Final Project 2

Consider the nonhomogeneous heat equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(\mu(x)\frac{\partial u(x,t)}{\partial x}\right) = f(x,t), \quad x \in (0,1), t > 0,$$
$$u(0,t) = u(1,t) = 0, \quad t > 0,$$
$$u(x,0) = g(x), \quad x \in (0,1),$$

where $\mu(x)$, $f(x,t)$, and $g(x)$ are some given functions.

1. Write down the weak form of the equation. What are the appropriate spaces of functions? Explain.
2. Let $\mathcal{T}_h$ be the partition
$$0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$$
and V_h be the space of continuous piecewise linear functions on the partition $\mathcal{T}_h$. Define the semidiscrete Galerkin solution $u_h(t) : [0,T] \to V_h$.
3. Define fully discrete Backward Euler method.
4. Solve the heat equation by backward Euler method.
5. To check your code, run the convergence tests with the exact solution $u(x,t) = t^3 \sin(\pi x)$, $(\mu \equiv 1 + x)$, by successive refining the mesh in space and time.

   Remember that now the error $e(x,t)$ between the exact and backward Euler solution is a function of space and time.

   The theory says that if the solution is smooth the error at the final time T should obey the estimate
$$\|e(T)\|_{L^2(0,1)} \leq C\left(h^2 + k\right),$$
   where h is the mesh size in space and k is the time step.

   To observe such convergence rates, first take the time step k very small and by successively refining the space mesh, observe the second order of convergence in space. After that by taking the mesh size h very small and by successively refining the time partition, observe the first order of convergence in time.
6. Repeat the problem for Crank-Nicolson method.