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Optimal error estimates for finite element discretization of elliptic optimal control problems with finitely many pointwise state constraints

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Abstract In this paper we consider a model elliptic optimal control problem with finitely many state constraints in two and three dimensions. Such problems are challenging due to low regularity of the adjoint variable. For the discretization of the problem we consider continuous linear elements on quasi-uniform and graded meshes separately. Our main result establishes optimal a priori error estimates for the state, adjoint, and the Lagrange multiplier on the two types of meshes. From our results, for example, it follows that in three dimensions the optimal second order convergence rate for all three variable can be expected only on properly refined meshes. Numerical examples at the end of the paper support our theoretical results.

1 Introduction

In this paper we consider the following optimal control problem

$$\text{Minimize } \frac{1}{2}\|u-u_d\|^2+\frac{\alpha}{2}\|q\|^2 \tag{1a}$$

subject to the state equation

$$\begin{aligned} -\Delta u &= q \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1b}$$

and to finitely many pointwise state constraints

$$u(x_i) = b_i \qquad i = 1,2,\ldots,n, \tag{1c}$$

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with mutually distinct points $x_1, x_2, \ldots, x_n \in \Omega$. Here q and u denote the control and the state variable respectively, $\alpha > 0$ is the regularization parameter and $\|\cdot\|$ denotes the L^2-norm on the domain Ω, see Section 2 for a precise functional analytic setting. Such optimal control problems with finitely many state constraints are motivated by technological applications and are analyzed in, e.g., [6,8,7,9] and also in [25] for the case of finite dimensional control variable. In addition, the investigation of finitely many state constraints is also helpful in the context of optimal control problems with pointwise constraints on the whole domain of type $u(x) \leq b$ almost everywhere in Ω. The corresponding active set often consists only of finitely many points, see, e.g., [24]. The main difference is that the active points are not known a priori in the contrast to the problem considered here.

The focus of the paper is the a priori error analysis for the finite element discretization of (1). We discretize the state and the control by linear finite elements, see Section 3 for details. We denote by $\bar{q}$, $\bar{u}$ and $\bar{\mu}$ the optimal control, optimal state and the Lagrangian multiplier for the state constrains. The corresponding discrete solutions are denoted by $\bar{q}_h$, $\bar{u}_h$ and $\bar{\mu}_h$ and the aim is to analyze the errors $\|\bar{q} - \bar{q}_h\|$, $\|\bar{u} - \bar{u}_h\|$ and $|\bar{\mu} - \bar{\mu}_h|$. We make the analysis for two types of families of meshes: quasi-uniform meshes and graded meshes, which are locally refined towards the points $x_1, x_2, \ldots, x_n$. The number of elements of such meshes with respect to the mesh size h are asymptotically the same. They are of order h^{-2} and h^{-3} for two and three dimensions, respectively, see, e.g., [4].

The orders of convergence with respect to the mesh size h for the case of quasi-uniform meshes are shown in Table 1 up to logarithmic terms, see Theorem 4 and Theorem 5 for precise results. These results improve the known estimates for the state and the multiplier, see [9], where the convergence only of the same order as for the control variable is shown. This improved convergence is also observed in the numerical examples, see Section 7. The main instrument to prove these improved error estimates is a duality argument for the whole optimality system. To our knowledge such a duality argument is only recently used in [23] and [21].

Table 1 Proved orders of convergence on quasi-uniform meshes

Dimension	$\|\bar{q}-\bar{q}_h\|$	$\|\bar{u}-\bar{u}_h\|$	$\lvert\bar{\mu}-\bar{\mu}_h\rvert$
$d=2$	1	2	2
$d=3$	1/2	1	1

The numerical examples illustrate that these estimates can not be further improved by using quasi-uniform meshes. This can be explained by the lack of regularity of the adjoint variable, which fulfills an elliptic equation with a linear combination of Dirac measures on the right-hand side, see Section 2 for details. However, the solutions to elliptic problems with such irregular data can be approximated with almost optimal order (up to a logarithmic term) if using graded meshes, which are locally refined towards the singularities, see [3,11, 14]. Our main contribution for the optimal control problem under consideration is showing that the optimal second order can be achieved on properly graded meshes for the errors in the control, state and in the Lagrangian multiplier in two and three dimensions, see Table 2 and the precise results in Theorem 7 and Theorem 8.

The proof of this result is on one hand based on a duality argument for the whole optimality system and on the other hand on a pointwise error estimate for the state equation of optimal order $\mathcal{O}(h^2|\ln h|)$ on properly graded meshes, see Theorem 6. An important feature of this result is the fact that only the $L^2(\Omega)$-norm of the right-hand side q enters the estimate,

Table 2 Proved orders of convergence on graded meshes

Dimension	$\|\bar{q}-\bar{q}_h\|$	$\|\bar{u}-\bar{u}_h\|$	$\|\bar{\mu}-\bar{\mu}_h\|$
$d=2$	2	2	2
$d=3$	2	2	2

whereas on uniform meshes only the first order $\mathcal{O}(h)$ can be expected for the pointwise error and a general right-hand side from $L^2(\Omega)$. A direct application of this theorem results in an estimate of order $\mathcal{O}(h^2|\ln h|)$ for the finite element approximation of a Green's function with respect to the $L^2(\Omega)$-norm, see Corollary 2. These estimates (Theorem 6 and Corollary 2) extend the results from [3] also for the three dimensional case. The proof is different from [3]. It uses the techniques from [29] and covers simultaneously the two and the three dimensional cases.

The paper is structured as follows. In the next section we provide the functional analytic setting for the problem (1) and discuss the optimality conditions as well as regularity issues. Section 3 is devoted to the finite element discretization of (1) and to the corresponding optimality conditions on the discrete level. In Section 4 we introduce a dual problem, which is essential for our error analysis, which is provided in Section 5 for quasi-uniform and in Section 6 for properly graded meshes. In Section 7 we discuss numerical examples illustrating our error estimates.

2 Continuous problem

Let $\Omega \subset \mathbb{R}^d$ for $d \in \{2,3\}$ be a convex domain with smooth boundary $\partial\Omega$ and

$$\frac{2d}{d+2} < s < \frac{d}{d-1}$$

The dual s' of s defined by $\frac{1}{s}+\frac{1}{s'}=1$ fulfills $s'>d$.

We define the control space as $Q := L^2(\Omega)$ and employ the usual notation for Lebesgue, Hilbert, and Sobolev spaces.

Then, the weak formulation of the state equation (1b) for a given control $q \in Q$ reads as follows: Find a state $u \in W_0^{1,s'}(\Omega)$ satisfying

$$(\nabla u, \nabla \varphi) = (q, \varphi) \quad \forall \varphi \in W_0^{1,s}(\Omega). \tag{2}$$

By standard arguments, one obtains the existence and uniqueness of a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of (2). By embedding, this implies $u \in W_0^{1,s'}(\Omega)$. Since $s' > d$, we have by the embedding $W_0^{1,s'}(\Omega) \hookrightarrow C(\bar{\Omega})$ that the pointwise state constraints given by (1c) are well-defined for the state $u \in W_0^{1,s'}(\Omega)$.

Remark 1 The solution u of (2) does not depend on the value of s and it holds $u \in H^2(\Omega)$ with the estimate

$$\|\nabla^2 u\| \le C\|q\|. \tag{3}$$

By introducing the operator $G\colon C(\bar{\Omega}) \to \mathbb{R}^n$ given by

$$G(u) = (u(x_1), u(x_2), \ldots, u(x_n))^T,$$

the state constraint (1c) can be formulated for $b = (b_1, b_2, \dots, b_n)^T \in \mathbb{R}^n$ as

$$G(u) = b. \tag{4}$$

With the cost functional $J\colon Q \times W_0^{1,s'}(\Omega) \to \mathbb{R}$ defined as

$$J(q,u) := \frac{1}{2}\|u - u_d\|^2 + \frac{\alpha}{2}\|q\|^2,$$

the weak formulation of the optimal control problem (1) reads as

$$\text{Minimize } J(q,u) \text{ for } (q,u) \in Q \times W_0^{1,s'}(\Omega) \text{ subject to (2) and (4),} \tag{5}$$

where $u_d \in L^2(\Omega)$ is the target state and $\alpha > 0$ is the regularization parameter.

Lemma 1 *The admissible set*

$$W_{ad} := \left\{ (q,u) \in Q \times W_0^{1,s'}(\Omega) \;\middle|\; u \text{ solves (2) for the given } q \text{ and fulfills (4)} \right\}$$

is nonempty.

Proof We construct an admissible pair (q,u) as follows. Since the points $x_1, x_2, \dots, x_n \in \Omega$ are mutually distinct, we have

$$\min_{\substack{i=1,2\dots,n \\ i \neq j}} \operatorname{dist}(x_i, x_j) = \eta_1 > 0 \quad \text{and} \quad \min_{i=1,2\dots,n} \operatorname{dist}(x_i, \partial\Omega) = \eta_2 > 0.$$

Let $\eta = \min(\eta_1, \eta_2)$. Since $\eta > 0$, it is possible to construct cut-off functions $\omega_i \in C^\infty(\Omega)$ such that

$$\omega_i = \begin{cases} 0, & \text{on } \Omega \setminus B_{\frac{\eta}{2}}(x_i) \\ 1, & \text{on } B_{\frac{\eta}{4}}(x_i). \end{cases}$$

By construction, we have that the function

$$u = \sum_{i=1}^{n} b_i \omega_i \in C_0^\infty(\Omega)$$

satisfies the state constraint (4). Hence the pair (q,u) with $q = -\Delta u \in C_0^\infty(\Omega)$ is in W_{ad}. □

By standard arguments, the existence of an admissible point for problem (5) from Lemma 1 ensures the existence and uniqueness of an optimal solution to the considered problem.

The existence result for the state equation ensures the existence of a control-to-state mapping $S\colon q \mapsto u = u(q)$ defined through (2). By means of this mapping, we introduce the reduced cost functional $j\colon Q \to \mathbb{R}$ as

$$j(q) := J(q, S(q)).$$

We now formulate the necessary and sufficient optimality conditions for the problem under the consideration:

Theorem 1 *A control $\bar{q} \in Q$ with associated state $\bar{u} = u(\bar{q}) \in W_0^{1,s'}(\Omega)$ is an optimal solution to problem* (5) *if and only if $G(u) = b$ and there exists an adjoint state $\bar{z} \in W_0^{1,s}(\Omega)$ and a Lagrange multiplier $\bar{\mu} \in \mathbb{R}^n$ such that*

$$(\nabla \bar{u}, \nabla \varphi) = (\bar{q}, \varphi) \qquad \forall \varphi \in W_0^{1,s}(\Omega), \tag{6}$$

$$(\nabla \varphi, \nabla \bar{z}) = (\varphi, \bar{u} - u_d) + \sum_{i=1}^{n} \bar{\mu}_i \varphi(x_i) \qquad \forall \varphi \in W_0^{1,s'}(\Omega), \tag{7}$$

$$(\alpha \bar{q} + \bar{z}, \varphi) = 0 \qquad \forall \varphi \in Q. \tag{8}$$

Proof We note that linearity and surjectivity of the constraints implies Robinson's constraint qualification, see [20]. Since both the state equation and the state constraint are linear, linearity is trivially fulfilled. The surjectivity of the state equation and of the state constraint is equivalent to the existence of an admissible pair (q,u) for any given $b \in \mathbb{R}^n$, which was already established in Lemma 1. Using the formal Lagrange technique, this implies the existence of $\bar{z} \in W_0^{1,s}(\Omega)$ and $\bar{\mu} \in \mathbb{R}^n$ fulfilling (6)-(8). □

Remark 2 Note that (8) is equivalent to the condition

$$\bar{q} = -\frac{1}{\alpha} \bar{z}.$$

Consequently, the optimal control exhibits an improved regularity $\bar{q} \in W_0^{1,s}(\Omega)$.

We denote by $\bar{z}_0 \in W_0^{1,s}(\Omega)$ the solution of

$$(\nabla \varphi, \nabla \bar{z}_0) = (\varphi, \bar{u} - u_d) \quad \forall \varphi \in W_0^{1,s'}(\Omega) \tag{9}$$

and by $\bar{z}_i \in W_0^{1,s}(\Omega)$ for $i = 1, 2, \ldots, n$ the solutions of

$$(\nabla \varphi, \nabla \bar{z}_i) = \varphi(x_i) \quad \forall \varphi \in W_0^{1,s'}(\Omega). \tag{10}$$

With this notation, we have the following splitting of the adjoint state $\bar{z}$:

$$\bar{z} = \bar{z}_0 + \sum_{i=1}^{n} \bar{\mu}_i \bar{z}_i.$$

Remark 3 The solutions $\bar{z}_i$ for $i = 0, 1, \ldots, n$ of (9) and (10) do not depend on the value of s. Since $\bar{u} - u_d \in L^2(\Omega)$ we have $\bar{z}_0 \in H^2(\Omega)$ with the following estimate

$$\|\nabla^2 \bar{z}_0\| \leq C \|\bar{u} + u_d\|. \tag{11}$$

Lemma 2 *The solutions $\bar{z}_i \in W_0^{1,s}(\Omega)$ of* (10) *for $i = 1, 2, \ldots, n$ satisfy the following the estimates*

$$\|\bar{z}_i\| \leq C \quad \text{and} \quad \|\nabla \bar{z}_i\|_{L^s(\Omega)} \leq \frac{C}{s' - d},$$

where the constant C depends only on the domain Ω and the points x_i.

