Best approximation results for elliptic problems in L^∞ in 2D

Dmitriy Leykekhman

1 Introduction

Let Ω be a bounded convex domain in $\mathbb{R}^N$, $N = 2,3$. With $0 < h < \frac{1}{2}$ a parameter, let $\mathscr{T}$ denote a triangulation of Ω with mesh size h, i.e., $\mathscr{T} = \{\tau\}$ is a partition of Ω into cells (triangles or tetrahedrons) τ of diameter h_τ such that for $h = \max_\tau h_\tau$. Let V_h be the set of all functions in $H_0^1(\Omega)$ that are polynomials of degree $r \in \mathbb{N}$ on each τ, i.e. V_h is the usual space of conforming finite elements. Consider the Ritz projection $R_h \colon H_0^1(\Omega) \to V_h$ defined by

$$(\nabla R_h u, \nabla \chi)_\Omega = (\nabla u, \nabla \chi)_\Omega, \quad \forall \chi \in V_h. \tag{1.1}$$

We will be concerned with the following

Question: Given a Banach space B. Under which assumptions

$$\|R_h u\|_B \le C(h)\|u\|_B, \tag{1.2}$$

for some constant $C(h)$ independent of u?

If $C(h)$ is independent of h, we say that the Ritz projection is stable in B and if $C(h) = o(h^{-\varepsilon})$ for any $\varepsilon > 0$, for example if $C(h) \sim |\ln h|$, we say that the Ritz projection is almost stable in B.

Writing

$$u - R_h u = (u - \chi) - R_h(u - \chi)$$

for any $\chi \in V_h$ and using the triangle inequality, we can see that from (1.2) follows the best approximations property

$$\|u - R_h u\|_B \le (1 + C(h)) \inf_{\chi \in V_h} \|u - \chi\|_B. \tag{1.3}$$

Taking $\chi = 0$ in (1.3) and using the the triangle inequality we obtain (1.2).

In the definition of the Ritz projection (1.1), by taking $\chi = R_h u$ and using the Cauchy-Schwarz inequality, we obtain

$$\|\nabla R_h u\|^2_{L^2(\Omega)} = (\nabla R_h u, \nabla R_h u)_\Omega = (\nabla u, \nabla R_h u)_\Omega \le \|\nabla u\|_{L^2(\Omega)} \|\nabla R_h u\|_{L^2(\Omega)}.$$

Canceling and using the Poincare inequality

$$\|u\|_{L^2(\Omega)} \le C_P \|\nabla u\|_{L^2(\Omega)},$$

Dmitriy Leykekhman
Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA. e-mail: dmitriy.leykekhman@uconn.edu

we obtain

$$\|R_h u\|_{H^1(\Omega)} \le C\|u\|_{H^1(\Omega)},$$

with no restriction on the mesh!.

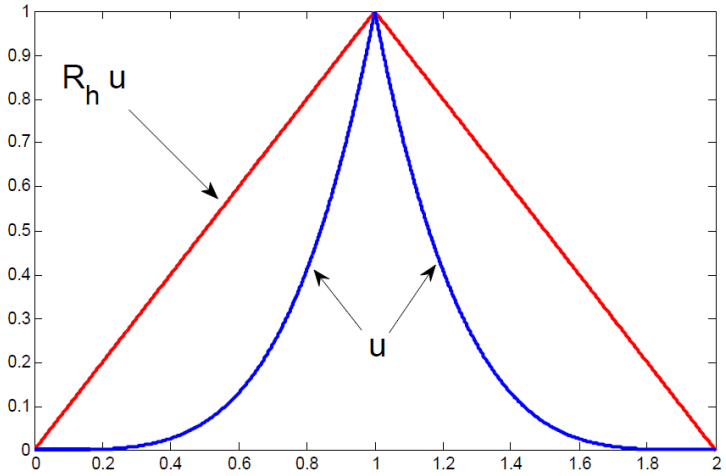
Surprisingly, the situation is very different for the L^2 norm [2]. Consider $\Omega = [0,2]$ and

$$u(x) = \begin{cases} x^p, & 0 \le x \le 1 \\ (2-x)^p, & 1 \le x \le 2, \end{cases}$$

for some $p > 1$. We partition Ω into just two elements $\tau_h^1 = [0,1]$ and $\tau_h^2 = [1,2]$ and V_h is the space of piecewise linear functions on $\tau_h^1 \cup \tau_h^2$.

Easy to see

$$R_h u(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2. \end{cases}$$



Direct computations show

$$\|R_h u\|_{L^2(\Omega)} = \sqrt{\frac{2}{3}} \quad \text{and} \quad \|u\|_{L^2(\Omega)} = \sqrt{\frac{2}{2p+1}},$$

hence

$$\|R_h u\|_{L^2(\Omega)} \le C\|u\|_{L^2(\Omega)}$$

can not hold as $p \to \infty$. This example also shows the Ritz projection is not stable in any L^q norm for $1 \le q < \infty$. However, it leaves the door open for L^∞ norm. Notice that using the Green's formula we can rewrite (1.1) as

$$(\nabla R_h u, \nabla \chi)_\Omega = (\nabla u, \nabla \chi)_\Omega = \sum_\tau (\nabla u, \nabla \chi)_\tau = -\sum_\tau (u, -\Delta \chi)_\tau + \sum_\tau (u, \frac{\partial \chi}{\partial n})_{\partial \tau}, \quad \forall \chi \in V_h.$$

Thus the Ritz projection is well defined even for $u \in C^0(\Omega)$.

Remark 1.1. In [4] was established an interesting result

$$\|u - R_h u\|_{L^2(\Omega)} \leq C \inf_{\chi \in V_h} \|u - \chi\|_{L^2(\Omega)} + osc(\mathscr{T}), \tag{1.4}$$

where

$$osc(\mathscr{T}) = \left(\sum_{\tau \in \mathscr{T}} h_\tau^4 \|f - f_\tau\|_{L^2(\tau)}^2 \right)^{\frac{1}{2}},$$

i.e. the stability of the Ritz projection in L^2 norm up to data oscillation.

One of the main goals of this notes is to provide a detailed proof in 2D of the following result.

Theorem 1.1 (Quasi-stability of Ritz projection in L^∞). *Let Ω be convex polyhedral domain and $R_h u$ is the Ritz projection of u on quasi-uniform meshes of size h. Then, there exists a constant C independent of h and u such that*

$$\|R_h u\|_{L^\infty(\Omega)} \leq C |\ln h| \|u\|_{L^\infty(\Omega)}.$$

However many **open problems** still remain:

1. Can the logarithmic term be removed in Theorem 1.1 for higher order elements (Such results exist for smooth domains)?
2. Can the result in Theorem 1.1 be extended for non-convex polyhedral domains (Such results exist in 2D [10])?
3. Can the result in Theorem 1.1 be extended to graded meshes (not quasi-uniform) (Such results exist for $W^{1\infty}$ norm)?
4. Can the result from [4], see Remark 1.1, namely (1.4), be generalized to L^p spaces?

