Problem Set 2

1. Give a counterexample that shows that

$$\|\pi_h v\|_{L_2(0,h)} \leq C\|v\|_{L_2(0,h)}$$

is not possible with a constant C independent of h and v .

2. In the first assignment you were asked to find the exact solution of the following problem:

$$\begin{aligned} -u''(x) &= \frac{1}{x}, \quad x \in (0,1), \\ u(0) = u(1) &= 0. \end{aligned}$$

Check that $u \in H_0^1$ but $u \notin H^2$, then show that $f = \frac{1}{x} \notin L_2(0,1)$ but $f \in H^{-1}(0,1)$. That will show that $H^{-1} \not\subset L_2(0,1)$.

Hint: Define linear functional $f(v) = (f, v)\ \forall v \in H_0^1(0,1)$ and prove the inequality

$$|(f,v)| \leq C\|v'\|_{L_2(0,1)}, \quad \forall v \in H_0^1(0,1).$$

3. In the previous assignment you had to find the local basis functions on the interval $[x_i, x_{i+1}]$, for the piecewise quadratic Lagrange elements. Construct a finite-dimensional subspace V_h of $H^1(0,1)$ consisting of functions which are quadratic on each subinterval I_j of a partition of $I = (0,1)$. Find the corresponding basis functions and define a quadratic interpolant $\pi_h^2 : C^0(0,1) \to V_h$. Show that for $u \in H^3(I_i)$

$$\|u - \pi_h^2 u\|_{L_2(x_i,x_{i+1})} \leq C h_i^3 \|u'''\|_{L_2(x_i,x_{i+1})}.$$

4. Modify `L2Projector1D` to use 2 point Gauss quadrature and **quadratic** elements instead and compute the L^2-projection $P_h f$ of the following functions f.

   (a) $f(x) = 2x^2$.

   (b) $f(x) = x^3(x-1)(1-2x)$.

   (c) $\arctan((x-0.5)/\epsilon)$, with $\epsilon = 0.1$ and 0.01.

   Plot the functions and the L^2 projections using a uniform mesh $\mathfrak{I}$ of the interval $I = [0,1]$ with $n = 5$ and 100 subintervals.

5. Write a matlab function `L2error1DP2` that computes the L^2 error between a continuous function f and a piecewise quadratic function $g \in V_h$ using 2 point Gauss quadrature. (The input should be the continuous function f and a n-vector g. The output should be the L^2-error.) Test your function with the function (b) from the previous problem.

Due at the beginning of class on Thursday, September 21.