Proof The first assertion follows immediately from the embedding $W_0^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$. To prove the second assertion, we note that for $v \in W_0^{1,s'}(\Omega) \hookrightarrow C(\bar{\Omega})$ and $y \in W_0^{1,s}(\Omega)$ the estimates

$$\|v\|_{L^\infty(\Omega)} \leq \frac{C}{s'-d} \|\nabla v\|_{L^{s'}(\Omega)} \quad \text{and} \quad \|\nabla y\|_{L^s(\Omega)} \leq C \sup_{\varphi \in W_0^{1,s'}(\Omega)} \frac{(\nabla y, \nabla \varphi)}{\|\nabla \varphi\|_{L^{s'}(\Omega)}}$$

hold, see [2] and [1]. By means of these estimates and (10), we obtain

$$\|\nabla z_i\|_{L^s(\Omega)} \leq C \sup_{\varphi \in W_0^{1,s'}(\Omega)} \frac{(\nabla z_i, \nabla \varphi)}{\|\nabla \varphi\|_{L^{s'}(\Omega)}} = C \sup_{\varphi \in W_0^{1,s'}(\Omega)} \frac{\varphi(x_i)}{\|\nabla \varphi\|_{L^{s'}(\Omega)}} \leq \frac{C}{s'-d}.$$

□

Based on this result, we can prove the boundedness of the remaining quantities in the optimality system:

Lemma 3 *It holds*

$$\|\bar{q}\| + \|\bar{u}\| + \|\bar{z}_0\| + |\bar{\mu}| \leq C\{\|u_d\| + |b|\},$$

where $b = (b_1, b_2, \ldots, b_n)^T$, C is independent of s, and $|\cdot|$ denotes the Euclidian norm on $\mathbb{R}^n$.

Proof From Remark 2, we have $\bar{q} \in W_0^{1,s}(\Omega)$. Hence, by using the optimality system (6) – (8), we obtain

$$\begin{aligned} 0 &= \alpha\|\bar{q}\|^2 + (\bar{z}, \bar{q}) = \alpha\|\bar{q}\|^2 + (\nabla \bar{u}, \nabla \bar{z}) = \alpha\|\bar{q}\|^2 + (\bar{u}, \bar{u} - u_d) + \sum_{i=1}^n \bar{\mu}_i \bar{u}(x_i) \\ &= \alpha\|\bar{q}\|^2 + \|\bar{u}\|^2 - (\bar{u}, u_d) + \sum_{i=1}^n \bar{\mu}_i b_i. \end{aligned}$$

This implies

$$\alpha\|\bar{q}\|^2 + \frac{1}{2}\|\bar{u}\|^2 \leq \frac{1}{2}\|u_d\|^2 + |\bar{\mu}||b|. \tag{12}$$

Testing the optimality condition (8) with $\varphi = \bar{z}_i \in W_0^{1,s}(\Omega) \hookrightarrow Q$ for $i = 1, 2, \ldots, n$, we get for $\bar{\mu}$ the system of linear equations

$$Z\bar{\mu} = f,$$

with $Z \in \mathbb{R}^{n \times n}$ and $f \in \mathbb{R}^n$ given by

$$Z_{ij} = (\bar{z}_i, \bar{z}_j) \text{ for } i, j = 1, 2, \ldots, n \quad \text{and} \quad f_i = -(\alpha\bar{q} + \bar{z}_0, \bar{z}_i) \text{ for } i = 1, 2, \ldots, n.$$

Since the points $x_1, x_2, \ldots, x_n$ are mutually distinct, the set of Green's functions $\{\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n\}$ is linearly independent in $L^2(\Omega)$. Consequently, the Gramian matrix Z of this set is regular and the norm of its inverse $\|Z^{-1}\|$ depends only on the position of $x_1, x_2, \ldots, x_n$ and Ω. It follows by means of the stability estimate

$$\|\bar{z}_0\| \leq C\{\|\bar{u}\| + \|u_d\|\} \tag{13}$$

and Lemma 2 that

$$|\bar{\mu}| \leq C|f| \leq C\{\alpha\|\bar{q}\| + \|\bar{z}_0\|\} \leq C\{\alpha\|\bar{q}\| + \|\bar{u}\| + \|u_d\|\}. \tag{14}$$

Inserting this into (12) and applying Young's inequality yields

$$\begin{aligned}\alpha\|\bar{q}\|^2+\frac{1}{2}\|\bar{u}\|^2 &\le \frac{1}{2}\|u_d\|^2+C|b|\big\{\alpha\|\bar{q}\|+\|\bar{u}\|+\|u_d\|\big\}\\ &\le \frac{1}{2}\|u_d\|^2+C\alpha|b|^2+\frac{\alpha}{2}\|\bar{q}\|^2+C|b|^2+\frac{1}{4}\|\bar{u}\|^2+C|b|^2+\frac{1}{2}\|u_d\|^2.\end{aligned}$$

Hence, we get the first part of the assertion

$$\|\bar{q}\|+\|\bar{u}\|\le C\big\{\|u_d\|+|b|\big\}.$$

Using this estimate in (13) and (14) implies the remaining estimate

$$|\bar{\mu}|+\|\bar{z}_0\|\le C\big\{\|u_d\|+|b|\big\}.$$

□

3 Discrete problem

To define the Galerkin finite element discretization, we consider a family of polygonal approximations $\Omega_h \subset \Omega$ of the computational domain Ω such that all corners of Ω_h lie on the boundary $\partial\Omega$.

Remark 4 Note that this condition implies that $|\Omega \setminus \Omega_h| \le Ch^2$.

On each Ω_h, we construct two or three dimensional meshes covering $\bar{\Omega}_h$, which satisfy the usual regularity conditions such as conformity and shape regularity (see, e.g., [10]). The meshes consist of triangular, quadrilateral, tetrahedral, or hexahedral cells K and are denoted by $\mathscr{T}_h = \{K\}$, where we define the discretization parameter h as a cellwise constant function by setting $h\big|_K = h_K$ with the diameter h_K of the cell K. We use the symbol h also for the maximum cell size, i.e., $h = \max h_K$.

In what follows, we will use two different kind of meshes: in Section 5, we derive error estimates on quasi-uniform and in Section 6 on graded meshes.

On the mesh $\mathscr{T}_h$, we construct a conforming finite element space V_h for the state variable in a standard way:

$$V_h = \left\{ v \in C(\bar{\Omega}) \;\middle|\; v\big|_K \in \mathscr{P}_1(K) \text{ for } K \in \mathscr{T}_h \text{ and } v\big|_{\Omega\setminus\Omega_h} = 0 \right\}.$$

Here, $\mathscr{P}_1(K)$ consists of shape functions obtained via linear transformations of linear polynomials defined on the reference cell.

Remark 5 Note, that by construction there holds $V_h \subset W_0^{1,s'}(\Omega)$ and $V_h \subset W_0^{1,s}(\Omega)$ for any value of $s \in \left(\frac{2d}{d+2}, \frac{d}{d-1}\right)$.

Then, the conforming Galerkin discretization of the state equation for a given control $q \in Q$ has the following form: Find a state $u_h = u_h(q) \in V_h$ satisfying

$$(\nabla u_h, \nabla\varphi) = (q, \varphi) \quad \forall \varphi \in V_h. \tag{15}$$

The state constraint on the discrete level is given as in Section 2 by

$$G(u_h) = b. \tag{16}$$

For discretizing the controls, we employ the same discretization as for the state variable. That is, we set $Q_h = V_h$. Then, the corresponding fully discrete optimal control problem reads as follows:

$$\text{Minimize } J(q_h, u_h) \text{ for } (q_h, u_h) \in Q_h \times V_h \text{ subject to (15) and (16),} \tag{17}$$

and the existence and uniqueness of solutions is implied by the following lemma:

Lemma 4 *For h small enough, the discrete admissible set*

$$W_{ad,h} := \{\, (q_h, u_h) \in Q_h \times V_h \mid u_h \textit{ solves } (15) \textit{ for the given } q_h \textit{ and fulfills } (16) \,\}$$

is nonempty.

Proof Each element $u_h \in V_h$ can by means of the nodal basis functions $\omega_{h,i} \in V_h$ be expressed as

$$u_h = \sum_{j=1}^{\dim V_h} \xi_j \omega_{h,j}.$$

In order that u_h fulfills the state constraints, the coefficients ξ_i have to satisfy the following linear system of equations:

$$\sum_{j=1}^{\dim V_h} \xi_j \omega_{h,j}(x_i) = b_i, \qquad i = 1, 2, \ldots, n. \tag{18}$$

Choosing h sufficiently small implies that $\dim V_h \geq n$ and also that the n equations above are linearly independent. Hence, system (18) admits at least one solution. Using the discrete Laplacian $\Delta_h \colon V_h \to V_h$ given by

$$(-\Delta_h u_h, \varphi) = (\nabla u_h, \nabla \varphi) \quad \forall \varphi \in V_h,$$

the pair (q_h, u_h) with $q_h = -\Delta_h u_h \in Q_h$ belongs to $W_{\mathrm{ad},h}$. □

Also in the discrete setting, the existence of an admissible point for problem (17) from Lemma 4 ensures the existence and uniqueness of an optimal solution to the considered problem. Furthermore, we introduce the discrete reduced cost functional $j_h \colon Q \to \mathbb{R}$ by

$$j_h(q) := J(q, u_h(q)),$$

where $S_h \colon q \mapsto u_h = u_h(q)$ denotes the discrete control-to-state mapping.

We now formulate the necessary and sufficient optimality conditions for the discrete problem (17):

Theorem 2 *A control $\bar{q}_h \in Q_h$ with associated state $\bar{u}_h = u_h(\bar{q}_h) \in V_h$ is an optimal solution of problem* (17) *if and only if $G(u_h) = b$ and there exists an adjoint state $\bar{z}_h \in V_h$ and a Lagrange multiplier $\bar{\mu}_h \in \mathbb{R}^n$ such that*

$$(\nabla \bar{u}_h, \nabla \varphi) = (\bar{q}, \varphi) \qquad \forall \varphi \in V_h \tag{19}$$

$$(\nabla \varphi, \nabla \bar{z}_h) = (\varphi, \bar{u}_h - u_d) + \sum_{i=1}^{n} \bar{\mu}_{h,i} \varphi(x_i) \qquad \forall \varphi \in V_h \tag{20}$$

$$(\alpha \bar{q}_h + \bar{z}_h, \varphi) = 0 \qquad \forall \varphi \in Q_h \tag{21}$$

Proof The assertion of the Theorem can be proved in the same way as the continuous analogue using Lemma 4 to ensure the surjectivity of the constraints. □

Remark 6 Note that (21) is equivalent to the condition

$$\bar{q}_h = -\frac{1}{\alpha}\bar{z}_h,$$

where the equality has to be understood in $Q_h = V_h$.

As in the continuous setting, we denote by $\bar{z}_{0,h} \in V_h$ the solution of

$$(\nabla\varphi, \nabla\bar{z}_{0,h}) = (\varphi, \bar{u}_h - u_d) \quad \forall\varphi \in V_h \tag{22}$$

and by $\bar{z}_{h,i} \in V_h$ for $i = 1,2,\dots,n$ the solutions of

$$(\nabla\varphi, \nabla\bar{z}_{h,i}) = \varphi(x_i) \quad \forall\varphi \in V_h \tag{23}$$

which yields the representation

$$\bar{z}_h = \bar{z}_{0,h} + \sum_{i=1}^{n} \bar{\mu}_{h,i}\bar{z}_{h,i}.$$

As stated in Lemma 3 for the continuous case, we have in the discrete setting the following estimate:

Lemma 5 *For h sufficiently small, the solutions $\bar{z}_{h,i} \in V_h$ of* (23) *for $i = 1,2,\dots,n$ are uniformly bounded:*

$$\|\bar{z}_{h,i}\| \le C.$$

Proof For both quasi-uniform refined and graded meshes, we have by Lemma 9 and Corollary 2 (see below), respectively, that $\|\bar{z}_i - \bar{z}_{h,i}\| \to 0$ for $h \to 0$. Lemma 2 provides the boundedness of the continuous solutions $\bar{z}_i$ for $i = 1,2,\dots,n$. Hence, we have for h sufficiently small that

$$\|\bar{z}_{h,i}\| \le \|\bar{z}_{h,i} - \bar{z}_i\| + \|\bar{z}_i\| \le C.$$

□

Lemma 6 *There holds*

$$\|\bar{q}_h\| + \|\bar{u}_h\| + \|\bar{z}_{0,h}\| + |\bar{\mu}_h| \le C\big\{\|u_d\| + |b|\big\}.$$

