MATH 5520 – Finite Element Methods.

Assignment 2. Solutions.

1. (Problem 2.3 from the textbook.) Consider the nonlinear boundary value problem

$$\begin{aligned} -u'' + u &= e^u \quad \text{in } \Omega = (0,1), \\ u(0) = u(1) &= 0. \end{aligned}$$

Use the maximum principle to show that all solutions are nonnegative, i.e., $u(x) \geq 0$ for all $x \in \bar{\Omega}$.

Solution:

Since $e^u \geq 0$, we have

$$-u'' + u \geq 0.$$

Applying directly Theorem 2.1 of the textbook we have

$$\min_{0 \leq x \leq 1} u(x) \geq \{u(0), u(1), 0\} = 0.$$

This show that $u(x) \geq 0$ for any x.

2. (Problem 2.4 from the textbook.) Consider the two point boundary value problem

$$\begin{aligned} \mathcal{L}u := -(a(x)u')' + cu &= f \quad \text{in } \Omega = (0,1), \\ u(0) = u(1) &= 0, \end{aligned}$$

and let $G(x,y)$ be the corresponding Green's function. Show that G is symmetric, i.e. $G(x,y) = G(y,x)$.

Let

$$A(u,v) = (au', v') + (cu, v)$$

be the corresponding bilinear form. Prove that

$$A(v, G(x,\cdot)) = v(x), \quad \forall v \in H_0^1, \ x \in \Omega.$$

This means that $\mathcal{L}G(x,\cdot) = \delta_x$, where δ_x is Dirac's delta function at x, defined as $(\delta_x, \phi) = \phi(x)$, for any $\phi \in C^0$.

Solution:

The Green's function is given by

$$G(x,y) = \begin{cases} \frac{1}{\kappa} U_0(x) U_1(y), & 0 \leq y \leq x \leq 1, \\ \\ \frac{1}{\kappa} U_1(x) U_0(y), & 0 \leq x \leq y \leq 1, \end{cases}$$

where

$$\kappa = a(x) \left(U_0(x) U_1'(x) - U_0'(x) U_1(x) \right) \equiv \textit{constant} > 0,$$

and U_0 and U_1 satisfy

$$\begin{aligned} \mathcal{L}U_0 := -(aU_0')' + cU_0 &= 0 \quad \text{in } \Omega, \\ U_0(0) = 1, \quad U_0(1) &= 0, \end{aligned}$$

and

$$\begin{aligned}\mathcal{L}U_1 := -(aU_1')' + cU_1 =& 0 \quad \text{in } \Omega,\\ U_1(0) = 0, \quad & U_1(1) = 1,\end{aligned}$$

respectively. From this definition it is clear that $G(x,y) = G(y,x)$.

To show

$$A(v, G(x,\cdot)) = v(x), \quad \forall v \in H_0^1,\ x \in \Omega,$$

we fist notice that since $G(x,y)$ is continuous

$$A(v, G(x,\cdot)) = \int_0^1 \left(a(y)v'(y)G_y(x,y) + c(y)v(y)G(x,y)\right) dy$$

is well defined. Using the definition of the Green's function, we split the domain of integration and integrate by parts,

$$\begin{aligned}\int_0^1 a(y)v'(y)G_y(x,y)dy &= \int_0^x a(y)v'(y)G_y(x,y)dy + \int_x^1 a(y)v'(y)G_y(x,y)dy\\ &= \int_0^x a(y)v'(y)\frac{1}{\kappa}U_0(x)U_1'(y)dy + \int_x^1 a(y)v'(y)\frac{1}{\kappa}U_1(x)U_0'(y)dy\\ &= \frac{1}{\kappa}U_0(x)\int_0^x (-a(y)U_1'(y))'v(y)dy + \frac{1}{\kappa}U_1(x)\int_x^1 (-a(y)U_0'(y))'v(y)dy\\ &+ \frac{1}{\kappa}U_0(x)a(x)v(x)U_1'(x) - \frac{1}{\kappa}U_1(x)a(x)v(x)U_0'(x)\\ &= v(x) + \frac{1}{\kappa}U_0(x)\int_0^x (-a(y)U_1'(y))'v(y)dy + \frac{1}{\kappa}U_1(x)\int_x^1 (-a(y)U_0'(y))'v(y)dy,\end{aligned}$$

