POINTWISE ERROR ESTIMATES FOR C^0 INTERIOR PENALTY APPROXIMATION OF BIHARMONIC PROBLEMS

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ABSTRACT. The aim of this paper is to derive pointwise global and local best approximation type error estimates for biharmonic problem using C^0 interior penalty method. The analysis uses the technique of dyadic decompositions of the domain, which assumed to be a convex polygon. The proofs require local energy estimate and new pointwise Green's function estimates for the continuous problem which have an independent interest.

1. INTRODUCTION

We consider the fourth order problem:

$$\begin{aligned} \Delta^2 u &= f && \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain, $f \in L^2(\Omega)$, and n denotes the outward unit normal of $\partial\Omega$. Finite element discretization of the above problem is not straightforward and various approaches to approximate the above problem were proposed over the years. However, they all have some drawbacks. For instance, conformal C^1 elements are rather complicated even in two dimensions [2, 7], the classical non-conformal elements [13, 20] must be altered in the presence of low order terms (cf. [14]). Furthermore the construction of higher order nonconforming finite elements for fourth order problems is also not easy. The C^0 interior penalty method is a sound alternative. This method is attractive since the finite elements consist of usual Lagrange elements of arbitrary order and straightforward to implement. A detailed description of the method with energy based error estimates on convex and nonconvex domains can be found in [4].

Pointwise error estimates is well developed area for the second order problems. However, there are few such results for fourth order problems. Many such pointwise error estimates are obtained via Sobolev embedding. This is not satisfactory since such results are usually not optimal and often the discrepancy between norms makes them hard to use for applications, for example for optimal control problems. In addition, it is hard to localize them. The only exceptions we are aware of are the papers by Rolf Rannacher [16] and Ming Wang [23], where pointwise error estimates were established for some non-conforming and mixed elements. Both papers used weighted technique, and the W^1_∞ error estimates there are in the form

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of quasi-optimal order error estimates with $H^3(\Omega)$ norm on the right hand side, which is consistent with regularity of the biharmonic problem on convex domains. However, the error estimate for the second derivatives in [16], which is important for computing bending moments, requires W^3_∞ and H^4 regularity for the solution, that can not be guaranteed on convex polygons. In addition, it is not clear how to localize those results.

In this paper we take a different approach and show the following best approximation type results.

Theorem 1.1. *Let u satisfy* (1) *and u_h be its C^0 interior penalty finite element approximation, and x_0 be an interior point of an element. Then there exists a constant C independent of h such that*

$$|D^2(u-u_h)(x_0)| \le C|\ln h|^{3/2} \min_{\chi\in S^r_h} |||u-\chi|||_{W^{2,\infty}_h(\Omega)},$$

where D^2 denotes a general second order differential operator and $W^{2,\infty}_h$ is a mesh dependent norm defined in (17).

In [10] it was shown that on convex domains the solution u to (1) is in $W^2_\infty(\Omega)$ and as a result the theorem makes sense without any additional smoothness assumptions on the solution or domain. In general, we can not expect even first order convergence since the solution is not in $W^3_\infty(\Omega)$. However, similarly to the second order elliptic equations [17, 18], we can establish the following interior results using which we can obtain first order convergence in the interior of the convex domains without any additional assumptions on the regularity.

Theorem 1.2 (Interior error estimate)**.** *Let $x_0 \in \Omega$ be an interior point of an element and $B_d = B_d(x_0)$ denote a ball of radius d centered at x_0. Then there exists a constant C independent of h and d such that*

$$|D^2(u-u_h)(x_0)| \le C|\ln h|^{3/2} \left(|||u-\chi|||_{W^{2,\infty}_h(B_d)} + d^{-1}|||u-\chi|||_{2,h,\Omega} \right),$$

where $|||\cdot|||_{W^{2,\infty}_h(B_d)}$ and $|||\cdot|||_{2,h,\Omega}$ are mesh dependent norms defined in the next section.

The analysis of the method uses a combination of well-established technique of dyadic decomposition together with local energy estimates for second order elliptic problem [22] and pointwise Green's function estimates. In this paper we only consider the two dimensional convex polygonal domain. However, even on such simple domains we had to overcome several technical difficulties in order to obtain the results. Our error analysis framework has a lot in common with pointwise error analysis for discontinuous Galerkin methods for the second order elliptic problem [5, 8]. The details are of course rather different and the main difficulty lies in the regularity results. In contrast to the second order elliptic problems on convex domains, where H^2 norm of the solution is bounded by the L^2 of the right hand side, for the biharmonic problem we can control H^3 norm of the solution by H^{-1} norm of the right hand side, which causes many difficulties due to the global nature of the H^{-1} norm. Although pointwise error analysis for the second order problem is available in three dimensions too, the pointwise Green's function estimates and the duality argument are the main obstacles for extending our main results to non-convex case or to three dimensions.

The rest of the paper is organized as follows. In the next chapter we introduce notation and some basic results for continuous problem. In chapter 3, we definite the C^0 interior penalty discretization, and establish key lemmas. The central result is the Local Energy Estimates, Lemma 3.9. In chapter 4 we provide a proof of the global pointwise error estimate, namely Theorem 1.1 and in the chapter 5, we establish a localized version, Theorem 1.2.

2. Notation and Preliminary results

In the paper we use the usual notation for Sobolev and Lebesgue spaces. We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$, and by $(\cdot,\cdot)_{\Omega_0}$ the L^2 inner product in over subdomain $\Omega_0 \subset \Omega$.

We let D denote a general first order differential operator, D^2 - second order differential operator, D^3 - third and etc. The partial derivatives we will denote by ∂. We will also use the multi-index notation when it is important.

The weak solution u to (1) given by

$$B(u,\varphi) := (D^2u, D^2\varphi) = \sum_{i,j=1}^{2} (\partial^2_{ij}u, \partial^2_{ij}\varphi) = (f,\varphi), \quad \forall \varphi \in H^2_0(\Omega), \tag{2}$$

is naturally in $H^2_0(\Omega)$ with the following estimate

$$\|u\|_{H^2(\Omega)} \le C\|f\|_{H^{-2}(\Omega)}. \tag{3}$$

On convex domains we also have that $u \in H^2_0(\Omega) \cap H^3(\Omega)$ with the following estimate (cf. [9] sec. 5.9)

$$\|u\|_{H^3(\Omega)} \le C\|f\|_{H^{-1}(\Omega)}. \tag{4}$$

2.1. **Green's function estimates.** In the proof of our main results we will make a heavy use of pointwise estimates for the Green's function and its derivatives. The following form of the estimate seems to be new for convex polygons, but follows rather easily from available results.

Lemma 2.1. *Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ and $G(x,y)$ be the corresponding Green's function for the problem* (1)*. Then for all multi-indices $\alpha = (\alpha_1,\alpha_2)$ and $\beta = (\beta_1,\beta_2)$ with the range $1 \le |\alpha|, |\beta| \le 2$ and $3 \le |\alpha| + |\beta| \le 4$ the following estimates hold,*

$$|D^\alpha_x D^\beta_y G(x,y)| \le C|x-y|^{2-|\alpha|-|\beta|}. \tag{5}$$

Proof. The estimate (5) essentially follows from the estimates from [12] (see also [9], sec. 5.8 or [11] p. 286). Let $\omega_1, \omega_2, \dots, \omega_M$ be the interior angles of the convex polygon Ω. Let ω be an arbitrary angle, and let $\rho(x)$ denotes a distance from x to the vertex of ω. Theorem 2.1 in [12] states that the for Green's function the following estimates hold

(6a)
$$|D^\alpha_x D^\beta_y G(x,y)| \le C|x-y|^{2-|\alpha|-|\beta|}, \quad \text{for} \quad \rho(x)/2 \le \rho(y) \le 2\rho(x), \quad 3 \le |\alpha|+|\beta|,$$
(6b)
$$|D^\alpha_x D^\beta_y G(x,y)| \le C\rho(x)^{\gamma-|\alpha|}\rho(y)^{-\gamma+2-|\beta|}, \quad \text{for} \quad 2\rho(x) \le \rho(y), \quad 0 \le |\alpha|+|\beta|,$$
(6c)
$$|D^\alpha_x D^\beta_y G(x,y)| \le C\rho(x)^{-\gamma+2-|\alpha|}\rho(y)^{\gamma-|\beta|}, \quad \text{for} \quad 2\rho(y) \le \rho(x), \quad 0 \le |\alpha|+|\beta|,$$

where γ is any real number that satisfies inequality $|\gamma-1|<C_\omega$. Since $\omega<\pi$, according to Lemma 1.2 in [12] (see also [3]), $C_\omega>1$ and as a result we may take $\gamma=2+\varepsilon$ for some $\varepsilon>0$.