Proof As in the proof of Lemma 3, we obtain

$$\alpha\|\bar{q}_h\|^2 + \frac{1}{2}\|\bar{u}_h\|^2 \le \frac{1}{2}\|u_d\|^2 + |\bar{\mu}_h||b| \tag{24}$$

and we get the the following system of linear equations for $\bar{\mu}_h$:

$$Z_h\bar{\mu}_h = f_h$$

with $Z_h \in \mathbb{R}^{n\times n}$ and $f_h \in \mathbb{R}^n$ given by

$$Z_{h,ij} = (\bar{z}_{h,i}, \bar{z}_{j,h}) \text{ for } i,j = 1,2,\dots,n \quad \text{and} \quad f_{h,i} = -(\alpha\bar{q}_h + \bar{z}_{0,h}, \bar{z}_{h,i}) \text{ for } i = 1,2,\dots,n.$$

Since the points $x_1, x_2, \dots, x_n$ are assumed to be mutually distinct, the set of discrete Green's functions $\{\bar{z}_{h,1}, \bar{z}_{h,2}, \dots, \bar{z}_{h,n}\}$ is linearly independent in $L^2(\Omega)$. From this and the fact that $z_{h,i} \to z_i$ in $L^2(\Omega)$ for $h \to 0$, we obtain that the Gramian matrix Z_h of this set is regular and the norm of its inverse $\|Z_h^{-1}\|$ can be bounded independently of h. It follows by means of the stability estimate

$$\|\bar{z}_{0,h}\| \le C\big\{\|\bar{u}_h\| + \|u_d\|\big\}$$

and Lemma 5 that

$$|\bar{\mu}_h| \le C|f_h| \le C\big\{\alpha\|\bar{q}_h\| + \|\bar{z}_{0,h}\|\big\} \le C\big\{\alpha\|\bar{q}_h\| + \|\bar{u}_h\| + \|u_d\|\big\}.$$

Inserting this into (24) yields as in the proof of Lemma 3 the desired estimates. □

Lemma 7 *For the optimal state $\bar{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ of* (5) *and the optimal discrete state $\bar{u}_h \in V_h$ of* (17)*, it holds*

$$\|\bar{u} - \bar{u}_h\| \le C\|\bar{q} - \bar{q}_h\| + Ch^2\big\{\|u_d\| + |b|\big\},$$
$$\|\nabla(\bar{u} - \bar{u}_h)\| \le C\|\bar{q} - \bar{q}_h\| + Ch\big\{\|u_d\| + |b|\big\}.$$

Proof By inserting the solution $u(\bar{q}_h)$ of (2) with $q = \bar{q}_h$, stability estimates for the state equations, and standard finite element error estimates, we get

$$\begin{aligned}\|\bar{u} - \bar{u}_h\| \le \|\bar{u} - u(\bar{q}_h)\| + \|u(\bar{q}_h) - \bar{u}_h\| &= \|u(\bar{q}) - u(\bar{q}_h)\| + \|u(\bar{q}_h) - u_h(\bar{q}_h)\| \\ &\le C\|\bar{q} - \bar{q}_h\| + Ch^2\|\nabla^2 u(\bar{q}_h)\| \le C\|\bar{q} - \bar{q}_h\| + Ch^2\|\bar{q}_h\|.\end{aligned}$$

By Lemma 6, we obtain the first assertion. The second assertion can be proved similarly. □

Lemma 8 *For the solution $\bar{z}_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ of* (9) *and the discrete solution $\bar{z}_{0,h} \in V_h$ of* (22)*, it holds*

$$\|\bar{z}_0 - \bar{z}_{0,h}\| \le C\|\bar{q} - \bar{q}_h\| + Ch^2\big\{\|u_d\| + |b|\big\},$$
$$\|\nabla(\bar{z}_0 - \bar{z}_{0,h})\| \le C\|\bar{q} - \bar{q}_h\| + Ch\big\{\|u_d\| + |b|\big\}.$$

Proof By inserting the solution $z_0(\bar{u}_h)$ of

$$(\nabla\varphi, \nabla z_0(\bar{u}_h)) = (\varphi, \bar{u}_h - u_d) \quad \forall \varphi \in V_h,$$

stability estimates for the state equations, and the standard finite element error estimates, we get

$$\begin{aligned}\|\bar{z}_0 - \bar{z}_{0,h}\| &\le \|\bar{z}_0 - z_0(\bar{u}_h)\| + \|z_0(\bar{u}_h) - \bar{z}_{0,h}\| \\ &\le \|\bar{u} - \bar{u}_h\| + Ch^2\|\nabla^2 z_0(\bar{u}_h)\| \le \|\bar{u} - \bar{u}_h\| + Ch^2\big\{\|\bar{u}_h\| + \|u_d\|\big\}.\end{aligned}$$

By the Lemmas 6 and 7, we obtain the first assertion. The second assertion follows in the same manner. □

4 Dual problem

To derive optimal error estimates, we will make use of a optimal control problem which is dual to the considered problem (5) in the following sense: We define for $x = (q,u,z,\mu) \in X = Q \times W_0^{1,s'}(\Omega) \times W_0^{1,s}(\Omega) \times \mathbb{R}^n$ and $\varphi = (\varphi_q, \varphi_u, \varphi_z, \varphi_\mu) \in X$ the bilinear form $A\colon X \times X \to \mathbb{R}$ as

$$\begin{aligned} A(x,\varphi) &= (q,\varphi_z) - (\nabla u, \nabla \varphi_z) + (u,\varphi_u) + \sum_{i=1}^n \mu_i \varphi_u(x_i) - (\nabla z, \nabla \varphi_u) \\ &\quad + \alpha(q,\varphi_q) + (z,\varphi_q) + \sum_{i=1}^n u(x_i)\varphi_\mu^i. \end{aligned}$$

Then, the optimality system of (5) from Theorem 1 for $\bar{x} = (\bar{q},\bar{u},\bar{z},\bar{\mu}) \in X$ is equivalent to

$$A(\bar{x},\varphi) = (u_d,\varphi_u) + \sum_{i=1}^n b_i \varphi_\mu^i \quad \forall \varphi \in X.$$

Let $\tilde{x} = (\tilde{q},\tilde{u},\tilde{z},\tilde{\mu}) \in X$ be the solution of the dual optimal control problem given as

$$A(\varphi,\tilde{x}) = (\hat{u}_d,\varphi_u) + \sum_{i=1}^n \hat{b}_i \varphi_\mu^i \quad \forall \varphi \in X. \tag{25}$$

with

$$\hat{u}_d = \frac{\bar{u} - \bar{u}_h}{\|\bar{u} - \bar{u}_h\|} \qquad \text{and} \qquad \hat{b} = \frac{\bar{\mu} - \bar{\mu}_h}{|\bar{\mu} - \bar{\mu}_h|}.$$

Since A is self-adjoint, this characterization is equivalent to $\tilde{x}$ being the solution of the optimality system for the following problem:

$$\text{Minimize } \frac{1}{2}\|u - \hat{u}_d\|^2 + \frac{\alpha}{2}\|q\|^2 \tag{26a}$$

subject to the constraints

$$\begin{aligned} -\Delta u &= q && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{26b}$$

and

$$u(x_i) = \hat{b}_i \qquad i = 1,2,\ldots,n. \tag{26c}$$

Hence, the dual problem (26) is identical to the original optimal control problem (5) for the particular choices $u_d = \hat{u}_d$ and $b = \hat{b}$.

As for the primal problem, we split $\tilde{z}$ in the regular part $\tilde{z}_0$ and the irregular parts $\tilde{z}_i$ defined as solutions of

$$(\nabla\varphi, \nabla\tilde{z}_0) = (\varphi, \tilde{u} - \hat{u}_d) \quad \forall \varphi \in W_0^{1,s'}(\Omega) \tag{27}$$

$$(\nabla\varphi, \nabla\tilde{z}_i) = \varphi(x_i) \qquad \forall \varphi \in W_0^{1,s'}(\Omega),\ i = 1,2,\ldots,n. \tag{28}$$

Using this notation, we have as for $\bar{z}$ the following splitting of $\tilde{z}$:

$$\tilde{z} = \tilde{z}_0 + \sum_{i=1}^n \tilde{\mu}_i \tilde{z}_i.$$

Remark 7 By definition, it holds $\tilde{z}_i = \bar{z}_i$ for $i = 1,2,\ldots,n$.

Additionally, we define the solution $\tilde{x}_h \in X_h = Q_h \times V_h \times V_h \times \mathbb{R}^n$ of the discrete analog of (26) by

$$A(\tilde{x} - \tilde{x}_h, \varphi) = 0 \quad \forall \varphi \in X_h. \tag{29}$$

Remark 8 Because of the same structure of the primal and dual problems (5) and (17) and the boundedness of $\hat{u}_d$ and $\hat{b}$, the proved assertions on the primal quantities $\bar{q}$, $\bar{u}$, $\bar{z}$, $\bar{\mu}$, and $\bar{q}_h$, $\bar{u}_h$, $\bar{z}_h$, $\bar{\mu}_h$ are also valid for the dual quantities $\tilde{q}$, $\tilde{u}$, $\tilde{z}$, $\tilde{\mu}$, and $\tilde{q}_h$, $\tilde{u}_h$, $\tilde{z}_h$, $\tilde{\mu}_h$. That is, it holds for $i = 1, 2, \ldots, n$

$$\|\tilde{q}\| + \|\tilde{u}\| + \|\tilde{z}_0\| + \|\tilde{z}_i\| + |\tilde{\mu}| \leq C, \tag{30}$$

$$\|\tilde{q}_h\| + \|\tilde{u}_h\| + \|\tilde{z}_{h,0}\| + \|\tilde{z}_{h,i}\| + |\tilde{\mu}_h| \leq C. \tag{31}$$

5 Error estimates on quasi-uniform meshes

Throughout this section, we assume the family of meshes $\mathscr{T}_h$ for $h \to 0$ to be quasi-uniform. That is, there exists a constant C independent of h such that

$$\max_{K \in \mathscr{T}_h} h_K \leq C \min_{K \in \mathscr{T}_h} h_K.$$

5.1 Error estimate for the state and adjoint equations

We start with the following estimate of the pointwise error between the continuous state $u(q)$ and its Ritz projection $u_h(q)$.

Theorem 3 *Let $q \in W^{1,s}(\Omega)$, $u = u(q) \in W_0^{1,s'}(\Omega)$ be the solution of* (2)*, and $u_h = u_h(q) \in V_h$ be the solution of* (15) *for a control $q \in Q$. Then, it holds for $i = 1, 2, \ldots, n$*

$$|u(x_i) - u_h(x_i)| \leq C\Big(\frac{ds}{d-s}\Big)^2 h^{3-\frac{d}{s}} \|\nabla q\|_{L^s(\Omega)},$$

where the constant C does not depend on s.

Proof Since by assumption, Ω is smooth and convex, it holds for $1 < t < \infty$

$$\|\nabla^2 u\|_{L^t(\Omega)} \leq C_t \|q\|_{L^t(\Omega)}, \tag{32}$$

where $C_t \sim \frac{1}{t-1}$ for $t \to 1$ and $C_t \sim t$ for $t \to \infty$. The exact form of the constant can be traced for example from [18, Theorem 9.9]. For $q \in W_0^{1,s}(\Omega)$, we have by embedding that $q \in L^p(\Omega)$ for $p = \frac{ds}{d-s} > 2$ and by (32), it follows

$$\|\nabla^2 u\|_{L^p(\Omega)} \leq Cp\|q\|_{L^p(\Omega)} \leq Cp\|\nabla q\|_{L^s(\Omega)}. \tag{33}$$

In [27, (1.9), p. 438] it was proved for $d = 2$ for $2 \leq t < \infty$ that

$$\|u - u_h\|_{L^t(\Omega)} \leq C_t h^2 \|\nabla^2 u\|_{L^t(\Omega)}, \tag{34}$$

where C_t is the constant in (32). Using the stability of the Ritz projection in $W_0^{1,t}$ from [5, Theorem 8.5.3], the proof in [27] can be repeated also for $d = 3$ yielding (34) for $d \in \{2, 3\}$.

