MATH 5520 – Finite Element Methods.

Assignment 3. Solutions.

1. (Problem 3.5 from the textbook.)

   Prove

$$\|v\|_{L^2(\Omega)} \le C \left(\|\nabla v\|^2_{L^2(\Omega)} + \big(\int_\Omega v(x)dx \big)^2 \right)^{1/2} \quad \text{for } v \in C^1(\Omega),$$

   where $\Omega = (0,1) \times (0,1)$ is the unit square in $\mathbb{R}^2$.

   Solution: Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ it is enough to consider $u \in C^1(\overline{\Omega})$. Denote $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. By the Fundamental Theorem of calculus we have an identity

$$(u(\mathbf{x}) - u(\mathbf{y}))^2 = \left(\int_{y_1}^{x_1} \frac{\partial u}{\partial s}(s, y_2)ds + \int_{y_2}^{x_2} \frac{\partial u}{\partial t}(x_1, t)dt \right)^2.$$

   Expanding the left hand side and using Cauchy-Schwarz inequality on the right hand side we obtain

$$u(\mathbf{x})^2 + u(\mathbf{y})^2 - u(\mathbf{x})u(\mathbf{y}) \le \left(|x_1 - y_1|^{1/2} \|\nabla u\|_{L^2(\Omega)} + |x_2 - y_2|^{1/2} \|\nabla u\|_{L^2(\Omega)} \right)^2.$$

   Using that $0 \le x_i \le 1$ and $0 \le y_i \le 1$, $i = 1, 2$ we have

$$u(\mathbf{x})^2 + u(\mathbf{y})^2 \le 8\|\nabla u\|^2_{L^2(\Omega)} + u(\mathbf{x})u(\mathbf{y}).$$

   Integrating with respect to $\mathbf{x}$ and $\mathbf{y}$ over Ω we obtain

$$2\|u\|^2_{L^2(\Omega)} \le 8\|\nabla u\|^2_{L^2(\Omega)} + \left(\int_\Omega u(\mathbf{x})d\mathbf{x} \right)^2.$$

   Dividing by 2 and taking the square root shows the desired inequality.

2. (Problem 3.9 from the textbook.)

   Consider the Neumann problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

   where $\Omega \subset \mathbb{R}^n$. Assume that $f \in L^2(\Omega)$ and show that the condition

$$\int_\Omega f(x)dx = 0$$

   is necessary for the existence of a solution.

   Notice that if u is a solution to (1), so is $u + c$ for any constant c. To obtain a uniqueness, we can add an extra condition

$$\int_\Omega u(x)dx = 0,$$

   i.e. requiring the mean value of u to be zero. Define a functional space

$$V = \{v \in H^1(\Omega) : \int_\Omega u dx = 0\}.$$

Show that V is a linear vector space and define a variational formulation to (1) using this space.

Solution:

By the Green's formula

$$\int_\Omega -\Delta u dx = \int_\Omega -1\Delta u dx = \int_\Omega \nabla u \cdot \nabla 1 dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} dx = 0.$$

Thus for consistency we need $\int_\Omega f dx = 0$.

It is also easy to see that if u satisfies (1) so does $u + c$, for any constant c. Thus we have a problem with uniqueness.

To show that

$$V = \{v \in H^1(\Omega) : \int_\Omega u dx = 0\}.$$

is a vector space it is enough to check that for $u \in V$ and $v \in V$, $u + v \in V$, since $\int_\Omega (u+v)dx = 0$ and for any constant c, $cu \in V$ since $\int_\Omega (cu)dx = c\int_\Omega u dx = 0$. To make V a norm vector space we equip V with H^1 norm. Notice that in view of the inequality

$$\|v\|^2_{L^2(\Omega)} \leq 4\left(\|\nabla v\|^2_{L^2(\Omega)} + \big(\int_\Omega v(x)dx\big)^2\right) \quad \text{for } v \in C^1(\Omega), \tag{2}$$

from the previous problem which is also valid for regular domains, H^1 norm and L^2 norm of the gradient are equivalent norms on V. Indeed (2) give us

$$\|u\|^2_{H^1} = \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} \leq 5\|\nabla u\|^2_{L^2} \quad \forall u \in V.$$

Hence

$$\|\nabla u\|_{L^2} \leq \|u\|_{H^1} \leq \sqrt{5}\|\nabla u\|_{L^2} \quad \forall u \in V.$$

