Final Project 1

1. Verify that the unbounded function

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$$

belongs to $H^1(B)$, where $B = B_1(0)$ is the unit ball in $\mathbb{R}^2$.

2. Let V_h denote the Lagrange piecewise linear subspace on a quasi-uniform triangulation. V_h possesses some fundamental properties, for example so called the Inverse Inequality

$$\|\chi\|_{H^1(D_h)} \le Ch^{-1}\|\chi\|_{L_2(D_h)} \text{ for } \chi \in V_h,$$

where D_h is the closure of the union of any set of triangles in quasi-uniform triangulation of size h. Prove this inequality.

Hint: First prove this inequality on a triangle K with vertices $(0,0)$, $(h,0)$, and $(0,h)$. This can be done by scaling to the reference triangle $\widehat{K}$ and using the fact that all norms on a fixed finite dimensional space are equivalent.

3. Give one dimensional counterexample to show that

$$\|I_h v\|_{L_2(0,h)} \le C\|v\|_{L_2(0,h)}$$

is not possible with a constant C independent of h and v .

4. Let $u(x,t)$ satisfies

$$\begin{aligned} u_t - \Delta u &= f(x,t) \quad \text{in } \Omega \times (0,T], \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T], \\ u(x,0) &= v(x), \quad x \in \Omega. \end{aligned}$$

By the energy argument show that there is a constant $C = C(T)$ such that

$$\int_0^T t\|u_t(t)\|^2_{L^2(\Omega)}dt \le C\left(\|v\|^2_{L^2(\Omega)} + \int_0^T \|f(t)\|^2_{L^2(\Omega)}dt\right) \tag{1}$$

$$\|\nabla u(T)\|^2_{L^2(\Omega)} \le CT^{-1}\left(\|v\|^2_{L^2(\Omega)} + \int_0^T \|f(t)\|^2_{L^2(\Omega)}dt\right) \tag{2}$$

and if $f \equiv 0$, then

$$\int_0^T \|\nabla u(t)\|^2_{L^2(\Omega)}dt \le C\|v\|^2_{L^2(\Omega)}. \tag{3}$$