Assignment 3 (T)

1. (Problem 3.5 from the textbook)

   Determine the stiffness matrix corresponding to the Poisson equation

   $$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

   when Ω is a square with side 1 and we use the bilinear element with $h = 1/4$.

2. (Problem 4.8 from the textbook)

   Let V_h be a finite element space on a triangulation T_h of the domain $\Omega \subset \mathbb{R}^N$ satisfying

   $$\|u - \pi_h u\|_{L^2(\Omega)} \leq Ch^{r+1}|u|_{H^{r+1}(\Omega)}.$$

   Given $u \in L^2(\Omega)$ and $u_h \in V_h$ be the L^2-projection of u onto V_h given by

   $$(u_h, v) = (u, v) \quad \text{for any } v \in V_h,$$

   where $(,)$ denotes the usual L^2 inner-product. Prove the error estimate

   $$\|u - u_h\|_{L^2(\Omega)} \leq \min_{v \in V_h} \|u - v\|_{L^2(\Omega)} \leq Ch^{r+1}|u|_{H^{r+1}(\Omega)}$$

   and that

   $$\|u_h\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}.$$

3. (Problem 5.1 from the textbook)

   Consider the elasticity problem

   $$\begin{aligned} \sigma_{ij} &= \lambda div(u)\delta_{ij} + \mu\varepsilon_{ij}(u), \\ -\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} &= f_i \quad \text{in } \Omega \quad i = 1, 2, 3, \\ u &= 0 \quad \text{on } \Gamma_2, \\ \sum_{j=1}^{3} \sigma_{ij} n_j &= g_i \quad \text{on } \Gamma_1 \quad i = 1, 2, 3, \end{aligned}$$

   where n is the other normal to Γ, $\partial\Omega = \Gamma_1 \cap \Gamma_2$, in a three-dimensional domain $\widehat{\Omega} = \Omega \times (-\varepsilon, \varepsilon)$ with $\Omega \subset \mathbb{R}^2$, ε small and $F_3 = f_3 = 0$. This corresponds to a thin elastic plate with middle surface Ω subject to in-plate loads only (no transversal loads).

   Assuming plane stresses (i.e. $\sigma_{i3} = 0$, $i = 1, 2, 3$) prove that in this case the above problem reduces to:

   $$\begin{aligned} \sigma_{ij} &= \overline{\lambda}(\varepsilon_{11}(u) + \varepsilon_{22}(u))\delta_{ij} + \overline{\mu}\varepsilon_{ij}(u), \quad i = 1, 2, \\ -\sum_{j=1}^{2} \frac{\partial \sigma_{ij}}{\partial x_j} &= f_i \quad \text{in } \Omega \quad i = 1, 2, \\ u_1 = u_2 &= 0 \quad \text{on } \Gamma_2, \\ \sum_{j=1}^{2} \sigma_{ij} n_j &= F_i \quad \text{on } \Gamma_1 \quad i = 1, 2, \end{aligned}$$

where

$$\overline{\lambda} = \frac{E\nu}{1-\nu^2}, \quad \overline{\mu} = \frac{E}{1+\nu},$$

f_i and F_i are given forces.

Give a variational formulation of this problem and formulate a corresponding finite element method. Determine the stiffness matrix in a problem with simple triangulation and piecewise linear displacements.