h)\right) + C\big(L^2_{\omega_+}(y)+L^2_{\omega_-}(\lambda)\big).
\end{aligned}$$

Thus, for δ small enough and K large enough we have

$$\begin{aligned}
&Q^2_{\omega_+}(y_h)+Q^2_{\omega_-}(\lambda_h)\\
&\le C\left(L^2_{\omega_+}(y)+L^2_{\omega_-}(\lambda) + \langle\lambda_h-\lambda, I(\omega_+^2 y_h)\rangle^s - \langle y_h - y, I(\omega_-^2\lambda_h)\rangle^a\right).
\end{aligned}\tag{4.24}$$

Thus, we only need to estimate the coupling terms,

$$\langle\lambda_h-\lambda, I(\omega_+^2 y_h)\rangle^s - \langle y_h - y, I(\omega_-^2\lambda_h)\rangle^a.\tag{4.25}$$

Step 2: Estimating coupled terms (4.25)
We split $\langle\lambda_h-\lambda, I(\omega_+^2 y_h)\rangle^s$ and $\langle y_h - y, I(\omega_-^2\lambda_h)\rangle^a$ over three subdomains Ω^-, Ω^0, and Ω^+, which are defined at the beginning of Section 4.4. Using that $\omega^+ = 1$ on Ω^- and Ω^0 and $I(y_h) = y_h$, we have

$$\langle\lambda_h-\lambda, I(\omega_+^2 y_h)\rangle^s = \langle\lambda_h-\lambda, y_h\rangle^s_{\Omega^-} + \langle\lambda_h-\lambda, y_h\rangle^s_{\Omega^0} + \langle\lambda_h-\lambda, I(\omega_+^2 y_h)\rangle^s_{\Omega^+} = I_1+I_2+I_3.$$

We start with

$$I_1 = \langle\lambda_h-\lambda, y_h\rangle_{\Omega^-} + \sum_{T_e\in\mathcal{T}_h\cap\Omega^-}\tau\langle\lambda_h-\lambda, y_h'\rangle_{T_e}.$$

Using $diam(\Omega^-) = Kh\ell_h$ we find $\|\lambda_h-\lambda\|_{L^1(\Omega^-)} \le (Kh\ell_h)^{1/2}\|\lambda_h-\lambda\|$. This inequality, inequality $\|y_h\|_{L^\infty(\Omega^-)} \le Ch\ell_h^3 + \|\lambda_h\|^2$ from Proposition 4.12, and using the Cauchy Schwarz inequality and Theorem 4.11 yield

$$\begin{aligned}
\langle\lambda_h-\lambda, y_h\rangle_{\Omega^-} &\le \|\lambda_h-\lambda\|_{0,1,\Omega^-}\|y_h\|_{0,\infty,\Omega^-}\\
&\le (Kh\ell_h)^{1/2}\|\lambda_h-\lambda\|(Ch\ell_h^3+\|\lambda_h\|^2) \le Ch\ell_h^2(Ch\ell_h^3+\|\lambda_h\|^2)\\
&\le Ch^2\ell_h^5 + Ch^2\ell^4 + C\|\lambda_h\|^4 \le Ch^2\ell_h^5 + C\|\lambda_h\|^4.
\end{aligned}$$

By the inverse inequality and the estimates above,

$$\begin{aligned}
\sum_{T_e\in\mathcal{T}_h\cap\Omega^-}\tau\langle\lambda_h-\lambda, y_h'\rangle_{T_e} &\le \sum_{T_e\in\mathcal{T}_h\cap\Omega^-}\|\lambda_h-\lambda\|_{0,1,T_e}C_{inv}\|y_h\|_{0,\infty,T_e}\\
&\le C\|y_h\|_{0,\infty,\Omega^-}\|\lambda_h-\lambda\|_{0,1,\Omega^-} \le C(h^2\ell_h^5+\|\lambda_h\|^4).
\end{aligned}$$

Thus,

$$I_1 \leq C(h^2\ell_h^5 + \|\lambda_h\|^4). \tag{4.26}$$

We postpone the estimate of I_2 for now and proceed with

$$I_3 \leq \langle \lambda_h - \lambda, I(\omega_+^2 y_h)\rangle_{\Omega^+} + \sum_{T_e \in \mathcal{T}_h \cap \Omega^+} \tau\langle \lambda_h - \lambda, I(\omega_+^2 y_h)'\rangle_{T_e}.$$

Notice that by the Mean Value Theorem and the assumption of the theorem,

$$\|\lambda\|_{0,\infty,\Omega^+} \leq Ch\ell_h\|\lambda'\|_{0,\infty,\Omega^+} \leq Ch\ell_h. \tag{4.27}$$