1.1 Notation

We will use the standard notation for Sobolev spaces, thus for $1 \leq p < \infty$, we denote by $L^p(\Omega)$ the Banach space

$$L^p(\Omega) := \{v : \Omega \to \mathbb{R} \ : \ \int_\Omega |v(x)|^p \, dx < \infty\}$$

with norm $\|v\|_{L^p(\Omega)} := (\int_\Omega |v(x)|^p \, dx)^{\frac{1}{p}}$. In particular when $p = 2$, we obtain $L^2(\Omega)$ which is a Hilbert space with inner product $(u, v) = \int_\Omega u(x)v(x) \, dx$. When $p = \infty$, we obtain $L^\infty(\Omega)$, a Banach space with norm $\|v\|_{L^\infty(\Omega)} := \text{ess sup}_\Omega |v|$.

Moving forward, we introduce the Sobolev spaces $W^{k,p}(\Omega)$ with norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \text{ess sup}_\Omega |D^\alpha u| & p = \infty. \end{cases}$$

If $p = 2$, we write

$$H^k(\Omega) = W^{k,2}(\Omega), \quad k = 0, 1, \ldots$$

Notice that $H^k(\Omega)$ is a Hilbert space and $H^0(\Omega) = L^2(\Omega)$. We denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to $W^{k,p}(\Omega)$-norm. Finally, we denote the dual of $H_0^1(\Omega)$ by $H^{-1}(\Omega)$. It is easy to see that $L^2(\Omega) \subset H^{-1}(\Omega)$.

2 Stability of the Ritz projection in L^∞ norm in $\mathbb{R}^2$

2.1 Regularity and Green's function estimates for the Laplace equation

As a simple model of a second order elliptic partial differential equation we consider,

$$\begin{aligned} -\Delta u(x) &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{2.1}$$

with a right-hand side $f \in L^r(\Omega)$, $r > 1$ on a convex polygonal domain $\Omega \subset \mathbb{R}^2$.

First result is a standard H^2 regularity on convex domains [6].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. Then the following estimate holds,*

$$\|u\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

Later, we will use pointwise estimates for the Green's function and its derivatives. The proof of the next lemma for a general second order elliptic equation can be found in [7].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. Let $\Gamma(x,y)$ denotes the elliptic Green's function of the Laplace operator on the domain Ω. Then for $x \neq y$ the following estimates hold,*

$$|\Gamma(x,y)| \le C|\ln|x-y||, \tag{2.2a}$$

$$|\nabla_x \Gamma(x,y)| \le C|x-y|^{-1}. \tag{2.2b}$$

2.2 Discretization and preliminary results

Let $\mathscr{T}$ denote a quasi-uniform triangulation of Ω with a mesh size h, i.e., $\mathscr{T} = \{\tau\}$ is a partition of Ω into triangles τ of diameter h_τ such that for $h = \max_\tau h_\tau$,

$$\operatorname{diam}(\tau) \le h \le C|\tau|^{\frac{1}{2}}, \quad \forall \tau \in \mathscr{T}$$

hold. Let V_h be the set of all functions in $H_0^1(\Omega)$ that are polynomial of degree k, $k \ge 1$ on each τ, i.e. V_h is the usual space of Lagrangian finite elements of degree k. The elements from V_h satisfy the inverse inequality

$$\|\chi\|_{W^{n,q}(\Omega)} \le Ch^{m-n+2\left(\frac{1}{q}-\frac{1}{p}\right)}\|\chi\|_{W^{m,p}(\Omega)}, \quad m \le n, \quad p \le q, \quad \forall \chi \in V_h. \tag{2.3}$$

For the space V_h we will utilize the L^2-Projection $P_h\colon L^2(\Omega) \to V_h$ defined by

$$(P_h v, \chi)_\Omega = (v, \chi)_\Omega, \quad \forall \chi \in V_h, \tag{2.4}$$

the Ritz-Projection $R_h\colon H_0^1(\Omega) \to V_h$ defined by

$$(\nabla R_h v, \nabla \chi)_\Omega = (\nabla v, \nabla \chi)_\Omega, \quad \forall \chi \in V_h, \tag{2.5}$$

and the usual nodewise interpolant $I_h\colon C_0(\Omega) \to V_h$ with usual approximation properties (cf., e. g., [3, Theorem 3.1.5])

$$\|u - I_h u\|_{L^q(\Omega)} \le Ch^{2+2(\frac{1}{q}-\frac{1}{p})}\|u\|_{W^{2,p}(\Omega)}, \quad \text{for} \quad q \ge p > 1, \tag{2.6}$$

and the stability in $W^{s,\infty}(\Omega)$ norm

$$\|I_h u\|_{W^{s,\infty}(\Omega)} \le C\|u\|_{W^{s,\infty}(\Omega)}, \quad \text{for} \quad s = 0,1. \tag{2.7}$$

Moreover we introduce the discrete Laplace operator $\Delta_h \colon V_h \to V_h$ by

$$(-\Delta_h v_h, \chi)_\Omega = (\nabla v_h, \nabla \chi)_\Omega, \quad \forall \chi \in V_h. \tag{2.8}$$

2.3 Basic properties of the weight function

For a fixed point $x_0 \in \Omega$, we introduce a weight function

$$\sigma(x) = \sqrt{|x - x_0|^2 + K^2 h^2}, \tag{2.9}$$

where $K > 0$ is sufficiently large constant. One can easily check that σ satisfies the following properties, (cf., e.g., [3], Sec. 3.3),

$$\|\sigma^{-1}\|_{L^2(\Omega)} \le C|\ln h|^{\frac{1}{2}}, \tag{2.10a}$$

$$|\nabla^l \sigma| \le C\sigma^{1-l}, \quad l = 1,2,\ldots, \tag{2.10b}$$

$$\max_\tau \sigma \le C \min_\tau \sigma, \quad \forall \tau \in \mathscr{T}. \tag{2.10c}$$

Next we require an estimate in weighted norms for the L^2-Projection P_h.