Let now be $x_i \in K_*$ for some cell $K_* \in \mathscr{T}_h$. Then, by the triangle inequality

$$|u(x_i) - u_h(x_i)| \le \|u - u_h\|_{L^\infty(K_*)} \le \|u - i_h u\|_{L^\infty(K_*)} + \|i_h u - u_h\|_{L^\infty(K_*)},$$

where i_h denotes the usual Lagrange interpolant. For $\|u - i_h u\|_{L^\infty(K_*)}$, we have by well-known interpolation estimates

$$\|u - i_h u\|_{L^\infty(K_*)} \le Ch^{2-\frac{d}{p}} \|\nabla^2 u\|_{L^p(K_*)}.$$

For $\|i_h u - u_h\|_{L^\infty(K_*)}$, it follows using an inverse estimate and estimate (34) for $t = p$ that

$$\begin{aligned}\|i_h u - u_h\|_{L^\infty(K_*)} &\le Ch^{-\frac{d}{p}} \|i_h u - u_h\|_{L^p(K_*)} \\ &\le Ch^{-\frac{d}{p}} \left\{ \|i_h u - u\|_{L^p(K_*)} + \|u - u_h\|_{L^p(K_*)} \right\} \\ &\le Cph^{2-\frac{d}{p}} \|\nabla^2 u\|_{L^p(K_*)}.\end{aligned}$$

Finally, by means of (33), it follows by the definition of p

$$|u(x_i) - u_h(x_i)| \le Cp^2 h^{2-\frac{d}{p}} \|\nabla q\|_{L^s(K_*)} \le Cp^2 h^{3-\frac{d}{s}} \|\nabla q\|_{L^s(\Omega)},$$

which concludes the proof. □

Lemma 9 *For the solutions $\bar{z}_i$ of* (28)*, it holds*

$$\|\bar{z}_i - \bar{z}_{h,i}\| \le Ch^{2-\frac{d}{2}}.$$

Proof Let i be arbitrary but fixed and let $v \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$(\nabla v, \nabla \varphi) = (\bar{z}_i - \bar{z}_{h,i}, \varphi) \quad \forall \varphi \in W_0^{1,s}(\Omega)$$

for some $s < \frac{d}{d-1}$ and $v_h \in V_h$ be its discrete analog defined by

$$(\nabla(v - v_h), \nabla \varphi) = 0 \quad \forall \varphi \in V_h.$$

Then, we have by the Galerkin orthogonality

$$\begin{aligned}\|\bar{z}_i - \bar{z}_{h,i}\|^2 &= (\bar{z}_i - \bar{z}_{h,i}, \bar{z}_i) - (\bar{z}_i - \bar{z}_{h,i}, \bar{z}_{h,i}) = (\nabla v, \nabla \bar{z}_i) - (\nabla v_h, \nabla \bar{z}_{h,i}) \\ &= v(x_i) - v_h(x_i).\end{aligned}$$

By the suboptimal L^∞ error estimate

$$\|v - v_h\|_{L^\infty(\Omega)} \le Ch^{2-\frac{d}{2}} \|\nabla^2 v\|$$

from [10, p. 168] and (3) applied to v, we get

$$\|\bar{z}_i - \bar{z}_{h,i}\|^2 \le \|v - v_h\|_{L^\infty(\Omega)} \le Ch^{2-\frac{d}{2}} \|\nabla^2 v\| \le Ch^{2-\frac{d}{2}} \|\bar{z}_i - \bar{z}_{h,i}\|,$$

which completes the proof. □

5.2 Error estimate for the optimal control problem

Throughout this section, let $s = s_\varepsilon$ with

$$s_\varepsilon = \frac{d+\varepsilon}{d+\varepsilon-1}$$

and ε from now on be chosen as $\varepsilon = |\log h|^{-1}$. It follows $s'_\varepsilon = d+\varepsilon$ and for h sufficiently small, it holds $0 < \varepsilon < 3$ and consequently

$$\frac{2d}{d+2} < s_\varepsilon < \frac{d}{d-1}.$$

Corollary 1 *Let $u(\bar{q})$ and $u(\bar{q}_h)$ be the solution of* (2) *and $u_h(\bar{q})$ and $u_h(\bar{q}_h)$ be the solution of* (15) *for the optimal controls $q = \bar{q}$ and $q = \bar{q}_h$, respectively. Then, it holds for $i = 1,2,\dots,n$*

$$|u(\bar{q})(x_i) - u_h(\bar{q})(x_i)| \le Ch^{4-d}|\log h|^{7-2d}\left\{\|u_d\| + |b|\right\},$$
$$|u(\bar{q}_h)(x_i) - u_h(\bar{q}_h)(x_i)| \le Ch^{4-d}|\log h|^{7-2d}\left\{\|u_d\| + |b|\right\}.$$

Proof Let $p_\varepsilon = \frac{ds_\varepsilon}{d-s_\varepsilon}$. To prove the first assertion, we note that $\bar{q} \in W_0^{1,s_\varepsilon}(\Omega)$. Hence, we get from Theorem 3 and (8) that

$$|u(\bar{q})(x_i) - u_h(\bar{q})(x_i)| \le Cp_\varepsilon^2 h^{3-\frac{d}{s_\varepsilon}}\left\{\|\nabla \bar{z}_0\|_{L^{s_\varepsilon}(\Omega)} + |\bar{\mu}|\sum_{i=1}^n \|\nabla \bar{z}_i\|_{L^{s_\varepsilon}(\Omega)}\right\}.$$

Using Lemma 3, it follows

$$\|\nabla \bar{z}_0\|_{L^{s_\varepsilon}(\Omega)} \le C\|\bar{u} - u_d\| \le C\left\{\|u_d\| + |b|\right\},$$

which leads to

$$|u(\bar{q})(x_i) - u_h(\bar{q})(x_i)| \le Cp_\varepsilon^2 h^{3-\frac{d}{s_\varepsilon}}\left\{\|u_d\| + |b| + |\bar{\mu}|\sum_{i=1}^n \|\nabla \bar{z}_i\|_{L^{s_\varepsilon}(\Omega)}\right\}$$

The Lemmas 2 and 3 and the estimates

$$-\frac{d}{s_\varepsilon} = 1-d-\frac{\varepsilon}{d+\varepsilon} > 1-d-\frac{\varepsilon}{d} \quad \text{and}$$
$$p_\varepsilon = \frac{d+\varepsilon}{d-2+\left(1-\frac{1}{d}\right)\varepsilon} \le \frac{d+\varepsilon}{d-2+\frac{\varepsilon}{2}} < \frac{C}{\varepsilon^{3-d}}$$

yield

$$|u(\bar{q})(x_i) - u_h(\bar{q})(x_i)| \le \frac{C}{\varepsilon^{7-2d}} h^{4-d-\frac{\varepsilon}{d}}\left\{\|u_d\| + |b|\right\}.$$

The choice $\varepsilon = |\log h|^{-1}$ finally implies the first assertion.

To prove the second assertion, we apply Theorem 3 for $q = \bar{q}_h$ and obtain using (21) that

$$|u(\bar{q}_h)(x_i) - u_h(\bar{q}_h)(x_i)| \le Cp_\varepsilon^2 h^{3-\frac{d}{s_\varepsilon}}\left\{\|\nabla \bar{z}_{h,0}\|_{L^{s_\varepsilon}(\Omega)} + |\bar{\mu}_h|\sum_{i=1}^n \|\nabla \bar{z}_{h,i}\|_{L^{s_\varepsilon}(\Omega)}\right\}.$$

Using the stability of the Ritz projection in $W_0^{1,s_\varepsilon}(\Omega)$ (cf. [5, Theorem 8.5.3])

$$\|\nabla \bar{z}_{h,i}\|_{L^{s_\varepsilon}(\Omega)} \le C\|\nabla \bar{z}_i\|_{L^{s_\varepsilon}(\Omega)}, \quad i = 1,2,\dots,n,$$

the stability estimate

$$\|\nabla \bar{z}_{h,0}\|_{L^{s\varepsilon}(\Omega)} \leq C\|\bar{u}_h - u_d\|$$

for $z_{h,0}$ and Lemma 6, it follows

$$|u(\bar{q}_h)(x_i) - u_h(\bar{q}_h)(x_i)| \leq C p_\varepsilon^2 h^{3-\frac{d}{s\varepsilon}} \left\{ \|u_d\| + |b| + |\bar{\mu}_h| \sum_{i=1}^{n} \|\nabla \bar{z}_i\|_{L^{s\varepsilon}(\Omega)} \right\}.$$

As above for the first assertion, this implies the second assertion. ☐

Theorem 4 *Let $\bar{q} \in Q$ and $\bar{q}_h \in Q_h$ be the solutions of* (5) *and* (17)*. Then, it holds*

$$\|\bar{q} - \bar{q}_h\| \leq C h^{2-\frac{d}{2}} |\log h|^{\frac{7}{2}-d} \big\{ \|u_d\| + |b| \big\}.$$

Proof We adapt the lines of the proof of Theorem 3.6 in [12]. We observe using the optimality conditions (6) – (8)

$$\begin{aligned}
(u(\bar{q}_h) - u(\bar{q}), u(\bar{q}) - u_d) &= (\nabla \bar{z}, \nabla(u(\bar{q}_h) - u(\bar{q}))) + \sum_{i=1}^{n} \bar{\mu}_i (u(\bar{q})(x_i) - u(\bar{q}_h)(x_i)) \\
&= (\bar{z}, \bar{q}_h - \bar{q}) + \sum_{i=1}^{n} \bar{\mu}_i (u(\bar{q})(x_i) - u(\bar{q}_h)(x_i)) \\
&= -\alpha(\bar{q}, \bar{q}_h - \bar{q}) + \sum_{i=1}^{n} \bar{\mu}_i (u(\bar{q})(x_i) - u(\bar{q}_h)(x_i)).
\end{aligned}$$

Using this equality, we obtain for the difference $j(\bar{q}_h) - j(\bar{q})$

$$\begin{aligned}
j(\bar{q}_h) - j(\bar{q}) &= \frac{1}{2}\|u(\bar{q}_h) - u(\bar{q})\|^2 + \frac{\alpha}{2}\|\bar{q}_h - \bar{q}\|^2 + (u(\bar{q}_h) - u(\bar{q}), u(\bar{q}) - u_d) \\
&\quad + \alpha(\bar{q}, \bar{q}_h - \bar{q}) \\
&= \frac{1}{2}\|u(\bar{q}) - u(\bar{q}_h)\|^2 + \frac{\alpha}{2}\|\bar{q} - \bar{q}_h\|^2 + \sum_{i=1}^{n} \bar{\mu}_i (u(\bar{q})(x_i) - u(\bar{q}_h)(x_i)).
\end{aligned} \tag{35}$$

Similarly for j_h, we obtain

$$j_h(\bar{q}) - j_h(\bar{q}_h) = \frac{1}{2}\|u_h(\bar{q}_h) - u_h(\bar{q})\|^2 + \frac{\alpha}{2}\|\bar{q}_h - \bar{q}\|^2 + \sum_{i=1}^{n} \bar{\mu}_{h,i}(u_h(\bar{q}_h)(x_i) - u_h(\bar{q})(x_i)). \tag{36}$$

By adding (35) and (36), we obtain

$$\begin{aligned}
\alpha\|\bar{q} - \bar{q}_h\|^2 &\leq j(\bar{q}_h) - j(\bar{q}) + j_h(\bar{q}) - j_h(\bar{q}_h) \\
&\quad - \sum_{i=1}^{n} \bar{\mu}_i (b_i - u(\bar{q}_h)(x_i)) - \sum_{i=1}^{n} \bar{\mu}_{h,i}(b - u(\bar{q})(x_i)) \\
&\leq |j(\bar{q}) - j_h(\bar{q})| + |j(\bar{q}_h) - j_h(\bar{q}_h)| \\
&\quad + |\bar{\mu}_h| \left(\sum_{i=1}^{n} |u_h(\bar{q}_h)(x_i) - u_h(\bar{q})(x_i)| \right)^{\frac{1}{2}} \\
&\quad + |\bar{\mu}| \left(\sum_{i=1}^{n} |u(\bar{q})(x_i) - u(\bar{q}_h)(x_i))| \right)^{\frac{1}{2}}.
\end{aligned} \tag{37}$$

For $|j(\bar{q})-j_h(\bar{q})|$ it holds by standard estimates

$$\begin{aligned}|j(\bar{q})-j_h(\bar{q})| &= \frac{1}{2}\|u(\bar{q})-u_d\|^2-\frac{1}{2}\|u_h(\bar{q})-u_d\|^2\\ &= \frac{1}{2}(u(\bar{q})-u_h(\bar{q}),u(\bar{q})-u_d+u_h(\bar{q})-u_d)\\ &\le \frac{1}{2}\|u(\bar{q})-u_h(\bar{q})\|\big\{\|u(\bar{q})\|+\|u_h(\bar{q})\|+2\|u_d\|\big\}\\ &\le Ch^2\big\{\|u_d\|+|b|\big\}^2.\end{aligned}$$

In the same manner, we obtain for $|j(\bar{q}_h)-j_h(\bar{q}_h)|$

$$|j(\bar{q}_h)-j_h(\bar{q}_h)| \le Ch^2\big\{\|u_d\|+|b|\big\}^2.$$

For $|u_h(\bar{q}_h)(x_i)-u_h(\bar{q})(x_i))|$ we obtain using the first assertion of Corollary 1

$$\begin{aligned}|u_h(\bar{q}_h)(x_i)-u_h(\bar{q})(x_i))| = |b_i-u_h(\bar{q})(x_i))| &= |u(\bar{q})(x_i)-u_h(\bar{q})(x_i))|\\ &\le Ch^{4-d}|\log h|^{7-2d}\big\{\|u_d\|+|b|\big\}\end{aligned}$$

and for $|u(\bar{q})(x_i)-u(\bar{q}_h)(x_i))|$ using the second assertion of Corollary 1

$$\begin{aligned}|u(\bar{q})(x_i)-u(\bar{q}_h)(x_i))| = |b_i-u(\bar{q}_h)(x_i))| &= |u_h(\bar{q}_h)(x_i)-u(\bar{q}_h)(x_i))|\\ &\le Ch^{4-d}|\log h|^{7-2d}\big\{\|u_d\|+|b|\big\}.\end{aligned}$$

Inserting the last four estimates into (37) finally yields the assertion. □

Using the standard techniques, the proved order of convergence for the error in the control carries over to the errors in the remaining variables. However, for $\|\bar{u}-\bar{u}_h\|$ and $|\bar{\mu}-\bar{\mu}_h|$ it is possible to show the following improved convergence order.