Using the Hölder, the triangle, and the Cauchy-Schwarz inequalities, the stability of the interpolant, Proposition 4.12, and (4.27), we have,

$$\begin{aligned}
\langle \lambda_h - \lambda, I(\omega_+^2 y_h)\rangle_{\Omega^+} &\leq \|\lambda - \lambda_h\|_{0,\infty,\Omega^+}\|I(\omega_+^2 y_h)\|_{0,1,\Omega^+} \\
&\leq (\|\lambda\|_{0,\infty,\Omega^+} + \|\lambda_h\|_{0,\infty,\Omega^+})Ch^{1/2}\ell_h^{1/2}\|I(\omega_+^2 y_h)\|_{\Omega^+} \\
&\leq C\big(h\ell_h^3 + \|y_h\|^2\big)Ch^{1/2}\ell_h^{1/2}\|y_h\| \leq C(h^2\ell_h^5 + \|y_h\|^4).
\end{aligned}$$

The other term in I_3 can be estimated similarly to obtain

$$I_3 \leq C(h^2\ell_h^5 + \|y_h\|^4). \tag{4.28}$$

Similarly, using that $\omega_- = 1$ on Ω^+ and Ω^0 and $I(\lambda_h) = \lambda_h$ we have

$$\langle y_h - y, I(\omega_-^2\lambda_h)\rangle^a = \langle y_h - y, I(\omega_-^2\lambda_h)\rangle^a_{\Omega^-} + \langle y_h - y, \lambda_h\rangle^a_{\Omega^0} + \langle y_h - y, \lambda_h\rangle^a_{\Omega^+} = J_1 + J_2 + J_3.$$

The estimates for the terms J_1 and J_3 are very similar to I_1 and I_3 and we can obtain

$$J_1 \leq C(h^2\ell_h^5 + \|y_h\|^4) \quad \text{and} \quad J_3 \leq C(h^2\ell_h^5 + \|\lambda_h\|^4). \tag{4.29}$$

Thus, we are only left to estimate $I_2 - J_2$. Using that $\langle y_h, \lambda_h\rangle^s = \langle y_h, \lambda_h\rangle^a$, we have

$$I_2 - J_2 = \langle \lambda_h, y_h\rangle^s_{\Omega^0} - \langle \lambda, y_h\rangle^s_{\Omega^0} - \langle y_h, \lambda_h\rangle^a_{\Omega^0} + \langle y, \lambda_h\rangle^a_{\Omega^0} = -\langle \lambda, y_h\rangle^s_{\Omega^0} + \langle y, \lambda_h\rangle^a_{\Omega^0}.$$

By the Cauchy-Schwarz and the arithmetic-geometric mean inequalities

$$\begin{aligned}
\langle \lambda, y_h\rangle^s_{\Omega^0} &= \langle \lambda, y_h\rangle_{\Omega^0} + \sum_{T_e \in \mathcal{T}_h \cap \Omega^0} \tau\langle \lambda, y_h'\rangle_{T_e} \\
&\leq \|\lambda\|_{\Omega^0}\|y_h\|_{\Omega^0} + C_1 C_{inv} \sum_{T_e \in \mathcal{T}_h \cap \Omega^0} \|\lambda\|_{T_e}\|y_h\|_{T_e} \\
&\leq Ch^{-1}\|\lambda\|^2_{\Omega^0} + Ch\|y_h\|^2_{\Omega^0} \leq C\left(L^2_{\omega_-}(\lambda) + h^2 + \|y_h\|^4\right).
\end{aligned}$$

Similarly,

$$\langle y, \lambda_h\rangle^a_{\Omega^0} \leq C\left(L^2_{\omega_+}(y) + h^2 + \|\lambda_h\|^4\right).$$

Step 3: Combining all estimates and using consistency of the method. Taking the above estimates into account we obtain,

$$Q^2_{\omega_+}(y_h) + Q^2_{\omega_-}(\lambda_h) \leq C\left(L^2_{\omega_+}(y) + L^2_{\omega_-}(\lambda) + Ch^2\ell_h^5 + \|y_h\|^4 + \|\lambda_h\|^4\right). \tag{4.30}$$

Now proceeding exactly like at the end of the proof of Theorem 4.11, the last estimate (4.30) implies

$$
\begin{aligned}
&Q^2_{\omega_+}(y-y_h)+Q^2_{\omega_-}(\lambda-\lambda_h)\\
&\le C\left(L^2_{\omega_+}(y-\chi_1)+L^2_{\omega_-}(\lambda-\chi_2)+Ch^2\ell_h^5+\|y-y_h\|^4+\|\lambda-\lambda_h\|^4\right),
\end{aligned}
$$

for any $\chi_1,\chi_2\in V_h$. The global L^2 error estimates of Theorem 4.11 finish the proof of Theorem 4.7.

