Assignment 1 (T)

1. (Problem 1.1 from the textbook)

   Show that if w is continuous on $[0,1]$ and

   $$\int_0^1 w(x)v(x)\,dx = 0, \quad \forall v \in V,$$

   then $w(x) = 0$ for $x \in [0,1]$. Here

   $$V = \{v : \ v \in C([0,1]),\ v' \text{ is piecewise continuous and bounded on } [0,1], \text{ and } v(0) = v(1) = 0\}.$$

2. (Problem 1.3 from the textbook)

   Construct a finite-dimensional subspace V_h of V consisting of functions which are quadratic on each subinterval I_j of a partition of $I = (0,1)$. How can one choose the parameters to describe such functions? Find the corresponding basis functions. Then formulate a finite element method for the problem

   $$\begin{aligned} -u''(x) &= f(x), \quad 0 < x < 1 \\ u(0) &= u(1) = 0. \end{aligned}$$

   using the space V_h and write down the corresponding linear system of equations in case of a uniform partition.

3. (Problem 1.5 from the textbook)

   Consider the boundary value problem

   $$\begin{aligned} &\frac{d^4u}{dx^4} = f, \quad 0 < x < 1 \\ &u(0) = u'(0) = u(1) = u'(1) = 0. \end{aligned} \tag{1}$$

   Here u represents e.g. the deflection of a clamped beam subject to a transversal force with intensity f.

   (a) Show that (1) can be given the following variational formulation: Find $u \in W$ such that

   $$(u'', v'') = (f, v) \quad \forall v \in W,$$

   where $W = \{v : \ v \text{ and } v' \text{ are continuous on } [0,1], v'' \text{ is piecewise continuous and } v(0) = v'(0) = v(1) = v'(1) = 0\}$.

   (b) For $I = [a,b]$ an interval, define

   $$\begin{aligned} \mathbb{P}_3(I) = \{v : \ & v \text{ is a polynomial of degree} \le 3 \text{ on } I, \text{ i.e., } v \text{ has the form} \\ & v(x) = a_3x^3 + a_2x^2 + a_1x + a_0, \ x \in I \text{ where } a_i \in \mathbb{R}\}. \end{aligned}$$

   Show that $v \in \mathbb{P}_3$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, $v'(b)$. Find the corresponding basis functions (the basis function corresponding to the value $v(a)$ is the cubic polynomial v such that $v(a) = 1$, $v'(a) = v(b) = v'(b) = 0$, etc).

   (c) Starting from (b) construct a finite-dimensional subspace W_h of W consisting of piecewise cubic functions. Specify suitable parameters to describe the functions in W_h and determine the corresponding basis functions.

   (d) Formulate a finite element method for (1) based on the space W_h. Find the corresponding linear system of equations in the case of a uniform partition. Determine the solution in e.g. the case of two intervals and f constant. Compare with the exact solution.