Assignment 4 (T)

1. Find the exact solution of the following problem:

$$-u''(x) = \frac{1}{x}, \quad x \in (0,1),$$
$$u(0) = u(1) = 0.$$

   Show that $u \in V$, where

$$V = \{v \in L_2(0,1) : \int_0^1 v'(x)^2 dx < \infty,\ v(0) = v(1) = 0\}.$$

   Is u a classical solution? Explain.

2. Verify that the unbounded function

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$$

   belongs to $H^1(B)$, where $B = B_1(0)$ is the unit ball in $\mathbb{R}^2$.

3. Let V_h denote the Lagrange piecewise linear subspace on a quasi-uniform triangulation. V_h possesses some fundamental properties, for example so called the Inverse Inequality

$$\|\chi\|_{H^1(D_h)} \le Ch^{-1}\|\chi\|_{L_2(D_h)} \text{ for } \chi \in V_h,$$

   where D_h is the closure of the union of any set of triangles in quasi-uniform triangulation of size h. Prove this inequality.

   Hint: First prove this inequality on a triangle K with vertices $(0,0)$, $(h,0)$, and $(0,h)$. This can be done by scaling to the reference triangle $\widehat{K}$ and using the fact that all norms on a fixed finite dimensional space are equivalent.

4. Consider the Burger's equation

$$-\mu u''(x) + u(x)u'(x) = f(x), \quad x \in (0,1),$$
$$u(0) = u(1) = 0.$$

   (a) Is this equation linear or nonlinear?

   (b) What would be the linearized version of the equation?

   (c) Write down the weak form of the equation. What are the appropriate spaces of function? Explain.

   (d) Let $\mathcal{T}_h$ be the partition

$$0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$$

   and V_h be the space of continuous piecewise linear functions on the partition $\mathcal{T}_h$. Define the Galerkin solution $u_h \in V_h$.

   (e) Is the resulting finite dimensional system of algebraic equations linear or nonlinear? How would you solve it?