5. Numerical Results. We illustrate our theoretical findings of the previous sections with a few simple examples.

5.1. Example 1. To contrast the convergence behavior of SUPG stabilized finite element methods for single PDE solves with those for optimal control problems, our first example applies the SUPG method to the solution of the single PDE

$$-\epsilon y''(x)+y'(x)=f(x)\quad \text{on } (0,1),\quad y(0)=y(1)=0. \tag{5.1a}$$

We set

$$\epsilon=10^{-6} \tag{5.1b}$$

and we select the function $f(x)$ such that the exact solution is

$$y(x)=x^3-\frac{e^{\frac{x-1}{\epsilon}}-e^{-\frac{1}{\epsilon}}}{1-e^{-\frac{1}{\epsilon}}}. \tag{5.1c}$$

We compute the L^2 and H^1 error between the computed solution and the exact solution over the subdomain

$$\Omega_0=(0.2,0.8). \tag{5.1d}$$

For small ϵ the exact solution has a boundary layer at $x=1$. Without any special mesh design, the SUPG method fails to resolve the boundary layer for meshes with $h>\epsilon$. The left plot in Figure 5.1 shows the exact solution (5.1c) and the solutions computed using SUPG and piecewise linear and piecewise quadratic elements on a uniform mesh with mesh size $h=1/20$. Note that, in contrast to the standard Galerkin methods, when SUPG is used, the error in the boundary layer does not pollute the numerical solution outside the boundary layer. The right plot in Figure 5.1 shows the L^2- and H^1-errors between the exact and computed solution on the subdomain $\Omega_0=(0.2,0.8)$, where the computed solution is obtained using SUPG with piecewise linear and piecewise quadratic elements. The numerical results confirm that these errors are of optimal order.

5.2. Example 2. In our second example we apply SUPG to the solution of the optimal control problem (1.1) on $\Omega=(0,1)$. The right hand side f and the desired solution $\widehat{y}$ are selected such that the optimal state y and adjoint λ are given by

$$y(x)=x^3-\frac{e^{\frac{x-1}{\epsilon}}-e^{-\frac{1}{\epsilon}}}{1-e^{-\frac{1}{\epsilon}}},\quad \lambda(x)=x^2(1-x). \tag{5.2a}$$

We set the diffusion and regularization parameters to

$$\epsilon=10^{-6},\quad \alpha=1. \tag{5.2b}$$

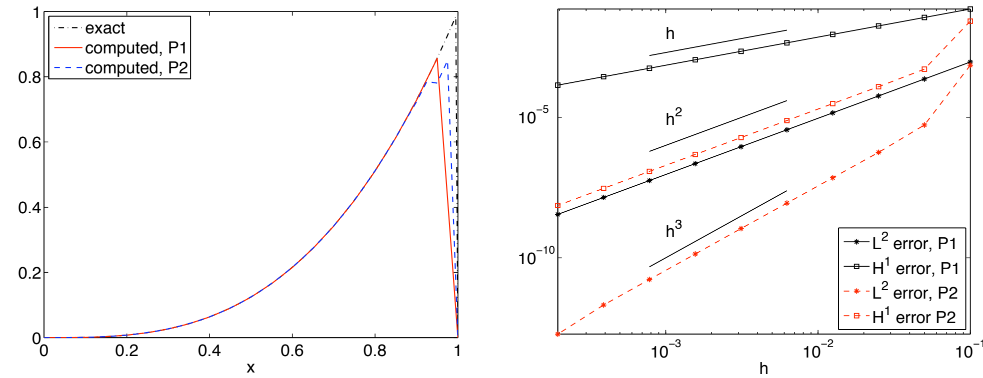

FIG. 5.1. *Results for Example 1. The left plot shows the exact solution (5.1c) and the solutions computed using SUPG and piecewise linear (P1) and piecewise quadratic (P2) elements on a uniform mesh with mesh size $h = 1/20$. The right plot shows the L^2- and H^1-errors between the exact and computed solution of the PDE (5.1a) on the subdomain $\Omega_0 = (0.2, 0.8)$, when the computed solution is obtained using SUPG with piecewise linear (P1) and piecewise quadratic (P2) elements.*

Note that with this choice of α, the optimal control u is identical to the optimal adjoint.