With this choice of γ from (6b) it follows that

$$|D_x^\alpha D_y^\beta G(x,y)|\le C\rho(x)^{2+\varepsilon-|\alpha|}\rho(y)^{2-\varepsilon+2-|\beta|}\le C\rho(y)^{2-|\alpha|-|\beta|},$$

where used that this is the case when $2\rho(x)\le\rho(y)$ and $2+\varepsilon-|\alpha|>0$. Using the triangle inequality

$$|x-y|\le\rho(y)+\rho(x)\le\frac{3}{2}\rho(y)$$

and that by the assumptions on $2-|\alpha|-|\beta|<0$ we have that in this case

$$|D_x^\alpha D_y^\beta G(x,y)|\le C\rho(y)^{2-|\alpha|-|\beta|}\le C|x-y|^{2-|\alpha|-|\beta|}. \tag{7}$$

The case $2\rho(y)\le\rho(x)$ is very similar. From (6c) with $\gamma=2+\varepsilon$ we have

$$|D_x^\alpha D_y^\beta G(x,y)|\le C\rho(x)^{-2-\varepsilon+2-|\alpha|}\rho(y)^{2+\varepsilon-|\beta|}\le C\rho(x)^{2-|\alpha|-|\beta|}.$$

Again using the triangle inequality

$$|x-y|\le\rho(x)+\rho(y)\le\frac{3}{2}\rho(x)$$

together with the assumption $2-|\alpha|-|\beta|<0$ we obtain

$$|D_x^\alpha D_y^\beta G(x,y)|\le C\rho(x)^{2-|\alpha|-|\beta|}\le C|x-y|^{2-|\alpha|-|\beta|}. \tag{8}$$

Combining (7), (8), and (6a), we obtain the lemma. □

3. Discretization

To define finite element approximation of the solution to (1), for $h\in(0,h_0]$; $h_0>0$, we let $\mathcal{T}_h$ denote a quasi-uniform triangulation of Ω with mesh size h, i.e., $\mathcal{T}_h=\{T\}$ is a partition of Ω into triangles T of diameter h_T such that for $h=\max_\tau h_T$,

$$\operatorname{diam}(T)\le h\le C|T|^{\frac{1}{2}},\quad \text{for all}\ \ T\in\mathcal{T}_h.$$

Let $S_h^r\subset H_0^1(\Omega)$ denote the Lagrange finite element space consisting of polynomials of degree $r\ge2$; that is,

$$S_h^r=\left\{v_h\in H_0^1(\Omega);\ v_h\big|_T\in\mathbb{P}^r(T)\ \forall T\in\mathcal{T}_h\right\}.$$

By $I_h: C^0(\Omega)\to S_h^r$ we denote the usual nodal interpolant which safisfies

$$\|v-I_hv\|_{W^{j,p}(T)}\le Ch^{s-j}\|v\|_{W^{s,p}(T)},\quad \forall v\in W^{s,p}(T), \tag{9}$$

for $1\le p\le\infty$, $j\le s\le r+1$, and $s>\frac{2}{p}$.

3.1. **Trace and inverse inequalities.** We will frequently use the following trace and inverse inequalities. For $T\in\mathcal{T}_h$ and $v\in H_0^2(\Omega)$, $v_h\in S_h^r$ there exist positive constants C_{tr} and C_{inv} independent of τ and v,v_h such that

$$\|v\|^2_{L^2(\partial T)}\le C_{tr}^2\|v\|_{L^2(T)}\|\nabla v\|_{L^2(T)}\le C_{tr}^2\left(h^{-1}\|v\|^2_{L^2(T)}+h\|\nabla v\|^2_{L^2(T)}\right), \tag{10a}$$

$$\|\nabla v_h\|_{L^2(T)}\le C_{inv}h^{-1}\|v_h\|_{L^2(T)}, \tag{10b}$$

$$\|v_h\|_{L^2(\partial T)}\le C_{tr}(1+C_{inv})h^{-1/2}\|v_h\|_{L^2(T)}. \tag{10c}$$

3.2. **The C^0 interior penalty method.** To define the method, we need some additional notation. Let $\mathcal{E}_h$ be the set of edges in $\mathcal{T}_h$. For $e \in \mathcal{E}_h$ and $v \in H^2(\Omega, \mathcal{T}_h)$, where

$$H^2(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega) : v \mid_T \in H^2(T) \quad \forall T \in \mathcal{T}_h\},$$

we define the jump $\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]$ of the normal derivative of v across an edge e and the average of the second normal derivate $\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}$ of v on an edge e as follows. If $e \subset \Omega$, we take n_e to be one of the two unit vectors normal to e. Then e is the common side of two triangles $T_+ \in \mathcal{T}_h$ and $T_- \in \mathcal{T}_h$, where n_e is pointing from T_- to T_+. Thus, on such e we define

$$\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] = \frac{\partial v_{T_+}}{\partial n} \mid_e - \frac{\partial v_{T-}}{\partial n} \mid_e \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{\partial^2 v_{T_+}}{\partial n^2} \mid_e + \frac{\partial^2 v_{T-}}{\partial n^2} \mid_e .$$

We note that the above definitions do not depend on the choice of n_e. If $e \subset \partial\Omega$, we take n_e to be the unit normal pointing outside Ω and

$$\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] = -\frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = -\frac{\partial^2 v}{\partial n_e^2}.$$

Next, we define the bilinear form $B_h(\cdot,\cdot)$ by

$$B_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T D^2 v : D^2 w \, dx + \sum_{e\in\mathcal{E}_h} \int_e \Big(\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} \left[\!\left[\frac{\partial w}{\partial n}\right]\!\right] + \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\} + \frac{\eta}{|e|} \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] \left[\!\left[\frac{\partial w}{\partial n}\right]\!\right] \Big) ds,$$

where

$$D^2 v : D^2 w = \sum_{i,j=1}^{2} \partial_{ij}^2 v \; \partial_{ij}^2 w.$$

The discrete problem is to find $u_h \in S_h^r$ s.t.

$$B_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in S_h^r. \tag{11}$$

Similarly to [4] we define the following mesh-dependent norm

$$|\!|\!| v |\!|\!|_{2,h}^2 = \sum_{T\in\mathcal{T}_h} \|D^2 v\|_{L^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \int_e \left(|e| \left| \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} \right|^2 + |e|^{-1} \left| \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] \right|^2 \right) ds.$$

Easy to see that the bilinear form $B_h(\cdot,\cdot)$ is bounded

$$B_h(v,w) \le C |\!|\!| v |\!|\!|_{2,h} |\!|\!| w |\!|\!|_{2,h}, \tag{12}$$

and for η sufficiently large, the bilinear form $B_h(\cdot,\cdot)$ is coersive (cf. [4]) on S_h^r,

$$B_h(v_h, v_h) \ge C |\!|\!| v_h |\!|\!|_{2,h}^2, \quad \forall v_h \in S_h^r. \tag{13}$$

As a result by the Lax-Milgram lemma, u_h is well-defined. From [4], eq. (4.9) we also have that the exact solution u satisfies

$$B_h(u, \chi) = (f, \chi), \quad \forall \chi \in S_h^r,$$

and as a result u and u_h satisfy the usual Galerkin orthogonality

$$B_h(u - u_h, \chi) = 0 \quad \forall \chi \in S_h^r. \tag{14}$$

In [4], Theorem 1, on convex domains the following error estimate was established

$$\|\!|u - u_h|\!\|_{2,h} \le Ch\|f\|_{L^2(\Omega)}. \tag{15}$$

Note that for $v_h \in S_h^r$ using the inequality (10c), we have

$$\sum_{e\in\mathcal{E}_h}\int_e |e|\left|\left\{\!\!\left\{\frac{\partial^2 v_h}{\partial n^2}\right\}\!\!\right\}\right|^2\, ds \le C\sum_{T\in\mathcal{T}_h}\|D^2 v_h\|^2_{L^2(T)},$$

and thus on S_h^r the $\|\!|\cdot|\!\|_{2,h}$ norm is equivalent to the following norm without the terms involving averages, namely

$$\|v\|^2_{2,h} := \sum_{T\in\mathcal{T}_h}\|D^2 v\|^2_{L^2(T)} + \sum_{e\in\mathcal{E}_h}\int_e |e|^{-1}\left|\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right|^2 ds.$$