Lemma 2.3. *There exist $\varepsilon > 0$ and a constant $C > 0$, such that for any positive function $\varphi \colon \Omega \to \mathbb{R}^+$ satisfying $h^l|\nabla^l \varphi| \le \varepsilon\varphi$ for all $l = 1,2,\ldots,k$ and any $v \in L^2(\Omega)$, we have*

$$\|\varphi P_h v\|_{L^2(\Omega)} \le C\|\varphi v\|_{L^2(\Omega)}.$$

Proof. We follow the proof in [5], appendix and their notation. Adding and subtracting $I_h(\varphi^2 P_h v)$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\varphi P_h v\|^2_{L^2(\Omega)} &= (P_h v, \varphi^2 P_h v)_\Omega = (P_h v, \varphi^2 P_h v - I_h(\varphi^2 P_h v))_\Omega + (v, I_h(\varphi^2 P_h v))_\Omega \\ &\le \|\varphi P_h v\|\|_{L^2(\Omega)} \|\varphi^{-1}\left(\varphi^2 P_h v - I_h(\varphi^2 P_h v)\right)\|_{L^2(\Omega)} + \|\varphi v\|_{L^2(\Omega)} \|\varphi^{-1} I_h(\varphi^2 P_h v)\|_{L^2(\Omega)}. \end{aligned} \tag{2.11}$$

Define

$$\underline{\varphi}_\tau = \min_\tau \varphi \quad \text{and} \quad \overline{\varphi}_\tau = \max_\tau \varphi$$

for each element τ. First we notice that $h|\nabla\varphi| \le \varepsilon\varphi$ implies

$$\overline{\varphi}_\tau \le C\underline{\varphi}_\tau, \tag{2.12}$$

for ε sufficiently small with the constant C independent of τ. Indeed, the inequality

$$\overline{\varphi}_\tau \le \underline{\varphi}_\tau + h\|\nabla\varphi\|_{L^\infty(\tau)},$$

and the assumption

$$h\|\nabla\varphi\|_{L^\infty(\tau)} \le \varepsilon\overline{\varphi}_\tau,$$

imply (2.12) for ε sufficiently small. Thus, using (2.12), the triangle inequality, and the property that for the nodal Lagrange interpolant holds

$$I_h(\varphi^2 P_h v) = I_h(I_h(\varphi^2)P_h v),$$

since

$$I_h(\varphi^2 P_h v)(x) = \sum_i \varphi^2(x_i)(P_h v)(x_i)\phi_i(x)$$

and

$$I_h(I_h(\varphi^2)P_h v) = I_h\left(\sum_i \varphi^2(x_i)(P_h v)(x_i)\phi_i(x)\right) = \sum_j \left(\sum_i \varphi^2(x_i)(P_h v)(x_i)\phi_i(x_j)\right)\phi_j(x) = \sum_j \varphi^2(x_j)(P_h v)(x_j)\phi_i(x)$$

Thus, we have

$$\begin{aligned}\|\varphi^{-1}\left(\varphi^2 P_h v - I_h(\varphi^2 P_h v)\right)\|_{L^2(\tau)} &\le \underline{\varphi}_\tau^{-1}\|\varphi^2 P_h v - I_h(\varphi^2 P_h v)\|_{L^2(\tau)} \\ &\le \underline{\varphi}_\tau^{-1}\|\left(\varphi^2 - I_h(\varphi^2)\right)P_h v)\|_{L^2(\tau)} + \underline{\varphi}_\tau^{-1}\|I_h(\varphi^2)P_h v - I_h(I_h(\varphi^2)P_h v)\|_{L^2(\tau)} \\ &:= J_1 + J_2.\end{aligned}$$

Using the approximation properties of I_h and the assumption that $h|\nabla\varphi| \le \varepsilon\varphi$, we obtain

$$J_1 \le C\underline{\varphi}_\tau^{-1} h\|\nabla(\varphi^2)\|_{L^\infty(\tau)}\|P_h v\|_{L^2(\tau)} \le C\varepsilon\|\varphi P_h v\|_{L^2(\tau)}. \tag{2.13}$$

To estimate J_2, we first notice that on each element $\partial^\gamma(P_h v) = 0$ and $\partial^\gamma(I_h(\varphi^2)) = 0$ for all multi-indices $\gamma \in \mathbb{N}^2$ with $|\gamma| = k+1$. Thus, using the approximations theory and the Leibniz's formula, we have

$$J_2 \le C\underline{\varphi}_\tau^{-1} h^{k+1}|I_h(\varphi^2)P_h v|_{H^{k+1}(\tau)} \le C\underline{\varphi}_\tau^{-1} h^{k+1}\sum_{l=1}^{k}|I_h(\varphi^2)|_{W^l_\infty(\tau)}|P_h v|_{H^{k+1-l}(\tau)}.$$

Using the stability of the interpolant in W^l_∞ and the inverse inequality, we obtain

$$J_2 \le C\underline{\varphi}_\tau^{-1}\|P_h v\|_{L^2(\tau)}\sum_{l=1}^{k} h^l|\varphi^2|_{W^l_\infty(\tau)}.$$

Finally, using the assumption $h^l|\nabla^l\varphi| \le \varepsilon\varphi$, we obtain

$$J_2 \le C\varepsilon\|\varphi P_h v\|_{L^2(\tau)}.$$

Thus combining the estimates for J_1 and J_2 and summing over the element, we have established that

$$\|\varphi^{-1}\left(\varphi^2 P_h v - I_h(\varphi^2 P_h v)\right)\|_{L^2(\Omega)} \le C\varepsilon\|\varphi P_h v\|_{L^2(\Omega)}. \tag{2.14}$$

The above estimate by the triangle inequality also implies that

$$\|\varphi^{-1} I_h(\varphi^2 P_h v)\|_{L^2(\Omega)} \le C(\varepsilon+1)\|\varphi P_h v\|_{L^2(\Omega)}. \tag{2.15}$$

Inserting the estimates (2.14) and (2.15) into (2.11), we obtain

$$\|\varphi P_h v\|^2_{L^2(\Omega)} \le C\varepsilon\|\varphi P_h v\|^2_{L^2(\Omega)} + C(\varepsilon+1)\|\varphi v\|_{L^2(\Omega)}\|\varphi P_h v\|_{L^2(\Omega)},$$

which for ε sufficiently small implies the lemma.

The next lemma provides error estimates for the projection and the interpolation errors in weighted norms. In what follows we will use the discrete L^2 norm $\|\cdot\|_{h,\Omega}$ for functions defined cellwise:

$$\|z\|_{h,\Omega}^2 = \sum_{\tau\in\mathscr{T}} \|z\|_{L^2(\tau)}^2.$$

Lemma 2.4. *Let $v \in C_0(\Omega)\cap H_0^1(\Omega)$ be cellwise in H^l, i.e. $v|_\tau \in H^l(\tau)$ for all $\tau\in\mathscr{T}$ and some $2\le l\le k+1$. Then the following estimates hold for any $\alpha\in\mathbb{R}$ and K large enough:*

$$\|\sigma^\alpha(v-I_hv)\|_{L^2(\Omega)} + h\|\sigma^\alpha\nabla(v-I_hv)\|_{L^2(\Omega)} + h^2\|\sigma^\alpha\nabla^2(v-I_hv)\|_{h,\Omega} \le Ch^l\|\sigma^\alpha\nabla^l v\|_{h,\Omega}, \tag{2.16a}$$

$$\|\sigma^\alpha(v-P_hv)\|_{L^2(\Omega)} + h\|\sigma^\alpha\nabla(v-P_hv)\|_{L^2(\Omega)} + h^2\|\sigma^\alpha\nabla^2(v-P_hv)\|_{h,\Omega} \le Ch^l\|\sigma^\alpha\nabla^l v\|_{h,\Omega}. \tag{2.16b}$$