The solution is constructed in such a way that the optimal state y has a boundary layer at $x = 1$, but the optimal adjoint λ is smooth and has no boundary layers. See also Figure 5.2. However, the failure to resolve the boundary layer in the state pollutes the numerical solution in the entire domain $\Omega = (0, 1)$. As a consequence, the error between the exact solution of the optimal control problem and the numerical solution of the optimal control problem using SUPG on the subinterval $\Omega_0 = (0.2, 0.8)$ behaves like $O(h)$ for both the L^2 and the H^1 norm and for both piecewise linear and piecewise quadratic elements. See Figure 5.3. This convergence behavior of the SUPG method for the optimal control problem is significantly different from that of the SUPG method for a single equation, as illustrated in Example 1.

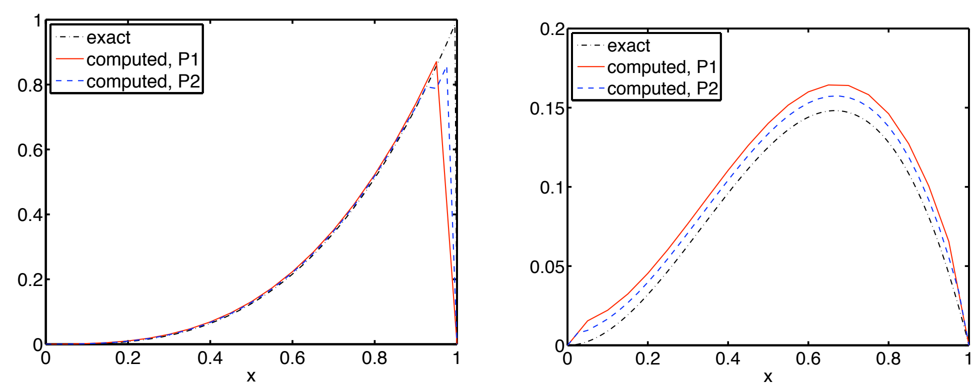

FIG. 5.2. *Exact and computed states (left plot) and adjoints (right plot) for Example 2. The computed states and adjoint are obtained using SUPG with piecewise linear (P1) or piecewise quadratic (P2) elements, respectively, on a uniform mesh with mesh size $h = 1/20$.*

One way to recover the optimal convergence rates is to consider special meshes. We rerun the same problem on a Shishkin mesh, which is uniform on each interval

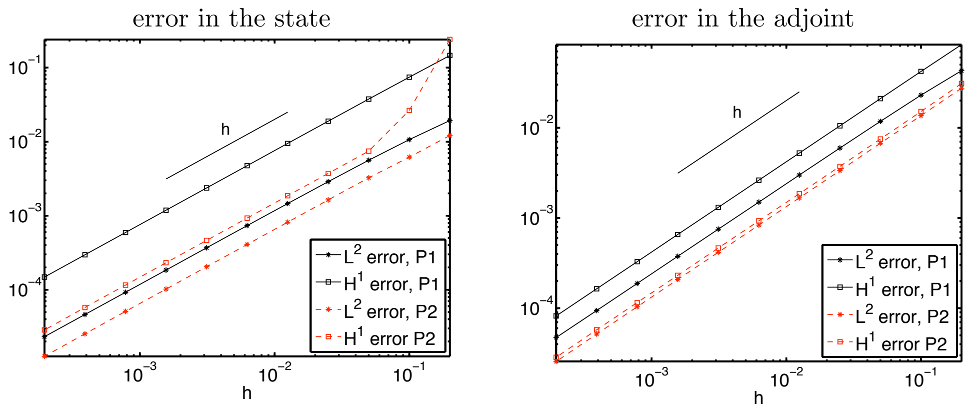

FIG. 5.3. *The left [right] plot shows the L^2- and H^1-errors between the exact and computed state [adjoint] for Example 2 on the subdomain $\Omega_0 = (0.2, 0.8)$, when the computed solution is obtained using SUPG with piecewise linear (P1) and piecewise quadratic (P2) elements.*

$[0,\ \sigma]$ and $[\sigma,\ 1]$, where we take $\sigma = 5\epsilon|\log h|$. We observe the optimal order of convergence for both the state and the adjoint in L^2 and H^1 norms in the interior for piecewise linear as well as piecewise quadratic elements. See Figure 5.4.