We will also require the following norms on subsets $D \subset \Omega$

$$\|\!|v|\!\|_{W_h^{2,1}(D)} = \sum_{T\in\mathcal{T}_h\cap D}|v|_{W^{2,1}(T)} + \sum_{e\in\mathcal{E}_h\cap D}\int_e\left(|e|\left|\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\right| + \left|\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right|\right) ds, \tag{16}$$

and

$$\|\!|v|\!\|_{W_h^{2,\infty}(D)} = \max_{T\in\mathcal{T}_h\cap D}|v|_{W^{2,\infty}(T)} + \max_{e\in\mathcal{E}_h\cap D}\left\|\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\right\|_{L^\infty(e)} + \max_{e\in\mathcal{E}_h\cap D}|e|^{-1}\left\|\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right\|_{L^\infty(e)}. \tag{17}$$

Using the above norms we also have

$$B_h(v,w) \le C\|\!|v|\!\|_{W_h^{2,1}(\Omega)}\|\!|w|\!\|_{W_h^{2,\infty}(\Omega)}. \tag{18}$$

3.3. **Superapporximation.** Superapproximation is an essential tool in pointwise finite element error estimates [15]. To describe them, we introduce a smooth cut-off function $\omega \in C^\infty(\Omega)$ with the properties that

$$\omega(x) \equiv 1, \quad x \in B_{d/2}(x_0) \tag{19a}$$

$$\omega(x) \equiv 0, \quad x \in \Omega\backslash B_d(x_0) \tag{19b}$$

$$|\omega|_{W^j_\infty} \le Cd^{-j}, \quad j = 0,1,2. \tag{19c}$$

In [6] an improved superapproximation result was obtained

$$\|\omega^2\chi - I_h(\omega^2\chi)\|_{H^1(T)} \le C\frac{h_T}{d}\left(\|\nabla(\omega\chi)\|_{L^2(T)} + d^{-1}\|\chi\|_{L^2(T)}\right), \quad \chi \in S_h^r.$$

We will follow the ideas from that paper. Let $P_1(\omega)$ and $P_0(\omega)$ be linear and constant order approximations to ω on element T, respectively, such that

$$\|\omega - P_0(\omega)\|_{L^\infty(T)} \le Ch_T\|\omega\|_{W^1_\infty(T)} \le Ch_T d^{-1} \tag{20}$$

$$\|\omega - P_1(\omega)\|_{W^s_\infty(T)} \le Ch_T^{2-s}\|\omega\|_{W^2_\infty(T)} \le Ch_T^{2-s}d^{-2},\ s = 0,1. \tag{21}$$

We take $P_0(\omega) = \widehat{\omega} = |T|^{-1}\int_T \omega$ and $P_1(\omega)$ a linear interpolant of w.

Lemma 3.1 (Superapproximation)**.** *Let ω be as above and $\chi \in S_h^r$ with $r \ge 2$. Then there exists a constant C independent of ω, d and h such that for $s = 1,2,3$*

$$|\omega^4\chi - I_h(\omega^4\chi)|_{H^s(T)} \le C\frac{h_T^{3-s}}{d}\left(|\omega^2\chi|_{H^2(T)} + d^{-1}|\omega\chi|_{H^1(T)} + d^{-2}\|\chi\|_{L^2(T)}\right).$$

Proof. We give a detailed proof for piecewise quadratic elements $r = 2$ only. The proof for general $r \geq 2$ is very similar. By the standard approximation theory we have

$$|\omega^4\chi - I_h(\omega^4\chi)|_{H^s(T)} \leq Ch_T^{3-s}|\omega^4\chi|_{H^3(T)}, \quad \text{for } s = 1, 2, 3. \tag{22}$$

Then

$$D^3(\omega^4\chi) = D^3(\omega^4)\chi + 3D^2(\omega^4)D\chi + 3D(\omega^4)D^2\chi + \omega^4 D^3\chi := I_1 + I_2 + I_3 + I_4. \tag{23}$$

Since we are dealing with quadratic elements, $I_4 = 0$. Using (19c), I_1 can easily be estimated as

$$I_1 = \|D^3(\omega^4)\chi\|_{L^2(T)} \leq Cd^{-3}\|\chi\|_{L^2(T)}. \tag{24}$$

Using

$$D(\omega^4) = 4\omega^3 D\omega \quad \text{and} \quad D^2(\omega^4) = 4D(\omega^3 D\omega) = 4(3\omega^2|D\omega|^2 + \omega^3 D^2\omega), \tag{25}$$

we have

$$I_2 = \|3D(\omega^4)D^2\chi\|_{L^2(T)} \leq \|12\omega^2|D\omega|^2 D\chi\|_{L^2(T)} + \|\omega^3 D^2\omega D\chi\|_{L^2(T)} = I_{21} + I_{22}. \tag{26}$$

Using (19c) and adding and subtracting $\widehat{\omega}$, where $\widehat{\omega} = |T|^{-1}\int_T \omega$, we have

$$\begin{aligned} I_{21} &\leq Cd^{-2}\|\omega D\chi\|_{L^2(T)} \\ &\leq Cd^{-2}\|(\omega - \widehat{\omega})D\chi\|_{L^2(T)} + Cd^{-2}\|\widehat{\omega}D\chi\|_{L^2(T)} \\ &= I_{211} + I_{212}. \end{aligned} \tag{27}$$

Using the approximation and the inverse estimate

$$I_{211} \leq Cd^{-2}h_T d^{-1}\|D\chi\|_{L^2(T)} \leq Cd^{-3}\|\chi\|_{L^2(T)}.$$

By using that $\widehat{\omega}$ is constant on T and using the triangle inequality, properties of ω and the inverse estimates, we have

$$\begin{aligned} I_{212} &= Cd^{-2}\|D(\widehat{\omega}\chi)\|_{L^2(T)} \\ &\leq Cd^{-2}\left(\|D((\widehat{\omega} - \omega)\chi)\|_{L^2(T)} + \|D(\omega\chi)\|_{L^2(T)}\right) \\ &\leq Cd^{-2}\left(\|(D\omega)\chi\|_{L^2(T)} + \|(\widehat{\omega} - \omega)D\chi\|_{L^2(T)} + |\omega\chi|_{H^1(T)}\right) \\ &\leq Cd^{-3}\|\chi\|_{L^2(T)} + Cd^{-2}h_T d^{-1}\|D\chi\|_{L^2(T)} + Cd^{-2}|\omega\chi|_{H^1(T)} \\ &\leq Cd^{-3}\|\chi\|_{L^2(T)} + Cd^{-2}|\omega\chi|_{H^1(T)}. \end{aligned}$$

Similarly to the analysis above, by the properties of ω

$$\begin{aligned} I_{22} &\leq Cd^{-2}\|\omega^3 D\chi\|_{L^2(T)} \leq Cd^{-2}\|\omega D\chi\|_{L^2(T)} \\ &\leq Cd^{-2}\|(\widehat{\omega} - \omega)D\chi\|_{L^2(T)} + Cd^{-2}\|D(\widehat{\omega}\chi)\|_{L^2(T)} \\ &\leq Cd^{-3}\|\chi\|_{L^2(T)} + Cd^{-2}|\omega\chi|_{H^1(T)}. \end{aligned}$$

The next term,

$$I_3 = \|3D(\omega^4)D^2\chi\|_{L^2(T)} = \|12\omega^3 D\omega D^2\chi\|_{L^2(T)}. \tag{28}$$

Thus adding and subtracting $P_1(\omega^2)$, we have

$$\begin{aligned} I_3 &\leq Cd^{-1}\|\omega^2 D^2\chi\|_{L^2(T)} \\ &\leq Cd^{-1}\left(\|(\omega^2 - P_1(\omega^2))D^2\chi\|_{L^2(T)} + \|P_1(\omega^2)D^2\chi\|_{L^2(T)}\right) \\ &= I_{31} + I_{32}. \end{aligned}$$

By the fmation and the inverse estimates

$$I_{31} \le Cd^{-1} h_T^2 |D^2(\omega^2)|_{L^\infty(T)} \|D^2\chi\|_{L^2(T)} \le Cd^{-3}\|\chi\|_{L^2(T)}.$$

Since $D^2(P_1(\omega^2)) = 0$, by the triangle inequality,

$$I_{32} \le Cd^{-1}\left(\|D^2(P_1(\omega^2)\chi)\|_{L^2(T)} + \|D(P_1(\omega^2))D\chi\|_{L^2(T)}\right) = I_{321} + I_{322}.$$