Theorem 5 *For the optimal states $\bar{u}\in H^2(\Omega)$ and $\bar{u}_h\in V_h$ and the Lagrangian multipliers $\bar{\mu}\in\mathbb{R}^n$ and $\bar{\mu}_h\in\mathbb{R}^n$ of the continuous and discrete problem* (5) *and* (17)*, respectively, it holds*

$$\|\bar{u}-\bar{u}_h\|+|\bar{\mu}-\bar{\mu}_h| \le Ch^{4-d}|\log h|^{7-2d}\big\{\|u_d\|+|b|\big\}.$$

Proof Choosing $\varphi=\bar{x}-\bar{x}_h$ in (25) yields by Galerkin orthogonality

$$\|\bar{u}-\bar{u}_h\|+|\bar{\mu}-\bar{\mu}_h| = A(\bar{x}-\bar{x}_h,\tilde{x}) = A(\bar{x}-\bar{x}_h,\tilde{x}-\tilde{x}_h),$$

where $\tilde{x}_h\in X_h=V_h\times V_h\times V_h\times\mathbb{R}^n$ is the discrete analog of $\tilde{x}$ defined by

$$A(x-x_h,\varphi)=0 \quad \forall\varphi\in X_h.$$

By inspection of the definition of A we have

$$\begin{aligned}A(\bar{x}-\bar{x}_h,\tilde{x}-\tilde{x}_h) &= (\bar{q}-\bar{q}_h,\tilde{z}-\tilde{z}_h)-(\nabla(\bar{u}-\bar{u}_h),\nabla(\tilde{z}-\tilde{z}_h))\\ &\quad+(\bar{u}-\bar{u}_h,\tilde{u}-\tilde{u}_h)+\sum_{i=1}^{n}(\bar{\mu}_i-\bar{\mu}_{h,i})(\tilde{u}(x_i)-\tilde{u}_h(x_i))\\ &\quad-(\nabla(\bar{z}-\bar{z}_h),\nabla(\tilde{u}-\tilde{u}_h))+\alpha(\bar{q}-\bar{q}_h,\tilde{q}-\tilde{q}_h)\\ &\quad+(\bar{z}-\bar{z}_h,\tilde{q}-\tilde{q}_h)+\sum_{i=1}^{n}(\bar{u}(x_i)-\bar{u}_h(x_i))(\tilde{\mu}_i-\tilde{\mu}_{h,i}).\end{aligned} \tag{38}$$

Now, we estimate the terms on the right-hand side of (38) separately:

(i) $(\bar{q}-\bar{q}_h, \tilde{z}-\tilde{z}_h)$: Using Theorem 4 and the Lemmas 9 and 8 applied to the dual problem, we get

$$(\bar{q}-\bar{q}_h, \tilde{z}-\tilde{z}_h) \le \|\bar{q}-\bar{q}_h\|\|\tilde{z}-\tilde{z}_h\| \le Ch^{4-d}|\log h|^{7-2d}\{\|u_d\|+|b|\}.$$

(ii) $(\nabla(\bar{u}-\bar{u}_h), \nabla(\tilde{z}-\tilde{z}_h))$: We split

$$\begin{aligned}(\nabla(\bar{u}-\bar{u}_h), \nabla(\tilde{z}-\tilde{z}_h)) &= (\nabla(\bar{u}-\bar{u}_h), \nabla(\tilde{z}_0-\tilde{z}_{h,0})) \\ &\quad + \sum_{i=1}^{n} (\nabla(\bar{u}-\bar{u}_h), \nabla(\tilde{\mu}_i\tilde{z}_i-\tilde{\mu}_{h,i}\tilde{z}_{h,i})).\end{aligned} \tag{39}$$

For the first term in (39), we have by the Lemmas 7 and 8 together with Theorem 4 applied to the primal and the dual problem

$$\begin{aligned}(\nabla(\bar{u}-\bar{u}_h), \nabla(\tilde{z}_0-\tilde{z}_{h,0})) &\le \|\nabla(\bar{u}-\bar{u}_h)\|\|\nabla(\tilde{z}_0-\tilde{z}_{h,0})\| \\ &\le Ch^{4-d}|\log h|^{7-2d}\{\|u_d\|+|b|\}.\end{aligned}$$

For the second term in (39), we proceed

$$\begin{aligned}(\nabla(\bar{u}-\bar{u}_h), \nabla(\tilde{\mu}_i\tilde{z}_i-\tilde{\mu}_{h,i}\tilde{z}_{h,i})) &= (\tilde{\mu}_i-\tilde{\mu}_{h,i})(\nabla(\bar{u}-\bar{u}_h), \nabla\tilde{z}_i) \\ &\quad + \tilde{\mu}_{h,i}(\nabla(\bar{u}-\bar{u}_h), \nabla(\tilde{z}_i-\tilde{z}_{h,i})).\end{aligned} \tag{40}$$

For the first term in (40), we have due to $\bar{u}(x_i) = b_i = \bar{u}_h(x_i)$ that

$$(\nabla(\bar{u}-\bar{u}_h), \nabla\tilde{z}_i) = \bar{u}(x_i)-\bar{u}_h(x_i) = 0.$$

For the second term in (40), we have by the Galerkin orthogonality and Corollary 1

$$\begin{aligned}(\nabla(\bar{u}-\bar{u}_h), \nabla(\tilde{z}_i-\tilde{z}_{h,i})) &= (\nabla(\bar{u}-u_h(\bar{q})), \nabla(\tilde{z}_i-\tilde{z}_{h,i})) = (\nabla(\bar{u}-u_h(\bar{q})), \nabla\tilde{z}_i) \\ &= u(\bar{q})(x_i)-u_h(\bar{q})(x_i) \\ &\le Ch^{4-d}|\log h|^{7-2d}\{\|u_d\|+|b|\}.\end{aligned}$$

Collecting the previous estimates and using (31), we obtain

$$(\nabla(\bar{u}-\bar{u}_h), \nabla(\tilde{z}-\tilde{z}_h)) \le Ch^{4-d}|\log h|^{7-2d}\{\|u_d\|+|b|\}.$$

(iii) $(\bar{u}-\bar{u}_h, \tilde{u}-\tilde{u}_h)$: Using Lemma 7 together with Theorem 4 for $\bar{u}$ and $\tilde{u}$, we estimate

$$(\bar{u}-\bar{u}_h, \tilde{u}-\tilde{u}_h) \le \|\bar{u}-\bar{u}_h\|\|\tilde{u}-\tilde{u}_h\| \le Ch^{4-\frac{d}{2}}|\log h|^{7-2d}\{\|u_d\|+|b|\}.$$

(iv) $(\bar{\mu}_i-\bar{\mu}_{h,i})(\tilde{u}(x_i)-\tilde{u}_h(x_i))$: Since $\tilde{u}(x_i) = \hat{b}_i = \tilde{u}_h(x_i)$, we have that

$$(\bar{\mu}_i-\bar{\mu}_{h,i})(\tilde{u}(x_i)-\tilde{u}_h(x_i)) = 0.$$

(v) $(\nabla(\bar{z}-\bar{z}_h), \nabla(\tilde{u}-\tilde{u}_h))$: This term is treated as the term in (ii) leading to

$$(\nabla(\bar{z}-\bar{z}_h), \nabla(\tilde{u}-\tilde{u}_h)) \le Ch^{4-d}|\log h|^{7-2d}\{\|u_d\|+|b|\}.$$

(vi) $\alpha(\bar{q}-\bar{q}_h, \tilde{q}-\tilde{q}_h)+(\bar{z}-\bar{z}_h, \tilde{q}-\tilde{q}_h)$: By the continuous and discrete optimality conditions $\alpha\bar{q}+\bar{z} = 0$ and $\alpha\bar{q}_h+\bar{z}_h = 0$, which are fulfilled pointwise in Ω, we get

$$\alpha(\bar{q}-\bar{q}_h, \tilde{q}-\tilde{q}_h)+(\bar{z}-\bar{z}_h, \tilde{q}-\tilde{q}_h) = 0.$$

(vii) $(\bar{u}(x_i)-\bar{u}_h(x_i))(\tilde{\mu}_i-\tilde{\mu}_{h,i})$: Since $\bar{u}(x_i) = b_i = \bar{u}_h(x_i)$, we have that

$$(\bar{u}(x_i)-\bar{u}_h(x_i))(\tilde{\mu}_i-\tilde{\mu}_{h,i}) = 0.$$

By collecting all the estimates from (i) – (vii), we end up with the assertion. □

6 Error estimates on graded meshes

To derive an improved estimate not only for $\bar{u}$ and $\bar{\mu}$ but also for $\bar{q}$, we need a mesh that is graded towards the set of points $\{x_1, x_2, \ldots, x_n\}$. Let

$$r_K = \operatorname{dist}(K, \{x_1, x_2, \ldots, x_n\}) = \min_{i=1,2,\ldots,n} \operatorname{dist}(K, x_i)$$

denote the distance from a cell $K \in \mathscr{T}_h$ to the set $\{x_1, x_2, \ldots, x_n\}$. We consider the graded mesh given in term of the cell size as

$$h_K = \begin{cases} h^{2d-2}, & \text{if } r_K = 0, \\ h r_K^{\frac{d}{4}}, & \text{if } r_K > 0. \end{cases} \tag{41}$$

Notice that the number of elements of such a triangulation is of order h^{-d}, see, e.g., [4]. Such meshes can be constructed via a coordinate transformation, see [26], by dyadic refinement, see, e.g., [15], or a combination of both, see [22].

6.1 Error estimate for the state equation

In our argument, we will often deal with functions that are harmonic on some parts of the domain. The following inverse-type inequality significantly simplifies many arguments.

Lemma 10 *Let $D \subset \Omega$ and for $r > 0$ let $D_r = \{x \in \Omega \mid \operatorname{dist}(x, D) \le r\}$. Assume that $v \colon D_r \to \mathbb{R}$ is harmonic on D_r, i.e.*

$$(\nabla v, \nabla \varphi) = 0 \quad \forall \varphi \in H_0^1(D_r).$$

Then, it holds

$$\|\nabla^2 v\|_D \le C r^{-1} \|\nabla v\|_{D_r}.$$

Proof We first note that a partial derivative of a harmonic function is also a harmonic function. Let $v_m = \partial_m v$ for $m = 1, 2, \ldots, d$ and let $\omega \in C^\infty(\Omega)$ be a cutoff function such that

$$\omega = \begin{cases} 0, & \text{on } \Omega \setminus D_r, \\ 1, & \text{on } D \end{cases},$$

and $|\nabla \omega| \le C r^{-1}$. Using that v_m is harmonic on D_r, we have

$$\begin{aligned} \|\omega \nabla v_m\|^2 &= (\omega \nabla v_m, \omega \nabla v_m) = (\nabla v_m, \nabla(\omega^2 v_m)) - (\nabla v_m, v_m \nabla(\omega^2)) \\ &= -2(\omega \nabla v_m, v_m \nabla \omega) \le C \|\omega \nabla v_m\| \|v_m \nabla \omega\| \le C r^{-1} \|\omega \nabla v_m\| \|v_m\|_{D_r}, \end{aligned}$$

which implies the assertion by

$$\|\nabla^2 v\|_D \le C \max_m \|\nabla v_m\|_D \le C \|\omega \nabla v_m\| \le C r^{-1} \|v_m\|_{D_r} \le C r^{-1} \|\nabla v\|_{D_r}.$$

□

As in Section 5, we start with the derivation of an estimate of the pointwise error between the state $u(q)$ and its Ritz projection $u_h(q)$. In [3], the following estimate was established for $d = 2$,

$$|u(q)(x_0) - u_h(q)(x_0)| \le C h^2 |\log h|^{\frac{3}{2}} \|q\|$$

In the following theorem, we derive a pointwise error estimate for $d \in \{2, 3\}$. In additional we obtain a slight improvement in terms of the power of the logarithmic term.

Theorem 6 *Let $u(q) \in H^2(\Omega)$ be the solution of* (2) *and $u_h(q)$ be the solution of* (15) *for a given $q \in Q$. Then, it holds for $i = 1, 2, \ldots, n$*

$$|u(q)(x_i) - u_h(q)(x_i)| \leq Ch^2 |\log h| \|q\|.$$

Proof Let for simplicity $i = 1$ be fixed and $K_* \in \mathscr{T}_h$ be the cell containing x_1. For abbreviation, we set $u = u(q)$ and $u_h = u_h(q)$. Denote $r_j = 2^{-j} \operatorname{diam}(\Omega)$ for $j = 0, 1, \ldots$ and define

$$\Omega_* = \{ x \in \Omega \mid |x - x_1| \leq C_* h_* \}$$

for C_* to be determined later and

$$\Omega_j = \{ x \in \Omega \mid r_{j+1} < |x - x_1| \leq r_j \}.$$

Let J be chosen such that $r_{J+1} \leq C_* h_* \leq r_J$. Note that by construction $\operatorname{diam}(\Omega_j) \leq r_j$ and $J \leq C |\log h|$. Furthermore, it holds for $k > j+1$ that $r_j > r_k$ and

$$\frac{1}{2} r_j \leq \operatorname{dist}(\Omega_k, \Omega_j) \leq r_j. \tag{42}$$

Then we have the decomposition

$$\Omega = \Omega_* \cup \bigcup_{j=0}^{J} \Omega_j.$$

The setting $h_* = h^{2d-2}$ and $h_j = h r_j^{\frac{d}{4}}$ implies

$$\frac{h_j}{r_j} = \frac{h}{r_j^{1-\frac{d}{4}}} \leq \frac{h}{r_J^{1-\frac{d}{4}}} \leq \frac{2h}{(C_* h_*)^{1-\frac{d}{4}}} = \frac{2}{C_*^{1-\frac{d}{4}}}, \tag{43}$$

since $(2d - d)(1 - \frac{d}{4}) = 1$ for $d \in \{2, 3\}$.