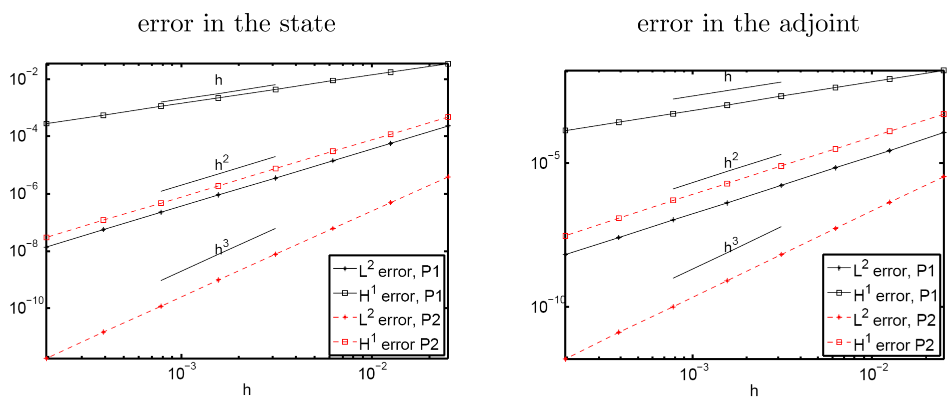

FIG. 5.4. *The left [right] plot shows the L^2- and H^1-errors between the exact and computed state [adjoint] for Example 2 on the subdomain $\Omega_0 = (0.2, 0.8)$, when the computed solution is obtained using SUPG with piecewise linear (P1) and piecewise quadratic (P2) elements on a Shishkin mesh.*

5.3. Example 3. The third example also applies the SUPG to the solution of an optimal control problem. In the previous example, we selected the optimal control and adjoint and constructed the other problem data from the optimality conditions. This may seem artificial. Therefore, we now specify right hand side f and desired state $\widehat{y}$ rather than the solution of the optimal control problem. We consider the optimal control problem (1.1) on $\Omega = (0, 1)$ with data

$$f \equiv 1, \quad \widehat{y} \equiv 1, \quad \epsilon = 10^{-4}, \quad \alpha = 10^{-2}.$$

The optimal state, control, and adjoint for this problem are not known analytically. Instead we compute the solution of the optimal control problem using SUPG on a fine grid with mesh size $h = 1/(5 * 2^{10})$. We refer to this solution as the 'exact' solution.

We compare this 'exact' solution with the computed solution on meshes with mesh sizes $h = 1/5$ to $h = 1/(5 * 2^8)$.

We note that the solution of the state equation with $u = 0$ exhibits a boundary layer at $x = 1$. The control tries to move the state as close as possible to the desired state $\widehat{y} \equiv 1$, while obeying the homogeneous Dirichlet boundary conditions. Furthermore, the control regularization penalizes excessively large (measured in the L^2 norm) control inputs. The exact and computed states (left plot) and adjoints (right plot) for Example 3 are shown in Figure 5.5.

The error between the exact solution of the optimal control problem and the numerical solution of the optimal control problem using SUPG on the subinterval $\Omega_0 = (0.2, 0.8)$ behaves like $O(h)$ for both the L^2 and the H^1 norm and for both piecewise linear and piecewise quadratic elements. Since the optimal control and adjoint are related via $\alpha u = \lambda$, we do not show the controls.

Figure 5.6 shows the L^2 and the H^1 errors between the exact and the computed states and adjoints on the subinterval $\Omega_0 = (0.2, 0.8)$ for various mesh sizes. Although the states, controls and adjoints are smooth on $\Omega_0 = (0.2, 0.8)$, the errors again behave like $O(h)$ for both the L^2 and the H^1 norm and for both piecewise linear and piecewise quadratic elements. Again, this convergence behavior of the SUPG method for the optimal control problem is significantly different from that of the SUPG method for a single equation, as illustrated in Example 1.

