Assignment 4

1. Theorem 4.1 predicts second order convergence for the difference scheme considered in Chapter 4.1 with the error estimate

$$\max_{0\le j\le N} |u(x_j) - U_j| \le Ch^2 \|u\|_{C^4}, \quad \text{where } h = \frac{1}{N}.$$

In general if the exact solution u is known and the error $e_h = \max_{0\le j\le N} |x_j - U_j|$ is of the form

$$e_h = C(u)h^\alpha + o(h^\alpha)$$

for some constant $C(u)$ that may depend on u but not on h, then it is possible to estimate the convergence rate α by comparing e_h and $e_{h/2}$. Since by assumption $e_{h/2} = C(u)(h/2)^\alpha + o(h^\alpha)$, we have

$$\frac{e_{h/2}}{e_h} = 2^\alpha + o(1),$$

we can estimate α from

$$\alpha \approx \log_2 \frac{e_{h/2}}{e_h}. \tag{1}$$

Assume now that we want to estimate the error at one node x_j, but the exact solution u is not known. Assuming that the error at the node satisfies

$$e_h^j = u(x_j) - U_j = Ch^\alpha + o(h^\alpha),$$

for some constant C that may depend on u but not on h. Develop a formula that will allow you to estimate the rate of convergence α without any knowledge of u.

2. Using the finite difference method, solve the problem 5 from the Assignment 2:

$$-u'' = \frac{1}{x}, \quad x \in (0,1),$$
$$u(0) = u(1) = 0.$$

Use the formula (1) to estimate the convergence rate.

3. Modify your finite difference code to solve Neumann boundary value problem

$$-a(x)u'' + b(x)u' + c(x)u = f(x), \quad x \in (0,1),$$
$$u'(0) = u_0, \quad u'(1) = u_1.$$

Check your code by taking

$$u(x) = \cos x, \quad a(x) = 1 + x, \quad b(x) = x^2, \quad c(x) = e^x, \quad f(x) = (1+x)\cos x - x^2 \sin x + e^x \cos x.$$

Use formula (1) to estimate the convergence rate.

4. Using the finite difference method, solve the problem

$$-a(x)u'' + b(x)u' + c(x)u = f(x), \quad x \in (0,1),$$
$$u(0) = (1) = 0,$$

where

$$a(x) = \begin{cases} 1, & 0 \le x \le 0.5 \\ 2, & 0.5 < x \le 1, \end{cases} \qquad b(x) = 0, \quad c(x) = 1, \quad f(x) = x.$$

Plot the solution for $N = 100$. Use the formula developed in the problem 1 to estimate the convergence rate at $x = 1/4$ and $x = 1/2$.