Let $\tilde{\delta} \in C_0^\infty(K_*)$ be an approximative delta function fulfilling

$$(\tilde{\delta}, \varphi) = \varphi(x_1) \quad \forall \varphi \in V_h$$

and

$$\|\tilde{\delta}\|_{L^p(\Omega)} \leq C h_*^{-d\left(1-\frac{1}{p}\right)}, \quad 1 \leq p \leq \infty. \tag{44}$$

The explicit construction of such a function is given for instance in the appendix of [29].

Then, we define $g \in H^2(\Omega) \cap H_0^1(\Omega)$ as the solution of

$$(\nabla g, \nabla \varphi) = (\tilde{\delta}, \varphi) \quad \forall \varphi \in H_0^1(\Omega)$$

and the Ritz projection $g_h \in V_h$ as the solution of

$$(\nabla g_h, \nabla \varphi) = (\tilde{\delta}, \varphi) \quad \forall \varphi \in V_h.$$

Note that by construction $(\nabla g_h, \nabla \varphi) = \varphi(x_1)$ holds for all $\varphi \in V_h$.

Using the properties of g and g_h, the nodal interpolation i_h, and Galerkin orthogonality, we have the estimate

$$\begin{aligned}(u-u_h)(x_1) &= (u-i_hu)(x_1)+(i_hu-u_h)(x_1)\\ &= (u-i_hu)(x_1)+(\nabla g,\nabla(i_hu-u_h))\\ &= (u-i_hu)(x_1)+(\nabla g,\nabla(i_hu-u))+(\nabla(g-g_h),\nabla(u-u_h))\\ &= (u-i_hu)(x_1)+(\tilde{\delta},u-i_hu)+(\nabla(g-g_h),\nabla(u-i_hu))\\ &\le C\|u-i_hu\|_{L^\infty(K_*)}\left\{1+\|\tilde{\delta}\|_{L^1(\Omega)}\right\}+(\nabla(g-g_h),\nabla(u-i_hu)).\end{aligned}$$

For the first term, by the interpolation estimate on K_*, estimate (44) for $p=1$, and the elliptic regularity, we have

$$\|u-i_hu\|_{L^\infty(K_*)}\|\tilde{\delta}\|_{L^1(\Omega)} \le Ch_*^{2-\frac{d}{2}}\|\nabla^2 u\|_{K_*} \le Ch^2\|\nabla^2 u\| \le Ch^2\|q\|.$$

We split the second term as

$$\begin{aligned}(\nabla(g-g_h),\nabla(u-i_hu)) &= (\nabla(g-g_h),\nabla(u-i_hu))_{\Omega_*}+(\nabla(g-g_h),\nabla(u-i_hu))_{\Omega\setminus\Omega_*}\\ &= I_1+I_2.\end{aligned}$$

For I_1, we have by the Cauchy-Schwarz inequality and the interpolation estimate

$$I_1 \le \|\nabla(g-g_h)\|_{\Omega_*}\|\nabla(u-i_hu)\|_{\Omega_*} \le CC_*h_*\|\nabla g\|\|\nabla^2 u\|_{\Omega_*}.$$

For all $1\le p<\infty$ in 2D and $1\le p\le 6$ in 3D and $\frac{1}{p'}=1-\frac{1}{p}$, we have that

$$\|\nabla g\|^2=(g,\tilde{\delta}) \le \|g\|_{L^p(\Omega)}\|\tilde{\delta}\|_{L^{p'}(\Omega*)} \le Cp\|\nabla g\|\|\tilde{\delta}\|_{L^{p'}(\Omega*)}.$$

Consequently, by (44), we get

$$\|\nabla g\| \le Cp\|\tilde{\delta}\|_{L^{p'}(\Omega*)} \le Cph_*^{-\frac{d}{p}}$$

For $d=2$, we choose $p=|\log h_*|$ and obtain

$$\|\nabla g\| \le C|\log h_*| \le C|\log h|.$$

For $d=3$, we obtain by choosing $p=6$

$$\|\nabla g\| \le Ch_*^{-\frac{1}{2}} \le Ch^{-2}.$$

Inserting these estimates and using the definition $h_*=h^{2d-2}$ yields

$$I_1 \le Ch^2|\log h|\|\nabla^2 u\| \le Ch^2|\log h|\|q\|.$$

For I_2, we have

$$\begin{aligned}I_2 &\le \sum_{j=0}^{J}(\nabla(g-g_h),\nabla(u-i_hu))_{\Omega_j} \le \sum_{j=0}^{J}\|\nabla(g-g_h)\|_{\Omega_j}\|\nabla(u-i_hu)\|_{\Omega_j}\\ &\le \sum_{j=0}^{J}Ch_j\|\nabla(g-g_h)\|_{\Omega_j}\|\nabla^2 u\|_{\Omega_j} \le Ch\|\nabla^2 u\|_{\Omega_j}\sum_{j=0}^{J}r_j^{\frac{d}{4}}\|\nabla(g-g_h)\|_{\Omega_j}\\ &\le CSh\|q\|,\end{aligned}$$

where S is given by

$$S = \sum_{j=0}^{J} r_j^{\frac{d}{4}} \|\nabla(g-g_h)\|_{\Omega_j}. \tag{45}$$

Our goal is now to establish $S \le Ch|\log h|$. Let $0 < \theta < \frac{1}{4}$ be some fixed constant. We define the following sets:

$$\Omega_j' = \{\, x \in \Omega \mid (1-\theta^2) r_{j+1} < |x-x_1| \le r_j(1+\theta^2) \,\},$$
$$\Omega_j'' = \{\, x \in \Omega \mid (1-\theta) r_{j+1} < |x-x_1| \le r_j(1+\theta) \,\}.$$

This implies $\Omega_j \subset \Omega_j' \subset \Omega_j''$ and by the local energy estimate from [13, Thm.3.4] we have

$$\|\nabla(g-g_h)\|_{\Omega_j} \le C\big\{ \|\nabla(g-i_h g)\|_{\Omega_j'} + r_j^{-1}\|g-i_h g\|_{\Omega_j'} + r_j^{-1}\|g-g_h\|_{\Omega_j'} \big\}. \tag{46}$$

By interpolation estimates, we get for the first two terms

$$\|\nabla(g-i_h g)\|_{\Omega_j'} + r_j^{-1}\|g-i_h g\|_{\Omega_j'} \le Ch_j \|\nabla^2 g\|_{\Omega_j'} = Chr_j^{\frac{d}{4}} \|\nabla^2 g\|_{\Omega_j'}. \tag{47}$$

Because of the local support of $\tilde{\delta}$, we have that g is harmonic on Ω_j' and by Lemma 10 we have

$$\|\nabla^2 g\|_{\Omega_j'} \le Cr_j^{-1} \|\nabla g\|_{\Omega_j''}. \tag{48}$$

Using the pointwise estimates of the derivatives of Green's function $G(\cdot,\cdot)$ for the Laplacian, see, e.g., [19] for $d \ge 3$ and [16] for $d = 2$

$$|\nabla_x G(x,y)| \le C\,\mathrm{dist}(x,y)^{1-d} \tag{49}$$

and that $\mathrm{dist}(K_*, \Omega_j'') \ge Cr_j$ for C_* large enough, we have for $x \in \Omega_j''$ that

$$\nabla g(x) = \int_{K_*} \nabla_x G(x,y)\tilde{\delta}(y)\,dy \le Cr_j^{1-d} \|\tilde{\delta}\|_{L^1(\Omega)} \le Cr_j^{1-d}.$$

This implies

$$\|\nabla g\|_{\Omega_j''} \le C|\Omega_j''|^{\frac{1}{2}} r_j^{1-d} = Cr_j^{1-\frac{d}{2}}. \tag{50}$$

Combining the estimates (46)-(50), we have

$$\|\nabla(g-g_h)\|_{\Omega_j} \le Chr_j^{-\frac{d}{4}} + Cr_j^{-1}\|g-g_h\|_{\Omega_j'}.$$

Inserting this in (45) implies by using $J \le C|\log h|$

$$S \le C\sum_{j=0}^{J} \big\{ h + r_j^{\frac{d}{4}-1} \|g-g_h\|_{\Omega_j'} \big\} \le Ch|\log h| + \sum_{j=0}^{J} r_j^{\frac{d}{4}-1} \|g-g_h\|_{\Omega_j'}. \tag{51}$$

To treat the last term in (51), we employ a duality argument to show that

$$\sum_{j=0}^{J} r_j^{\frac{d}{4}-1} \|g-g_h\|_{\Omega_j'} \le Ch|\log h| + \frac{S}{2}$$

for C_* sufficiently large.

It holds

$$\|g-g_h\|_{\Omega_j'} = \sup_{\psi \in C_0^\infty(\Omega_j')} \frac{(g-g_h,\psi)}{\|\psi\|}. \tag{52}$$

To estimate the right-hand side $(g-g_h,\psi)$, we introduce the following dual problem: For a given $\psi \in C_0^\infty(\Omega_j')$, let $v \in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of

$$(\nabla v, \nabla \varphi) = (\psi, \varphi) \quad \forall \varphi \in H_0^1(\Omega).$$

Then, it holds

$$\begin{aligned}
(g-g_h,\psi) &= (\nabla(g-g_h), \nabla(v-i_h v)) \\
&= (\nabla(g-g_h), \nabla(v-i_h v))_{\Omega_*} + (\nabla(g-g_h), \nabla(v-i_h v))_{\Omega\setminus\Omega_*} \\
&= I_3 + I_4.
\end{aligned}$$

For I_3, we get as for I_1 before

$$I_3 \le Ch^2|\log h|\|\nabla^2 v\|_{\Omega_*} \le Ch^2|\log h|\|\psi\|. \tag{53}$$

For I_4, we have

$$\begin{aligned}
I_4 &\le \sum_{k=0}^{J} (\nabla(g-g_h), \nabla(v-i_h v))_{\Omega_k} \le C \sum_{k=0}^{J} h_k \|\nabla(g-g_h)\|_{\Omega_k} \|\nabla^2 v\|_{\Omega_k} \\
&= \sum_{k=0}^{j-2} \ldots + \sum_{k=j-1}^{j+1} \ldots + \sum_{k=j+2}^{J} \ldots = M_1 + M_2 + M_3.
\end{aligned}$$

To estimate M_2, we notice that here $k \approx j$ and hence $r_k \approx r_j$. Thus,

$$M_1 \le Ch_j \|\nabla(g-g_h)\|_{\tilde{\Omega}_j} \|\nabla^2 v\| \le Ch_j \|\nabla(g-g_h)\|_{\tilde{\Omega}_j} \|\psi\| \tag{54}$$

with $\tilde{\Omega}_j = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$. To estimate M_1 and M_3, we notice that v is harmonic on Ω_k. Thus, we have by Lemma 10 that

$$\|\nabla^2 v\|_{\Omega_k} \le Cr_k^{-1} \|\nabla v\|_{\Omega_k'}.$$

Using estimate (49) for Green's function $G(x,y)$ for $x \in \Omega_k'$ and $y \in \Omega_j'$

$$\begin{aligned}
\nabla v(x) = \int_{\Omega_j'} \nabla_x G(x,y) \psi(y)\, dy &\le C \operatorname{dist}(\Omega_k', \Omega_j')^{1-d} \|\psi\|_{L^1(\Omega_j')} \\
&\le C \operatorname{dist}(\Omega_k', \Omega_j')^{1-d} r_j^{\frac{d}{2}} \|\psi\|,
\end{aligned}$$

we get

$$\|\nabla^2 v\|_{\Omega_k} \le Cr_k^{-1} |\Omega_k'|^{\frac{1}{2}} \operatorname{dist}(\Omega_k', \Omega_j')^{1-d} r_j^{\frac{d}{2}} \|\psi\| \le Cr_k^{\frac{d}{2}-1} \operatorname{dist}(\Omega_k', \Omega_j')^{1-d} r_j^{\frac{d}{2}} \|\psi\|.$$

For M_1, we notice that $j > k+1$. Thus, it holds by (42) that $\operatorname{dist}(\Omega_k', \Omega_j') \ge Cr_k$, which implies

$$\|\nabla^2 v\|_{\Omega_k} \le Cr_k^{-\frac{d}{2}} r_j^{\frac{d}{2}} \|\psi\|$$

and consequently

$$M_1 \le C r_j^{\frac{d}{2}} \|\psi\| \sum_{k=0}^{j-2} r_k^{-\frac{d}{2}} h_k \|\nabla(g-g_h)\|_{\Omega_k}. \tag{55}$$