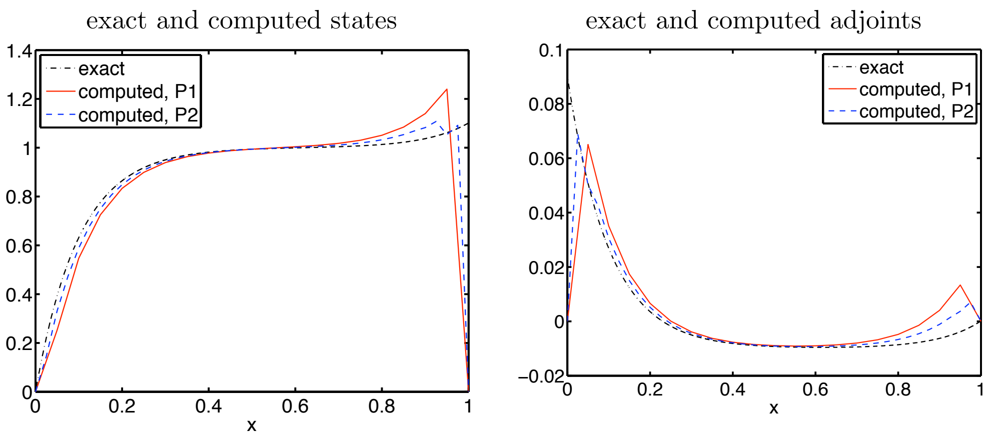

FIG. 5.5. *Exact and computed states (left plot) and adjoints (right plot) for Example 3. The computed states and adjoint are obtained using SUPG with piecewise linear (P1) or piecewise quadratic (P2) elements, respectively, on a uniform mesh with mesh size* $h = 1/20$.

5.4. Example 4. Theorem 3.2 among other things shows that interior layers do not pollute the solution. To illustrate this statement numerically we consider the system (2.11) with

$$\Omega = (0,1)^2, \quad \epsilon = 10^{-5}, \alpha = 1, \quad \mathbf{c} = (1,0)^T.$$

The functions f and $\widehat{y}$ are computed such that the exact solution is

$$y(x_1, x_2) = (1 - x_1)^3 \tan^{-1}\left(\frac{x_2 - 0.5}{\epsilon}\right) \tag{5.3}$$

$$\lambda(x_1, x_2) = x_1(1 - x_1)x_2(1 - x_2). \tag{5.4}$$

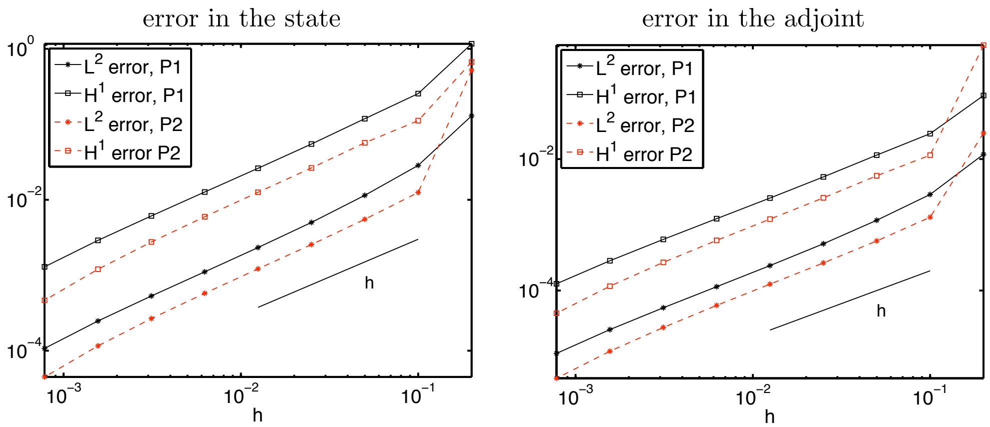

FIG. 5.6. *The left [right] plot shows the L^2- and H^1-errors between the exact and computed state [adjoint] for Example 3 on the subdomain $\Omega_0 = (0.2, 0.8)$, when the computed solution is obtained using SUPG with piecewise linear (P1) and piecewise quadratic (P2) elements.*

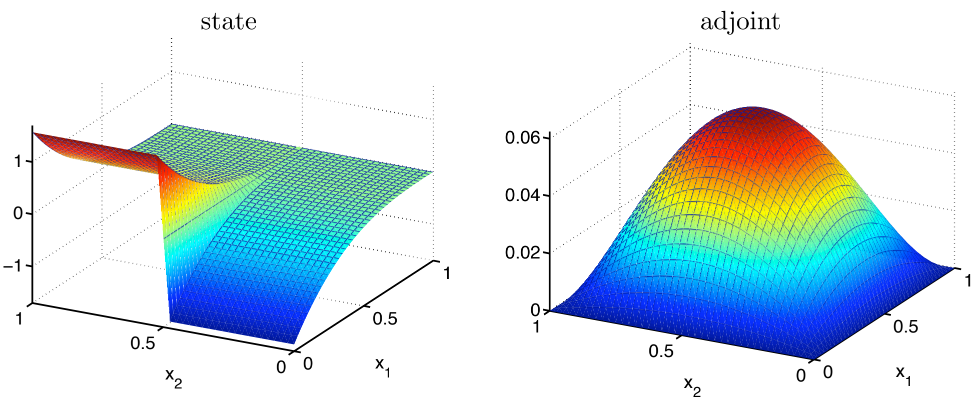

FIG. 5.7. *Exact state and adjoint for Example 4.*

For small ϵ the exact state has an interior layer along the line $x_2 = 0.5$. The SUPG method without special treatment does not resolve the interior layer even in the case of a single equation. Because of the coupling the computed adjoint is not resolved along the location of the interior layer, the line $x_2 = 0.5$, despite the fact that the exact adjoint is smooth (cf. Figures 5.8-5.8). On the other hand Theorem 3.2 says that the interior layers do not pollute the SUPG solutions into the subdomains of smoothness. This fact we observe numerically in Figure 5.9.