Using $D^2(P_1(\omega^2)) = 0$, the approximation and the inverse inequality,

$$\begin{aligned} I_{321} \le & Cd^{-1}\left(\|D^2((P_1(\omega^2)-\omega^2)\chi)\|_{L^2(T)} + \|D^2(\omega^2\chi)\|_{L^2(T)}\right) \\ \le & Cd^{-1}\left(\|D^2(\omega^2)\chi\|_{L^2(T)} + \|D(P_1(\omega^2)-\omega^2)D\chi\|_{L^2(T)}\right) \\ & + Cd^{-1}\left(\|(P_1(\omega^2)-\omega^2)D^2\chi\|_{L^2(T)} + |\omega^2\chi|_{H^2(T)}\right) \\ \le & Cd^{-1}\left(d^{-2}\|\chi\|_{L^2(T)} + h_T d^{-2}\|D\chi\|_{L^2(T)} + h_T^2 d^{-2}\|D^2\chi\|_{L^2(T)} + |\omega^2\chi|_{H^2(T)}\right) \\ \le & Cd^{-3}\|\chi\|_{L^2(T)} + Cd^{-1}|\omega^2\chi|_{H^2(T)}. \end{aligned}$$

Adding and subtracting ω^2, and using the estimate for I_{21} (27), we have

$$\begin{aligned} I_{322} &\le Cd^{-1}\left(\|D(P_1(\omega^2)-\omega^2)D\chi\|_{L^2(T)} + \|D(\omega^2)D\chi\|_{L^2(T)}\right) \\ &\le Cd^{-1}\left(h_T d^{-2}\|D\chi\|_{L^2(T)} + 2\|\omega D\omega D\chi\|_{L^2(T)}\right) \\ &\le Cd^{-3}\|\chi\|_{L^2(T)} + Cd^{-2}\|\omega D\chi\|_{L^2(T)} \\ &\le Cd^{-3}\|\chi\|_{L^2(T)} + Cd^{-2}|\omega\chi|_{H^1(T)}. \end{aligned}$$

Combining all the estimates we complete the proof. □

Remark 3.2. *The straightforward approach* [15] *gives the following superapproximation result*

$$|\omega^4\chi - I_h(\omega^4\chi)|_{H^s(T)} \le C\frac{h_T^{3-s}}{d}\left(|\chi|_{H^2(T)} + d^{-1}|\chi|_{H^1(T)} + d^{-2}\|\chi\|_{L^2(T)}\right),\ s = 1,2,3.$$

Although, such a result would be sufficient for our goals in this paper, it would make the analysis of local energy error estimates in Lemma 3.9 more cumbersome.

Corollary 3.3. *Let ω be as above and $\chi \in S_h^r$ with $r \ge 2$. Then there exists a constant C independent of ω, d and h such that*

(29a)

$$|||\omega^4\chi - I_h(\omega^4\chi)|||_{2,h} \le C\frac{h_T}{d}\left(|\omega^2\chi|_{H^2(\Omega)} + d^{-1}|\omega\chi|_{H^1(\Omega)} + d^{-2}\|\chi\|_{L^2(B_d)}\right)$$

(29b)

$$|||\omega^4\chi - I_h(\omega^4\chi)|||_{2,h} \le C\frac{h_T}{d}\left(\|\omega^2 D^2\chi\|_{L^2(\Omega)} + d^{-1}\|\omega\nabla\chi\|_{L^2(\Omega)} + d^{-2}\|\chi\|_{L^2(B_d)}\right).$$

Proof. The proof follows from Lemma 3.1, the trace and inverse inequalities and the product rule. □

3.4. **Preliminary weighted results.** First we show the following supplementary result.

Lemma 3.4. *Let ω be as at the beginning of section 3.3 with the properties* (19a)-(19c). *Then for any $\varepsilon > 0$ and any $v_h \in S_h^r$*

$$\begin{aligned} \|\omega\nabla v_h\|^2_{L^2(\Omega)} \le & \varepsilon d^2\left(\sum_{T\in\mathcal{T}_h}\|\omega^2 D^2 v_h\|^2_{L^2(T)} + \sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|\omega^2\left[\!\!\left[\frac{\partial v_h}{\partial n}\right]\!\!\right]\right\|^2_{L^2(e)}\right) \\ & + C_\varepsilon d^{-2}\|v_h\|^2_{L^2(B_d)}. \end{aligned}$$

Proof. Integrating by parts on an element T we have

$$\begin{aligned}\|\omega\nabla v_h\|^2_{L^2(T)} &= (\omega^2\nabla v_h, \nabla v_h)_T = -(\nabla\cdot(\omega^2\nabla v_h), v_h)_T + \left(\omega^2\frac{\partial v_h}{\partial n}, v_h\right)_{\partial T}\\ &= -(\nabla(\omega^2)\cdot\nabla v_h, v_h)_T - (\omega^2\Delta v_h, v_h)_T + \left(\omega^2\frac{\partial v_h}{\partial n}, v_h\right)_{\partial T}.\end{aligned}$$

Summing over elements we obtain

$$\begin{aligned}\|\omega\nabla v_h\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}_h} -(\nabla(\omega^2)\cdot\nabla v_h, v_h)_T - (\omega^2\Delta v_h, v_h)_T + \left(\omega^2\frac{\partial v_h}{\partial n}, v_h\right)_{\partial T}\\ &:= J_1 + J_2 + J_3.\end{aligned}$$

Using the Cauchy-Schwarz and Young's inequalities

$$\begin{aligned}J_1 &= -\sum_{T\in\mathcal{T}_h} 2(\omega\nabla\omega\cdot\nabla v_h, v_h)_T\\ &\le \sum_{T\in\mathcal{T}_h\cap B_d} Cd^{-1}\|\omega\nabla v_h\|_{L^2(T)}\|v_h\|_{L^2(T)}\\ &\le \frac{1}{4}\sum_{T\in\mathcal{T}_h}\|\omega\nabla v_h\|^2_{L^2(T)} + Cd^{-2}\sum_{T\in\mathcal{T}_h\cap B_d}\|v_h\|^2_{L^2(T)}\\ &\le \frac{1}{4}\|\omega\nabla v_h\|^2_{L^2(\Omega)} + Cd^{-2}\|v_h\|^2_{L^2(B_d)}.\end{aligned}$$

Similarly using the Cauchy-Schwarz and Young's inequalities

$$\begin{aligned}J_2 &= -\sum_{T\in\mathcal{T}_h}(\omega^2\Delta v_h, v_h)_T\\ &\le \sum_{T\in\mathcal{T}_h}\|\omega^2\Delta v_h\|_{L^2(T)}\|v_h\|_{L^2(T)}\\ &\le \frac{\varepsilon d^2}{2}\sum_{T\in\mathcal{T}_h}\|\omega^2 D^2 v_h\|^2_{L^2(T)} + C_\varepsilon d^{-2}\sum_{T\in\mathcal{T}_h\cap B_d}\|v_h\|^2_{L^2(T)}\\ &\le \frac{\varepsilon d^2}{2}\sum_{T\in\mathcal{T}_h}\|\omega^2 D^2 v_h\|^2_{L^2(T)} + C_\varepsilon d^{-2}\|v_h\|^2_{L^2(B_d)}.\end{aligned}$$

Finally, summing over the elements and using the trace and inverse inequalities we obtain

$$\begin{aligned}J_3 &= \sum_{T\in\mathcal{T}_h}\left(\omega^2\frac{\partial v_h}{\partial n}, v_h\right)_{\partial T} = \sum_{e\in\mathcal{E}_h}\left(\omega^2\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right], v_h\right)_e\\ &\le \frac{\varepsilon d^2}{2}\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|\omega^2\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]\right\|^2_{L^2(e)} + C_\varepsilon d^{-2}\sum_{e\in\mathcal{E}_h\cap B_d}|e|\|v_h\|^2_{L^2(e)}\\ &\le \frac{\varepsilon d^2}{2}\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|\omega^2\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]\right\|^2_{L^2(e)} + C_\varepsilon C_{inv}C_{tr}d^{-2}\sum_{T\in\mathcal{T}_h\cap B_d}\|v_h\|^2_{L^2(T)}\\ &\le \frac{\varepsilon d^2}{2}\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|\omega^2\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]\right\|^2_{L^2(e)} + C_\varepsilon d^{-2}\|v_h\|^2_{L^2(B_d)}.\end{aligned}$$