Proof. The estimates (2.16a) are straightforward due to the local nature of the nodewise interpolant I_h and due to estimate (2.10c). The estimate for the first term in (2.16b) is standard for $\alpha=0$. For $\alpha\neq 0$ we consider $\varphi=\sigma^\alpha$. There holds for any $l=1,2,\dots,k$

$$|\nabla^l\varphi| = |\nabla^l(\sigma^\alpha)| \le C_\alpha\sigma^{\alpha-l} \le C_\alpha\sigma^{-l}\varphi \le C_\alpha K^{-l}h^{-l}\varphi,$$

where we used (2.10b) and the fact that $\sigma\ge Kh$. For K large enough this φ fulfills the condition of Lemma 2.3 and we get

$$\|\sigma^\alpha(P_hv-I_hv)\|_{L^2(\Omega)} = \|\sigma^\alpha P_h(v-I_hv)\|_{L^2(\Omega)} \le C\,\|\sigma^\alpha(v-I_hv)\|_{L^2(\Omega)}.$$

Therefore

$$\|\sigma^\alpha(v-P_hv)\|_\Omega \le C\|\sigma^\alpha(v-I_hv)\|_\Omega$$

and we get the estimate of the first term (2.16b) using (2.16a). The estimate for the second term in (2.16b) is obtained by the inverse inequality, which holds in weighted norms again due to (2.10c).

The next lemma is a superconvergence result in weighted norms.

Lemma 2.5. *Let $v_h\in V_h$. Then the following estimates hold for any $\alpha,\beta\in\mathbb{R}$ and K large enough:*

$$\|\sigma^\alpha(\mathrm{Id}-I_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} + h\|\sigma^\alpha\nabla(\mathrm{Id}-I_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} \le ch\|\sigma^{\alpha+\beta-1}v_h\|_{L^2(\Omega)}, \tag{2.17}$$

$$\|\sigma^\alpha(\mathrm{Id}-P_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} + h\|\sigma^\alpha\nabla(\mathrm{Id}-P_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} \le Ch\|\sigma^{\alpha+\beta-1}v_h\|_{L^2(\Omega)}. \tag{2.18}$$

Proof. We prove the estimate for the first term in (2.17). By the estimate (2.16a) from the previous lemma we obtain,

$$\|\sigma^\alpha(\mathrm{Id}-I_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} \le Ch^{k+1}\|\sigma^\alpha\nabla^{k+1}(\sigma^\beta v_h)\|_{h,\Omega}.$$

Using the multi-index notation with $\gamma\in\mathbb{N}^2$, we have

$$\|\sigma^\alpha\nabla^{k+1}(\sigma^\beta v_h)\|_{h,\Omega}^2 = \sum_{|\gamma|=k+1}\|\sigma^\alpha\partial^\gamma(\sigma^\beta v_h)\|_{h,\Omega}^2.$$

Using the Leibniz's formula for $\partial^\gamma(\sigma^\beta v_h)$ and noticing that $\partial^\gamma v_h=0$ on a cell τ for $|\gamma|=k+1$ (since v_h is a polynomial of degree at most k on τ), for each $\tau\in\mathscr{T}$ we obtain:

$$\partial^\gamma(\sigma^\beta v_h) = \sum_{|\delta|>0,\delta\le\gamma}\binom{\gamma}{\delta}\partial^\delta(\sigma^\beta)\partial^{\gamma-\delta}v_h,$$

where the inequality $\delta\le\gamma$ for multi-indices is understood as $\delta_i\le\gamma_i$ for $i=1,2$ and the binomial coefficient is defined as

$$\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1}\binom{\gamma_2}{\delta_2}.$$

By the property of σ, namely (2.10b), we have

$$|\partial^\delta(\sigma^\beta)| \le c\sigma^{\beta-|\delta|}.$$

Therefore we get for any $\tau \in \mathscr{T}$ and any γ with $|\gamma| = k+1$,

$$h^{k+1}\|\sigma^\alpha \partial^\gamma(\sigma^\beta v_h)\|_{L^2(\tau)} \le Ch^{k+1}\sum_{|\delta|>0,\delta\le\gamma}\|\sigma^\alpha\partial^\delta(\sigma^\beta)\partial^{\gamma-\delta}v_h\|_{L^2(\tau)} \le Ch^{k+1}\sum_{|\delta|>0,\delta\le\gamma}\|\sigma^{\alpha+\beta-|\delta|}\partial^{\gamma-\delta}v_h\|_{L^2(\tau)}.$$

Using property (2.10c) of σ and the inverse inequality $\|\partial^{\gamma-\delta}v_h\|_{L^2(\tau)} \le ch^{|\delta|-k-1}\|v_h\|_{L^2(\tau)}$, we obtain

$$h^{k+1}\sum_{|\delta|>0,\delta\le\gamma}\|\sigma^{\alpha+\beta-|\delta|}\partial^{\gamma-\delta}v_h\|_{L^2(\tau)} \le Ch\sum_{|\delta|>0,\delta\le\gamma}h^{|\delta|-1}\|\sigma^{\alpha+\beta-|\delta|}v_h\|_{L^2(\tau)}.$$

Finally using $h \le \sigma$, we obtain the desired estimate.

The estimates for the second term in (2.17) and estimates (2.18) follow by similar arguments as in the proof of Lemma 2.4.

2.4 Regularized Delta function

Let $x_0 \in \Omega$ be a fixed (but arbitrary) point. Associated to this point we introduce a smooth Delta function. We will follow [11, Lemma 2.2].

Lemma 2.6. *There exists a function $\tilde\delta = \tilde\delta^h_{x_0}$ supported in one cell τ_0 and satisfies*

$$(\chi, \tilde\delta)_{\tau_0} = \chi(x_0), \quad \forall \chi \in \mathbb{P}^k(\tau_0).$$

Furthermore, there are constants $c > 0$ and C_k independent of x_0 and h such that

$$\operatorname{dist}(\operatorname{supp}\tilde\delta, \partial\tau_0) \ge ch$$

and

$$\|\tilde\delta\|_{W^{s,p}(\Omega)} \le Ch^{-s-2(1-\frac{1}{p})}, \quad 1 \le p \le \infty, \quad s = 0,1,2. \tag{2.19}$$

Thus in particular $\|\tilde\delta\|_{L^1(\Omega)} \le C$, $\|\tilde\delta\|_{L^2(\Omega)} \le Ch^{-1}$, and $\|\tilde\delta\|_{L^\infty(\Omega)} \le Ch^{-2}$.