For M_3, we notice that $k > j+1$. Thus, it holds by (42) that $\operatorname{dist}(\Omega_k', \Omega_j') \ge C r_j$, which implies

$$\|\nabla^2 v\|_{\Omega_k} \le C r_k^{\frac{d}{2}-1} r_j^{1-\frac{d}{2}} \|\psi\|$$

and consequently

$$M_3 \le C r_j^{1-\frac{d}{2}} \|\psi\| \sum_{k=j+2}^{J} r_k^{\frac{d}{2}-1} h_k \|\nabla(g-g_h)\|_{\Omega_k}. \tag{56}$$

Using (52), we have by collecting the estimates (53)-(56) that

$$\begin{aligned} \|g-g_h\|_{\Omega_j'} \le{} & C h^2 |\log h| + C r_j^{\frac{d}{2}} \sum_{k=0}^{j-2} r_k^{-\frac{d}{2}} h_k \|\nabla(g-g_h)\|_{\Omega_k} \\ & + C h_j \|\nabla(g-g_h)\|_{\tilde{\Omega}_j} \\ & + C r_j^{1-\frac{d}{2}} \sum_{k=j+2}^{J} r_k^{\frac{d}{2}-1} h_k \|\nabla(g-g_h)\|_{\Omega_k}. \end{aligned} \tag{57}$$

To estimate the last term in (51), we have to sum up these terms weighted by $r_j^{\frac{d}{4}-1}$:

The contribution of the first term on the right-hand side of (57) is

$$\sum_{j=0}^{J} r_j^{\frac{d}{4}-1} h^2 |\log h| \le C r_{J+1}^{\frac{d}{4}-1} h^2 |\log h| \le C C_*^{\frac{d}{4}-1} h_*^{\frac{d}{4}-1} h^2 |\log h| \le \frac{C}{C_*^{1-\frac{d}{4}}} h |\log h|,$$

since $(2d-2)(\frac{d}{4}-1) = -1$ for $d \in \{2,3\}$. For the third term in (57), we get by (43)

$$\sum_{j=0}^{J} r_j^{\frac{d}{4}-1} h_j \|\nabla(g-g_h)\|_{\tilde{\Omega}_j} \le C \sum_{j=0}^{J} r_j^{\frac{d}{4}} \|\nabla(g-g_h)\|_{\Omega_j} \frac{h_j}{r_j} \le \frac{C}{C_*^{1-\frac{d}{4}}} S.$$

For the second term in (57), it follows by changing the order of summation and using the properties of the geometric series that

$$\begin{aligned} \sum_{j=0}^{J} r_j^{\frac{d}{4}-1} r_j^{\frac{d}{2}} \sum_{k=0}^{j-2} r_k^{-\frac{d}{2}} h_k \|\nabla(g-g_h)\|_{\Omega_k} & \le C \sum_{j=0}^{J} r_j^{3\frac{d}{4}-1} \sum_{k=0}^{j} r_k^{-\frac{d}{2}} h_k \|\nabla(g-g_h)\|_{\Omega_k} \\ & \le C \sum_{k=0}^{J} r_k^{-\frac{d}{2}} h_k \|\nabla(g-g_h)\|_{\Omega_k} \sum_{j=k}^{J} r_j^{3\frac{d}{4}-1} \\ & \le C \sum_{k=0}^{J} r_k^{-\frac{d}{2}} r_k^{3\frac{d}{4}-1} h_k \|\nabla(g-g_h)\|_{\Omega_k} = C \sum_{k=0}^{J} r_k^{\frac{d}{4}} \|\nabla(g-g_h)\|_{\Omega_k} \frac{h_k}{r_k} \\ & \le \frac{C}{C_*^{1-\frac{d}{4}}} S. \end{aligned}$$

Finally, for the last term in (57), we obtain similarly

$$\begin{aligned}\sum_{j=0}^{J} r_j^{\frac{d}{4}-1} r_j^{1-\frac{d}{2}} \sum_{k=j+2}^{J} r_k^{\frac{d}{2}-1} h_k \|\nabla(g-g_h)\|_{\Omega_k} &\leq C \sum_{j=0}^{J} r_j^{-\frac{d}{4}} \sum_{k=j}^{J} r_k^{\frac{d}{2}-1} h_k \|\nabla(g-g_h)\|_{\Omega_k} \\ &\leq C \sum_{k=0}^{J} r_k^{\frac{d}{2}-1} h_k \|\nabla(g-g_h)\|_{\Omega_k} \sum_{j=0}^{k} r_j^{-\frac{d}{4}} \\ &\leq C \sum_{k=0}^{J} r_k^{\frac{d}{2}-1} r_k^{-\frac{d}{4}} h_k \|\nabla(g-g_h)\|_{\Omega_k} = C \sum_{k=0}^{J} r_k^{\frac{d}{4}} \|\nabla(g-g_h)\|_{\Omega_k} \frac{h_k}{r_k} \\ &\leq \frac{C}{C_*^{1-\frac{d}{4}}} S.\end{aligned}$$

This implies for C_* large enough that

$$\sum_{j=0}^{J} r_j^{\frac{d}{4}-1} \|g-g_h\|_{\Omega_j'} \leq \frac{C}{C_*^{1-\frac{d}{4}}} h|\log h| + \frac{C}{C_*^{1-\frac{d}{4}}} S \leq Ch|\log h| + \frac{1}{2} S.$$

Inserting this into (51) yields

$$S \leq Ch|\log h|,$$

which completes the proof. □

Corollary 2 *For the solutions $\bar{z}_i$ of* (10)*, it holds*

$$\|\bar{z}_i - \bar{z}_{h,i}\| \leq Ch^2|\log h|.$$

Proof The assertion can be proved as Lemma 9 using Theorem 6 instead of the suboptimal L^∞ error estimate. □

6.2 Error estimate for the optimal control problem

The following Corollary is the counterpart of Corollary 1 for uniform meshes.

Corollary 3 *Let $u(\bar{q})$ and $u(\bar{q}_h) \in H^2(\Omega)$ be the solution of* (2) *and $u_h(\bar{q})$ and $u_h(\bar{q}_h)$ be the solution of* (15) *for $q = \bar{q}$ and $q = \bar{q}_h$, respectively. Then, it holds for $i = 1, 2, \ldots, n$*

$$\begin{aligned}|u(\bar{q})(x_i) - u_h(\bar{q})(x_i)| &\leq Ch^2|\log h| \{ \|u_d\| + |b| \}, \\ |u(\bar{q}_h)(x_i) - u_h(\bar{q}_h)(x_i)| &\leq Ch^2|\log h| \{ \|u_d\| + |b| \}.\end{aligned}$$

Proof From Theorem 6, we have

$$|u(\bar{q})(x_i) - u_h(\bar{q})(x_i)| \leq Ch^2|\log h| \|\bar{q}\|.$$

Then, by Lemma 3 we directly obtain the first assertion. The second assertion can be proved in the same way. □

In the following Lemma, we prove a first estimate for the error in the control, which will be improved later on.

Lemma 11 *Let $\bar q \in Q$ and $\bar q_h \in Q_h$ be the solutions of* (5) *and* (17)*. Then, it holds*

$$\|\bar q - \bar q_h\| \le Ch|\log h|^{\frac{1}{2}}\big\{\|u_d\| + |b|\big\}.$$

Proof We proceed as in the proof of Theorem 4 to derive (37). The first two terms of the right-hand side of (37) are estimated as before yielding

$$|j(\bar q) - j_h(\bar q)| + |j(\bar q_h) - j_h(\bar q_h)| \le Ch^2\big\{\|u_d\| + |b|\big\}^2.$$

For $|u(\bar q)(x_i) - u_h(\bar q)(x_i))|$ and $|u_h(\bar q_h)(x_i) - u(\bar q_h)(x_i))|$, we obtain by Corollary 3

$$|u(\bar q)(x_i) - u_h(\bar q)(x_i))| + |u_h(\bar q_h)(x_i) - u(\bar q_h)(x_i))| \le Ch^2|\log h|\big\{\|u_d\| + |b|\big\}.$$

The rest of the proof goes along the line of the proof of Theorem 4. □

Theorem 7 *For the optimal states $\bar u \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\bar u_h \in V_h$ and the Lagrangian multipliers $\bar\mu \in \mathbb{R}^n$ and $\bar\mu_h \in \mathbb{R}^n$ of the continuous and discrete problem* (5) *and* (17)*, respectively, it holds on a mesh graded as defined by* (41)

$$\|\bar u - \bar u_h\| + |\bar\mu - \bar\mu_h| \le Ch^2|\log h|\big\{\|u_d\| + |b|\big\}.$$

Proof As in the proof of Theorem 5, we get

$$\begin{aligned}\|\bar u - \bar u_h\| + |\bar\mu - \bar\mu_h| = {} & (\bar q - \bar q_h, \tilde z - \tilde z_h) - (\nabla(\bar u - \bar u_h), \nabla(\tilde z - \tilde z_h)) \\ & + (\bar u - \bar u_h, \tilde u - \tilde u_h) + \sum_{i=1}^n (\bar\mu_i - \bar\mu_{h,i})(\tilde u(x_i) - \tilde u_h(x_i)) \\ & - (\nabla(\bar z - \bar z_h), \nabla(\tilde u - \tilde u_h)) + \alpha(\bar q - \bar q_h, \tilde q - \tilde q_h) \\ & + (\bar z - \bar z_h, \tilde q - \tilde q_h) + \sum_{i=1}^n (\bar u(x_i) - \bar u_h(x_i))(\tilde\mu_i - \tilde\mu_{h,i}).\end{aligned} \tag{58}$$

Now, we estimate the terms on the right-hand side of (58) separately:

(i) $(\bar q - \bar q_h, \tilde z - \tilde z_h)$: Using the Lemmas 11 and 8 and Corollary 2 applied to the dual problem, we get

$$(\bar q - \bar q_h, \tilde z - \tilde z_h) \le \|\bar q - \bar q_h\|\|\tilde z - \tilde z_h\| \le Ch^2|\log h|\big\{\|u_d\| + |b|\big\}.$$

(ii) $(\nabla(\bar u - \bar u_h), \nabla(\tilde z - \tilde z_h))$: We split as in the proof of Theorem 5:

$$\begin{aligned}(\nabla(\bar u - \bar u_h), \nabla(\tilde z - \tilde z_h)) = {} & (\nabla(\bar u - \bar u_h), \nabla(\tilde z_0 - \tilde z_{h,0})) \\ & + \sum_{i=1}^n (\nabla(\bar u - \bar u_h), \nabla(\tilde\mu_i \tilde z_i - \tilde\mu_{h,i}\tilde z_{h,i})).\end{aligned} \tag{59}$$

For the first term in (59), we have by the Lemmas 7 and 8 together with Lemma 11

$$\begin{aligned}(\nabla(\bar u - \bar u_h), \nabla(\tilde z_0 - \tilde z_{h,0})) &\le \|\nabla(\bar u - \bar u_h)\|\|\nabla(\tilde z_0 - \tilde z_{h,0})\| \\ &\le Ch^2|\log h|\big\{\|u_d\| + |b|\big\}.\end{aligned}$$

For the second term in (59), we proceed as in the proof of Theorem 5:

$$\begin{aligned}(\nabla(\bar u - \bar u_h), \nabla(\tilde\mu_i \tilde z_i - \tilde\mu_{h,i}\tilde z_{h,i})) = {} & (\tilde\mu_i - \tilde\mu_{h,i})(\nabla(\bar u - \bar u_h), \nabla\tilde z_i) \\ & + \tilde\mu_{h,i}(\nabla(\bar u - \bar u_h), \nabla(\tilde z_i - \tilde z_{h,i})).\end{aligned} \tag{60}$$

For the first term in (60), we have

$$(\nabla(\bar{u}-\bar{u}_h),\nabla\tilde{z}_i) = \bar{u}(x_i) - \bar{u}_h(x_i) = 0.$$

For the second term in (60), we have by Galerkin orthogonality and Corollary 3 as in the proof of Theorem 5

$$(\nabla(\bar{u}-\bar{u}_h),\nabla(\tilde{z}_i-\tilde{z}_{h,i})) = u(\bar{q})(x_i) - u_h(\bar{q})(x_i) \le Ch^2|\log h|\big\{\|u_d\|+|b|\big\}.$$

Collecting the previous estimates and using (31), we obtain

$$(\nabla(\bar{u}-\bar{u}_h),\nabla(\tilde{z}-\tilde{z}_h)) \le Ch^2|\log h|\big\{\|u_d\|+|b|\big\}.$$

(iii) $(\bar{u}-\bar{u}_h,\tilde{u}-\tilde{u}_h)$: Using Lemma 7 together with Lemma 11 for $\bar{u}$ and $\tilde{u}$, we estimate

$$(\bar{u}-\bar{u}_h,\tilde{u}-\tilde{u}_h) \le \|\bar{u}-\bar{u}_h\|\|\tilde{u}-\tilde{u}_h\| \le Ch^2|\log h|\big\{\|u_d\|+|b|\big\}.$$