Combining the above estimates we obtain the lemma. □

We also need the following result

Lemma 3.5. *Let ω be as above. There exists a constant C independent of d and h such that for any $v_h \in S_h^r$*

$$\sum_{e\in\mathcal{E}_h} |e| \left\| \omega^2 \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \right\|^2_{L^2(e)} \le C \sum_{T\in\mathcal{T}_h} \left\| \omega^2 D^2 v_h \right\|^2_{L^2(T)} + Cd^{-4}\|v_h\|^2_{L^2(B_d)}.$$

Proof. Adding and subtracting $P_1(\omega^2)$ we have

$$\left\| \omega^2 \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \right\|^2_{L^2(e)} \le \left\| P_1(\omega^2) \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \right\|^2_{L^2(e)} + \left\| (\omega^2 - P_1(\omega^2)) \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \right\|^2_{L^2(e)}.$$

By the trace and inverse inequalities

$$\begin{aligned}\sum_{e\in\mathcal{E}_h} |e| \left\| P_1(\omega^2) \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \right\|^2_{L^2(e)} &\le C \sum_{T\in\mathcal{T}_h} \left\| P_1(\omega^2) D^2 v_h \right\|^2_{L^2(T)} \\ &\le C \sum_{T\in\mathcal{T}_h} \left\| \omega^2 D^2 v_h \right\|^2_{L^2(T)} + \left\| (P_1(\omega^2) - \omega^2) D^2 v_h \right\|^2_{L^2(T)}.\end{aligned}$$

By the approximation and the inverse inequality, we have

$$\begin{aligned}\|(P_1(\omega^2) - \omega^2) D^2 v_h\|_{L^2(T)} &\le \|P_1(\omega^2) - \omega^2\|_{L^\infty(T)} \|D^2 v_h\|_{L^2(T)} \\ &\le Ch^2 \|D^2(\omega^2)\|_{L^\infty(T)} \|D^2 v_h\|_{L^2(T)} \le Cd^{-2}\|v_h\|_{L^2(T)}\end{aligned}$$

and as a result

$$\sum_{e\in\mathcal{E}_h} |e| \left\| P_1(\omega^2) \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \right\|^2_{L^2(e)} \le C \sum_{T\in\mathcal{T}_h} \left\| \omega^2 D^2 v_h \right\|^2_{L^2(T)} + Cd^{-4}\|v_h\|^2_{L^2(B_d)}.$$

Similarly, by the approximation, properties of ω, trace and inverse inequalities,

$$\sum_{e\in\mathcal{E}_h} |e| \left\| (P_1(\omega^2) - \omega^2) \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \right\|^2_{L^2(e)} \le C \sum_{T\in\mathcal{T}_h} \left\| \omega^2 D^2 v_h \right\|^2_{L^2(T)} + Cd^{-4}\|v_h\|^2_{L^2(B_d)}.$$

□

From the above lemma it follows that

Corollary 3.6. *For any $v_h \in S_h^r$ there exists a constant C independent of h such that*

$$|||\omega^2 v_h|||_{2,h} \le C\|\omega^2 v_h\|_{2,h} + Cd^{-2}\|v_h\|_{L^2(B_d)}.$$

Next we show the following coersivity type result.

Lemma 3.7. *Let ω be defined as above. Then, for any $v_h \in S_h^r$*

$$\|\omega^2 v_h\|^2_{2,h} \le CB_h(v_h, \omega^4 v_h) + Cd^{-4}\|v_h\|^2_{L^2(B_d)}.$$

Proof. From the definition

$$\begin{aligned} B_h(v_h,\omega^4 v_h) = & \sum_{T\in\mathcal{T}_h}\int_T D^2 v_h : D^2(\omega^4 v_h)\,dx + \\ & + \sum_{e\in\mathcal{E}_h}\int_e \left(\left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial(\omega^4 v_h)}{\partial n} \right]\!\!\right] + \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] \left\{\!\!\left\{ \frac{\partial^2(\omega^4 v_h)}{\partial n^2} \right\}\!\!\right\} \right) ds \\ & + \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|}\int_e \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial(\omega^4 v_h)}{\partial n} \right]\!\!\right] ds \\ := & J_1 + J_2 + J_3 + J_4. \end{aligned}$$

First we will address

$$J_1 = \sum_{T\in\mathcal{T}_h}\int_T D^2 v_h : D^2(\omega^4 v_h)\; dx := \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{2} (\partial^2_{ij} v_h, \partial^2_{ij}(\omega^4 v_h))_T.$$

Since

$$\begin{aligned} \partial^2_{ij}(\omega^4 v_h) &= \partial_i(4\omega^3\partial_j\omega v_h + \omega^4\partial_j v_h) \\ &= 12\omega^2\partial_i\omega\partial_j\omega v_h + 4\omega^3\partial^2_{ij}\omega v_h + 4\omega^3\partial_j\omega\partial_i v_h + 4\omega^3\partial_i\omega\partial_j v_h + \omega^4\partial^2_{ij} v_h, \end{aligned}$$

we have

$$J_1 = \sum_{T\in\mathcal{T}_h} \|\omega^2 D^2 v_h\|^2_{L^2(T)} + J_{12} + J_{13},$$

where

$$\begin{aligned} J_{12} &= \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{2} (\omega^2\partial^2_{ij} v_h, (12\partial_i\omega\partial_j\omega + 4\omega\partial^2_{ij}\omega) v_h)_T \\ J_{13} &= \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{2} (\omega^2\partial^2_{ij} v_h, 4\omega\partial_j\omega\partial_i v_h + 4\omega\partial_i\omega\partial_j v_h)_T. \end{aligned}$$

Since $v_h \in C^0(\Omega)$ we have

$$\left[\!\!\left[\frac{\partial(\omega^4 v_h)}{\partial n} \right]\!\!\right] = \omega^4 \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] \quad \text{and} \quad \left\{\!\!\left\{ \frac{\partial^2(\omega^4 v_h)}{\partial n^2} \right\}\!\!\right\} = \omega^4 \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\}$$

and as a result

$$J_2 = J_3 = \sum_{e\in\mathcal{E}_h} \left(\omega^2 \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\}, \omega^2 \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] \right)_e$$

and

$$J_4 = \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|}\int_e \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial(\omega^4 v_h)}{\partial n} \right]\!\!\right] ds = \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|} \left\| \omega^2 \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] \right\|^2_{L^2(e)}.$$

Combining the above estimates we obtain

$$\begin{aligned} \|\omega^2 v_h\|^2_{2,h} = B(v_h,\omega^4 v_h) - J_{12} - J_{13} - \sum_{e\in\mathcal{E}_h} \left(\omega^2 \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\}, \omega^2 \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] \right)_e \\ - (\eta-1)\sum_{e\in\mathcal{E}_h} |e|^{-1} \left\| \omega^2 \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] \right\|^2_{L^2(e)}. \end{aligned}$$

Using the Cauchy-Schwarz and Young's inequalities

$$\begin{aligned}(\omega^2\partial_{ij}^2 v_h, (12\partial_i\omega\partial_j\omega + 4\omega\partial_{ij}^2\omega)v_h)_T &\leq Cd^{-2}\|\omega^2 D^2 v_h\|_{L^2(T)}\|v_h\|_{L^2(T)}\\ &\leq \frac{1}{4}\|\omega^2 D^2 v_h\|_{L^2(T)}^2 + Cd^{-4}\|v_h\|_{L^2(T\cap B_d)}^2\end{aligned}$$

and summing up we have

$$J_{12} \leq \frac{1}{4}\sum_{T\in\mathcal{T}_h}\|\omega^2 D^2 v_h\|_{L^2(T)}^2 + Cd^{-4}\|v_h\|_{L^2(B_d)}^2.$$

Again using the Cauchy-Schwarz and Young's inequalities

$$\begin{aligned}4(\omega^2\partial_{ij}^2 v_h, \omega\partial_j\omega\partial_i v_h + \omega\partial_i\omega\partial_j v_h)_T &\leq Cd^{-1}\|\omega^2 D^2 v_h\|_{L^2(T)}\|\omega\nabla v_h\|_{L^2(T)}\\ &\leq \frac{1}{4}\|\omega^2 D^2 v_h\|_{L^2(T)}^2 + Cd^{-2}\|\omega\nabla v_h\|_{L^2(T)}^2.\end{aligned}$$