5.5. Example 5. For our final numerical example we consider (1.1) with

$$\Omega = (0,1)^2, \quad \epsilon = 10^{-5}, \alpha = 10^{-2}, \quad \mathbf{c} = (\cos\theta,\ \sin\theta)^T \text{ with } \theta = 47.3^o$$

The boundary conditions for the state are shown in Figure 5.10. The coarsest grid is obtained by subdividing Ω into squares of size $h \times h$ with $h = 1/40$, and then subdividing (southwest to northeast) each square into two triangles. Subsequent meshes are obtained by regularly refining each triangle into four triangles. We refer to the coarse grid as the level 1 grid and to the fine grid as the level 7 grid.

The analytical solution for this control problem is not available. Therefore we

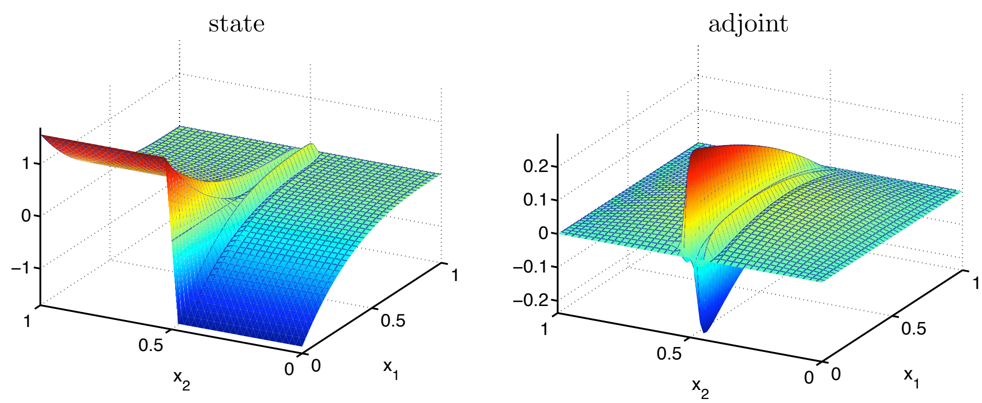

FIG. 5.8. *Computed state and adjoint for Example 4 using SUPG with piecewise linear elements on a uniform mesh with mesh size $h = 1/40$.*

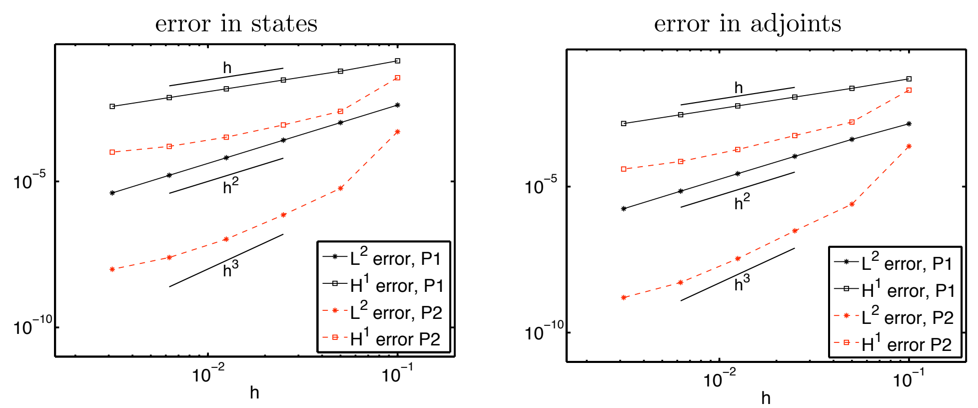

FIG. 5.9. *The left [right] plot shows the L^2- and H^1-errors between the exact and computed state [adjoint] for Example 4 on the subdomain $\Omega_0 = [0.8,\ 1] \times [0,\ 1]$, when the computed solution is obtained using SUPG with piecewise linear (P1) and piecewise quadratic (P2) elements.*