(iv) $(\bar{\mu}_i-\bar{\mu}_{h,i})(\tilde{u}(x_i)-\tilde{u}_h(x_i))$: Since $\tilde{u}(x_i) = \hat{b}_i = \tilde{u}_h(x_i)$, we have that

$$(\bar{\mu}_i-\bar{\mu}_{h,i})(\tilde{u}(x_i)-\tilde{u}_h(x_i)) = 0.$$

(v) $(\nabla(\bar{z}-\bar{z}_h),\nabla(\tilde{u}-i_h\tilde{u}))$: This term is treated as the term in ii) leading to

$$(\nabla(\bar{z}-\bar{z}_h),\nabla(\tilde{u}-\tilde{u}_h)) \le Ch^2|\log h|\big\{\|u_d\|+|b|\big\}.$$

(vi) $\alpha(\bar{q}-\bar{q}_h,\tilde{q}-\tilde{q}_h)+(\bar{z}-\bar{z}_h,\tilde{q}-\tilde{q}_h)$: By the continuous and discrete optimality conditions $\alpha\bar{q}+\bar{z}=0$ and $\alpha\bar{q}_h+\bar{z}_h=0$, we get

$$\alpha(\bar{q}-\bar{q}_h,\tilde{q}-\tilde{q}_h)+(\bar{z}-\bar{z}_h,\tilde{q}-\tilde{q}_h) = 0.$$

(vii) $(\bar{u}(x_i)-\bar{u}_h(x_i))(\tilde{\mu}_i-\tilde{\mu}_{h,i})$: Since $\bar{u}(x_i) = b_i = \bar{u}_h(x_i)$, we have that

$$(\bar{u}(x_i)-\bar{u}_h(x_i))(\tilde{\mu}_i-\tilde{\mu}_{h,i}) = 0.$$

Collecting all estimates from (i) – (vii), we end up with the assertion. □

Finally, the proved order of convergence for $\bar{u}$ and $\bar{\mu}$ can on graded meshes be used to prove the following optimal estimate for the error in the control variable.

Theorem 8 *Let $\bar{q}\in Q$ and $\bar{q}_h\in Q_h$ be the solutions of* (5) *and* (17). *Then, it holds on a mesh graded as defined by* (41)

$$\|\bar{q}-\bar{q}_h\| \le Ch^2|\log h|\big\{\|u_d\|+|b|\big\}.$$

Proof For any $p\in Q$, it holds

$$\begin{aligned}
\alpha\|\bar{q}-\bar{q}_h\|^2 &\le j_h''(p)(\bar{q}-\bar{q}_h,\bar{q}-\bar{q}_h) = j_h'(\bar{q})(\bar{q}-\bar{q}_h) - j_h'(\bar{q}_h)(\bar{q}-\bar{q}_h)\\
&= (\alpha\bar{q}+z_{h,0}(\bar{q}),\bar{q}-\bar{q}_h) - (\alpha\bar{q}_h+\bar{z}_{h,0},\bar{q}-\bar{q}_h)\\
&= -\sum_{i=1}^n \bar{\mu}_i(\bar{z}_i,\bar{q}-\bar{q}_h) - (\bar{z}_0 - z_{h,0}(\bar{q}),\bar{q}-\bar{q}_h) + \sum_{i=1}^n \bar{\mu}_{h,i}(\bar{z}_{h,i},\bar{q}-\bar{q}_h)\\
&\le \|\bar{z}_0 - z_{h,0}(\bar{q})\|\|\bar{q}-\bar{q}_h\| + \sum_{i=1}^n |\bar{\mu}_{h,i}-\bar{\mu}_i|\|\bar{z}_i\|\|\bar{q}-\bar{q}_h\|\\
&\quad + \sum_{i=1}^n |\bar{\mu}_{h,i}|\|\bar{z}_{h,i}-\bar{z}_i\|\|\bar{q}-\bar{q}_h\|.
\end{aligned}$$

We estimate $\|\bar{z}_0 - z_{h,0}(\bar{q})\|$ in standard way by inserting the Ritz projection $R_h\bar{z}_0 \in V_h$ of $\bar{z}_0$ as

$$\begin{aligned}\|\bar{z}_0 - z_{h,0}(\bar{q})\| &\leq \|\bar{z}_0 - R_h\bar{z}_0\| + \|R_h\bar{z}_0 - z_{h,0}(\bar{q})\| \\ &\leq Ch^2\|\bar{u} - u_d\| + \|\bar{u} - u_h(\bar{q})\| \leq Ch^2\big\{\|\bar{u} - u_d\| + \|\bar{q}\|\big\} \\ &\leq Ch^2\big\{\|u_d\| + |b|\big\}.\end{aligned}$$

Using this, Theorem 7, and Corollary 2 complete the proof by

$$\begin{aligned}\|\bar{q} - \bar{q}_h\| &\leq C\big\{\|\bar{z}_0 - z_{h,0}(\bar{q})\| + |\bar{\mu} - \bar{\mu}_h| + \max_{i=1,2,\dots,n}\|\bar{z}_i - \bar{z}_{h,i}\|\big\} \\ &\leq Ch^2|\log h|\big\{\|u_d\| + |\bar{\mu}|\big\}.\end{aligned}$$

□

7 Numerical results

In this section, we are going to validate the a priori error estimates for the error in the control, and state variable es well as in the Lagrangian multiplier. To this end, we consider the following concrete optimal control problem (5) with known exact solution on $\Omega = B_1(0) \subset \mathbb{R}^d$ with $d \in \{2,3\}$ with $n = 1$ state constraints and $x_1 = 0 \in \mathbb{R}^d$. For this choice of Ω. one can directly calculate $\bar{z}_1$ as

$$\bar{z}_1(x) = -\frac{1}{4\pi}\cdot\begin{cases}2\ln|x|, & \text{for } d = 2,\\ \big(1 - \frac{1}{|x|}\big), & \text{for } d = 3,\end{cases}$$

where $|x|$ denotes the Euclidian norm of $x \in \mathbb{R}^d$. We further choose $\bar{\mu}_1 = 1$ and $u_d = \bar{u}$. This implies $\bar{z} = \bar{z}_1$ and $\bar{q} = -\frac{1}{\alpha}\bar{z}$. A direct calcuation shows that

$$\bar{u}(x) = -\frac{1}{8\pi\alpha}\cdot\begin{cases}\big(1 - |x|^2 + |x|^2\ln|x|\big), & \text{for } d = 2,\\ \big(\frac{2}{3} - |x| + \frac{1}{3}|x|^2\big), & \text{for } d = 3\end{cases}$$

fulfills the state equation for the choosen $\bar{q}$. The value $b_1 = \bar{u}(0)$ for the state constraint is then

$$b_1 = \begin{cases}-\frac{1}{8\pi\alpha}, & \text{for } d = 2,\\ -\frac{1}{12\pi\alpha}, & \text{for } d = 3.\end{cases}$$

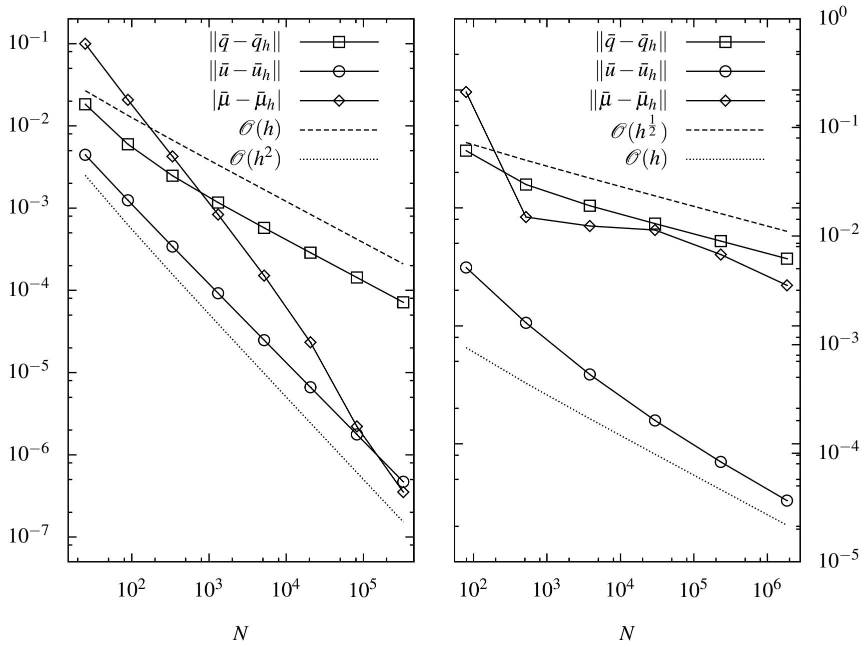
For the computations, we choose $\alpha = 1$.

The optimal control problems are solved by the optimization library RoDoBo [28] and the finite element toolkit GASCOIGNE [17] using a two-step procedure. For the first step, we write the discrete optimality system (19) – (21) in the case $n = 1$ in terms of the discrete control-to-state-operator $S_h\colon q \mapsto u_h$ as

$$\bar{u}_h = S_h\bar{q}_h, \qquad \bar{z}_h = S_h(\bar{u} - u_d + \bar{\mu}_{h,1}\delta_{x_1}), \qquad \bar{q}_h = -\alpha^{-1}\bar{z}_h.$$

Combining these equations yields

$$(\alpha I + S_h^2)\bar{u}_h = S_h^2 u_d - \bar{\mu}_{h,1}S_h^2\delta_{x_1}.$$



(a) $d=2$ dimensions (b) $d=3$ dimensions

Fig. 1 Discretization errors on uniform meshes

Noting that $H_h := (\alpha I + S_h^2)$ coincides with the Hessian of the discrete reduced functional j_h, we obtain

$$b_1 = \bar{u}_h(x_1) = (H_h^{-1} S_h^2 u_d)(x_1) - \bar{\mu}_{h,1}(H_h^{-1} S_h^2 \delta_{x_1})(x_1),$$

from which the unknown Lagrangian multiplier $\bar{\mu}_1$ can directly be computed. Thereafter, using the computed value of $\bar{\mu}_{h,1}$ the optimal control $\bar{q}_h$ and the associated state $\bar{u}_h$ are computed classically by a conjugate gradient method solving the reduced optimality system.

Figure 1 depicts the development of the errors in the control and state variables and in the Lagrangian multiplier under refinement of the cell size h using uniform mesh refinement for $d=2$ (a) and $d=3$ (b). Here, $N=\mathcal{O}(h^{-d})$ denotes the number of nodes. The discretization error $\|\bar{q}-\bar{q}_h\|$ and $\|\bar{u}-\bar{u}_h\|$ clearly exhibit the proved orders of convergence $\mathcal{O}(h^{2-\frac{d}{2}})$ and $\mathcal{O}(h^{4-d})$, respectively. For $d=2$, the error in term of the Lagrangian multiplier decreases much faster than predicted, whereas for $d=3$ the proved $\mathcal{O}(h)$ convergence can be observed not until the mesh gets finer than approx 10^4 nodes.

In Figure 2 the development of the errors in the control, state and Lagrangian multiplier on a series of graded meshes for $d=2$ (a) and $d=3$ (b) are shown. Here, for all errors the expected order $\mathcal{O}(h^2)$ is observed.

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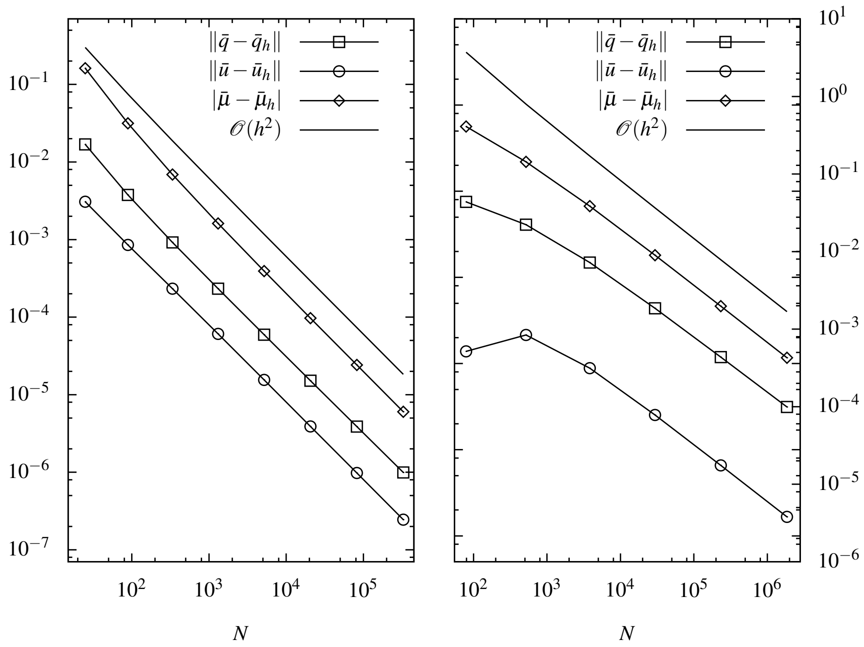

(a) $d = 2$ dimensions (b) $d = 3$ dimensions

Fig. 2 Discretization errors on graded meshes

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