Summing and using Lemma 3.4, we have

$$\begin{aligned}J_{13} \leq& \frac{1}{4}\sum_{T\in\mathcal{T}_h}\|\omega^2 D^2 v_h\|_{L^2(T)}^2 + Cd^{-2}\sum_{T\in\mathcal{T}_h}\|\omega\nabla v_h\|_{L^2(T)}^2\\ \leq& \left(\frac{1}{4}+\varepsilon C\right)\sum_{T\in\mathcal{T}_h}\|\omega^2 D^2 v_h\|_{L^2(T)}^2 + \varepsilon C\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|\omega^2\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]\right\|_{L^2(e)}^2\\ &+ C_\varepsilon d^{-4}\|v_h\|_{L^2(B_d)}^2.\end{aligned}$$

Using the Cauchy-Schwarz and Young's inequalities and Lemma 3.5

$$\begin{aligned}\sum_{e\in\mathcal{E}_h}\left(\omega^2\left\{\!\!\left\{\frac{\partial^2 v_h}{\partial n^2}\right\}\!\!\right\}, \omega^2\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]\right)_e \leq& \frac{1}{8}\sum_{T\in\mathcal{T}_h}\|\omega^2 D^2 v_h\|_{L^2(T)}^2 + Cd^{-4}\|v_h\|_{L^2(B_d)}^2\\ &+ C\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|\omega^2\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]\right\|_{L^2(e)}^2.\end{aligned}$$

Choosing ε such that $C\varepsilon \leq \frac{1}{8}$ and providing η is sufficiently large we obtain the lemma. $\square$

In view of Corollary 3.6, we also have

Corollary 3.8. *Under the assumption of Lemma 3.7, we have*

$$|||\omega^2 v_h|||_{2,h}^2 \leq CB(v_h, \omega^4 v_h) + Cd^{-4}\|v_h\|_{L^2(B_d)}^2.$$

3.5. **Local energy estimates.** Next we establish the following key result.

Lemma 3.9 (Local Energy Estimate). *Let u and u_h satisfy $B_h(u-u_h,\chi)=0$ for any $\chi\in S_h^r$. Given $D\subset\Omega$, $d\geq\kappa h>0$ for sufficiently large κ, and $D_d\subset D$ with $dist(D_d,\partial D\setminus\partial\Omega)\geq d$, there exists a constant C independent of d and h such that*

$$|||u-u_h|||_{2,h,D} \leq C\inf_{\chi\in S_h^r}|||u-\chi|||_{2,h,D_d} + Cd^{-2}\|u-u_h\|_{L^2(D_d)}.$$

Proof. To show the lemma, it is sufficient to establish the following estimate

$$|||u_h|||_{2,h,B} \leq C|||u|||_{2,h,B_d} + Cd^{-2}\|u_h\|_{L^2(B_d)} \tag{30}$$

for a ball B. The lemma then would follow by standard covering argument and by replacing u with $u-\chi$ and $u_h-\chi$ and the triangle inequality.

Let ω be the cut-off function as in (19). Then by Corollary 3.8,

$$|||u_h|||^2_{2,h,B} \leq \left|\left|\left|\omega^2 u_h\right|\right|\right|^2_{2,h} \leq CB_h(u_h,\omega^4 u_h) + Cd^{-4}\|u_h\|^2_{L^2(B_d)}.$$

We use the identity

$$\begin{aligned} B_h(u_h,\omega^4 u_h) &= B_h(u_h,\omega^4 u_h - I_h(\omega^4 u_h)) + B_h(u,\omega^4 u_h) + B_h(u,\omega^4 u_h - I_h(\omega^4 u_h)) \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

Now we treat all three terms separately. Using the boundedness of the bilinear from and Young's inequality

$$J_2 \leq C|||u|||_{2,h,B_d}\left|\left|\left|\omega^4 u_h\right|\right|\right|_{2,h} \leq C|||u|||^2_{2,h,B_d} + \frac{1}{4}\left|\left|\left|\omega^2 u_h\right|\right|\right|^2_{2,h}.$$

Using the superapproximation result Lemma 3.1 and Lemma 3.4 and similarly to the above

$$\begin{aligned} J_3 &\leq C|||u|||_{2,h,B_d}\left|\left|\left|\omega^4 u_h - I_h(\omega^4 u_h)\right|\right|\right|_{2,h} \\ &\leq C\frac{h}{d}|||u|||_{2,h,B_d}\left(|\omega^2 u_h|_{H^2(B_d)} + d^{-1}|\omega u_h|_{H^1(B_d)} + d^{-2}\|u_h\|_{L^2(B_d)}\right) \\ &\leq C|||u|||_{2,h,B_d}\left(\left|\left|\left|\omega^2 u_h\right|\right|\right|_{2,h} + Cd^{-2}\|u_h\|_{L^2(B_d)}\right) \\ &\leq C|||u|||^2_{2,h,B_d} + Cd^{-4}\|u_h\|^2_{L^2(B_d)} + \frac{1}{4}\left|\left|\left|\omega^2 u_h\right|\right|\right|^2_{2,h}. \end{aligned}$$

To estimate J_1 we use the superapproximation result Corollary 3.3 to obtain

$$\begin{aligned} J_1 &\leq C|||u_h|||_{2,h,B_d}\left|\left|\left|\omega^4 u_h - I_h(\omega^4 u_h)\right|\right|\right|_{2,h} \\ &\leq C\frac{h}{d}|||u|||_{2,h,B_d}\left(|\omega^2 u_h|_{H^2(B_d)} + d^{-1}|\omega u_h|_{H^1(B_d)} + d^{-2}\|u_h\|_{L^2(B_d)}\right) \\ &\leq C\frac{h}{d}|||u_h|||_{2,h,B_d}\left(\left|\left|\left|\omega^2 u_h\right|\right|\right|_{2,h} + Cd^{-2}\|u_h\|_{L^2(B_d)}\right) \\ &\leq C\frac{h^2}{d^2}|||u_h|||^2_{2,h,B_d} + Cd^{-4}\|u_h\|^2_{L^2(B_d)} + \frac{1}{4}\left|\left|\left|\omega^2 u_h\right|\right|\right|^2_{2,h}. \end{aligned}$$

Combining the estimates and kicking back the terms with $\left|\left|\left|\omega^2 u_h\right|\right|\right|^2_{2,h}$ we obtain

$$|||u_h|||^2_{2,h,B} \leq C|||u|||^2_{2,h,B_d} + C\frac{h^2}{d^2}|||u_h|||^2_{2,h,B_d} + Cd^{-4}\|u_h\|^2_{L^2(B_d)}.$$

Iterating the argument once again and using the inverse inequality we obtain

$$\begin{aligned} |||u_h|||^2_{2,h,B} \leq & C|||u|||^2_{2,h,B_{2d}} + C\frac{h^4}{d^4}|||u_h|||^2_{2,h,B_{2d}} + Cd^{-4}\|u_h\|^2_{L^2(B_{2d})} \leq C|||u|||^2_{2,h,B_{2d}} \\ &+ Cd^{-4}\|u_h\|^2_{L^2(B_{2d})}, \end{aligned}$$

which establishes (30) with insignificant difference of having subset B_{2d} instead of B_d on the right hand side. □

4. Pointwise error estimates

Let $x_0 \in \Omega$ be a fixed (but arbitrary) point. Associated to this point we introduce a smooth Delta function [21, Lemma 2.2], which we will denote by $\tilde{\delta} = \tilde{\delta}_{x_0}$, cf. also [19]. This function is supported in one cell, denoted by T_0, and satisfies

$$(\chi, \tilde{\delta})_{T_0} = \chi(x_0), \quad \forall \chi \in \mathbb{P}^k(T_0).$$

In addition from [21, Lemma 2.2] we also have

$$\|\tilde{\delta}\|_{W^{s,p}(\Omega)} \le Ch^{-s-2(1-\frac{1}{p})}, \quad 1 \le p \le \infty, \quad s = 0, 1, 2. \tag{31}$$

Thus in particular $\|\tilde{\delta}\|_{L^1(\Omega)} \le C$, $\|\tilde{\delta}\|_{L^2(\Omega)} \le Ch^{-1}$, and $\|\tilde{\delta}\|_{L^\infty(\Omega)} \le Ch^{-2}$.