Proof. Let $\hat\tau$ be the open unit size reference triangle in R^2 and let $\omega \in C_0^\infty(B)$ for $B \subset\subset \Omega$ be fixed and nonnegative satisfying $\int_B \omega(x)dx = 1$. Set

$$(\hat v, \hat w)_\omega = \int_{\hat\tau} \hat v\hat w\omega \, d\hat x$$

and let $\hat\phi_1, \hat\phi_2, \dots, \hat\phi_M$ be an orthonormal basis for the finite dimensional space $\mathbb{P}^k$ under this inner-product. Then

$$\tilde\delta(\hat x) = \sum_{i=1}^M \phi_i(\hat x_0)\phi_i(\hat x)\omega(\hat x)$$

has the required properties on $\hat\tau$, with $h = 1$. A standard affine mapping argument using quasi-uniformity establishes the lemma.

Remark 2.1. The construction in the above lemma is valid for any dimension N. The support of the $\tilde{\delta}$ can be chosen arbitrary.

The next lemma shows how the weight σ compensates the singularity of the regularized Delta function $\tilde{\delta}$.

Lemma 2.7. *There holds*

$$\|\sigma\tilde{\delta}\|_{L^2(\Omega)} + h\|\sigma\nabla\tilde{\delta}\|_{L^2(\Omega)} + h^2\|\sigma\nabla^2\tilde{\delta}\|_{L^2(\Omega)} + \|\sigma P_h\tilde{\delta}\|_{L^2(\Omega)} \leq C. \tag{2.20}$$

Proof. Using that the support of $\tilde{\delta}_{x_0}$ is in a single element τ_0 and using (2.19), we have

$$\|\sigma\tilde{\delta}\|^2_{L^2(\Omega)} = \int_{\tau_0} |\sigma\tilde{\delta}|^2 dx \leq \|\tilde{\delta}\|^2_{L^\infty(\Omega)} \int_{\tau_0} (|x-x_0|^2 + K^2h^2) dx \leq Ch^{-4}h^2|\tau_0| \leq C.$$

Similarly we get

$$\|\sigma\nabla\tilde{\delta}\|^2_{L^2(\Omega)} = \int_{\tau_0} |\sigma\nabla\tilde{\delta}|^2 dx \leq \|\nabla\tilde{\delta}\|^2_{L^\infty(\Omega)} \int_{\tau_0} (|x-x_0|^2 + K^2h^2) dx \leq Ch^{-6}h^2|\tau_0| \leq Ch^{-2}$$

and thus $\|\sigma\nabla\tilde{\delta}\|_{L^2(\Omega)} \leq Ch^{-1}$.

Analogously we obtain the estimate $\|\sigma\nabla^2\tilde{\delta}\|_{L^2(\Omega)} \leq Ch^{-2}$. For the last term in (2.20) we obtain by (2.16b)

$$\|\sigma P_h\tilde{\delta}\|_{L^2(\Omega)} \leq \|\sigma\tilde{\delta}\|_{L^2(\Omega)} + \|\sigma(P_h - I)\tilde{\delta}\|_{L^2(\Omega)} \leq \|\sigma\tilde{\delta}\|_{L^2(\Omega)} + Ch^2\|\sigma\nabla^2\tilde{\delta}\|_{L^2(\Omega)} \leq C.$$

2.5 Three technical lemmas

Before giving a proof of the stability of the Ritz projection for $N = 2$ we need the following technical lemmas.The first lemma, which is a result from [1], allows us to "trade" negative powers of σ for stronger norms.

Lemma 2.8. *Let $v \in H^1_0(\Omega)$, then there exists a constant C independent of v and h such that*

$$\|\sigma^{-1}v\|_{L^2(\Omega)} \leq C|\ln h|\|\nabla v\|_{L^2(\Omega)}.$$

Proof. In the proof we use polar coordinates centered at x_0, i.e. $r = |x-x_0|$. Then $\sigma(x) = \sigma(r) = \sqrt{r^2 + K^2h^2}$, $\sigma'(r) = \frac{r}{\sigma}$ and as a result

$$\frac{d}{dr}\left(\ln(\sigma(r)) - \ln(\sigma(0))\right) = \frac{r}{\sigma^2}.$$

Let B_R be ball of radius R centered at x_0, with R so large that $\Omega \subset B_R$. We extend v by zero outside of Ω. Then, using polar coordinates and integrating by parts, we have

$$\begin{aligned}
\|\sigma^{-1}v\|^2_{L^2(\Omega)} = \int_0^{2\pi}\int_0^R \sigma(r)^{-2}v^2(r,\theta)r\,dr\,d\theta &= \int_0^{2\pi}\int_0^R \frac{d}{dr}\left(\ln(\sigma(r)) - \ln(\sigma(0))\right)v^2(r,\theta)\,dr\,d\theta \\
&= -2\int_0^{2\pi}\int_0^R \left(\ln(\sigma(r)) - \ln(\sigma(0))\right)\partial_r v(r,\theta)v(r,\theta)\,dr\,d\theta \\
&= -2\int_0^{2\pi}\int_0^R \frac{\ln(\sigma(r)) - \ln(\sigma(0))}{r}\partial_r v(r,\theta)v(r,\theta)r\,dr\,d\theta.
\end{aligned}$$

Using that

$$\frac{\ln(\sigma(r)) - \ln(\sigma(0))}{r} \leq \frac{C}{\sigma(r)}|\ln h|, \tag{2.21}$$

we obtain

$$\|\sigma^{-1}v\|^2_{L^2(\Omega)} \le C|\ln h| \int_0^{2\pi}\int_0^R |\nabla v||\sigma^{-1}v| r\, dr\, d\theta \le C|\ln h|\|\nabla v\|_{L^2(\Omega)}\|\sigma^{-1}v\|_{L^2(\Omega)},$$

from which the lemma follows. The inequality (2.21) follows from considering two cases: $r \le h$ and $r > h$. In the first case by the mean value theorem for some $0 < s < r \le h$

$$\frac{\ln(\sigma(r)) - \ln(\sigma(0))}{r} \le \frac{1}{\sigma(s)} \le \frac{C}{h} \le \frac{1}{\sigma(r)}.$$

In the second case

$$\frac{\ln(\sigma(r)) - \ln(\sigma(0))}{r} \le 2 \max_{0\le r\le R} \frac{|\ln(\sigma(r))|}{r} \le C\frac{|\ln h|}{\sigma(r)}.$$

Next result is Lemma 2.4 in [8].

