Assignment 1

1. (Problem 1.4 from the textbook. The Laplace operator in spherical symmetry.)
   Introduce spherical coordinates (r, θ, ϕ) defined by

$$\begin{aligned} x_1 &= r \sin\theta \cos\phi, \\ x_2 &= r \sin\theta \sin\phi, \\ x_3 &= r \cos\theta. \end{aligned}$$

   Assume that the function u does not depend on θ and ϕ, i.e. $u = u(r)$. Show that

$$\Delta u = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{du}{dr} \right).$$

2. (Problem 1.5 from the textbook. The Laplace operator in cylindrical symmetry.)
   Introduce cylindrical coordinates (ρ, φ, z) defined by

$$\begin{aligned} x_1 &= \rho \cos\varphi, \\ x_2 &= \rho \sin\varphi, \\ x_3 &= z. \end{aligned}$$

   Assume that the function u does not depend on φ and z, i.e. $u = u(\rho)$. Show that

$$\Delta u = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{du}{d\rho} \right).$$

3. (Problem 1.6 from the textbook.) Let

$$\Omega = \{x \in \mathbb{R}^3 : |x| < 1\},$$

   be the unit open ball. Determine an explicit solution of the boundary value problem

$$\begin{aligned} -\Delta u + c^2 u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

   assuming spherical symmetry and that c, f, g are constants. That is solve

$$\begin{aligned} -(r^2 u'(r))' + c^2 r^2 u(r) &= r^2 f \quad \text{for } r \in (0,1), \\ u(1) = g, &\quad u(0) \text{ finite.} \end{aligned}$$

   Hint: Set $v(r) = r u(r)$.