Now we define the function $g \in H_0^2(\Omega)$, by the equation

$$B(g, \varphi) = (D^2\tilde{\delta}, \varphi), \quad \forall \varphi \in H_0^2(\Omega), \tag{32}$$

which also satisfy

$$B_h(g, \chi) = (D^2\tilde{\delta}, \chi) = D^2\chi(x_0), \quad \forall \chi \in S_h^r,$$

and correspondingly we define $g_h \in S_h^r$ by

$$B_h(g - g_h, \chi) = 0, \quad \forall \chi \in S_h^r. \tag{33}$$

Thus, for any $\chi \in S_h^r$ using (33), (14), (18) and that $\tilde{\delta}$ is supported on T_0, we have

$$\begin{aligned}
D^2(u_h - \chi)(x_0) &= B_h(u_h - \chi, g) \\
&= B_h(u_h - \chi, g_h) \\
&= B_h(u - \chi, g_h) \\
&= B_h(u - \chi, g) - B_h(u - \chi, g - g_h) \\
&= (u - \chi, D^2\tilde{\delta})_{T_0} - B_h(u - \chi, g - g_h) \\
&= (D^2(u - \chi), \tilde{\delta})_{T_0} - B_h(u - \chi, g - g_h) \\
&\le \|D^2(u - \chi)\|_{L^\infty(T_0)} \|\tilde{\delta}\|_{L^1(T_0)} + C|||u - \chi|||_{W_h^{2,\infty}(\Omega)} |||g - g_h|||_{W_h^{2,1}(\Omega)} \\
&\le C|||u - \chi|||_{W_h^{2,\infty}(\Omega)} (1 + |||g - g_h|||_{W_h^{2,1}(\Omega)}).
\end{aligned}$$

The main result would follow once we establish the following result.

Lemma 4.1. *There exists a constant C independent of h such that*

$$|||g - g_h|||_{W_h^{2,1}(\Omega)} \le C|\ln h|^{3/2}.$$

4.1. **Proof of Lemma 4.1.** In the proof we will use a dyadic decomposition of Ω. Let $j_0 \in \mathbb{Z}$ be the largest integer such that $d_{j_0} := 2^{-j_0} \ge \operatorname{diam}(\Omega)$. We have

$$\Omega = \Omega_* \cup \bigcup_{j=0}^{J} \Omega_j, \tag{34}$$

with

$$\Omega_* = \{x \in \Omega : \ |x - x_0| \le d_*\}, \quad d_* = C_*h$$
$$\Omega_j = \{x \in \Omega : \ d_j \le |x - x_0| \le d_{j+1}\}, \quad d_j = 2^{-j},$$

where C_* is a sufficiently large constant and J is the integer such that $C_*h \le d_J \le 2C_*h$. Note that $J \approx |\ln h|$. We will also use $\Omega_j' = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$ and $\Omega_j'' = (\Omega_j')'$. Thus by the triangle inequality

$$|||g - g_h|||_{W_h^{2,1}(\Omega)} \le |||g - g_h|||_{W_h^{2,1}(\Omega_*)} + \sum_{j=j_0}^{J} |||g - g_h|||_{W_h^{2,1}(\Omega_j)}.$$

By the Cauchy-Schwarz inequality and (15), we obtain

(35)

$$|||g - g_h|||_{W_h^{2,1}(\Omega_*)} \le C_*h|||g - g_h|||_{2,h} \le Ch^2\|D^2\tilde{\delta}\|_{H^{-1}(\Omega)} \le Ch^2\|\nabla\tilde{\delta}\|_{L^2(\Omega)} \le C.$$

Similarly by the Cauchy-Schwarz inequality

$$\sum_{j=j_0}^{J} |||g-g_h|||_{W_h^{2,1}(\Omega_j)} \le \sum_{j=j_0}^{J} d_j |||g-g_h|||_{2,h,\Omega_j}.$$

Thus we have

$$|||g-g_h|||_{W_h^{2,1}(\Omega)} \le C + \sum_{j=j_0}^{J} d_j |||g-g_h|||_{2,h,\Omega_j}. \tag{36}$$

By the local energy estimate Lemma 3.9

$$|||g-g_h|||_{2,h,\Omega_j} \le C|||g-\chi|||_{2,h,\Omega'_j} + Cd_j^{-2}\|g-g_h\|_{L^2(\Omega'_j)}.$$

Taking $\chi = I_h g$, where I_h is the usual nodal interpolant with properties (9), we obtain

$$\begin{aligned}
|||g-I_hg|||^2_{2,h,\Omega'_j} \le & \sum_{T\in\mathcal{T}_h\cap\Omega'_j} \|D^2(g-I_hg)\|^2_{L^2(T)} \\
&+ \sum_{e\in\mathcal{E}_h\cap\Omega'_j} \int_e \left(|e| \left| \left\{\!\!\left\{ \frac{\partial^2(g-I_hg)}{\partial n^2} \right\}\!\!\right\} \right|^2 + |e|^{-1} \left| \left[\!\!\left[\frac{\partial(g-I_hg)}{\partial n} \right]\!\!\right] \right|^2 \right) ds \\
&= E_1 + E_2 + E_3.
\end{aligned}$$

Using the Hölder inequality and (9), we have

$$\begin{aligned}
E_1 &\le \sum_{T\in\mathcal{T}_h\cap\Omega'_j} Ch^2\|D^2(g-I_hg)\|^2_{L^\infty(T)} \\
&\le Ch^2\|D^2g\|^2_{L^\infty(\Omega'_j)} \sum_{T\in\mathcal{T}_h\cap\Omega'_j} 1 \le Cd_j^2\|D^2g\|^2_{L^\infty(\Omega'_j)},
\end{aligned}$$

where in the last step we used that for the quasi-uniform triangulation, the number of triangle in Ω'_j is of order $(d_j/h)^2$. Similarly, using the Hölder inequality and stability of the interpolant, we have

$$\begin{aligned}
E_2 &\le C \sum_{e\in\mathcal{E}_h\cap\Omega_j} \int_e |e|^2\|D^2(g-I_hg)\|^2_{L^\infty(e)} \\
&\le Ch^2\|D^2g\|^2_{L^\infty(\Omega'_j)} \sum_{e\in\mathcal{E}_h\cap\Omega'_j} 1 \le Cd_j^2\|D^2g\|^2_{L^\infty(\Omega'_j)}.
\end{aligned}$$

To estimate E_3 we use the inverse inequality together with the approximation property and the stability of the interpolant (9). Thus, we obtain

$$\begin{aligned}
E_3 &\le C \sum_{T\in\mathcal{E}_h\cap\Omega'_j} \left(h^{-2}\|D(g-I_hg)\|^2_{L^2(T)} + \|D^2(g-I_hg)\|^2_{L^2(T)} \right) \\
&\le C \sum_{T\in\mathcal{E}_h\cap\Omega'_j} \left(\|D(g-I_hg)\|^2_{L^\infty(T)} + h^2\|D^2(g-I_hg)\|^2_{L^\infty(T)} \right) \\
&\le Ch^2\|D^2g\|^2_{L^\infty(T)} \sum_{T\in\mathcal{E}_h\cap\Omega'_j} 1 \le Cd_j^2\|D^2g\|^2_{L^\infty(\Omega'_j)}.
\end{aligned}$$

Using the Green's function representation and the integration by part, we obtain

$$D^2 g(x) = \int_{T_0} D_x^2 G(x,y) D^2 \tilde{\delta}(y)\ dy = \int_{T_0} D_x^2 D_y^2 G(x,y) \tilde{\delta}(y)\ dy.$$

Using the pointwise Green's function estimate from Lemma 2.1 and that $\operatorname{dist}(\Omega'_j, T_0) \approx d_j$, for any $x \in \Omega'_j$ we have

$$|D^2 g(x)| \le \int_{T_0} |D_x^2 D_y^2 G(x,y)||\tilde{\delta}(y)|\ dy \le \int_{T_0} \frac{|\tilde{\delta}(y)|}{|x-y|^2}\ dy \le C d_j^{-2}$$

and as a result

$$\sum_{j=j_0}^{J} d_j |||g - I_h g|||_{2,h,\Omega'_j} \le C \sum_{j=j_0}^{J} d_j^2 \|g\|_{W^{2,\infty}(\Omega'_j)} \le C \sum_{j=j_0}^{J} 1 \le CJ \le C|\ln h|.$$

Thus so far we have established

$$|||g - g_h|||_{W_h^{2,1}(\Omega)} \le C|\ln h| + \sum_{j=j_0}^{J} d_j^{-1} \|g - g_h\|_{L^2(\Omega'_j)}. \tag{37}$$

To complete the proof of Lemma 4.1, we need to establish

$$\sum_{j=j_0}^{J} d_j^{-1} \|g - g_h\|_{L^2(\Omega'_j)} \le C|\ln h|^{3/2}, \tag{38}$$

which we will accomplish in the next section via a duality argument.

