MATH 5520 – Finite Element Methods.

Assignment 5. Solutions.

Problem 2. Let u_I be the piecewise linear interpolant of u defined in class. Prove that

$$\|u' - u_I'\|_{L_2(0,1)} \leq Ch\|u''\|_{L_2(0,1)} \tag{1}$$

and

$$\|u - u_I\|_{L_2(0,1)} \leq Ch^2\|u''\|_{L_2(0,1)}, \tag{2}$$

where h is the mesh size.

Solution:
Let $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ be the partition of the interval $(0,1)$. Since

$$\|u' - u_I'\|^2_{L_2(0,1)} = \sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} (u' - u_I')^2(x)dx,$$

it is sufficient to show that

$$\int_{x_j}^{x_{j+1}} (u' - u_I')^2(x)dx \leq Ch_j^2 \int_{x_j}^{x_{j+1}} u''(x)^2 dx, \tag{3}$$

where $h_j = x_{j+1} - x_j$. Since u_I is linear on x_j, x_{j+1}, $u_I'' = 0$ and as a result

$$\int_{x_j}^{x_{j+1}} u''(x)^2 dx = \int_{x_j}^{x_{j+1}} (u'' - u_I'')^2(x)dx.$$

By density argument we can assume that u is smooth. Put $e(x) = u(x) - u_I(x)$. Thus we want to show

$$\int_{x_j}^{x_{j+1}} e'(x)^2 dx \leq Ch_j^2 \int_{x_j}^{x_{j+1}} e''(x)^2 dx.$$

By the change of variables $y = \frac{x - x_j}{x_{j+1} - x_j} = \frac{x - x_j}{h_j}$ we have

$$\int_{x_j}^{x_{j+1}} e'(x)^2 dx = \frac{1}{h_j} \int_0^1 \tilde{e}'(y)^2 dy, \tag{4}$$

where $\tilde{e}'(y) = e(x_j) + yh_j$. Similarly,

$$\int_{x_j}^{x_{j+1}} e''(x)^2 dx = \frac{1}{h_j^3} \int_0^1 \tilde{e}''(y)^2 dy. \tag{5}$$

Thus the problem is reduced to showing

$$\int_{x_j}^{x_{j+1}} \tilde{e}'(x)^2 dx \leq C \int_{x_j}^{x_{j+1}} \tilde{e}''(x)^2 dx. \tag{6}$$

Notice that by scaling to a unit size interval we removed the dependence on h in the above estimate. To show (6), we notice that $\tilde{e}(0) = \tilde{e}(1) = 0$, hence by the Rolle's Theorem there is point $\xi \in (0,1)$ such that $\tilde{e}'(\xi) = 0$. As a result for any $x \in (0,1)$ by the Fundamental Theorem of Calculus,

$$\tilde{e}'(x) = \int_\xi^x \tilde{e}''(x)dx.$$

Using the Cauchy-Schwarz inequality

$$\tilde{e}'(x) = \int_{\xi}^{x} \tilde{e}''(x)dx \leq \|\tilde{e}''\|_{L^2(0,1)}.$$

Squaring both sides and integrating from 0 to 1 we have

$$\int_0^1 \tilde{e}'(x)^2 dx \leq \|\tilde{e}''\|^2_{L^2(0,1)} = \int_0^1 \tilde{e}''(x)^2 dx.$$

That establishes (6). Combining it with (4) and (5) we have (3) and as a result (1).

The argument for (2) is similar and follows easily by scaling to unit interval and noticing that since $\tilde{e}(0) = \tilde{e}(1) = 0$, again by the Fundamental Theorem of Calculus

$$\tilde{e}(x) = \int_{\xi}^{x} \tilde{e}'(x)dx \leq \|\tilde{e}'\|_{L^2(0,1)}.$$

Squaring both sides and integrating from 0 to 1 we have

$$\int_0^1 \tilde{e}(x)^2 dx \leq \|\tilde{e}'\|^2_{L^2(0,1)} = \int_0^1 \tilde{e}'(x)^2 dx.$$

Now (2) follows easily by scaling back to (x_j, x_{j+1}), summing, and using (1).