Assignment 2

1. (Problem 2.3 from the textbook.) Consider the nonlinear boundary value problem

$$\begin{aligned} -u'' + u &= e^u \quad \text{in } \Omega = (0,1), \\ u(0) = u(1) &= 0. \end{aligned}$$

   Use the maximum principle to show that all solutions are nonnegative, i.e., $u(x) \geq 0$ for all $x \in \bar{\Omega}$.

2. (Problem 2.4 from the textbook.) Consider the two point boundary value problem

$$\begin{aligned} \mathcal{L}u := -(a(x)u')' + cu &= f \quad \text{in } \Omega = (0,1), \\ u(0) = u(1) &= 0, \end{aligned}$$

   and let $G(x,y)$ be the corresponding Green's function. Show that G is symmetric, i.e. $G(x,y) = G(y,x)$.

   Let

$$A(u,v) = (au', v') + (cu, v)$$

   be the corresponding bilinear form. Prove that

$$A(v, G(x,\cdot)) = v(x), \quad \forall v \in H_0^1, \ x \in \Omega.$$

   This means that $\mathcal{L}G(x,\cdot) = \delta_x$, where δ_x is Dirac's delta function at x, defined as $(\delta_x, \phi) = \phi(x)$, for any $\phi \in C^0$.

3. (Problem 2.6 from the textbook.) Give variational formulations and prove existence of solution of

$$-u'' = f \quad \text{in } \Omega = (0,1),$$

   with the following boundary condtions

   (a) $u(0) = u(1) = 0$,

   (b) $u(0) = u'(1) = 0$,

   (c) $-u'(0) + u(0) = u'(1) = 0$.

4. (Problem A.11 from the textbook.) Let $\Omega = (0,1)$ and $f = 1/x$. Show that $f \notin L_2(\Omega)$ but $f \in H^{-1}(\Omega)$. That will show that $H^{-1} \not\subset L_2(\Omega)$.

   Hint: Define linear functional $f(v) = (f,v)\ \forall v \in H_0^1(\Omega)$ and prove the inequality

$$|(f,v)| \leq C\|v'\|_{L_2(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

5. (Problem 2.8 from the textbook.) Find an explicit solution of

$$\begin{aligned} -u'' &= \frac{1}{x} \quad \text{in } \Omega = (0,1), \\ u(0) = u(1) &= 0. \end{aligned}$$

   Check that $u \in H_0^1$ but $u \notin H^2$.