Lemma 2.9. *There exists a constant $C > 0$ independent on h, such that for any $v_h \in V_h$,*

$$\sum_{\tau\in\mathscr{T}} \left(\|\Delta v_h\|_{L^1(\tau)} + \|[\![\partial_n v_h]\!]\|_{L^1(\partial\tau)} \right) \le C|\ln h|^{\frac{3}{2}} \left(\|\sigma\Delta_h v_h\|_{L^2(\Omega)} + \|\nabla v_h\|_{L^2(\Omega)} \right).$$

Proof. We define v as the solution of

$$\begin{aligned} -\Delta v &= -\Delta_h v_h \quad \text{in } \Omega, \\ v &= 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{2.22}$$

Thus, by construction v_h is the Ritz projection of v, i.e., $R_h v = v_h$ and $v \in H^2(\Omega)\cap H^1_0(\Omega)$ with

$$\|v\|_{H^2(\Omega)} \le C\|\Delta v\|_{L^2(\Omega)} = C\|\Delta_h v_h\|_{L^2(\Omega)}.$$

Using the fact that jumps of v are zero, the trace inequality and the inverse inequality we have

$$\begin{aligned} \sum_\tau \|[\![\partial_n v_h]\!]\|_{L^1(\partial\tau)} &= \sum_\tau \|[\![\partial_n [v_h - v]]\!]\|_{L^1(\partial\tau)} \\ &\le C\sum_\tau \left(h^{-1}\|\nabla(v-v_h)\|_{L^1(\tau)} + \|\nabla^2(v-v_h)\|_{L^1(\tau)} \right) \\ &\le C\sum_\tau \left(h^{-1}\|\nabla(v-v_h)\|_{L^1(\tau)} + \|\nabla^2(v-I_h v)\|_{L^1(\tau)} + \|\nabla^2(v_h - I_h v)\|_{L^1(\tau)} \right) \\ &\le C\sum_\tau \left(h^{-1}\|\nabla(v-v_h)\|_{L^1(\tau)} + \|\nabla^2(v-I_h v)\|_{L^1(\tau)} + h^{-1}\|\nabla(v_h - I_h v)\|_{L^1(\tau)} \right) \\ &\le C\sum_\tau \left(h^{-1}\|\nabla(v-v_h)\|_{L^1(\tau)} + h^{-1}\|\nabla(v-I_h v)\|_{L^1(\tau)} + \|\nabla^2(v-I_h v)\|_{L^1(\tau)} \right). \end{aligned} \tag{2.23}$$

Similarly, we obtain for the Laplacian of v_h

$$\begin{aligned} \sum_\tau \|\Delta v_h\|_{L^1(\tau)} &\le \sum_\tau \left(\|\Delta v\|_{L^1(\tau)} + \|\Delta(v_h - v)\|_{L^1(\tau)} \right) \\ &\le C\sum_\tau \left(\|\nabla^2 v\|_{L^1(\tau)} + h^{-1}\|\nabla(v-v_h)\|_{L^1(\tau)} + h^{-1}\|\nabla(v-I_h v)\|_{L^1(\tau)} + \|\nabla^2(v-I_h v)\|_{L^1(\tau)} \right). \end{aligned} \tag{2.24}$$

Combining (2.23) and (2.24), using the properties of σ, we obtain

$$\begin{aligned}
&\sum_\tau \left(\|\Delta v_h\|_{L^1(\tau)} + \|[\![\partial_n [v_h - v]]\!]\|_{L^1(\partial\tau)} \right) \\
&\quad \le C \sum_\tau \left(\|\nabla^2 v\|_{L^1(\tau)} + h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + h^{-1}\|\nabla(v - I_h v)\|_{L^1(\tau)} + \|\nabla^2(v - I_h v)\|_{L^1(\tau)} \right) \\
&\quad \le C\|\sigma^{-1}\|_\Omega \left(\|\sigma\nabla^2 v\|_\Omega + h^{-1}\|\sigma\nabla(v - v_h)\|_\Omega + h^{-1}\|\sigma\nabla(v - I_h v)\|_\Omega + \|\sigma\nabla^2(v - I_h v)\|_{h,\Omega} \right) \\
&\quad \le C|\ln h|^{\frac{1}{2}} \left(h^{-1}\|\sigma\nabla(v_h - v)\|_\Omega + \|\sigma\nabla^2 v\|_\Omega \right).
\end{aligned}$$

To conclude the proof, we need to establish that

$$h^{-1}\|\sigma\nabla(v_h - v)\|_{L^2(\Omega)} + \|\sigma\nabla^2 v\|_{L^2(\Omega)} \le C|\ln h| \left(\|\sigma\Delta_h v_h\|_{L^2(\Omega)} + \|\nabla v_h\|_{L^2(\Omega)} \right). \tag{2.25}$$

We start with $\|\sigma\nabla^2 v\|_{L^2(\Omega)}$. Since

$$\partial_{ij}^2(\sigma v) = \sigma\partial_{ij}^2 v + (\partial_i\sigma\partial_j v + \partial_j\sigma\partial_i v) + \partial_{ij}^2\sigma v,$$

and by properties of σ, $|\nabla(\sigma)| \le C$ and $|\nabla^2(\sigma)| \le C\sigma^{-1}$, we obtain

$$\|\sigma\nabla^2 v\|_\Omega \le \|\nabla^2(\sigma v)\|_\Omega + C\|\nabla v\|_\Omega + C\|\sigma^{-1} v\|_\Omega. \tag{2.26}$$

Using Lemma 2.8 we have

$$\|\sigma^{-1} v\|_\Omega \le C|\ln h|\|\nabla v\|_\Omega. \tag{2.27}$$

By the global $H^2(\Omega)$ regularity, we have

$$\|\nabla^2(\sigma v)\|_\Omega \le C\|\Delta(\sigma v)\|_\Omega.$$

For $\Delta(\sigma v)$ we obtain

$$\Delta(\sigma v) = \Delta\sigma v + 2\nabla\sigma\cdot\nabla v + \sigma\Delta v$$

and thus by the properties of σ we get

$$\|\nabla^2(\sigma v)\|_\Omega \le C\|\sigma^{-1} v\|_\Omega + C\|\nabla v\|_\Omega + C\|\sigma\Delta v\|_\Omega$$

and by using (2.27) we obtain

$$\|\nabla^2(\sigma v)\|_\Omega \le C|\ln h| \left(\|\nabla v\|_\Omega + \|\sigma\Delta v\|_\Omega \right) = C|\ln h| \left(\|\nabla v\|_\Omega + \|\sigma\Delta_h v_h\|_\Omega \right), \tag{2.28}$$

where we applied the definition of v (2.22) in the last step. It remains to estimate $\|\nabla v\|_\Omega$. By the triangle inequality and the properties of the Ritz projection, H^2 regularity and that $h \le \sigma$, we obtain

$$\begin{aligned}
\|\nabla v\|_{L^2(\Omega)} &\le \|\nabla v_h\|_{L^2(\Omega)} + \|\nabla(v - v_h)\|_{L^2(\Omega)} \le \|\nabla v_h\|_{L^2(\Omega)} + \|\nabla(v - I_h v)\|_{L^2(\Omega)} \\
&\le \|\nabla v_h\|_{L^2(\Omega)} + Ch\|\nabla^2 v\|_{L^2(\Omega)} \le \|\nabla v_h\|_{L^2(\Omega)} + Ch\|\Delta v\|_{L^2(\Omega)} = \|\nabla v_h\|_{L^2(\Omega)} + C\|\sigma\Delta_h v_h\|_{L^2(\Omega)}.
\end{aligned}$$

