Problem Set 3

1. Verify that the unbounded function

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$$

belongs to $H^1(B)$, where $B = B_1(0)$ is the unit ball in $\mathbb{R}^2$.

2. Determine the basis functions for piecewise linear functions on an arbitrary triangle with corner coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3).

3. Let V_h denote the Lagrange piecewise linear subspace on a quasi-uniform triangulation. V_h possesses some fundamental properties, for example so called the Inverse Inequality

$$\|\chi\|_{H^1(D_h)} \leq Ch^{-1}\|\chi\|_{L_2(D_h)} \text{ for } \chi \in V_h,$$

where D_h is the closure of the union of any set of triangles in quasi-uniform triangulation of size h. Prove this inequality.

Hint: First prove this inequality on a triangle K with vertices $(0,0)$, $(h,0)$, and $(0,h)$. This can be done by scaling to the reference triangle $\widehat{K}$ and using the fact that all norms on a fixed finite dimensional space are equivalent.

4. The error estimates shown in class, namely

$$\|f - P_h f\|_{L^2(a,b)} \leq Ch^2\|f''\|_{L^2(a,b)}$$

predicts second order convergence for the L^2-projection computed with piecewise linear elements. In general if the error $e_h = \|f - P_h f\|_{L^2(a,b)}$ is of the form

$$e_h = C(f)h^\alpha + o(h^\alpha)$$

for some constant $C(f)$ that may depend on f but not on h, then it is possible to estimate the convergence rate α by comparing e_h and $e_{h/2}$. Since by assumption $e_{h/2} = C(f)(h/2)^\alpha + o(h^\alpha)$, we have

$$\frac{e_{h/2}}{e_h} = 2^\alpha + o(1),$$

we can estimate α from

$$\alpha \approx \log_2 \frac{e_{h/2}}{e_h}. \tag{1}$$

Using a matlab function `L2error1D` from the previous assignment that computes the L^2 error between a continuous function f and a piecewise linear function $g \in V_h$ using 2 point Gauss quadrature. Estimate the convergence rate α both for piecewise linear and piecewise quadratic elements for the functions $f(x) = sin(x)$ and $f(x) = x\ln x$ on the interval $(0,1)$ for $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{1024}$.

Due at the beginning of class on Thursday, October 5.