4.2. **Duality argument.** To estimate $\sum_{j=j_0}^{J} d_j^{-1} \|g - g_h\|_{L^2(\Omega'_j)}$ we use a duality argument. However, in contrast to the second order elliptic problems, the regularity shift results do not hold on polygonal domains and estimating the L^2 norm of the error is problematic. To avoid this difficulty, we first transform $\sum_{j=j_0}^{J} d_j^{-1} \|g - g_h\|_{L^2(\Omega'_j)}$ to a gradient in L^2 norm.

Lemma 4.2. *There exists a constant C independent of h such that*

$$\sum_{j=j_0}^{J} d_j^{-1} \|g - g_h\|_{L^2(\Omega'_j)} \le C|\ln h|^{3/2} \|\nabla(g - g_h)\|_{L^2(\Omega)}.$$

Proof. Define a weight function

$$\sigma(x) = \sqrt{|x - x_0|^2 + h^2}.$$

By the definition of σ it is easy to see that $\sigma \approx d_j$ on Ω_j. Thus, by the Cauchy-Schwarz inequality and using that $J \le C|\ln h|$, we have

$$\begin{aligned}
\sum_{j=j_0}^{J} d_j^{-1} \|g - g_h\|_{L^2(\Omega'_j)} &\le C \left(\sum_{j=j_0}^{J} 1 \right)^{1/2} \left(\sum_{j=j_0}^{J} d_j^{-2} \|g - g_h\|^2_{L^2(\Omega'_j)} \right)^{1/2} \\
&\le C|\ln h|^{1/2} \left(\sum_{j=j_0}^{J} \|\sigma^{-1}(g - g_h)\|^2_{L^2(\Omega'_j)} \right)^{1/2} \\
&\le C|\ln h|^{1/2} \|\sigma^{-1}(g - g_h)\|_{L^2(\Omega)}.
\end{aligned}$$

Since $g - g_h$ is zero on $\partial\Omega$, by the argument in Lemma 3.4 in [1], we have

$$\|\sigma^{-1}(g - g_h)\|_{L^2(\Omega)} \leq C|\ln h|\|\nabla(g - g_h)\|_{L^2(\Omega)},$$

which gives us the lemma. □

Now we proceed with a duality argument.

Lemma 4.3. *There exists a constant C such that*

$$\|\nabla(g - g_h)\|_{L^2(\Omega)} \leq C.$$

Proof. We proceed similarly to [16], section 4, and define a bounded linear functional on $H_0^1(\Omega)$ by

$$F(\varphi) = (\nabla(g - g_h), \nabla\varphi) \tag{39}$$

and the corresponding solution $w \in H_0^2(\Omega)$ by

$$B(w, \varphi) = (D^2 w, D^2 \varphi) = F(\varphi), \quad \forall \varphi \in H_0^2(\Omega). \tag{40}$$

Since Ω is convex, $w \in H_0^2(\Omega) \cap H^3(\Omega)$ and by the regularity estimate (4), we have

$$\|w\|_{H^3(\Omega)} \leq C\|F\|_{H^{-1}(\Omega)} = C \sup_{\varphi \in H_0^1(\Omega)} \frac{(\nabla(g - g_h), \nabla\varphi)}{\|\nabla\varphi\|_{L^2(\Omega)}} \leq C\|\nabla(g - g_h)\|_{L^2(\Omega)}. \tag{41}$$

From Lemma 5 in [4], w also satisfies

$$B_h(w, \chi) = F(\chi), \quad \forall \chi \in S_h^r.$$

From the definition of w above, a priori error estimate (15), and using (41) and (4), we have

$$\begin{aligned}
\|\nabla(g - g_h)\|_{L^2(\Omega)}^2 = F(g - g_h) = B_h(w, g - g_h) &= B_h(w - I_h w, g - g_h) \\
&\leq |||w - I_h w|||_{2,h} |||g - g_h|||_{2,h} \\
&\leq Ch\|w\|_{H^3(\Omega)} Ch\|g\|_{H^3(\Omega)} \\
&\leq Ch^2 \|\nabla(g - g_h)\|_{L^2(\Omega)} \|D^2 \tilde{\delta}\|_{H^{-1}(\Omega)} \\
&\leq Ch^2 \|\nabla(g - g_h)\|_{L^2(\Omega)} \|\nabla\tilde{\delta}\|_{L^2(\Omega)} \\
&\leq C\|\nabla(g - g_h)\|_{L^2(\Omega)}.
\end{aligned}$$

Canceling, we obtain the result. □

Combining Lemma 4.2 and Lemma 4.3, which establishes (38) and as a result Lemma 4.1.

5. Interior error estimates

To show interior error estimates we use again the weight function

$$\sigma(x) = \sqrt{|x - x_0|^2 + h^2} > 0. \tag{42}$$

One can easily check that σ satisfies the following properties,

$$\|\sigma^{-1}\|_{L^2(\Omega)} \leq C|\ln h|^{\frac{1}{2}}, \tag{43a}$$

$$|\nabla\sigma| \leq C, \tag{43b}$$

$$|\nabla^2\sigma| \leq C|\sigma^{-1}| \tag{43c}$$

$$\max_T \sigma \leq C \min_T \sigma, \quad \forall T \in \mathcal{T}_h. \tag{43d}$$

Using σ we define the following weighted norms

$$|||v|||_{2,h,\sigma}^2 = \sum_{T\in\mathcal{T}_h} \|\sigma D^2 v\|_T^2 + \sum_{e\in\mathcal{E}_h} \int_e \left(\sigma^2 |e| \left|\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\right|^2 + \sigma^2 |e|^{-1} \left|\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right|^2\right) ds$$

and

$$|||v|||_{2,h,\sigma^{-1}}^2 = \sum_{T\in\mathcal{T}_h} \|\sigma^{-1} D^2 v\|_T^2 + \sum_{e\in\mathcal{E}_h} \int_e \left(\sigma^{-2} |e| \left|\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\right|^2 + \sigma^{-2} |e|^{-1} \left|\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right|^2\right) ds.$$

Lemma 5.1. *There exists a constant C independent of h such that*

$$|||g - g_h|||_{2,h,\sigma} \le C|\ln h|.$$

Proof. Using dyadic decomposition (34) and the property that $\sigma \approx d_j$ on each Ω_j we have

$$|||g - g_h|||_{2,h,\sigma} \le C_* h |||g - g_h|||_{2,h} + \sum_{j=j_0}^{J} d_j |||g - g_h|||_{2,h,\Omega_j}.$$

The rest of the proof is identical to the proof of Lemma 4.1. □

5.1. **Proof of Theorem 1.2.**

Proof. Similarly to the proof of Theorem 1.1, for any $\chi \in S_h^r$ and using that $\tilde{\delta}$ is supported on T_0

$$\begin{aligned} D^2(u_h - \chi)(x_0) &= B_h(u_h - \chi, g) \\ &= B_h(u_h - \chi, g_h) \\ &= B_h(u - \chi, g_h) \\ &= B_h(u - \chi, g) - B_h(u - \chi, g - g_h) \\ &= (u - \chi, D^2\tilde{\delta})_{T_0} - B_h(u - \chi, g - g_h) \\ &= (D^2(u - \chi), \tilde{\delta})_{T_0} - B_h(u - \chi, g - g_h)_{B_d} - B_h(u - \chi, g - g_h)_{\Omega\setminus B_d} \\ &= J_1 + J_2 + J_3, \end{aligned}$$

where by $B(\cdot,\cdot)_D$ we denote the part of the bilinear form restricted to a set $D \subset \Omega$. Exactly as in Theorem 1.1

$$J_1 \le \|D^2(u - \chi)\|_{L^\infty(T_0)} \|\tilde{\delta}\|_{L^1(T_0)} \le C\|D^2(u - \chi)\|_{L^\infty(B_d)}.$$

Using the Hölder inequality and Lemma 4.1

$$J_2 \le C|||u - \chi|||_{W_h^{2,\infty}(B_d)} |||g - g_h|||_{W_h^{2,1}(\Omega)} \le C|\ln h| |||u - \chi|||_{W_h^{2,\infty}(B_d)}.$$

Using the Cauchy-Schwarz inequality, Lemma 5.1, and using that $\sigma^{-1} \le Cd^{-1}$ on $\Omega\setminus B_d$, we have

$$J_3 \le C|||u - \chi|||_{2,h,\Omega\setminus B_d,\sigma^{-1}} |||g - g_h|||_{2,h,\Omega,\sigma} \le C|\ln h| d^{-1} |||u - \chi|||_{2,h,\Omega}.$$

□

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