Now the variational problem: find $u \in V$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in V$$

has a unique solution by Lax-Milgram lemma. The bilinear form $a(u, v) = (\nabla u, \nabla v)$ is coercive over V since

$$a(u, u) = (\nabla u, \nabla u) = \|\nabla u\|^2_{L^2} \geq \frac{1}{5}\|u\|^2_{H^1},$$

bounded

$$a(u, v) = (\nabla u, \nabla v) \leq \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} \leq \|u\|_{H^1}\|v\|_{H^1},$$

and the linear functional by the Cauchy-Schwarz inequality is bounded,

$$\ell(v) = (f, v) \leq \|f\|_{L^2}\|u\|_{L^2} \leq \|f\|_{L^2}\|u\|_{H^1}.$$

In addition from

$$\|u\|^2_{H^1} \leq 5\|\nabla u\|^2_{L^2} = 5(\nabla u, \nabla u) = 5(f, u) \leq 5\|f\|_{L^2}\|u\|_{H^1},$$

we have the following stability result

$$\|u\|_{H^1} \leq 5\|f\|_{L^2}.$$

3. (Problem 3.13 from the textbook. Stability with respect to the coefficient.)

Let u_1 and u_2 be two weak solutions of the problems

$$\begin{aligned} -\nabla \cdot (a_1(x)\nabla u_1) &= f & -\nabla \cdot (a_2(x)\nabla u_2) &= f \quad \text{in } \Omega, \\ u_1 &= 0, & u_2 &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $f \in L^2(\Omega)$. The coefficients a_1 and a_2 are smooth functions such that

$$a_i(x) \geq a_0 > 0, \quad \forall x \in \Omega, \quad i = 1, 2.$$

Prove the stability estimate

$$\|\nabla(u_1 - u_2)\|_{L^2(\Omega)} \leq \frac{C}{a_0^2}\|a_1 - a_2\|_{C(\Omega)}\|f\|_{L^2(\Omega)}. \tag{3}$$

Solution: Notice that since $u_i = 0$ on $\partial\Omega$, the Poincare inequality holds

$$\|u_i\|_{L^2(\Omega)} \leq C\|\nabla u_i\|_{L^2(\Omega)}, \quad i = 1, 2. \tag{4}$$

Thus u_i satisfy the following stability estimates

$$\|\nabla u_i\|_{L^2(\Omega)} \leq \frac{C}{a_0}\|f\|_{L^2(\Omega)}, \quad i = 1, 2. \tag{5}$$

Indeed, since u_i satisfy

$$(a_i \nabla u_i, \nabla v) = (f, v) \quad \forall v \in H_0^1,$$

taking $v = u_i$ we have

$$a_0\|\nabla u_i\|^2_{L^2(\Omega)} \leq (a_i \nabla u_i, \nabla u_i) = (f, u_i) \leq \|f\|_{L^2(\Omega)}\|u_i\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}\|\nabla u_i\|_{L^2(\Omega)}.$$

The last two inequalities follow from the Cauchy-Schwarz and Poincare inequalities. On the other hand the difference $u_1 - u_2$ satisfies

$$(a_1 \nabla u_1 - a_2 \nabla u_2, \nabla v) = 0 \quad \forall v \in H_0^1. \tag{6}$$

Taking $v = u_1 - u_2$ in (6) we have

$$(a_1 \nabla u_1 - a_2 \nabla u_2, \nabla(u_1 - u_2)) = 0.$$

Now adding and subtracting $a_1 u_2$ we obtain

$$(a_1 \nabla u_1 - a_1 \nabla u_2 + a_1 \nabla u_2 - a_2 \nabla u_2, \nabla(u_1 - u_2)) = 0,$$

which is equivalent to

$$\begin{aligned} a_0\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} &\leq (a_1(\nabla(u_1 - u_2), \nabla(u_1 - u_2)) \\ &= ((a_1 - a_2)\nabla u_2, (u_1 - u_2)) \\ &\leq \|a_1 - a_2\|_{C(\Omega)}\|\nabla u_2\|_{L^2(\Omega)}\|u_1 - u_2\|_{L^2(\Omega)}. \end{aligned}$$

The desired stability estimate (3) now follows from (5) and the Poincare inequality for $u_1 - u_2$.