where we used the definition of κ. Thus using that $\mathcal{L}U_0 = 0$ and $\mathcal{L}U_1 = 0$, we have

$$\begin{aligned}A(v, G(x,\cdot)) &= v(x) + \frac{1}{\kappa}U_0(x)\int_0^x \left((-a(y)U_1'(y))' + c(y)U_1(y)\right)v(y)dy\\ &\quad + \frac{1}{\kappa}U_1(x)\int_0^x \left((-a(y)U_0'(y))' + c(y)U_0(y)\right)v(y)dy\\ &= v(x).\end{aligned}$$

3. (Problem 2.6 from the textbook.) Give variational formulations and prove existence of solution of

$$-u'' = f \quad \text{in } \Omega = (0,1),$$

with the following boundary condtions

(a) $u(0) = u(1) = 0$,
(b) $u(0) = u'(1) = 0$,
(c) $-u'(0) + u(0) = u'(1) = 0$.

Solution:

(a) $u(0) = u(1) = 0$. Multiplying both sides of the equation by arbitrary $v \in H_0^1(0,1)$ and integrating by parts we obtain

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx.$$

Thus the variational problem is to find $u \in H_0^1(0,1)$ such that

$$(u', v') = (f, v) \quad \forall v \in H_0^1(0,1).$$

Using the Poincare inequality $\|u\|_{L^2(0,1)} \leq \|u'\|_{L^2(0,1)}$ we have

$$\|u\|^2_{H^1(0,1)} = \|u\|^2_{L^2(0,1)} + \|u'\|^2_{L^2(0,1)} \leq 2\|u'\|^2_{L^2(0,1)} = 2(u', u').$$

Thus the bilinear form is coersive over H^1_0 and by Cauchy-Schwarz inequality bounded,

$$(u', v') \leq \|u'\|_{L^2(0,1)}\|v'\|_{L^2(0,1)} \leq \|u\|_{H^1(0,1)}\|v\|_{H^1(0,1)}.$$

Thus for $f \in L^2(0,1)$ all the conditions of Lax-Milgram Lemma are satisfied and the problem has a unique solution in H^1_0.

(b) $u(0) = u'(1) = 0$. Define a space

$$V = \{v \in H^1(0,1) : \ v(0) = 0\}.$$

Equip the space V with H^1 norm. Multiplying both sides of the equation by arbitrary $v \in V$ and integrating by parts we obtain

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx.$$

Thus the variational problem is to find $u \in V$ such that

$$(u', v') = (f, v) \quad \forall v \in V.$$

Since the Poincare inequality $\|u\|_{L^2(0,1)} \leq \|u'\|_{L^2(0,1)}$ again holds for any $u \in V$, we have

$$\|u\|^2_V = \|u\|^2_{H^1(0,1)} = \|u\|^2_{L^2(0,1)} + \|u'\|^2_{L^2(0,1)} \leq 2\|u'\|^2_{L^2(0,1)} = 2(u', u').$$

Thus the bilinear form is coersive over V and by Cauchy-Schwarz inequality bounded,

$$(u', v') \leq \|u'\|_{L^2(0,1)}\|v'\|_{L^2(0,1)} \leq \|u\|_V\|v\|_V.$$

Thus for $f \in L^2(0,1)$ all the conditions of Lax-Milgram Lemma are satisfied and the problem has unique solution in V.