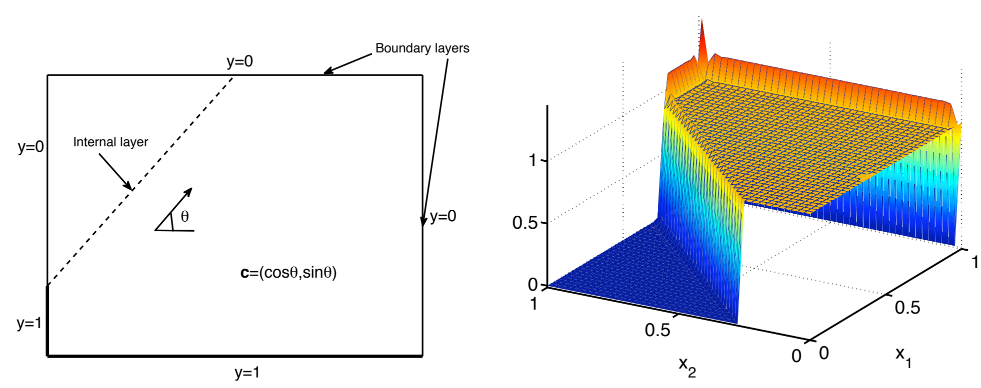

FIG. 5.10. *Left plot: Problem set-up for Example 5. Right plot: Computed solution of the state equation with $u = 0$ using SUPG on a uniform mesh with mesh size $h = 1/40$ and piecewise linear elements.*

refine our coarse grid six times and use the computed solution on the fine grid as our exact solution. We compute the errors between the computed solution on the grids with level 1 to 6 with the 'exact' solution on the fine grid (level 7). We compute the $L^2(\Omega_0)$ and $H^1(\Omega_0)$ errors, where $\Omega_0 = (0.5, 0.7) \times (0.2, 0.8)$. The right plot in Figure 5.10 indicates that the solution of the optimal control problem is smooth in Ω_0. The observed convergence rates for the local L^2 and H^1 errors are close to linear (cf., Figure 5.11) for both linear and quadratic finite elements.

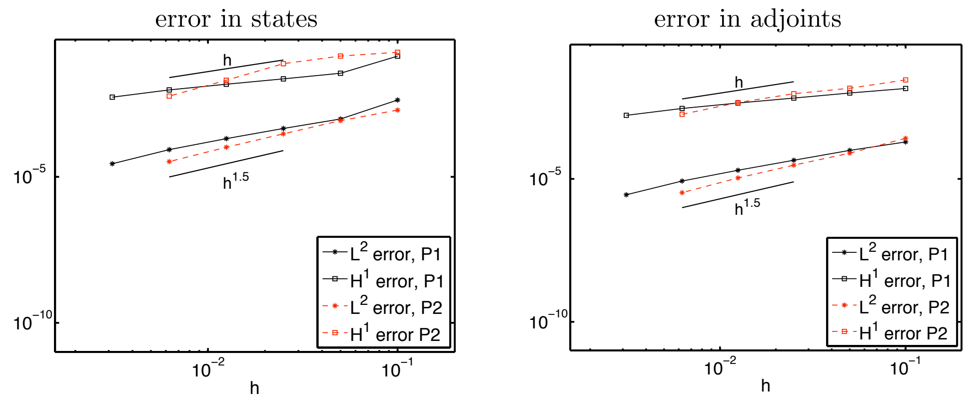

FIG. 5.11. *The left [right] plot shows the L^2- and H^1-errors between the exact and computed state [adjoint] for Example 5 on the subdomain $\Omega_0 = (0.5, 0.7) \times (0.2, 0.8)$, when the computed solution is obtained using SUPG with piecewise linear (P1) elements.*

6. Conclusion. In this paper we have shown that the behavior of the SUPG method applied to PDE constrained optimal control problems can be very different than the behavior of SUPG method applied to a single equation. In particular we have shown that when the governing PDE is an advection dominated second order elliptic equation, then the presence of the boundary layers pollutes the numerical solution everywhere even into subdomains where the exact solution is very smooth. In general in such situations only first order convergence rates are the best possible on unstructured quasi-uniform meshes regardless the order of approximating polynomials. Hence, for such problems, it does not payoff to use high order elements. This is in sharp contrast to a case of a single equation.

Our numerical examples strongly support the conjecture that the reason for the reduction of the error between the SUPG solution and the solution of the infinite dimensional optimal control problem to first order is that the boundary layers are not sufficiently resolved. The discretization errors in the boundary layers are transported via the adjoint equation and the state equation into the domain. Hence, we expect that this order reduction cannot be avoided by using other stabilizations than SUPG, such as local projection based stabilizations. To regain the optimal convergence rates one must resolve the layers, typically by using special meshes, as indicated by one of our numerical examples.

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