Which establishes

$$\|\sigma\nabla^2 v\|_\Omega \le C|\ln h| \left(\|\nabla v_h\|_\Omega + \|\sigma\Delta_h v_h\|_\Omega \right). \tag{2.29}$$

Next we will treat $h^{-1}\|\sigma\nabla(v_h - v)\|_\Omega$. Denote $e = v - v_h$, then

$$\|\sigma\nabla e\|_\Omega^2 = (\nabla e, \sigma^2\nabla e)_\Omega = (\nabla e, \nabla(\sigma^2 e))_\Omega - 2(\nabla e, \sigma\nabla\sigma e))_\Omega = I_1 + I_2.$$

Using the Cauchy-Schwartz inequality and the Young's inequality, approximation theory in L^2 norm, H^2 regularity and that $h \le \sigma$, we obtain

$$I_2 \le C\|e\|^2_{L^2(\Omega)} + \frac{1}{4}\|\sigma\nabla e\|^2_{L^2(\Omega)} \le Ch^4\|\nabla^2 v\|^2_{L^2(\Omega)} + \frac{1}{4}\|\sigma\nabla e\|^2_{L^2(\Omega)} \le Ch^2\|\sigma\Delta_h v_h\|^2_{L^2(\Omega)} + \frac{1}{4}\|\sigma\nabla e\|^2_{L^2(\Omega)}.$$

Using the Galerkin orthogonality we have

$$I_1 = (\nabla e, \nabla(\sigma^2 e - I_h(\sigma^2 e))_\Omega = \sum_\tau (\nabla e, \nabla(\sigma^2 e - I_h(\sigma^2 e))_\tau.$$

On each element τ using the Cauchy-Schwartz inequality and (2.16a), we have

$$\begin{aligned}
(\nabla e, \nabla(\sigma^2 e - I_h(\sigma^2 e))_\tau &= (\sigma\nabla e, \sigma^{-1}\nabla(\sigma^2 e - I_h(\sigma^2 e))_\tau \\
&\le \|\sigma\nabla e\|_{L^2(\tau)}\|\sigma^{-1}\nabla(\sigma^2 e - I_h(\sigma^2 e)\|_{L^2(\tau)} \\
&\le \|\sigma\nabla e\|_{L^2(\tau)} Ch\|\sigma^{-1}\nabla^2(\sigma^2 e)\|_{L^2(\tau)} \\
&\le \|\sigma\nabla e\|_{L^2(\tau)} Ch\left(\|\sigma\nabla^2 e\|_{L^2(\tau)} + \|\nabla e\|_{L^2(\tau)} + \|\sigma^{-1} e\|_{L^2(\tau)}\right) \\
&\le \frac{1}{4}\|\sigma\nabla e\|^2_{L^2(\tau)} + Ch^2\|\sigma\nabla^2 e\|^2_{L^2(\tau)} + \frac{Ch^2}{K}\|\sigma\nabla e\|^2_{L^2(\tau)} + \frac{C}{K}\|e\|^2_{L^2(\tau)} \\
&\le \left(\frac{1}{4} + \frac{C}{K}\right)\|\sigma\nabla e\|^2_{L^2(\tau)} + Ch^2\|\sigma\nabla^2 e\|^2_{L^2(\tau)} + Ch^4\|\nabla^2 e\|^2_{L^2(\tau)} \\
&\le \left(\frac{1}{4} + \frac{C}{K}\right)\|\sigma\nabla e\|^2_{L^2(\tau)} + Ch^2\|\sigma\nabla^2 e\|^2_{L^2(\tau)}.
\end{aligned}$$

Summing over the elements we obtain

$$I_1 \le \left(\frac{1}{4} + \frac{C}{K}\right)\|\sigma\nabla e\|^2_{L^2(\Omega)} + Ch^2\|\sigma\nabla^2 e\|^2_{L^2(\Omega)} \le \left(\frac{1}{4} + \frac{C}{K}\right)\|\sigma\nabla e\|^2_{L^2(\tau)} + Ch^2|\ln h|^2\left(\|\nabla v_h\|^2_\Omega + \|\sigma\Delta_h v_h\|^2_\Omega\right),$$

where in the last step we used (2.29). Thus, combining the estimates for I_1 and I_2 and choosing K large enough we obtain

$$\|\sigma\nabla e\|^2_\Omega \le Ch^2|\ln h|^2\left(\|\nabla v_h\|^2_\Omega + \|\sigma\Delta_h v_h\|^2_\Omega\right),$$

which together with (2.29) establishes (2.25).

Remark 2.2. The power of the logarithm is not optimal. Using a more sophisticated argument, the power of $|\ln h|$ can be reduced to $\frac{1}{2}$.

We will need the following discrete version of the Sobolev type inequality.

Lemma 2.10. *Let $B_d(x_0)$ denote a ball of radius d centered at x_0. Then for any $\chi \in V_h$ and $h \le d$, there exists a constant C independent of h such that*

$$\chi(x_0) \le C|\ln h|^{\frac{1}{2}}\left(\|\nabla\chi\|_{L^2(B_d(x_0))} + d^{-1}\|\chi\|_{L^2(B_d(x_0))}\right).$$

Proof. The proof goes along the lines of [9, Lemma 1.1]. Let $\omega(x)$ be a smooth cut-off function with properties

$$\omega(x) \equiv 1, \quad x \in B_{d/2}(x_0) \tag{2.30a}$$

$$\omega(x) \equiv 0, \quad x \in \Omega\backslash B_d(x_0) \tag{2.30b}$$

$$|\nabla\omega| \le Cd^{-1}. \tag{2.30c}$$

Furthermore, let $\Gamma_{x_0}(x)$ denote the Green's function for the Laplacian on $B_d(x_0)$ with homogeneous Dirichlet boundary conditions. Then

$$\begin{aligned}\chi(x_0) = (\omega\chi)(x_0) &= \int_{B_d(x_0)} \nabla_x \Gamma_{x_0}(x)\cdot\nabla(\omega\chi)(x)dx \\ &\le \int_{B_h(x_0)} \nabla_x \Gamma_{x_0}(x)\cdot\nabla\chi(x)dx + \int_{B_d(x_0)\setminus B_h(x_0)} \nabla_x \Gamma_{x_0}(x)\cdot\nabla(\omega\chi)(x)dx \\ &:= J_1 + J_2.\end{aligned}$$

Using the estimate $|\nabla_x \Gamma_{x_0}(x)| \le \frac{C}{|x-x_0|}$ and the inverse inequality we have

$$J_1 \le C\|\nabla\chi\|_{L^\infty(B_h(x_0))}\int_{B_h(x_0)}\frac{dx}{|x-x_0|} \le Ch^{-1}\|\nabla\chi\|_{L^2(B_h(x_0))}h \le C\|\nabla\chi\|_{L^2(B_d(x_0))}.$$

Similarly we have

$$\begin{aligned}J_2 &\le \|\nabla\Gamma_{x_0}\|_{L^2(B_d(x_0)\setminus B_h(x_0))}\left(|\omega|\|\nabla\chi\|_{L^2(B_d(x_0))} + |\nabla\omega|\|\chi\|_{L^2(B_d(x_0))}\right) \\ &\le C|\ln h|^{\frac{1}{2}}\left(\|\nabla\chi\|_{L^2(B_d(x_0))} + d^{-1}\|\chi\|_{L^2(B_d(x_0))}\right).\end{aligned}$$

This completes the proof.