(c) $-u'(0) + u(0) = u'(1) = 0$.
Multiplying both sides of the equation by arbitrary $v \in H^1(0,1)$ and integrating by parts we obtain

$$\int_0^1 u'(x)v'(x)dx - u'(0)v(0) = \int_0^1 f(x)v(x)dx.$$

Using that $u(0) = u'(0)$, we have

$$\int_0^1 u'(x)v'(x)dx + u(0)v(0) = \int_0^1 f(x)v(x)dx.$$

Define the bilinear form

$$a(u, v) = \int_0^1 u'(x)v'(x)dx + u(0)v(0).$$

Thus the variational problem is to find $u \in H^1(0,1)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H^1(0,1).$$

To apply Lax-Milgram Lemma we need to show that the bilinear form is coersive and bounded on $H^1(0,1)$. To show coersivity we use

$$u(x) = u(0) + \int_0^x u'(t)dt.$$

Thus by Cauchy-Schwarz inequality

$$|u(x)| \le |u(0)| + \|u'\|_{L^2(0,1)},$$

and as a result

$$\|u\|^2_{L^2(0,1)} \le 2(u(0)^2 + \|u'\|^2_{L^2(0,1)}) = 2a(u,u).$$

This shows coersivity. To show boundedness we notice that $|u(0)| \le \|u\|_{H^1(0,1)}$, which follows from

$$u(0) = u(x) - \int_0^x u'(t)dt.$$

Thus

$$a(u,v) \le \|u\|_{H^1(0,1)}\|v\|_{H^1(0,1)},$$

and for $f \in L^2(0,1)$ all the conditions of Lax-Milgram Lemma are satisfied and the problem has a unique solution in H^1.

4. (Problem A.11 from the textbook.) Let $\Omega = (0,1)$ and $f = 1/x$. Show that $f \notin L_2(\Omega)$ but $f \in H^{-1}(\Omega)$. That will show that $H^{-1} \not\subset L_2(\Omega)$.

   Hint: Define linear functional $f(v) = (f,v)\ \forall v \in H^1_0(\Omega)$ and prove the inequality

   $$|(f,v)| \le C\|v'\|_{L_2(\Omega)}, \quad \forall v \in H^1_0(\Omega).$$

   Solution: Simple computations show that

   $$\|f\|^2_{L_2(\Omega)} = \int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{x^2} = \lim_{\varepsilon\to 0^+} -\frac{1}{x}\Big|_\varepsilon^1 = \lim_{\varepsilon\to 0^+}(1/\varepsilon - 1) = \infty.$$

   Thus $f \notin L_2(\Omega)$.

   On the other hand

   $$\|f\|_{H^{-1}(\Omega)} = \sup_{\substack{v\in H^1_0(\Omega)\\ \|v\|_{H^1}\le 1}} (f,v) = \sup_{\substack{v\in H^1_0(\Omega)\\ \|v\|_{H^1}\le 1}} \int_0^1 \frac{v(x)}{x}dx.$$

   Since C_0^∞ is dense in H^1_0 it is sufficient to assume that $v \in C^1_0$. We have

   $$\int_0^1 \frac{v(x)}{x}dx = v(x)\log x\Big|_0^1 - \int_0^1 v'(x)\log x dx = -\int_0^1 v'(x)\log x dx \le \|v'\|_{L_2(\Omega)}\|\log x\|_{L_2(\Omega)}.$$

   Using that

   $$\|\log x\|^2_{L_2(\Omega)} = \int_0^1 (\log x)^2 dx = x(\log x)^2\Big|_0^1 - 2\int_0^1 \log x dx = -x\log x\Big|_0^1 + 2\int_0^1 dx = 2,$$

   we have

   $$\|f\|_{H^{-1}(\Omega)} \le \sqrt{2}$$

   and as a result $f \in H^{-1}(\Omega)$.

5. (Problem 2.8 from the textbook.) Find an explicit solution of

   $$\begin{aligned} -u'' &= \frac{1}{x} \quad \text{in } \Omega = (0,1),\\ u(0) = u(1) &= 0. \end{aligned}$$

   Check that $u \in H^1_0$ but $u \notin H^2$.

Solution: Easy to see that

$$u = -x \log x$$

solves the equation. Also

$$u'(x) = -1 - \log x \in L^2(0,1),$$

but as we have seen in the previous problem

$$u''(x) = -\frac{1}{x} \notin L_2(0,1).$$