Assignment 3 (C)

1. (Problem 3.5 from the textbook)

   Determine the stiffness matrix corresponding to the Poisson equation

   $$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

   when Ω is a square with side 1 and we use the bilinear element with $h = 1/4$.

2. (Problem 4.8 from the textbook)

   Let V_h be a finite element space on a triangulation T_h of the domain $\Omega \subset \mathbb{R}^N$ satisfying

   $$\|u - \pi_h u\|_{L^2(\Omega)} \le Ch^{r+1}|u|_{H^{r+1}(\Omega)}.$$

   Given $u \in L^2(\Omega)$ and $u_h \in V_h$ be the L^2-projection of u onto V_h given by

   $$(u_h, v) = (u, v) \quad \text{for any } v \in V_h,$$

   where $(,)$ denotes the usual L^2 inner-product. Prove the error estimate

   $$\|u - u_h\|_{L^2(\Omega)} \le \min_{v \in V_h} \|u - v\|_{L^2(\Omega)} \le Ch^{r+1}|u|_{H^{r+1}(\Omega)}$$

   and that

   $$\|u_h\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}.$$

3. Modify your code from HW 2 to include piecewise quadratic element, i.e. your code should be able to solve the problem

   $$\begin{aligned} -(a(x)u'(x))' &= f(x), \quad 0 \le x \le 1 \\ u(0) &= u(1) = 0. \end{aligned}$$

   where $a(x)$ and $f(x)$ some functions with the choice of piecewise linear or quadratic Lagrange elements.

   Once you accomplish it, run the convergence test for both linear and quadratic elements with the exact solution $u(x) = \sin \pi x$ by successive refining the mesh by a factor of 2 (i.e. your partition should have $10 * 2^j$ intervals at the step j) and computing

   $$\log_2 \left(\frac{e_j}{e_{j+1}} \right),$$

   where

   $$e_j = \left(\int_0^1 (u(x) - u_{h,j}(x))^2 \, dx \right)^{1/2}$$

   is the error in the L^2 norm between the exact solution and the finite element solution at step j (you will need numerical quadrature to evaluate the integral).