Assignment 3

1. (Problem 3.5 from the textbook.)

   Prove

$$\|v\|_{L^2(\Omega)} \leq C \left(\|\nabla v\|_{L^2(\Omega)} + \big(\int_\Omega v(x)dx \big)^2 \right)^{1/2} \quad \text{for } v \in C^1(\Omega),$$

   where $\Omega = (0,1) \times (0,1)$ is the unit square in $\mathbb{R}^2$.

2. (Problem 3.9 from the textbook.)

   Consider the Neumann problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

   where $\Omega \subset \mathbb{R}^n$. Assume that $f \in L^2(\Omega)$ and show that the condition

$$\int_\Omega f(x)dx = 0$$

   is necessary for the existence of a solution.

   Notice that if u is a solution to (1), so is $u + c$ for any constant c. To obtain a uniqueness, we can add an extra condition

$$\int_\Omega u(x)dx = 0,$$

   i.e. requiring the mean value of u to be zero. Define a functional space

$$V = \{v \in H^1(\Omega) : \int_\Omega u dx = 0\}.$$

   Show that V is a linear vector space and define a variational formulation to (1) using this space.

3. (Problem 3.13 from the textbook. Stability with respect to the coefficient.)

   Let u_1 and u_2 be two weak solutions of the problems

$$\begin{aligned} -\nabla \cdot (a_1(x)\nabla u_1) &= f \qquad & -\nabla \cdot (a_2(x)\nabla u_2) &= f \quad \text{in } \Omega, \\ u_1 &= 0, & u_2 &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

   where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $f \in L^2(\Omega)$. The coefficients a_1 and a_2 are smooth functions such that

$$a_i(x) \geq a_0 > 0, \quad \forall x \in \Omega, \quad i = 1, 2.$$

   Prove the stability estimate

$$\|\nabla(u_1 - u_2)\|_{L^2(\Omega)} \leq \frac{C}{a_0^2} \|a_1 - a_2\|_{C(\Omega)} \|f\|_{L^2(\Omega)}.$$