Corollary 2.1 (Discrete Sobolev). *Choosing d so large that $\Omega \subset B_{d/2}$ and taking the supremum over x_0, we immediately obtain*

$$\|\chi\|_{L^\infty(\Omega)} \le C|\ln h|^{\frac{1}{2}}\|\nabla\chi\|_{L^2(\Omega)}, \quad \forall\chi \in V_h.$$

2.6 Proof of the stability of the Ritz projection in L^∞ norm

Finally we establish one of the main results in this section the stability (modulo logarithm) of the Ritz projection in L^∞-norm.

Theorem 2.1. *There exists a constant C independent of h such that for the solution u of* (2.1) *and its Ritz projection $u_h = R_h u \in V_h$ holds*

$$\|u_h\|_{L^\infty(\Omega)} \le C|\ln h|^2\|u\|_{L^\infty(\Omega)}.$$

Proof. Let $x_0 \in \bar{\tau}_0$ such that $\|u_h\|_{L^\infty(\Omega)} = |u_h(x_0)|$. For such x_0 define a regularized Green's function g, that satisfies

$$\begin{aligned}-\Delta g(x) &= \tilde{\delta}(x), \quad x \in \Omega, \\ g(x) &= 0, \qquad x \in \partial\Omega,\end{aligned} \tag{2.31}$$

where $\tilde{\delta}(x) := \tilde{\delta}_{x_0}(x)$ is the regularized Delta function from (2.19). We define $g_h = R_h g \in V_h$, i.e.

$$(\nabla g_h, \nabla\chi) = (\tilde{\delta}, \chi) \quad \forall\chi \in V_h. \tag{2.32}$$

Alternatively we can rewrite equation (2.32) using the discrete Laplacian and the L^2-projection as

$$-\Delta_h g_h = P_h\tilde{\delta}. \tag{2.33}$$

Then using the Galerkin orthogonality, integration by parts, and Lemma 2.9, we obtain

$$\begin{aligned}u_h(x_0) = (\nabla u_h, \nabla g_h) = (\nabla u, \nabla g_h) &= \sum_{\tau\in\mathscr{T}}\left((u, [\![\partial_n g_h]\!])_{\partial\tau} + (u, -\Delta g_h)_\tau\right) \\ \le \|u\|_{L^\infty(\Omega)}\sum_{\tau\in\mathscr{T}}\left(\|[\![\partial_n g_h]\!]\|_{L^1(\partial\tau)} + \|\Delta g_h\|_{L^1(\tau)}\right) &\le C|\ln h|^{\frac{3}{2}}\|u\|_{L^\infty(\Omega)}\left(\|\sigma\Delta_h g_h\|_{L^2(\Omega)} + \|\nabla g_h\|_{L^2(\Omega)}\right).\end{aligned} \tag{2.34}$$

From (2.33) and then using Lemma 2.20 we easily obtain

$$\|\sigma \Delta_h g_h\|_{L^2(\Omega)} = \|\sigma P_h \tilde{\delta}\|_{L^2(\Omega)} \le C. \tag{2.35}$$

To estimate $\|\nabla g_h\|_\Omega$, we set $\chi = g_h$ in (2.32) and using the Hölder inequality and the discrete Sobolev embedding result from Lemma 2.10, we obtain

$$\|\nabla g_h\|^2_{L^2(\Omega)} = (\nabla g_h, \nabla g_h) = (\tilde{\delta}, g_h) \le \|\tilde{\delta}\|_{L^1(\Omega)} \|g_h\|_{L^\infty(\Omega)} \le C|\ln h|^{\frac{1}{2}} \|\nabla g_h\|_{L^2(\Omega)}.$$

Canceling $\|\nabla g_h\|_{L^2(\Omega)}$ we obtain

$$\|\nabla g_h\|^2_{L^2(\Omega)} \le C|\ln h|^{\frac{1}{2}}$$

and hence the theorem.

References

1. Thomas Apel, Olaf Benedix, Dieter Sirch, and Boris Vexler. A priori mesh grading for an elliptic problem with Dirac right-hand side. *SIAM J. Numer. Anal.*, 49(3):992–1005, 2011.
2. I. Babuˇska and J. Osborn. Analysis of finite element methods for second order boundary value problems using mesh dependent norms. *Numer. Math.*, 34(1):41–62, 1980.
3. Philippe G. Ciarlet. *The finite element method for elliptic problems*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. Studies in Mathematics and its Applications, Vol. 4.
4. Alan Demlow and Rob Stevenson. Convergence and quasi-optimality of an adaptive finite element method for controlling L_2 errors. *Numer. Math.*, 117(2):185–218, 2011.
5. Kenneth Eriksson and Claes Johnson. Adaptive finite element methods for parabolic problems. II. Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$. *SIAM J. Numer. Anal.*, 32(3):706–740, 1995.
6. Pierre Grisvard. *Elliptic problems in nonsmooth domains*, volume 69 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683], With a foreword by Susanne C. Brenner.
7. Michael Grüter and Kjell-Ove Widman. The Green function for uniformly elliptic equations. *Manuscripta Math.*, 37(3):303–342, 1982.
8. Rolf Rannacher. L^∞-stability estimates and asymptotic error expansion for parabolic finite element equations. In *Extrapolation and defect correction (1990)*, volume 228 of *Bonner Math. Schriften*, pages 74–94. Univ. Bonn, Bonn, 1991.
9. A. H. Schatz, V. C. Thomée, and L. B. Wahlbin. Maximum norm stability and error estimates in parabolic finite element equations. *Comm. Pure Appl. Math.*, 33(3):265–304, 1980.
10. Alfred H. Schatz. A weak discrete maximum principle and stability of the finite element method in L_∞ on plane polygonal domains. I. *Math. Comp.*, 34(149):77–91, 1980.
11. V. Thomée and L. B. Wahlbin. Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions. *Numer. Math.*, 87(2):373–389, 2000.